# Antenna Fundamentals

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## SECTION 2.1: ELECTROMAGNETIC FUNDAMENTALS

Maxwell's equations in phasor form (time-harmonic fields or permanent response):

$$\nabla \times \vec{E} = -jw\vec{B} \tag{2-7}$$

$$\nabla \times \vec{H} = jw\vec{D} + \vec{J}_T \tag{2-8}$$

where the total current density $\vec{J}_T = \sigma\vec{E} + \vec{J}$ [ampere/meter$^2$ or A/m$^2$] (i.e., conduction current density plus the impressed/source current). $\vec{E}$ is the electric field intensity [volt/meter or V/m] and $\vec{D}$ is the electric flux density [coulomb/meter$^2$ or C/m$^2$]:

$$\vec{D} = \varepsilon\vec{E} \qquad \text{with } \varepsilon = \varepsilon_r\varepsilon_0, \text{ with } \varepsilon_0 = 8.854\times10^{-12} \tag{*}$$

where ε is the electric permittivity [faraday/meter or F/m]: **"Indicates/measures how friendly to the electric field is the medium/dielectric: A higher ε indicates less fringe fields, or more concentrated fields/higher density".**

$\vec{H}$ is the magnetic field intensity [ampere/meter or A/m] and $\vec{B}$ is the magnetic flux density [tesla or weber/meter$^2$ or T, Wb/m$^2$]:

$$\vec{B} = \mu\vec{H} \qquad \text{with } \mu = \mu_r\mu_0, \text{ with } \mu_0 = 4\pi 10^{-7} \qquad (**)$$

where μ is the magnetic permeability [henry/meter or H/m]: **"Indicates/measures how friendly to the magnetic field is the medium: A higher μ indicates less fringe fields, or more concentrated fields/higher density".**

The remaining Maxwell's equations are:

$$\nabla \bullet \vec{D} = \rho_T \tag{2-9}$$

"The divergence –origin or source– of $\vec{D}$ is the electric charge: The electric field start and end in electric charges". The total volumetric electric charge density is given in [coulomb/meter$^3$ or C/m$^3$].

$$\nabla \bullet \vec{B} = 0 \tag{2-10}$$

"The divergence -origin or source- of $\vec{B}$ is zero (i.e., there is no magnetic charge): There are no discrete points or quantities determining the start and end of magnetic fields".

$$\nabla \bullet \vec{J}_T = -jw\rho_T \qquad (2\text{-}11)$$

“The source of the current density is the ratio of variation –i.e., derivative– of the charge density”.

Time derivatives have been replaced by a *jw* factor in the equations (*w* is the angular frequency in radians, $w = 2\pi f$, also related to the wave number β through $w = \frac{\beta}{\sqrt{\mu\varepsilon}} = \frac{2\pi}{\lambda}$; where the wavelength can be determined from $v = \lambda f$.

Using (*) and (**), and **considering** that for antenna problems we normally solve for the **fields in air surrounding the antenna** where **σ = 0**, Maxwell’s equations (2-7) through (2-11) become:

$$\nabla \times \vec{E} = -jw\mu\vec{H} \tag{2-16}$$

$$\nabla \times \vec{H} = jw\varepsilon\vec{E} + \vec{J} \tag{2-17}$$

$$\nabla \bullet \vec{E} = \frac{\rho}{\varepsilon} \tag{2-18}$$

$$\nabla \bullet \vec{H} = 0 \tag{2-19}$$

$$\nabla \bullet \vec{J} = -jw\rho \tag{2-20}$$

It is convenient sometimes to introduce a **fictitious** magnetic current density $\vec{M}$, which is useful as an equivalent source replacing more complex electric field distributions (see Chapter 7 for more details). Then (2-16) becomes:

$$\nabla \times \vec{E} = -jw\mu\vec{H} - \vec{M} \tag{2-21}$$

## SECTION 2.2: SOLUTION OF MAXWELL'S EQUATIONS FOR RADIATION PROBLEMS ("σ = 0")

Equation (2-10), or (2-19), combined with the vector property (C-9), $\nabla \bullet \left(\nabla \times \vec{A}\right) = 0$, yields

$$\vec{H} = \frac{1}{\mu} \nabla \times \vec{A} \qquad (2\text{-}36)$$

where $\vec{A}$ is the *magnetic vector potential*. Substituting (2-36) into (2-16) gives

$$\nabla \times \vec{E} = -jw\mu \frac{1}{\mu} \nabla \times \vec{A} \quad \Rightarrow \quad \nabla \times \vec{E} + jw\nabla \times \vec{A} = 0$$

$$\nabla \times (\vec{E} + jw\vec{A}) = 0 \qquad (2\text{-}37)$$

Using the vector property (C-10), $\nabla \times (-\nabla\Phi) = 0$ — "the negative sign has been introduced for convenience", we can conclude from (2-37):

$$\vec{E} + jw\vec{A} = -\nabla\Phi \quad \text{or} \quad \vec{E} = -jw\vec{A} - \nabla\Phi \tag{2-39}$$

where $\Phi$ is the *electric scalar potential*. "Note from (2-36) and (2-39) that both the electric and magnetic fields are now written in terms of the scalar and vector potentials. Once the potential functions are determined, we can easily compute the fields".

We now use (2-36) with the second Maxwell's equation (2-17) leading to

$$\frac{1}{\mu}\nabla\times\nabla\times\vec{A}=jw\varepsilon\vec{E}+\vec{J} \tag{2-40}$$

Using the vector property (C-17), $\nabla\times\nabla\times\vec{A}=\nabla\left(\nabla\bullet\vec{A}\right)-\nabla^2\vec{A}$, the following is obtained with (2-40)

$$\nabla\left(\nabla\bullet\vec{A}\right)-\nabla^2\vec{A}=jw\mu\varepsilon\vec{E}+\mu\vec{J}$$

Replacing the electric field by the result found in (2-39) leads to

$$\nabla\left(\nabla\bullet\vec{A}\right)-\nabla^2\vec{A}=jw\mu\varepsilon\left(-jw\vec{A}-\nabla\Phi\right)+\mu\vec{J} \tag{2-42}$$

which can be rearranged as

$$\nabla\left(\nabla\bullet\vec{A}\right)-\nabla^2\vec{A}=w^2\mu\varepsilon\vec{A}-jw\mu\varepsilon\nabla\Phi+\mu\vec{J}$$

$$\nabla^2 \vec{A} + w^2 \mu\varepsilon \vec{A} - \nabla\left(jw\mu\varepsilon\Phi + \nabla \bullet \vec{A}\right) = -\mu\vec{J} \qquad (2\text{-}43)$$

“There still remains, however, one degree of freedom: The divergence of magnetic vector potential has not yet been specified (any vector is completely defined by **both** its curl and divergence). In order to simplify (2-43), it is convenient to specify the divergence such that the term between parenthesis cancels out:”

$$\nabla \bullet \vec{A} = -jw\mu\varepsilon\Phi \qquad (2\text{-}44)$$

which is known as the *Lorentz condition*. With (2-44), (2-43) reduces to the *vector wave equation*

$$\nabla^2 \vec{A} + w^2 \mu \varepsilon \vec{A} = -\mu \vec{J} \tag{2-45}$$

"Equation (2-45) is a differential equation that can be solved once the currents are known (source). **The magnetic field can then be computed from (2-36)**, and "combining (2-39) and (2-44)"

$$\vec{E} = -jw\vec{A} - j\frac{\nabla\left(\nabla \bullet \vec{A}\right)}{w\mu\varepsilon} \tag{2-46}$$

**"Normally it is easier to find the electric field directly from (2-17) using the magnetic field computed from (2-36)"**

“The differential equation in (2-45) can be split into three identical, independent scalar differential equations by decomposing the vector $\vec{A}$ into its three rectangular components $A_x$, $A_y$, and $A_z$. The same must be done with $\vec{J}$ and it is possible only in rectangular coordinates. Once any of the equations is solved, the other two solutions easily follows.”

Within this context, let's pick any of the three scalar equations and represent the $\vec{A}$ component by ψ. In addition, let's consider that the source current is a point unitary source at the origin (impulse). Thus

$$\nabla^2\psi + \beta^2\psi = -\delta(x)\delta(y)\delta(z) \qquad (2\text{-}55)$$

where $\beta^2 = w^2\mu\varepsilon$ or $\beta = w\sqrt{\mu\varepsilon}$ (2-50). "Outside the origin, the right-hand side of (2-55) is zero, which leads to a well-known second order differential equation called *Helmholtz equation* (2-57)." The physically meaningful solution represents a spherical wave propagating away from the point source and given by the *free-space Green's function*

Mathematically represents a spherical wave propagating from the origin to infinity:

$$\psi = \frac{e^{-j\beta r}}{4\pi r} \quad \Rightarrow \quad \frac{e^{-j\beta R}}{4\pi R} \text{ if the point source is not at the origin}$$

where "R" is the distance to the observation point "P". (2-59)

“If instead of an unitary point source, we have a discrete array of non-unitary point sources, $\psi$ can be determined by solving a weighted sum of Green’s functions (2-59), where the weights are the complex excitations of each point source (i.e., amplitude and phase of the point source); See the class notes of the introductory lecture. When the discrete array of point sources approach a continuous current distribution, the weighted sum becomes an integral. If we consider the generic, three-dimensional vector situation, $\psi$ returns to $\vec{A}$ (before decomposition) and

$$\vec{A} = \iiint_{v'} \mu \vec{J} \frac{e^{-j\beta R}}{4\pi R} dv' \qquad (2\text{-}60)$$

**"which is the "fundamental" equation in our course, as commented in the first class. Note that (2-60) is generic, no approximations or restrictions have been made (see Sec. 2.4.3: far-field ≈).** As already pointed out before, once (2-60) is solved, the magnetic field can be computed from (2-36) and then the electric field can be determined from Maxwell's equation (2-17)."