Solutions for MATH2901 - 2011 paper.

Edward O'Callaghan

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2 Question 2

Let some random univariate (**r.v.**) $X \sim \mathcal{P}(\lambda)$ be *Poisson* distributed with parameter λ. Then X has density,

$$f_X(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} : x = 0, 1, 2, \ldots$$

2.1 a.)

2.1.1 i.)

Suppose $\{X_i\}_{i=1}^n$ denotes an **i.i.d** random sample and $\mathcal{L}_i(\lambda|x) = f_{X_i}(x;\lambda)$ denotes the likelihood function of the **r.v.** X with any $x \in \{X_i\}_{i=1}^n$. Then the loglikelihood $\ell(\lambda; x)$ defined on the random sample is given by,

$$\begin{aligned}
\ell(\lambda; x) &= \log_2 \mathcal{L}(\lambda|x) \\
&= log_2 \prod_{i=1}^n \mathcal{L}_i(\lambda|x) && \text{(by indepenance of the sample)} \\
&= \log_2 \prod_{i=1}^n f_{X_i}(x;\lambda) \\
&= \sum_{i=1}^n \log_2 f_{X_i}(x;\lambda) \\
&= \sum_{i=1}^n \log_2 \left\{ \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right\} \\
&= \sum_{i=1}^n \left\{ \log_2 e^{-\lambda} + \log_2 \frac{\lambda^{x_i}}{x_i!} \right\} \\
&= \sum_{i=1}^n \{ x_i \log_2 \lambda - \log_2 x_i! - \lambda \} \\
\Rightarrow \ell(\lambda; x) &= -n\lambda + \sum_{i=1}^n \{ x_i \log_2 \lambda - \log_2 x_i! \} .
\end{aligned}$$

In particular, we may ignore the constant term that does not depend on λ,

$$\Rightarrow \ell(\lambda; x) = n\bar{X}\log_2(\lambda) - n\lambda.$$

3 Question 3

3.1 a.)

Suppose that $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = \frac{1}{|X|}$.

3.1.1 i.)

We wish to discern if the function $f(x) = \frac{1}{|x|} : x \neq 0$ has monotonicity.

Clearly we have weak-monotonicity, since if $0 < x_1 \leq x_2$, we have that $f(x_1) \geq f(x_2)$ and if $x_1 \leq x_2 < 0$ we then have $f(x_1) \leq f(x_2)$.

3.1.2 ii.)

Show that $F_Y(y) = F_X(\frac{1}{y}) - F_X(-\frac{1}{y})$.

Given $f(x) = \frac{1}{|x|} : x \neq 0$. Then

$$g(y) = \begin{cases} \frac{1}{y} & x > 0, \\ -\frac{1}{y} & x < 0. \end{cases}$$

By $X = g(Y)$ the result trivally follows by composition.

3.2 b.)

Consider the two **r.v.** X and Y with bivariate density given by,

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}.$$

3.2.1 i.)

We call the random vector $(X, Y) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ density the multivate (in particular, bivariate) standard normal Gaussian. Where

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}, \boldsymbol{\mu} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and by fixing $\sigma_x = \sigma_y = 1$ and taking $\det(\Sigma) = 1 - \rho^2$.