Gluing Polygons Final Draft

Kaushik Damle, Yiwei Fu, Zachary Halberstam

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Abstract

Gluing the edges of polygons can produce a variety of surfaces. We formalize the process and discuss how to determine which surfaces are obtained by glued n-gons, as well as the frequencies with which different surfaces occur.

1 Introduction

A popular exploration at math circles is gluing the edges of a square. The process is easy enough to understand and communicate intuitively, and the results are quite interesting. Often, an image like Figure 1.1 is used to represent a gluing. The two sides with one arrow each, and the two sides with two arrows each, are to be "glued together" so that the arrows line up. Students then investigate which surfaces are obtained as the result of these gluing diagrams. As shown in Figure 1.2, the gluing diagram in Figure 1.1 yields a torus when glued. Other surfaces, including the Mobius strip and Klein bottle, can be obtained as gluings of squares, and math circles proceed in a variety of different directions, including cutting and pasting physical surfaces or attempting to play simple games such as Tic-Tac-Toe or chess on these glued squares. They rarely, however, formalize the process of gluing or investigate which surfaces are obtained for gluing n-gons for $n > 4$.

This paper investigates the general question of gluing the edges of an n-gon together. Section 2 formalizes the process of gluing an n-gon and addresses topological questions. Section 3 presents results that allow us to reduce gluings of the n-gon to gluings of polygons with fewer sides. Section 4 is devoted to non-orientable glued polygons, which are significantly less intuitive. Section 5 discusses which surfaces can be obtained as glued n-gons. Section 6 explores the combinatorial properties of orientable fully glued n-gons. Section 7 gives directions for future study. Section 8 is the appendix, which includes Java code that determines the genus of an orientable glued polygon.

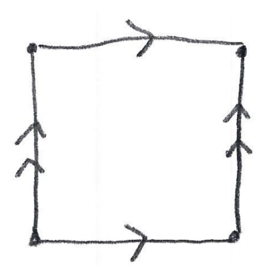


Figure 1.1: Sample gluing diagram

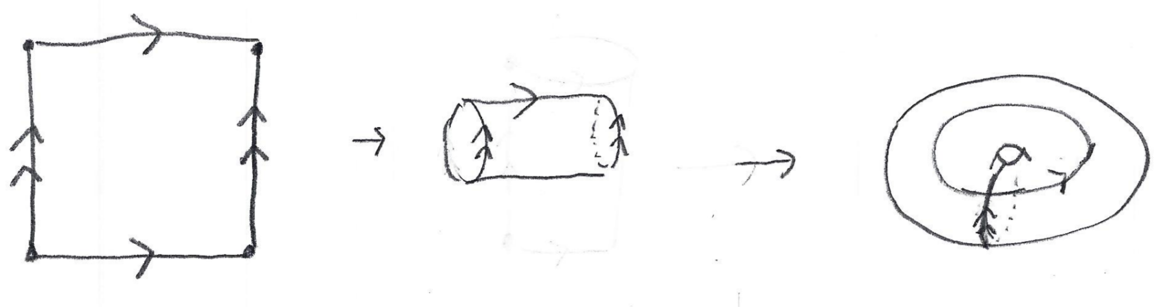


Figure 1.2: The procedure for obtaining a torus from the diagram in Figure 1.1

2 Initial Definitions and Topological Preliminaries

In this section, we formalize the process of gluing the edges of polygons. We must make careful definitions to obtain a reasonable set of glued polygons. Proceeding intuitively would lead us with several questions that might affect our results. What if we tried to glue the edges of a self-intersecting polygon? What topology do we have on a polygon, before it is glued? What does it mean, topologically, to glue? What if we only glued some points of two edges together, or glued three edges all together, or glued two edges in a non-continuous manner?

We define the n-gon, gluings, and glued polygons. We then discuss some relevant topological results. We give a simple sufficient condition for two glued polygons to be homeomorphic [1] and a necessary and sufficient condition for glued polygons to be orientable. [2]

Definition 2.1. For $n \in \mathbb{N}$, the **topological** n**-gon** P_n is the disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$, with the subspace topology inherited from $\mathbb{R}^2$. The **vertices** are $v_i := (\cos(2\pi i/n),\ \sin(2\pi i/n))$, and the **edges** are defined by $e_i := \{(\cos x,\ \sin x) : 2\pi(i-1)/n \leqslant x \leqslant 2\pi i/n\}$, for $1 \leqslant i \leqslant n$.

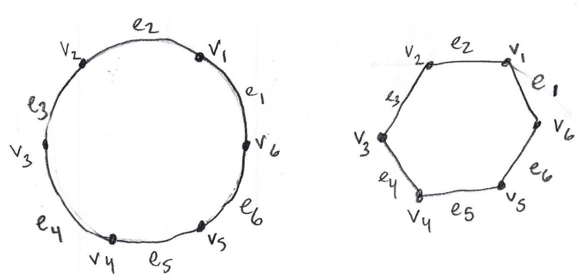


Figure 2.1: The topological hexagon and the regular hexagon

In other words, P_n is the disc, and its edges are arcs that partition the perimeter of the disc. This may seem strange at first, but this is homeomorphic to a regular polygon with n edges. We use the disc since it is

[1]Two topological spaces are **homeomorphic** when there is a continuous bijection with continuous inverse between the two topological spaces. This is an isomorphism in the category of topological spaces.

[2]A topological manifold is **orientable** when there is a continuous selection of outward normal vectors along the manifold.

easier to define for general n and to refer to arbitrary edges. Many of our pictures in this document will draw the topological n-gon as an n-gon with edges as straight lines, which is, again, homeomorphic to the topological n-gon but somewhat easier to draw and visualize. We do not give an explicit homeomorphism between the topological n-gon and the regular n-gon. Figure 2.1 shows the topological hexagon and the regular hexagon, with edges and vertices labeled.

Next, we seek to define a glued polygon. We first define a term for a set of choices of edges paired together, which will be useful to have a name for throughout the paper.

Definition 2.2. Given the topological n-gon P_n, a **grouping** of its edges is a set of ordered pairs of edges $\{(e_{i_1}, e_{j_1}), \ldots, (e_{i_m}, e_{j_m})\}$, for some $m \in \mathbb{N}$, such that every edge of P_n is included in at most one ordered pair. A **complete grouping** is a grouping that includes every edge of P_n.

With this definition, we define a glued polygon.

Definition 2.3. Given even n, the topological n-gon P_n, and a grouping of edges $\{(e_{i_1}, e_{j_1}), ..., (e_{i_m}, e_{j_m})\}$ with continuous bijections (**gluing functions**) $f_{i_k j_k} : e_{i_k} \to e_{j_k}$ with continuous inverses $f_{j_k i_k}$, the **glued polygon** is defined as $P_n/\sim$, where $\sim$ is defined so that $a \sim a$ and $a \sim b$ if there exists k such that $f_{i_k j_k}(a) = b$ or $f_{j_k i_k}(a) = b$. The polygon is called **fully glued** if the grouping of edges is a complete grouping.

We devote the vast majority of our attention to fully glued polygons. Note that a topological n-gon being fully glued implies n must be even, since it means the edges must be able to all be paired up. However, glued polygons which are not fully glued are helpful as intermediates in proving some theorems, especially in the "Reduction Approach" section.

Now, we show there is a nice condition on the gluing functions that tells us when two different sets of gluing functions yield the same surface. This condition also tells us when the glued polygon is orientable.

Let $h : [0, 2\pi] \to \partial P_n$ be given by $h(x) = (\cos x, \sin x)$, and note $h(x)$ is continuous and that the restrictions of h with domain $[2\pi(i-1)/n, 2\pi i/n]$ and range e_i are invertible.

Definition 2.4. Given a continuous bijection $f_{ij} : e_i \to e_j$, call f_{ij} **twisted** if $h^{-1} \circ f_{ij} \circ h : [2\pi(i-1)/n, 2\pi i/n] \to [2\pi(j-1)/n, 2\pi j/n]$ is increasing, and **untwisted** otherwise (i.e. it is decreasing). If f_{ij} is twisted, we also call the pair (e_i, e_j) twisted.

We now show that, given a grouping of edges, and choices for which pairs should be twisted and which pairs should be untwisted, we can find gluing functions to realize those choices. For this, it helps to define a special kind of grouping with extra information.

Definition 2.5. A **signed grouping** is a set of the form $\{(e_{i_1}, e_{j_1}, s_1), ..., (e_{i_m}, e_{j_m}, s_m)\}$, where $s_i \in \{-1, 1\}$ for all i and $\{(e_{i_1}, e_{j_1}), ..., (e_{i_m}, e_{j_m})\}$ is a grouping.

We use a sign of -1 to represent a twisted gluing function, and a sign of 1 to represent an untwisted gluing function. In what follows, we show that the signs are all that matters topologically for a gluing.

Theorem 2.1. *Given P_n and a signed grouping of edges $\{(e_{i_1}, e_{j_1}, s_1), ..., (e_{i_m}, e_{j_m}, s_m)\}$, there exists a family of continuous bijections $f_{i_k j_k} : e_{i_k} \to e_{j_k}$ such that $f_{i_k j_k}$ is twisted if and only if $s_k = -1$.*

Proof. We define $f_{i_k j_k}$. We will define $f_{i_k j_k}$ so that $h^{-1} \circ f \circ h$ is linear, with slope 1 if $f_{i_k j_k}$ is untwisted and slope -1 if $f_{i_k j_k}$ is twisted. Let $f_{i_k j_k} = h \circ g \circ h^{-1}$, where $g : [2\pi(i_k - 1)/m, 2\pi i_k/m] \to [2\pi(j_k - 1)/m, 2\pi j_k/m]$ is defined as $g(x) = x + (2\pi(j_k - i_k)/m)$ if $s_k = 1$ and $g(x) = -x + (2\pi(j_k - i_k + 1)/m)$ if $s_k = -1$. Then $h^{-1} \circ f \circ h$ is g, which is decreasing if $s_k = -1$ and increasing otherwise, so $f_{i_k j_k}$ is indeed twisted if and only

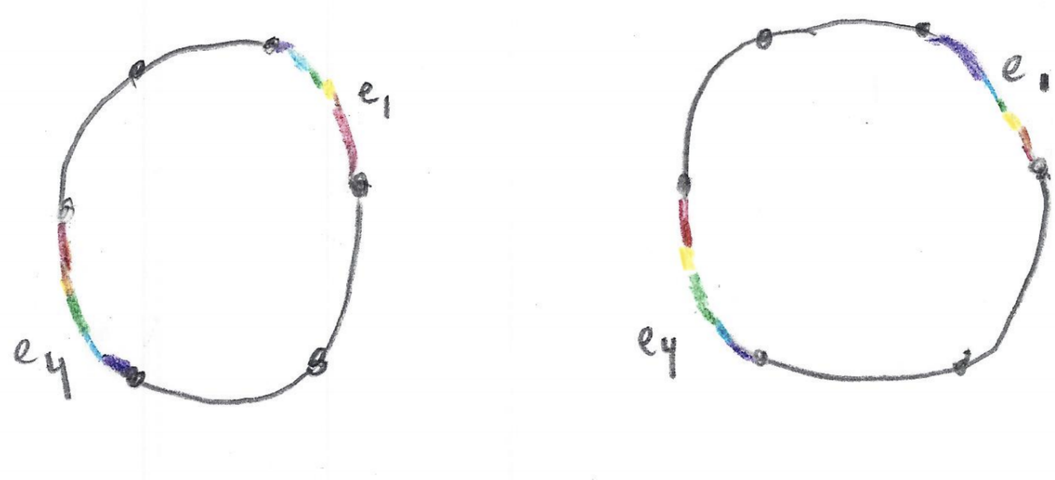


Figure 2.2: Two twisted gluing functions in the 6-gon between e_1 and e_4. Let the functions be such that the points of the same color are mapped to each other.

if $s_k = -1$.

The next theorem states that choices of gluing functions do not matter, as long as the gluing functions are all either twisted or untwisted for the same pairs of edges.

Theorem 2.2. *Given P_n and a grouping of edges $\{(e_{i_1}, e_{j_1}), ..., (e_{i_m}, e_{j_m})\}$, and two families of continuous bijections $f_{i_k j_k}, g_{i_k j_k} : e_{i_k} \to e_{j_k}$, such that $f_{i_k j_k}$ is twisted if and only if $g_{i_k j_k}$ is, the glued polygons resulting from the $f_{i_k j_k}$'s and the $g_{i_k j_k}$'s are homeomorphic.*

Proof. We show that replacing $f_{i_1 j_1}$ with $g_{i_1 j_1}$ does not change the surface topologically. This will show the desired result, since it shows we can replace the $f_{i_k j_k}$ with $g_{i_k j_k}$ one by one without affecting the homeomorphism class of the surface. Essentially, the continuous compressing and stretching involved by switching gluing functions results in a homeomorphism. Figure 2.2 gives a colorful visualization for two different gluing functions in the same directions. In this figure, points of the same color are mapped to each other (treating the colors used in the picture as a perfect spectrum). Continuously stretching and compressing portions of e_1 in the first hexagon to get the gluing function as depicted in the second hexagon results in a homeomorphism. The rest of the proof attempts to state this a bit more formally.

Let $S_1 = P_6/\sim_1$, where $a \sim_1 b$ if $a = b$ or $f_{i_k j_k}(a) = b$ or $f_{i_k j_k}(b) = a$ for some k. Then, let $S_2 = P_6/\sim_2$, where $\sim_2$ is defined by replacing $f_{i_1 j_1}$ with $g_{i_1 j_1}$. Define the function $\phi : S_1 \to S_2$ to be the identity away from $\{[p]_1 : p \in e_{i_1}\}$, where $[p]_1$ denotes the equivalence class of $p \in P_6$ in S_1. On $\{[p]_1 : p \in e_{i_1}\}$, define $\phi([p]) = [g_{i_1 j_1}^{-1}(f_{i_1 j_1}(p))]_2$. ($[q]_2$ is the equivalence class of $q \in P_6$ in S_2.) We have $[p]_1 = [f_{i_1 j_1}(p)]_1$, and $[g_{i_1 j_1}^{-1}(f_{i_1 j_1}(p))]_2 = [g_{i_1 j_1}(g_{i_1 j_1}^{-1}(f_{i_1 j_1}(p)))]_2 = [f_{i_1 j_1}(p)]_2$ so this gives $\phi([f_{i_1 j_1}(p)]_1) = [f_{i_1 j_1}(p)]_2$. Now, we extend ϕ so that ϕ is continuous and bijective, ϕ is the identity outside a specific neighborhood of $\{[p]_1 : p \in e_{i_1}\}$, and ϕ is as we've defined it on $\{[p]_1 : p \in e_{i_1}\}$. (This can be made completely rigorous by using a partition of unity.)

Finally, we show that twistedness is the only condition needed to determine when a glued polygon is

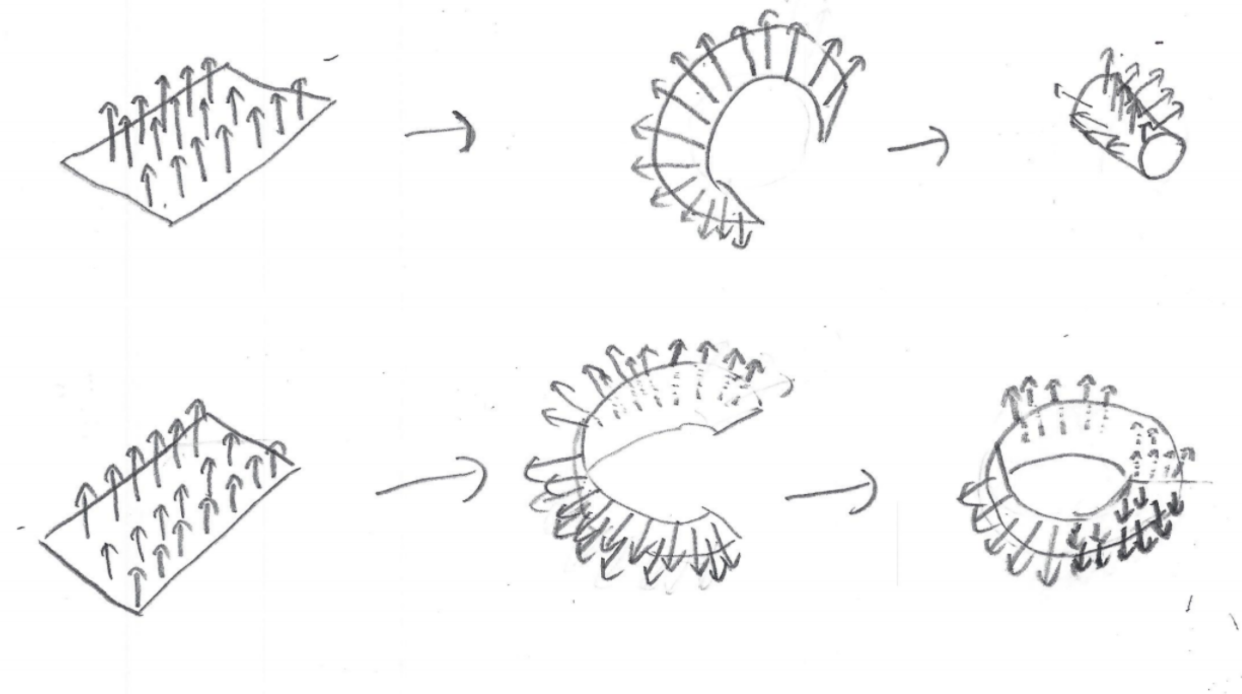


Figure 2.3: For the untwisted gluing, on top, the outward normal vectors agree over the seam, and the surface is orientable. For the twisted gluing, on the bottom, the outward normal vectors disagree across the seam, and the surface is not orientable.

orientable.

Theorem 2.3. *The glued polygon $P_n/\sim$ is orientable if and only if all of the gluing functions are untwisted.*

Proof. Outward normal vectors are well-defined away from the places of gluing ("seams"). At the seams, if any gluing function is twisted, passing a normal vector over that seam will reverse its direction. If every gluing function is untwisted, normal vectors may be passed over every seam without changing their directions, so outward normal vectors exist over the entire glued polygon Figure 2.3 shows how untwisted gluing functions respect orientation and twisted gluing functions do not.

Together, Theorem 2.1, Theorem 2.2, and Theorem 2.3 imply the following extremely important corollary.

Corollary 2.3.1. *Given P_n for n even, each pairing of the edges $\{(e_{i_1}, e_{j_1}), ..., (e_{i_{n/2}}, e_{j_{n/2}})\}$ determines exactly one orientable glued polygon, up to homeomorphism.*

Proof. By Theorem 2.3, a gluing with this edge pairing is orientable if and only if all the gluing functions are untwisted. By Theorem 2.1, there is a gluing with the edge pairing with all of the gluing functions untwisted. Finally, by Theorem 2.2, the surfaces obtained for any set of gluing functions such that all of them are untwisted are all homeomorphic.

Finally, we show an important property of fully glued polygons, which makes them the most compelling to study.

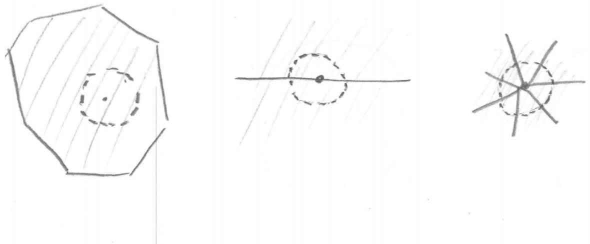


Figure 2.4: Charts of the image of an interior point of P_n, a point on an edge in P_n, and a vertex in P_n in a fully glued n-gon.

Figure 2.5: Some ways a point in a glued polygon can fail to have a neighborhood homeomorphic to a neighborhood in $\mathbb{R}^2$. None of these failures are possible in a fully glued n-gon.

Theorem 2.4. *Let $P_n/\sim$ be a fully glued polygon, for n even. Then, $P_n/\sim$ is a compact 2-manifold without boundary.[3].*

Proof. Any quotient space of a compact space is compact, and P_n is compact, so the fully glued polygon must be compact. To see that $P_n/\sim$ is a manifold, we show there are charts around $[p]$, where $[p] \in P_n/\sim$ is an equivalence. If p is in the interior of P_n, we can simply use a neighborhood of p in P_n as a chart. If p is on an edge e_i, the chart is obtained by stitching together neighborhoods on both sides of the glued edge, which correspond to neighborhoods of p and $f(p)$, where f is a gluing function on e_i. (f exists because the gluing function is fully glued.) If p is a vertex, the chart is obtained by stitching together neighborhoods of all of the vertices q with $[q] = [p]$, which has to produce a single neighborhood since the polygon is fully glued. The charts for each type of point are showed in Figure 2.4. Figure 2.5 shows some situations in which this result might fail for a polygon that is not fully glued.

[3]A ***k*-manifold without boundary** is a topological space where each point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^k$.

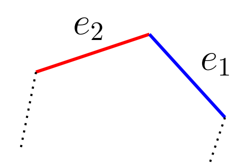


Figure 3.1: An adjacent pair.

3 Reduction Approach

In this section, we are going to focus on the orientable gluing. Since there is no twisted edges, we can simply identify a pair by the edges, say (e_1, e_2), without paying attention to the twistedness. Starting from a square, we already know that gluing adjacent edges of a square gives a sphere, while gluing opposite edges gives a torus. It turns out that these two scenarios are indeed the building block of all orientable gluings.

Firstly, we need to a way to describe our pairs in a grouping:

Definition 3.1. A pair in a grouping is called an adjacent pair if it pairs two neighboring edges.

Definition 3.2. A set of two pairs in a grouping is called a set of cross-pairs if two pairs are "diagonal" to each other. That is, if we connect between the midpoints of edges in each pair, two lines will intersect with each other.

Remark. See Figure 3.1 and Figure 3.2. Notice that since we are using topological n-gon, we can assume the polygon is convex.

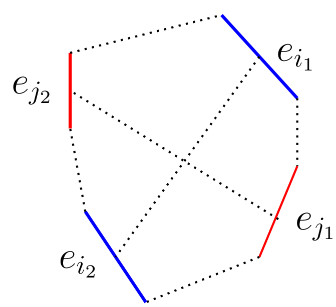


Figure 3.2: A set of cross-pairs.

Definition 3.3. Consider two topological surface S_1, S_2. The **connected sum** of surface S_1, S_2 is obtained by removing one disk from each surface, and gluing the surfaces after the cut along the border.

Remark. See Figure 3.3. For orientable surfaces, it is easy to visualize that two surfaces are just "attached" to each other.

Lemma 3.1. *The connected sum of any topological surface S and a sphere is S itself.*

Proof. We need to find a homeomorphism f between S and the connected sum S', which is not difficult to obtain. Firstly, we take a identity map id restricted to the $S \backslash D$ where D is the removed closed disk. In addition, after removing a disk, the sphere is homeomorphic to a disk. So we can choose any homeomorphism g between disks such that the map is continuous on $\partial S \backslash D$. Then we have

$$f = \mathrm{id}\,|_{S \backslash D} + g : S \to S'$$

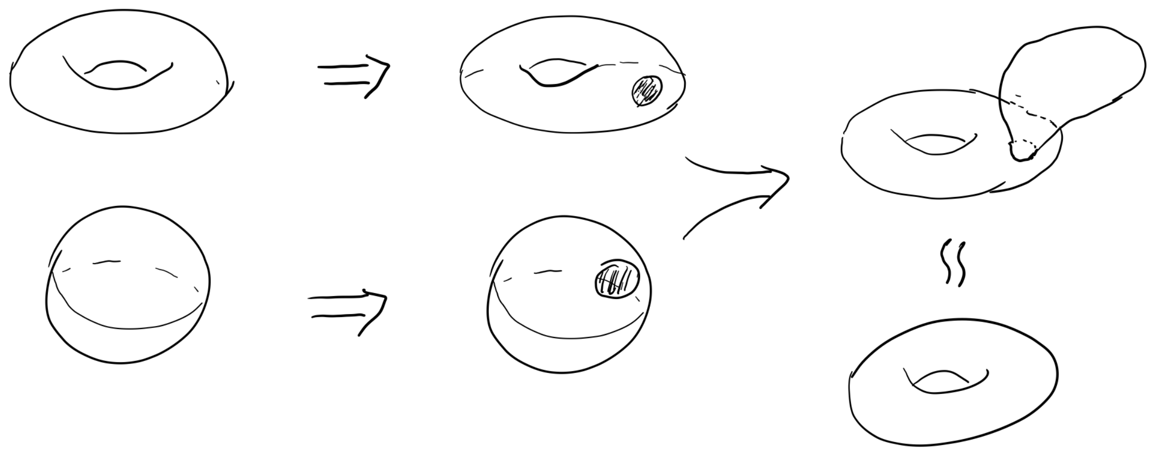


Figure 3.3: Connceted sum of a torus and a sphere.

a homeomorphism.

As Figure 3.3 shows, the connected sum of a torus and a sphere is a torus, since we can easily "shrink" the cap of the sphere into a disk.

Theorem 3.1 (Decomposition of Orientable Grouping)**.** *Any orientable surface obtained by a grouping G on an n-gon is homeomorphic to the connected sum of some number of tori.*

In order to prove this, we will use reduction approach to convert any orientable n-gon grouping into connected sum of a torus/sphere and the surface determined by a grouping on n'-gon, where $n' < n$.

Firstly, we look at the somewhat trivial case.

Lemma 3.2 (Adjacent pair reduces grouping trivially)**.** *An adjacent pair of edges gives a surface determined by sub-grouping P', where P' is obtained by removing the adjacent pair.*

Proof. This is best illustrated by a picture(Figure 3.4), that we can just simply glue the two adjacent edges together. Since the remaining edges forms a closed curve, we obtain a grouping of $(n-2)$-gon from the original n-gon.

Lemma 3.3 (Cross-pairs also reduce grouping)**.** *A set of cross-pairs in a n-gon gives a connected sum of a torus and the surface determined by P', where P' is a grouping on $(n-4)$-gon.*

Proof. Without loss of generality, we label the vertices of these edges

$$v_i, v_{i+1}, v_j, v_{j+1}, v_m, v_{m+1}, v_n, v_{n+1}$$

as in Figure 3.5.

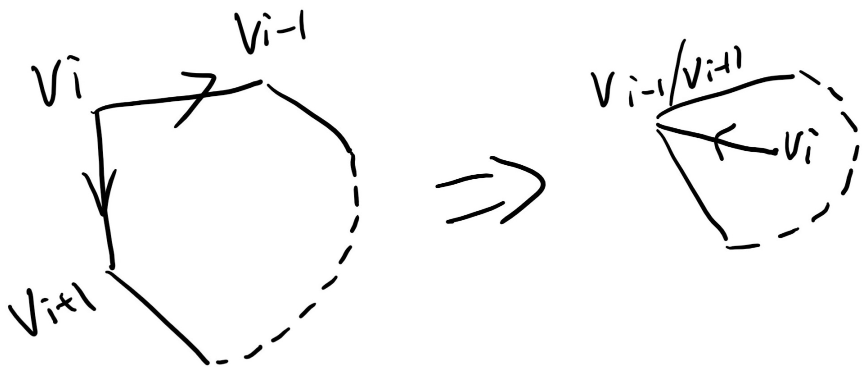


Figure 3.4: A simple reduction case.

In the first step we glue one pair of edges. Then we can see that $(v_{i+1}, \ldots, v_n, v_{n+1}, \ldots)$ forms a polygon that is the boundary of the a disk removed from a surface. Similarly, $(v_{j+1}, \ldots, v_m, v_{m+1}, \ldots)$ is also such polygon.

When we glue the second pair together, v_n coincides with v_{m+1}, v_{n+1} coincides with v_m. So the two polygons in step 1 become one polygon, which is the boundary of both disks removed from an gluing of cross-pairs and the gluing of polygon $(v_{i+1}, \ldots, v_n, \ldots, v_i, \ldots, v_m, \ldots, v_{i+1})$. Notice that since we goes from v_n to v_i since v_n coincides with v_{m+1}. Similarly we can go from v_n back to v_{i+1}, as in Figure 3.6. So by definition we have a connected sum of a torus and a surface obtained by a new grouping G' on a $(n-4)$-gon.

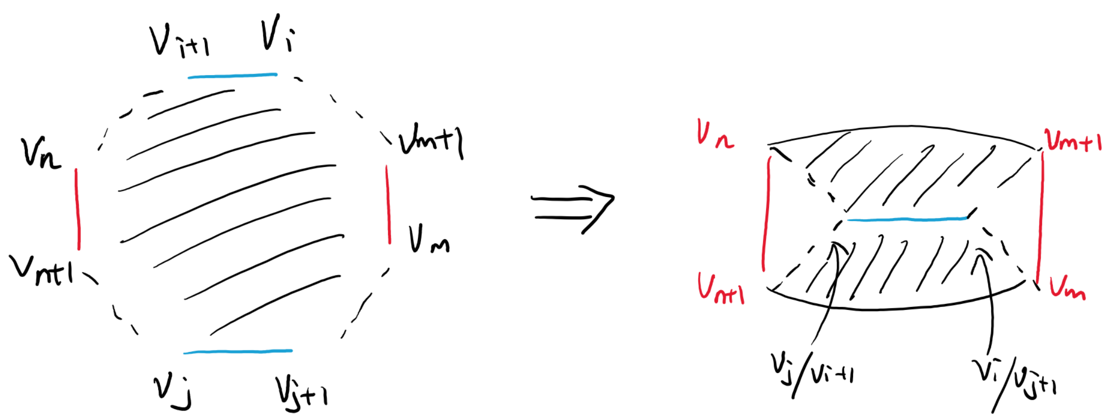


Figure 3.5: First step of gluing.

Remark. Notice that the grouping G' on the $(n-4)$-gon is not a simple case where we can just “ignore” the chosen cross-pairs and the connected the remaining segments in its original order. However, it is not hard to find how the new grouping G' is obtained. Suppose the cross-pairs partitions the polygon into four segments A, B, C, D respectively, the resulting grouping should be determined by $ADCB$, see Figure 3.7. In each section, the order of edges is preserved.

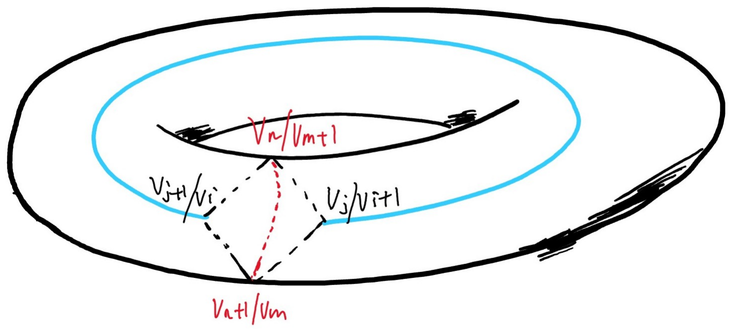


Figure 3.6: Second step of gluing.

Lemma 3.2 and Lemma 3.3 helps us to identify part of the resulting surface from a grouping G and leave the unsolved grouping G' to be performed on a polygon with lower number of sides. Hence, we call this a reduction. An adjacent pair and a set of cross-pairs are called reduction group. Now we generalize our reduction approach to all orientable edge groupings so that we can more easily analyze more complicated gluings of polygon with much larger number of edges.

Theorem 3.2 (Existence of Reduction Group)**.** *For a given orientable grouping, there exists at least an adjacent pair or a set of cross-pairs.*

Proof. Suppose we have no set of cross pairs. We consider an edge pair (e_m, e_n) with $m < n$. If there is no edge between e_m, e_n, we've discovered an adjacent pair. Otherwise, suppose an edge e_i where $m < i < n$. The edge should belong to a pair (e_i, e_j). Since there is no cross pairs, we can't have $j < m$ or $j > n$. As a result, we have $m < j < n$. Then we repeat the same process for e_i, e_j if there are edges between them. Since there is finite edges between e_m, e_n. We will have eventually stop the process. Then the pair $(e_{i'}, e_{j'})$ we obtain when we stop will be an adjacent pair.

In order to keep reduce the grouping, we need the following lemma:

Lemma 3.4. *Every grouping obtained by performing reduction operations of an orientable grouping is orientable.*

Proof. The lemma is easy to prove since an orientable grouping requires no twisted gluing, so any subset of the grouping has no twisted gluing. The reduction operation only changes the relative position these edges, so grouping obtained has no twisted gluing as well.

So we can prove Theorem 3.1 now.

Proof. Following Theorem 3.2, Lemma 3.2, and Lemma 3.3. For any orientable, we can find a reduction group, obtain another orientable, then continue to find a reduction group. Since the number of edges in the grouping is strictly decreasing, we will end up with just a square or a di-gon pairing, which is either a sphere or a torus. The reduction operations only add tori to the surface, so every surface obtained by orientable grouping is a connected sum of some tori.

We show a relatively simple yet non-trivial example of our reduction operation:

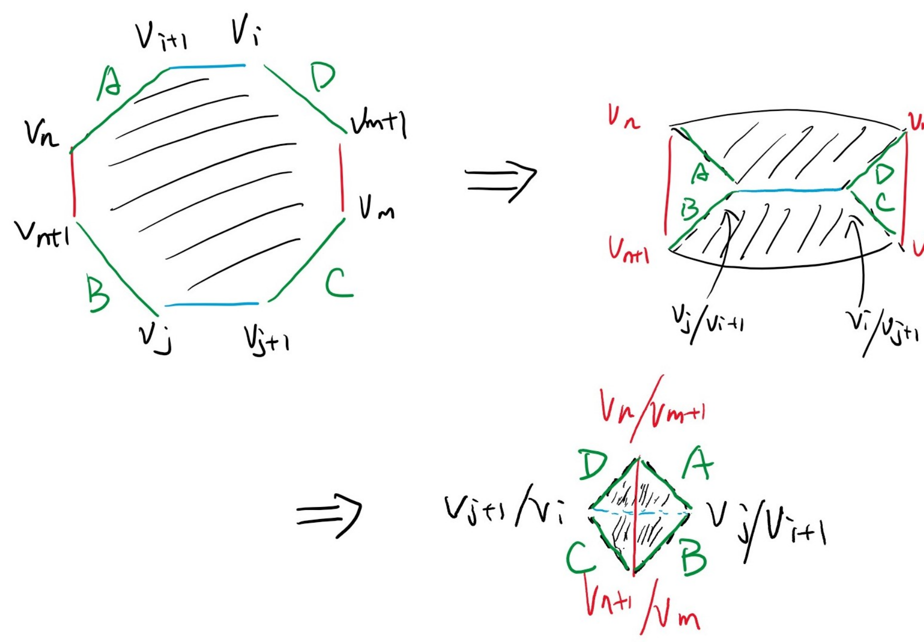


Figure 3.7: The change is in the order of the segments.

Example 3.1 (Octagon Gluing)**.** When the octagon is wrapped around, we can see that the removed disk on the torus has a boundary that is equivalent to a topological square (4−gon). So we can obtain the gluing surface by the connect sum of two square gluings.

4 Non-Orientable Gluings

4.1 Digons

Note: throughout this section, we will use $e_i \circledequal e_j$ to denote an untwisted gluing and $e_i \otimes e_j$ to denote a twisted gluing of edges e_i and e_j. The digon, or the two sided polygon is the simplest case we shall consider. If we were to use the traditional notion of a polygon, such an object would be impossible. Our definition of a topological n-gon however, permits such an object. It may be useful, in some instances to visualize the digon as shown in Figure 4.1.

It is not too difficult to see that the only two gluings possible for this polygon are: $\{e_1 \circledequal e_2\}$ and $\{e_1 \otimes e_2\}$. We shall refer to $\{e_1 \circledequal e_2\}$ as the untwisted digon gluing and $\{e_1 \otimes e_2\}$ as the twisted digon gluing.

4.1.1 Untwisted Digon Gluings

We know that $\{e_1 \circledequal e_2\}$ is simply S^2. In fact, an untwisted digon gluing can be viewed as the intermediate step in the gluing of adjacent edges of a square. The resulting object has genus 0. This gluing is demonstrated below in Figure 4.2.

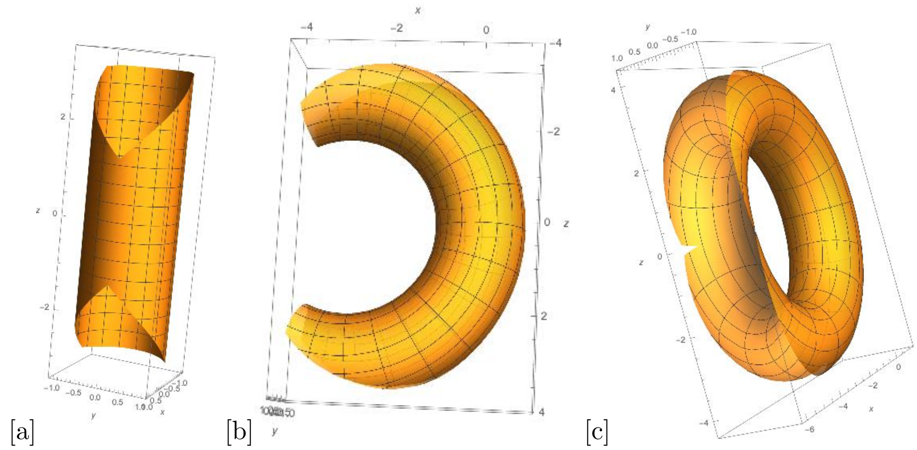


Figure 3.8: Wrapping for top/bottom edge pair results in square embedding in a torus

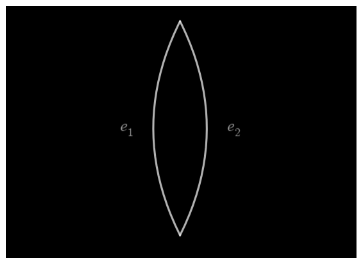


Figure 4.1: A digon

4.1.2 Twisted Digon Gluings

Twisted digon gluings are a far more interesting gluing, and leads to a non-orientable 2-manifold called the real projective plane, which we will denote by $P(\mathbb{R}^2)$. The projective plane can be defined as the quotient space obtained by the equivalence relation that identifies diametrically opposite points on a disk.

Theorem 4.1. *Any twisted digon forms* $\mathbb{R}P^2$

Proof. Let P be a topological 2-gon. Consider the twisted gluing function $f : e_1 \to e_2$ that maps a point on e_1 to the point diametrically opposite to it on e_2. The resulting polygon satisfies the definition of the projective plane stated above. The result holds even if we had chosen a different twisted gluing function as a consequence of Theorem 2.2

4.2 Visualizing Connected Sums

Although we do not have the framework to describe the gluings of the edge of polygon to the edge of a digon, the idea proves to be quite useful for determining the genus of non-orientable surfaces through connected sums. The purpose of this section is to describe the notion of a connected sum as described in Definition 3.3. Recall that to perform a connected sum of two 2-manifolds, we cut out a disk from each manifold and the glue the resulting manifolds-with-boundary to each other along the newly created boundary. We will now demonstrate how the connected sum operation can be represented using polygons. Let P_n be an n-gon with a gluing G defined on it. We cut a disk in the middle of the polygon. The result is not a topological n-go, but if we move the disk to the boundary of the n-gon, then we can "pop" it out of one of the vertices, leaving us with

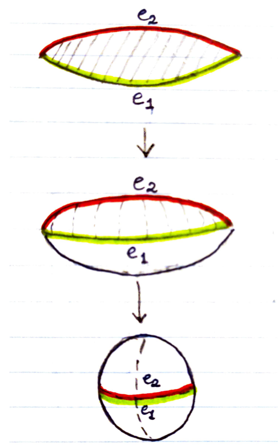


Figure 4.2: untwisted digons form spheres

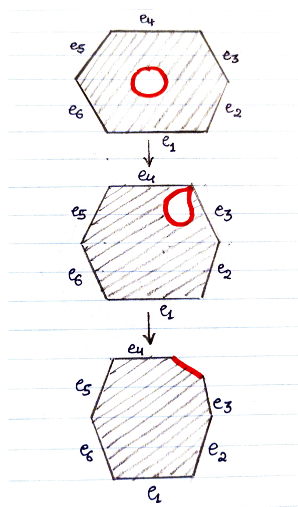


Figure 4.3: Cutting a disk out of a polygon

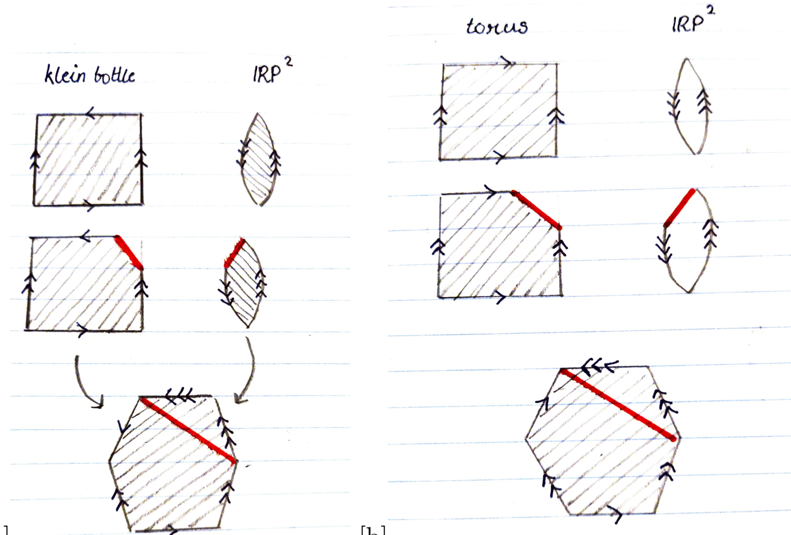
one edge e' that is not part of the gluing G. When all the other sides have been glued together, e' will be the boundary that we will glue the other polygon's boundary to. This procedure is illustrated in Figure 4.3.

4.3 Pitfalls

While this approach provides us with a nice visual representation of a connected sum of two manifolds, it is unfortunately rather difficult to ascertain the genus of a glued polygon, or to show that two gluings are yield the same result. This is demonstrated in Figure 4.4 where we have the connected sum of a torus and $\mathbb{R}P^2$ and the connected sum of a klein bottle and $\mathbb{R}P^2$. There is no good way to determine, using this method, whether the surfaces will be equivalent to other. We hope to resolve this issue by extending our reduction rules on orientable grouping onto non-orientable ones.

4.4 Reduction Approach, Revisited

We identify that the signature of a non-orientable grouping is a twisted pair. So we take a look into that.



[a] [b]

Figure 4.4: Connected sum of a klein bottle and $\mathbb{R}P^2$, torus and $\mathbb{R}P^2$

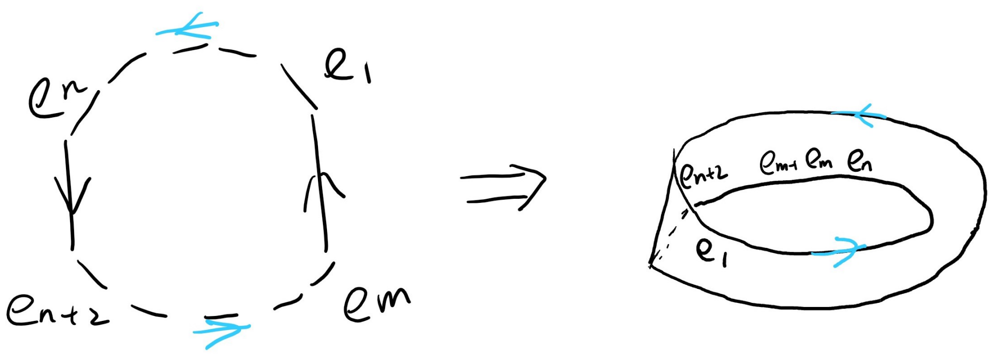


Figure 4.5: Twisted pair generates a Möbius strip

Theorem 4.2 (Twisted Pair Reduces Grouping)**.** *A twisted pair on a grouping G on a n-gon gives a connected sum of a projective plane $\mathbb{R}P^2$ and the surface determined by P', where P' is a grouping on $(n-2)$-gon.*

Proof. As Figure 4.5 has shown, the twisted pair generates a Möbius strip, which has single boundary. This would come to our aid, since the boundary is exactly a $(n-2)$-gon. In addition, the polygon is the boundary of a disk removed from a projective plane. We conclude that the surface can be expressed as a connected sum of a projective plane $\mathbb{R}P^2$ and the surface determined by P', where P' is a grouping on $(n-2)$-gon.

Remark. Notice that here unlike orientable groupings, the "twistedness" and relative edge order in a certain segment are revsered. This would result in uncertainty that whether the grouping G' would be orientable. However, by Theorem 2.3, the existence of twisted pair signals the orientablity. Thus, if G' is non-orientable, we can still find twisted pairs and continue reduction operations. If G' is orientable, we then apply orientable reduction operations.

5 Attainable Surfaces

The results from the reduction approach let us make very powerful statements about which glued polygons can be attained as gluings of the topological n-gon. In this section, we show those results, including an extremely powerful theorem that classifies all complete gluings of the topological n-gon.

Theorem 5.1. *If a surface S is homeomorphic to a glued k-gon, and $n > k$ is even, there is a glued n-gon homeomorphic to S.*

Proof. We induct on $n - k$. Let $\{(e_{i_1}, e_{j_1}, s_1), ..., (e_{i_m}, e_{j_m}, s_m)\}$ be a signed grouping that gives a gluing homeomorphic to S. We know from the results in Section 2 that a signed grouping determines a surface, since different choices of gluing functions do not affect the topological object obtained. For the base case $n - k = 0$, we may just use this signed grouping.

For the inductive step, suppose there is a gluing of the $(n-2)$-gon homeomorphic to S, given by $\{(e_{i_1}, e_{j_1}, s_1), ..., (e_{i_{m'}}, e_{j_{m'}}, s_{m'})\}$. Then the gluing of the n-gon given by $\{(e_{i_1}, e_{j_1}, s_1), ..., (e_{i_{m'}}, e_{j_{m'}}, s_{m'}), (e_{n-1}, e_n, 1)\}$ is homeomorphic to S, by reducing the final two edges. By induction, the claim is proved.

Next, we have a theorem about the maximal genus that can be obtained for an orientable gluing.

Theorem 5.2. *For n even, the maximal genus obtainable in an orientable fully glued n-gon is $\lfloor \frac{n}{4} \rfloor$, which is always obtainable.*

Proof. From Section 3, we know the genus for an orientable fully glued n-gon. Let the associated grouping be $\{(e_{i_1}, e_{j_1}), \ldots, (e_{i_{n/2}}, e_{j_{n/2}})\}$. To attain the maximal genus, choose this grouping to be such that we can reduce as many cross-pairs as possible. Every time we reduce cross-pairs, we add 1 to the genus, and, when there are no more cross-pairs to be reduced, we are left with a sphere with k handles, where k is the number of cross-pairs we reduced. We lose four edges with every reduction, so we can perform at most $\lfloor \frac{n}{4} \rfloor$ reductions, in which case the genus is $\lfloor \frac{n}{4} \rfloor$.

We can guarantee that we can perform $\lfloor \frac{n}{4} \rfloor$ reductions by working backwards: start with one cross-pair (and possibly two additional edges, if n is not divisible by 4) and repeatedly add four edges to get an edge grouping that produces the previous edge grouping after reducing a cross-pair.

These two theorems give us the following important corollary.

Corollary 5.2.1. *The genuses of orientable surfaces that can be obtained as fully glued n-gons are exactly the integers between 0 and $\lfloor \frac{n}{4} \rfloor$, inclusive.*

Proof. By Theorem 5.2, genus $\lfloor \frac{n}{4} \rfloor$ can be obtained as a gluing of the n-gon, and for all $m < n$ even, $\lfloor \frac{m}{4} \rfloor$ can be obtained as a gluing of the m-gon. Therefore, by Theorem 5.1, $\lfloor \frac{m}{4} \rfloor$ can be attained as a gluing of the n-gon. Since every integer between 0 and $\lfloor \frac{n}{4} \rfloor$ can be written as $\lfloor \frac{m}{4} \rfloor$ for some $0 \leqslant m \leqslant n$, all of these genuses are attainable. No lower genus can be attained, since negative genus is not possible, and no higher genus can be attained, since we showed $\lfloor \frac{n}{4} \rfloor$ is the maximum.

We now show a theorem that completely categorizes complete gluings of topological n-gons.

Theorem 5.3 (Classification of fully glued n-gons)**.** *Let S be a fully glued topological n-gon, for n even. Then S is homeomorphic to a sphere, a connected sum of k tori for some $k > 0$, or a connected sum of k projective planes for some $k > 0$.*

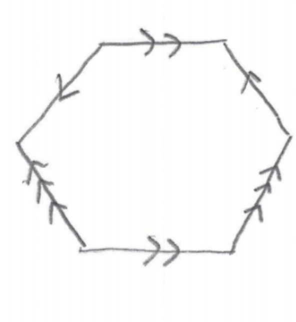


Figure 5.1: This glued polygon is homeomorphic to both $T^2\#\mathbb{R}P^2$ and $\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2$.

Remark. This theorem closely resembles the famous Classification of Closed Surfaces Theorem, which states that every connected closed surface is homeomorphic to a sphere, a connected sum of tori, or a connected sum of projective planes. This is not an accident. In a standard proof of the Classification of Closed Surfaces Theorem, one first shows every closed surface can be triangulated (informally, that it can be cut into triangles). This result is equivalent to the claim that every closed surface can be expressed as a glued polygon. (A glued polygon representation is obtained from a triangulation of a surface by making cuts that divide the triangles until we obtain a polygon.)

To prove Theorem 5.3, we first show a lemma, which is interesting in its own right.

Lemma 5.1. *The connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes.*

Note that this lemma is not in the language of gluing polygons. It is purely a topological result. Moreover, it is not particularly easy to show topologically. Our proof uses glued polygons, and shows that glued polygons may be a powerful tool in simplifying some topological arguments.

Proof. To prove this lemma, we show both the connected sum of a torus and a projective plane and the connected sum of three projective planes can be obtained by using the reduction approach on a specific glued polygon. Consider a glued hexagon, that glues e_1 to e_3 twisted, e_2 to e_5 untwisted, and e_4 to e_6 untwisted. (This glued hexagon is depicted in Figure 5.1.)

By reducing the cross-pairs $(e_2, e_5), (e_4, e_6)$, we obtain that this surface, when glued, is homeomorphic to the connected sum of a torus and a twisted digon, which is the connected sum of a torus and $\mathbb{R}P^2$. This reduction is shown in Figure 5.2.

We may also first reduce the twisted pair (e_1, e_3). When we do this, this twists the edge e_2 relative to the other edges, resulting in a gluing of the 4-gon with opposite edges glued together and one edge pair twisted. This reduction is shown in Figure 5.3. Then, reducing the remaining pair of twisted edges yields that the gluing of the 4-gon reduces to the connected sum of two copies of $\mathbb{R}P^2$. So, this reduction produces that Figure 5.1 is homeomorphic to the connected sum of three copies of $\mathbb{R}P^2$.

Therefore, the connected sums $\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2$ and $T^2\#\mathbb{R}P^2$ must yield homeomorphic surfaces, since they are obtained by the same glued polygon.

We can now prove Theorem 5.3, that a fully glued topological n-gon, for n even, is either a sphere, the connected sum of tori, or the connected sum of projective planes.

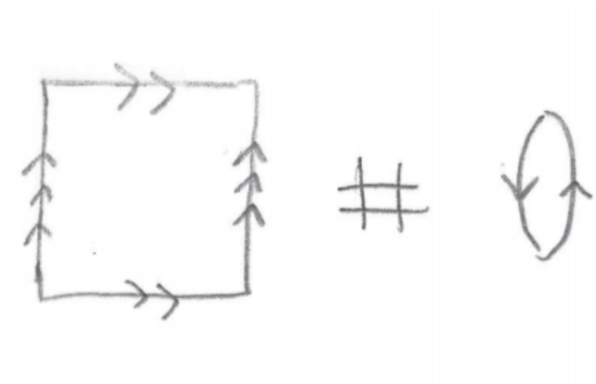


Figure 5.2: Reducing cross pairs in Figure 5.1 produces the connected sum of a torus and $\mathbb{R}P^2$.

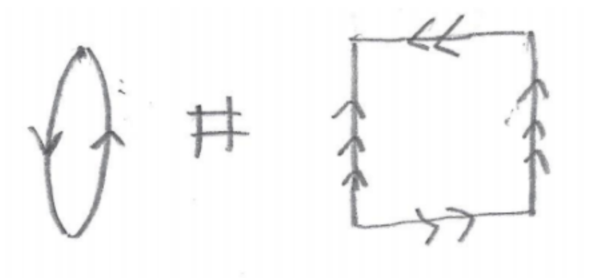


Figure 5.3: Reducing twisted pairs in Figure 5.1 produces the connected sum of $\mathbb{R}P^2$ and a Klein bottle.

Proof. We proceed by induction. The result is true for digons, for which there are two possibilities. A twisted digon results in $\mathbb{R}P^2$, as we know from the results of the "non-orientable gluings" section. An untwisted digon is a sphere.

Now, suppose the result is true for n-gons, and moreover for m-gons for all $m \leqslant n$. Then, we show the result is true for $(n+2)$-gons. Take a signed grouping G of the edges of the topological $(n+2)$-gon. Then, if G produces an orientable gluing, we know from the previous section there are either cross-pairs or an adjacent pair in G. If there is an adjacent pair in G, reduce them to obtain the connected sum of a sphere and a fully glued n-gon, which is homeomorphic to a fully glued n-gon, which by inductive hypothesis is of the desired form.

Otherwise, reduce two cross-pairs in G to obtain the connected sum of a torus with a fully glued $(n-2)$-gon. If the fully glued $(n-2)$-gon is a sphere, then G produces a gluing homeomorphic to the torus, and if the fully glued $(n-2)$-gon is a connected sum of k tori, then G produces a gluing homeomorphic to the connected sum of $k+1$ tori. If the fully glued $(n-2)$-gon is the sum of $k \geqslant 1$ projective planes, then we have that G produces a gluing homeomorphic to $T^2\#\mathbb{R}P^2\#\cdots\#\mathbb{R}P^2 = (T^2\#\mathbb{R}P^2)\#\mathbb{R}P^2\#\cdots\#\mathbb{R}P^2 = (\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2)\#\mathbb{R}P^2\#\cdots\#\mathbb{R}P^2$, the connected sum of $k+2$ projective planes, by applying Lemma 5.1. (The connected sum is associative.) However, this final case will not occur, since then G would produce a nonorientable surface.

If G produces a nonorientable surface, we know by Theorem 2.3 there is a twisted pair of edges in G. Then, reduce this twisted pair of edges to get the connected sum of $\mathbb{R}P^2$ and a fully glued n-gon. If the fully glued n-gon is a sphere, then we obtain the connected sum of a sphere and $\mathbb{R}P^2$, which is just $\mathbb{R}P^2$. If the fully glued n-gon is a sum of k copies of $\mathbb{R}P^2$, we obtain the connected sum of $k+1$ copies of $\mathbb{R}P^2$.

Finally, if the fully glued n-gon is the sum of k tori, we repeatedly apply Lemma 5.1. The surface defined by G is $(\mathbb{R}P^2 \# T^2) \# \cdots \# T^2 = (\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2) \# T^2 \# \cdots \# T^2 = \mathbb{R}P^2 \# \mathbb{R}P^2 \# (\mathbb{R}P^2 \# T^2) \# T^2 \# \cdots \# T^2 = \mathbb{R}P^2 \# \mathbb{R}P^2 \# (\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2) \# T^2 \# \cdots \# T^2 = \cdots$. Continuing in this manner, we ultimately obtain that the surface defined by G is $2k$ copies of $\mathbb{R}P^2$.

By induction, the claim is proved.

6 Probabilities and Averages

We can start to discuss the probabilities of various genuses occurring for an orientable gluing of the n-gon, as well as average genuses in an orientable glued n-gon. Since the genus of an orientable fully glued polygon is determined by the complete grouping of the edges, we can refer to orientable fully glued polygons by complete groupings. In what follows, a "random glued polygon" is an orientable fully glued polygon determined by a random complete grouping of the edges, and anything about "averages" refers to an average across all complete groupings of edges. We consider two complete groupings the same here if and only if the same pairs of edges are included in the ordered pairs (though the order of the pairs, and the order within each pair, can be permuted). Notably, two complete groupings that are different, but can be obtained from each other by rotation or reflection, are considered different here.

First, we count some objects we may determine exactly.

Theorem 6.1. *There are $\frac{n!}{2^{n/2}(\frac{n}{2})!}$ complete groupings of the edges of P_n, for n even.*

Proof. We convert permutations to complete groupings. Given a permutation of the edges of P_n, convert it to a complete grouping by pairing the first two edges, the third and fourth edge, and so on in the permutation. This overcounts by a factor of $2^{n/2}(\frac{n}{2})!$, since the pairs of edges can be freely shuffled with each other, and the edges within each pair can switch their order. So, the final count is $\frac{n!}{2^{n/2}(\frac{n}{2})!}$.

Now, we count how many complete groupings result in surfaces of genus 0.

Theorem 6.2. *There are $C_{n/2}$ complete groupings of the edges of P_n that result in genus-0 surfaces when the corresponding orientable fully glued n-gon is formed, where $C_{n/2}$ is the $n/2$th Catalan number.*[4]

Proof. Among other things, the $n/2$th Catalan number is the number of ways to pair the n vertices of a convex n-gon so that the lines joining the vertices do not intersect. Therefore, it also counts the number of ways to pair the n edges of the topological n-gon together so that the lines joining the edges do not intersect. This is the same as the condition that the n edges do not form any cross-pairs, so it is the condition for the pairing to result in a genus-0 surface.

The previous two results give us an immediate corollary.

Corollary 6.2.1. *The probability that a random orientable fully glued n-gon has genus 0 is $\frac{2^{n/2}C_{n/2}(\frac{n}{2})!}{n!}$.*

Note that the number of complete groupings resulting in genus-1 surfaces is not simply $\binom{n}{4}C_{(n-4)/2}$. A complete grouping corresponding to a genus-1 surface has a cross-pair, but no two disjoint cross-pairs. However, such a grouping may have many different cross-pairs, so we cannot count these groupings by simply

[4]The Catalan Numbers enumerate many different combinatorial objects, including the number of ways to properly arrange n open and n closed parantheses so that at no point in the sequence are there more closed than open parantheses. The nth Catalan number is $\frac{1}{n+1}\binom{2n}{n}$.

selecting four elements to form a cross-pair in $\binom{n}{4}$ ways and then pairing the rest of the elements to not intersect in $C_{(n-4)/2}$ ways. We do have a conjecture for the number of pairings of a topological n-gon that result in a genus-1 surface, but the question gets even more complicated for higher genus, and experimental results become a better option than trying to find a formula.

Conjecture 6.1. *The number of complete groupings of edges of P_n that results in a genus-1 surface is*

$$C_{\frac{n-4}{2}} \sum_{i=1}^{n-3} i(n-2-i) = \frac{1}{6} \frac{(n-1)!}{(\frac{n}{2}-2)!(\frac{n}{2}-1)!}$$

Intuition: suppose we have P_n, we fix one edge e_1 to be one of the edges in the cross pair, then we consider positions of the remaining edges of the cross pairs in the polygon.

Suppose e_1 is paired with e_j. To make cross pairs, e_i and e_j cannot be adjacent, so there are $n-3$ choices for e_j. So e_1 and e_j partition the remaining edges of the polygon into two parts of connected edges, with one part having k edges, the other part having $n-2-k$ edges. We notice that k takes value between 1 and $n-3$. Then we take randomly one edge from each part to fix a set of cross pairs. So in total, there are $\sum_{i=1}^{n-3} i(n-2-i)$ choices of cross pairs.

After we fix a set of cross pairs, we can perform reduction on the polygon grouping, which gives us a connected sum between a torus and a surface resulted from a grouping on P_{n-4}. Since we are looking for genus-1 surfaces, we should have the grouping on P_{n-4} rendering us a sphere. So for each choice of cross pairs, we have $C_{(n-4)/2}$ ways to match the rest of edges. Each such grouping would be mapped one-to-one with a grouping on P_n. Combined with the cross pairs we have in the first place, we have a complete grouping on P_n that results in a genus-1 surface

Interestingly, it turns out that fixing e_1 to be one of the edges in the cross pairs actually counts all the cases. Fixing e_1 in this way gives us the count in the conjecture. Table 1 suggests this count is accurate. It shows the probability of a genus-1 surface being obtained, as predicted by the conjecture, as well as experimentally-determined probabilities for some small values of n, showing that the count given by the conjecture is very likely to be accurate, at least for these n.

T_n: number of groupings (per conjecture) of the n-gon that result in a torus.
P_n: number of ways to pair up the edges of an n-gon.
Sim.: program simulated result of $\frac{T_n}{P_n}$ (1,000,000 trials.)

n	4	6	8	10	12	14
T_n	1	10	70	420	2310	12012
P_n	3	15	105	945	10395	135135
$\frac{T_n}{P_n}$	0.333	0.667	0.667	0.444	0.222	0.0889
Sim.	0.3326	0.666	0.667	0.445	0.223	0.0883

Table 1: Predicted frequency vs. simulation frequency

Now, we discuss experimental results for two very natural questions: what the probability a given glued n-gon has the maximum possible genus is, and what the average genus for a glued n-gon is. Figure 6.1 shows experimentally determined probabilities that a random $2k$-gon has maximal genus for $1 \leqslant k \leqslant 50$. We used a program (with code replicated in the appendix) to test one million random glued polygons for each value of k and to tabulate the experimental probability based on the number of these polygons with maximal genus. The graph clearly shows that the probability the genus is maximal is greater for odd k than even k of comparative

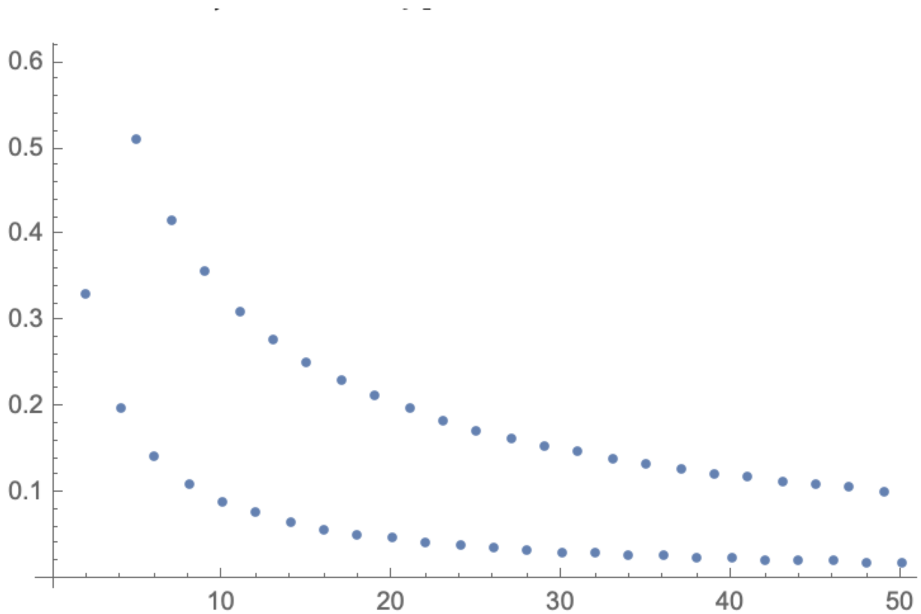


Figure 6.1: The probability that a randomly selected orientable fully glued $2k$-gon has maximal genus, $1 \leqslant k \leqslant 50$

magnitude, which is what we might predict, since the maximal genus of an orientable fully-glued $2k$-gon is greater relative to k when 4 divides $2k$.

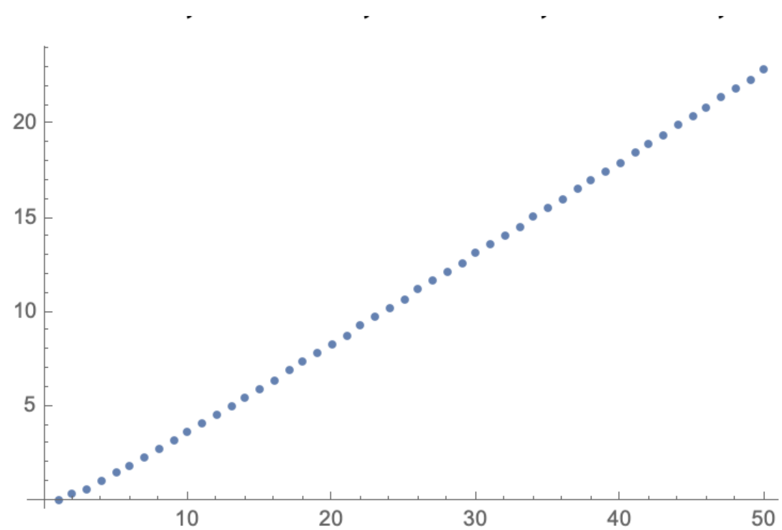


Figure 6.2: The average genus of an orientable fully glued $2k$-gon, for $1 \leqslant k \leqslant 50$

Figure 6.2 shows the experimentally determined average genus of an orientable fully glued $2k$-gon, for $1 \leqslant k \leqslant 50$. The average is estimated by taking the average genus across one million random orientable fully glued $2k$-gons, for each k. This figure shows the increase to be strongly linear, especially after the first few values.

7 Future Directions

7.1 Higher and Lower Dimensions

One natural question to ask is whether we can say anything about gluings of connected k-dimensional polytopes, for $k \neq 2$. These are gluings of the $(k-1)$-dimensional faces of the polytopes, with bijections between the faces and equivalence relations defined based on these bijections as we did for polygons.

For $k < 2$, the question is completely trivial: For $k = 0$, the only connected polytope is the single point. Gluings might or might not be well-defined under different definitions, but if they are well-defined, the only glued object we could obtain would be the single point itself. For $k = 1$, the only connected polytope is a single line segment. Gluing its $(k-1)$-dimensional faces, which are its endpoints, together gives the 1-sphere.

For $k \geqslant 3$, the question is quite complicated. One major obstacle here that we did not encounter for the polygons is the many different bijections there are between two-dimensional or higher-dimensional figures. We will not be able to summarize up the important information about gluing functions with a condition as simple as the twisted/untwisted condition. Also, manifolds of dimension > 2 are significantly more complicated than manifolds of dimension $\leqslant 2$. One could investigate whether any general conclusions about gluings of higher-dimensional figures can be drawn.

7.2 More edges glued together

In our definition for a glued polygon, we required every edge to be present exactly at most once in an edge grouping. If we allowed edges to be present more than once, we would obtain more classes of glued polygons, in which three or more edges might all be glued on top of each other. One could try to categorize the surfaces that are obtained by this broader class of gluings.

8 Appendix

8.1 Website

You can calculate the genus, given a grouping, or generate a number of random trials of n-gon on the website `https://victorfu14.github.io/parts/index.html`

8.2 Code

We have made a Java program that determines the genus of an orientable completely glued polygon. What follows is the polygon class from that program. This class represents an orientable completely glued polygon by an array signifying which edges are glued together. The genus method returns the genus of the glued polygon. The reduceCrossPairs method is a helper method for the genus method.

```java
import java.util.*;

class polygon
{
    int numSides; // number of sides of the polygon. precondition: numSides is even
    int[] sidePairings;
```

```java
/** array of integers which represents which sides are glued together
 * preconditions: sidePairings has length numSides, and contains the elements
 * 1, 1, 2, 2, ..., numSides/2, numSides/2 in some order.
 * two edges are glued together if the elements at the corresponding
 * indices in the sidePairings array are the same
 * for instance, the array [1, 2, 1, 2] represents the gluing of the 4-gon where the first and
 * third edges, and second and fourth edges, are glued together. This is a torus.
 * from our work in the paper, the sidePairings array describes a complete grouping of the
 * n-gon, and therefore describes all of the relevant information for an orientable gluing.
 */

public polygon (int n, int[] s) // constructor for polygon class
{
    numSides = n;
    sidePairings = s;
}

/** constructs a random polygon if only number of edges are given
 * uses the Fisher-Yates shuffle to generate a
 * polygon with n sides and random sidePairings array
 */
public polygon(int n)
{
    randomPolygon(n);
}

public void randomPolygon(int n) {
    numSides = n;
    int[] sideArray = new int[n];
    ArrayList<Integer> sidesLeft = new ArrayList<>();
    for (int i = n/2; i>0; i--) //this loop fills sidesLeft with two copies of 1, ..., n/2
    {
        sidesLeft.add(i);
        sidesLeft.add(i);
    }
    for (int i = 0; i<n; i++)
    // this loop fills sideArray with random entries of sidesLeft
    // that haven't already been chosen
    {
        int index = (int) (Math.random()*(n-i));
        sideArray[i] = sidesLeft.get(index);
        sidesLeft.remove(index);
    }
    sidePairings = sideArray;
```

```java
}

// returns the genus of the glued polygon determined by sidePairings
public int genus()
{
    int total = 0;
    while (this.reduceCrossPairs()) total++;
    return total;
}

/** returns false if there are no cross-pairs present.
 * if there are cross-pairs present, this method returns true and
 * reduces the edges of the cross-pairs
 */
private boolean reduceCrossPairs()
{
    int currIdx = 0;
    while (currIdx < numSides) {
        int currPair = currIdx + 1;
        // find the pair
        while (currPair < numSides) {
            if (sidePairings[currIdx] == sidePairings[currPair]) break;
            currPair++;
        }

        // currPair going over last index means we already searched this pair before
        if (currPair == numSides) {
            currIdx += 1;
            continue;
        }

        /** get the "unpaired" edge "within" (currIdx, currPair)
         ** pairTwo is -1 if all the edges between (currIdx, currPair)
         ** are paired within this range
         */
        int pairTwo = searchForSingle(currIdx, currPair);

        if (pairTwo == -1) {
            currIdx += 1;
            // we guarantee that no cross-pairs involving currIdx
            // continue searching...
        } else {
            // cross-pairs found
            int indexTwo = findPair(pairTwo);
```

```java
            // update the grouping to the new ones obtained
            updatePolygon(currIdx, currPair, pairTwo, indexTwo);
            return true;
        }
    }
    return false;
}

// should have start < finish < numSides
private int searchForSingle(int start, int finish) {
    boolean[] tempData = new boolean[finish - start + 1];
    for (int i = 0; i < tempData.length; i++) {
        if (!tempData[i]) {
            boolean hasPair = false;
            for (int j = i + 1; j < tempData.length; j++) {
                if (sidePairings[j + start] == sidePairings[i + start]) {
                    // set value to true so it won't be considered again
                    tempData[j] = true; tempData[i] = true;
                    hasPair = true;
                    break;
                }
            }
            if (!hasPair) return i + start;
        }
    }
    return -1;
}

private int findPair(int idx) {
    for (int i = 0; i < numSides; i++) {
        if (sidePairings[i] == sidePairings[idx]
        && i != idx) return i;
    }
    return -1;
}

/** we should have guarantee that idx00 < idx10 < idx01 < idx11
 * the polygon is segmented as idx00, A, idx10, B, idx01, C, idx11, D
 * we should return an array of ADCB
 */
private void updatePolygon(int idx00, int idx01, int idx10, int idx11) {
    int[] arr = new int[this.numSides - 4];

    // this loop fills in the first segment, A
```

```
        int idx1 = 0; int idx2 = idx00 + 1;
        while (idx2 < idx10) {
            arr[idx1] = this.sidePairings[idx2];
            idx2++; idx1++;
        }

        // second segment, D
        idx2 = idx11 + 1;
        while (idx2 < numSides + idx00) {
            if (idx2 >= numSides) {
                arr[idx1] = this.sidePairings[idx2 - numSides];
            } else arr[idx1] = this.sidePairings[idx2];
            idx2++; idx1++;
        }

        // third segment, C
        idx2 = idx01 + 1;
        while (idx2 < idx11) {
            arr[idx1] = this.sidePairings[idx2];
            idx2++; idx1++;
        }

        // fourth segment, B
        idx2 = idx10 + 1;
        while (idx2 < idx01) {
            arr[idx1] = this.sidePairings[idx2];
            idx2++; idx1++;
        }

        numSides -= 4;
        sidePairings = arr;
    };
}
```