Chapter 8

8.1 Provided that the channel bandwidth is not smaller than the reciprocal of the transmitted pulse duration T, the received pulse is recognizable at the channel output. With $T = 1\,\mu s$, a small enough value for the channel bandwidth is $1/T = 10^6$ Hz $= 1$ MHz

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8.2 The integrator output is

$$y(t) = \int_{t-T}^{t} x(\tau)\, d\tau \qquad (1)$$

Let $x(t) \overset{FT}{\longleftrightarrow} X(j\omega)$. We may therefore reformulate the expression for $y(t)$ as

$$y(t) = \int_{t-T}^{t} \underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega\tau}\, d\omega\right)}_{x(\tau)} d\tau$$

Interchanging the order of integration:

$$y(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(j\omega) \left(\int_{t-T}^{t} e^{j\omega\tau}\, d\tau \right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot \frac{T}{2\pi} \operatorname{sinc}\left(\frac{\omega T}{2\pi}\right) e^{j\omega\left(t - \frac{T}{2}\right)} d\omega \qquad (2)$$

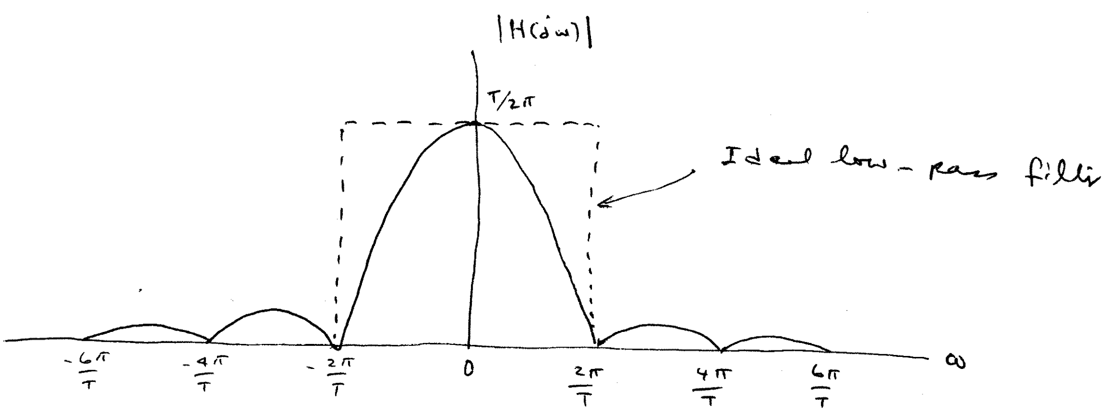
(a) Invoking the formula for the inverse Fourier transform, we immediately deduce from Eq. (2) that the Fourier transform of the integrator output $y(t)$ is given by

$$Y(j\omega) = \frac{T}{2\pi} X(j\omega) \operatorname{sinc}\left(\frac{\omega T}{2\pi}\right) e^{-j\omega T/2} \qquad (3)$$

Examining this formula, we also readily see that $y(t)$ can be equivalently obtained by passing the input signal $x(t)$ through a filter whose frequency response is defined by

$$H(j\omega) = \frac{T}{2\pi} \operatorname{sinc}\left(\frac{\omega T}{2\pi}\right) e^{-j\omega T/2}$$

The magnitude response of the filter is depicted below:



(b) The figure at the bottom of the previous page also includes the magnitude response of an "approximating" ideal low-pass filter. This latter filter has a cutoff frequency $\omega_0 = 2\pi/T$ and passband gain of $T/2\pi$. Moreover, the filter has a constant delay of $T/2$. The response of this ideal filter to a step function applied at time $t=0$ is given by

$$y'(t) = \frac{T}{\pi} \int_{-\infty}^{\frac{2\pi}{T}(t-\frac{T}{2})} \frac{\sin\lambda}{\lambda}\, d\lambda$$

At $t = T$, we therefore have

$$y'(T) = \frac{T}{\pi} \int_{-\infty}^{\pi} \frac{\sin\lambda}{\lambda}\, d\lambda$$

$$= \frac{T}{\pi}\left(\int_{-\infty}^{0} \frac{\sin\lambda}{\lambda}\, d\lambda + \int_{0}^{\pi} \frac{\sin\lambda}{\lambda}\, d\lambda \right)$$

$$= \frac{T}{\pi}\left(Si(\infty) + Si(\pi) \right)$$

$$= \frac{T}{\pi}\left(\frac{\pi}{2} + 1.85 \right)$$

$$= 1.09\, T$$

From Eq. (1) we find that the ideal integrator output at time $t = T$ in response to the step function $x(t) = u(t)$ is given by

$$y(T) = \int_0^T u(\tau)\, d\tau$$

$$= T$$

It follows therefore that the output of the "approximating" ideal low-pass filter exceeds the output of the ideal integrator by 9%. It is noteworthy that this overshoot is indeed a manifestation of the Gibbs' phenomenon.

8.3 For a low-pass filter of the Butterworth type, the squared magnitude response is defined by

$$|H(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_c)^{2N}} \qquad (1)$$

At the edge of the pass band, $\omega = \omega_p$, we have by definition

$$|H(j\omega_s)| = 1 - \epsilon$$

We may therefore write

$$(1-\epsilon)^2 = \frac{1}{1+(\omega_p/\omega_c)^{2N}} \qquad (2)$$

Define

$$\epsilon_0 = 2\epsilon - \epsilon^2$$

Then solving Eq. (2) for ω_p:

$$\omega_p = \left(\frac{\epsilon_0}{1-\epsilon_0}\right)\omega_c$$

Next, by definition, at the edge of the stop band, $\omega = \omega_s$, we have

$$|H(j\omega_s)| = \delta$$

Hence

$$\delta^2 = \frac{1}{1+(\omega_s/\omega_c)^{2N}} \qquad (3)$$

Define

$$\delta_0 = \delta^2$$

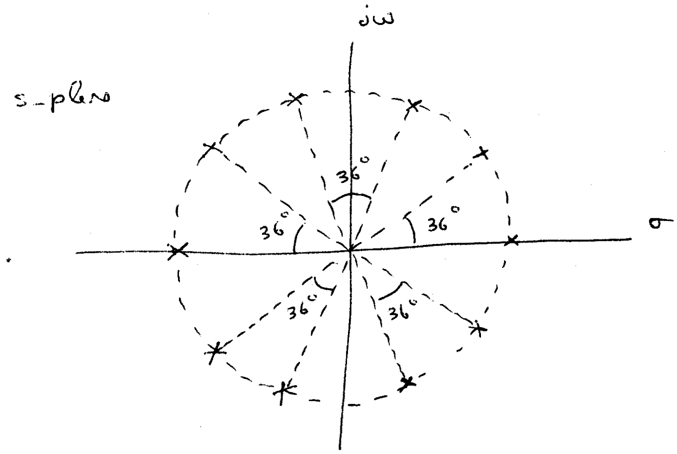
Hence solving Eq. (3) for ω_s:

$$\omega_s = \left(\frac{1-\delta_0}{\delta_0}\right)\omega_c$$

8.4 We start with the relation

$$\left| H(j\omega) \right|^2_{j\omega = s} = H(s)\, H(-s)$$

For a Butterworth low-pass filter of order 5, the 10 poles of $H(s)\,H(-s)$ are uniformly distributed around the unit circle in the s-plane as follows:



Let $D(s)\,D(-s)$ denote the denominator polynomial of $H(s)\,H(-s)$. Hence,

$$\begin{aligned} D(s)\,D(-s) = {} & (s+1)(s + \cos 144^\circ + j \sin 144^\circ)(s + \cos 144^\circ - j \sin 144^\circ) \\ & (s + \cos 108^\circ + j \sin 108^\circ)(s + \cos 108^\circ - j \sin 108^\circ) \\ & (s-1)(s - \cos 36^\circ + j \sin 36^\circ)(s - \cos 36^\circ - j \sin 36^\circ) \\ & (s - \cos 72^\circ + j \sin 72^\circ)(s - \cos 72^\circ - j \sin 72^\circ) \end{aligned}$$

Identifying the zeros of $D(s)\,D(-s)$ in the left-half plane with $D(s)$ and those in the right-half plane with $D(-s)$, we may express $D(s)$ as

$$D(s) = (s+1)(s+\cos 144^\circ + j\sin 144^\circ)(s+\cos 144^\circ - j\sin 144^\circ)$$
$$(s+\cos 108^\circ + j\sin 108^\circ)(s+\cos 108^\circ - j\sin 108^\circ)$$
$$= s^5 + 3.2361 s^4 + 5.2361 s^3 + 5.2361 s^2 + 3.2361 s + 1$$

Hence,

$$H(s) = \frac{1}{D(s)}$$

$$= \frac{1}{s^5 + 3.2361 s^4 + 5.2361 s^3 + 5.2361 s^2 + 3.2361 s + 1}$$

8.5

(a) For filter order N that is odd, the transfer function $H(s)$ of the filter must have a real pole in the left-half plane. Let this pole be $s = -a$ where $a > 0$. We may then write

$$H(s) = \frac{1}{(s+a)\,D'(s)}$$

where $D'(s)$ is the remainder of the denominator polynomial. For a Butterworth low-pass filter of cutoff frequency ω_c, all the poles of $H(s)$

lie on a circle of radius ω_c in the left-half plane. Hence we must have $a = -\omega_c$.

(b) For a Butterworth low-pass filter of even order N, all the poles of the transfer function $H(s)$ are complex. They all lie on a circle of radius ω_c in the left-half plane. Let $s = -a - jb$, with $a > 0$ and $b > 0$, denote a complex pole of $H(s)$. All the coefficients of $H(s)$ are real. This condition can only be satisfied if we [have] a complex conjugate pole at $s = -a + jb$. We may then express the contribution of this pair of poles as

$$\frac{1}{(s+a+jb)(s+a-jb)} = \frac{1}{(s+a)^2 + b^2}$$

whose coefficients are all real. We therefore conclude that for even filter order N, all the poles of $H(s)$ occur in complex-conjugate pairs.

8.6 The transfer function of a Butterworth low-pass filter of order 5 is

$$H(s) = \frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)} \qquad (1)$$

The low pass to high-pass transformation is defined by

$$s \rightarrow \frac{1}{s}$$

where it is assumed that the cutoff frequency of the high-pass filter is unity. Hence replacing s with $1/s$ in Eq. (1), we find that the transfer function of a Butterworth high-pass filter of order 5 is

$$H(s) = \frac{1}{\left(\frac{1}{s}+1\right)\left(\frac{1}{s^2}+\frac{0.618}{s}+1\right)\left(\frac{1}{s^2}+\frac{1.618}{s}+1\right)}$$

$$= \frac{s^5}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$$

The magnitude response of this high-pass filter is plotted on the next page.

8.7 We are given the transfer function

$$H(s) = \frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)} \qquad (1)$$

To modify this low-pass filter so as to assume a cutoff frequency ω_c, we use the transformation

$$s \longrightarrow \frac{s}{\omega_c}$$

Hence replacing s with s/ω_c in Eq. (1), we obtain

$$H(s) = \frac{1}{\left(\frac{s}{\omega_c}+1\right)\left(\frac{s^2}{\omega_c^2}+0.618\frac{s}{\omega_c}+1\right)\left(\frac{s^2}{\omega_c^2}+1.618\frac{s}{\omega_c}+1\right)}$$

$$= \frac{\omega_c^5}{(s+\omega_c)(s^2+0.618\omega_c s+\omega_c^2)(s^2+1.618\omega_c s+\omega_c^2)}$$

8.8 We are given the transfer function

$$H(s) = \frac{1}{(s+1)(s^2+s+1)} \qquad (1)$$

To transform this low-pass filter to a band-pass filter with bandwidth B centered on ω_0, we use the following transformation:

$$s \longrightarrow \frac{s^2 + \omega_0^2}{B\, s}$$

With $\omega_0 = 1$ and $B = 0.1$ the transformation takes the value

$$s \longrightarrow \frac{s^2 + 1}{0.1 s} \qquad (2)$$

Substituting Eq. (2) in (1):

$$H(s) = \frac{1}{\left(\frac{s^2+1}{0,1s} + 1\right)\left(\left(\frac{s^2+1}{0,1s}\right)^2 + \left(\frac{s^2+1}{0,1s}\right) + 1\right)}$$

$$= \frac{0,001\, s^3}{\left(s^2 + 0,1s + 1\right)\left(s^4 + 2s^2 + 1 + 0,1s^3 + 0,1s + 0,01s^2\right)}$$

$$= \frac{0,001\, s^3}{(s^2 + 0,1s + 1)(s^4 + 0,1s^3 + 2.01s^2 + 0,1s + 1)}$$

The magnitude response of this band-pass filter of the Butterworth type is shown plotted on the next page.

8.9 To transform a prototype low-pass filter into a band stop filter of midband rejection frequency ω_0 and bandwidth B, we may use the transformation

$$s \longrightarrow \frac{Bs}{s^2 + \omega_0^2} \qquad (1)$$

As an illustrative example, consider the low-pass filter:

$$H(s) = \frac{1}{s+1}$$

Using the transformation of Eq. (1), we obtain a band pass filter defined by

$$H(s) = \frac{1}{\dfrac{Bs}{s^2+\omega_0^2} + 1}$$

$$= \frac{s^2 + \omega_0^2}{s^2 + Bs + \omega_0^2}$$

which is characterized as follows:

$$H(s)\Big|_{s=0} = H(s)\Big|_{s=\infty} = 1$$

$$H(s)\Big|_{s=\pm j\omega_0} = 0$$

8.10 An FIR filter of type 1 has an even length M and is symmetric about $M/2$ in that its coefficients satisfy the condition

$$h[n] = h[M-n] \quad \text{for } n = 0, 1, \ldots, M$$

The frequency response of the filter is

$$H(e^{j\Omega}) = \sum_{n=0}^{M} h[n]\, e^{-jn\Omega}$$

which may be reformulated as follows:

$$H(e^{j\Omega}) = \sum_{n=0}^{\frac{M}{2}-1} h[n]\, e^{-jn\Omega} + h[\tfrac{M}{2}]\, e^{-jM\Omega/2} + \sum_{n=\frac{M}{2}+1}^{M} h[n]\, e^{-jn\Omega}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h[n]\, e^{-jn\Omega} + h[\tfrac{M}{2}]\, e^{-jM\Omega/2} + \sum_{n=0}^{\frac{M}{2}-1} h[M-n]\, e^{-j\Omega(M-n)}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h[n]\, e^{-jn\Omega} + h[\tfrac{M}{2}]\, e^{-jM\Omega/2} + \sum_{n=0}^{\frac{M}{2}-1} h[n]\, e^{-j\Omega(M-n)} \qquad (1)$$

Define

$$a[k] = 2\, h[\tfrac{M}{2} - k], \qquad k = 1, 2, \ldots, \tfrac{M}{2}$$

$$a[0] = h[\tfrac{M}{2}]$$

and let

$$n = \frac{M}{2} - k$$

We may then rewrite Eq. (1) in the equivalent form:

$$H(e^{j\Omega}) = e^{-jM\Omega/2}\left\{ \sum_{k=1}^{\frac{M}{2}} h\left[\tfrac{M}{2}-k\right]\left(e^{jk\Omega} + e^{-jk\Omega}\right) + h\left[\tfrac{M}{2}\right]\right\}$$

$$= e^{-jM\Omega/2}\left\{ \sum_{k=1}^{\frac{M}{2}} 2a[k]\left(e^{jk\Omega} + e^{-jk\Omega}\right) + a(0)\right\}$$

$$= e^{-jM\Omega/2}\left\{ \sum_{k=1}^{M/2} a[k]\cos(k\Omega) + a(0)\right\}$$

$$= e^{-jM\Omega/2} \sum_{k=0}^{M/2} a[k]\cos(k\Omega) \qquad (2)$$

From Eq. (4) we may make the following observations for an FIR filter of type I:

1. The frequency response $H(e^{j\Omega})$ has a linear phase component exemplified by the exponential $e^{-jM\Omega/2}$.
2. At $\Omega = 0$,

$$H(e^{j0}) = \sum_{k=0}^{M/2} a[k]$$

At $\Omega = \pi$,

$$H(e^{j\pi}) = \sum_{k=0}^{M/2} a[k]\,(-1)^{\frac{M}{2}+k}$$

The implications are that there are no restrictions on $H(e^{j\Omega})$ at $\Omega = 0$ and $\Omega = \pi$.

8.11 For an FIR filter of type II, the filter length M is even and it is antisymmetric in that its coefficients satisfy the condition

$$h[n] = -h[M-n], \qquad 0 \le n \le \frac{M}{2} - 1$$

The frequency response of the filter is

$$H(e^{j\Omega}) = \sum_{n=0}^{M} h[n] e^{-jn\Omega}$$

which may be reformulated as follows:

$$H(e^{j\Omega}) = \sum_{n=0}^{\frac{M}{2}-1} h[n] e^{-jn\Omega} + h[\tfrac{M}{2}] e^{-jM\Omega/2} + \sum_{n=\frac{M}{2}+1}^{M} h[n] e^{-jn\Omega}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h[n] e^{-jn\Omega} + h[\tfrac{M}{2}] e^{-jM\Omega/2} + \sum_{n=0}^{\frac{M}{2}-1} h[M-n] e^{-j(M-n)\Omega}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h[n] e^{-jn\Omega} + h[\tfrac{M}{2}] e^{-jM\Omega/2} - \sum_{n=0}^{\frac{M}{2}-1} h[n] e^{-j(M-n)\Omega} \qquad (1)$$

Define

$$a[k] = 2h[\tfrac{M}{2} - k], \qquad k = 1, 2, \ldots, \tfrac{M}{2}$$

$$a[0] = h[\tfrac{M}{2}]$$

and let

$$k = \frac{M}{2} - n$$

We may then rewrite Eq. (1) in the equivalent form:

$$H(e^{j\Omega}) = e^{-jM\Omega/2} \sum_{k=1}^{M/2} h[\tfrac{M}{2}-k]\, e^{jk\Omega} + h[\tfrac{M}{2}]\, e^{-jM\Omega/2}$$

$$- e^{-jM\Omega/2} \sum_{k=1}^{M/2} h[\tfrac{M}{2}-k]\, e^{-jk\Omega}$$

$$= e^{-jM\Omega/2} \left\{ 2\sum_{k=1}^{M/2} a[k] \left(e^{jk\Omega} - e^{-jk\Omega} \right) + a[0] \right\}$$

$$= e^{-jM\Omega/2} \left\{ j \sum_{k=1}^{M/2} a[k] \sin(k\Omega) + a[0] \right\}$$

$$= j\, e^{-jM\Omega/2} \sum_{k=0}^{M/2} a[k] \sin(k\Omega) \qquad (2)$$

From Eq. (2) we may make the following observations on the frequency response of an FIR filter of type II:

1. The phase response includes a linear component exemplified by the exponential $e^{-jM\Omega/2}$.
2. At $\Omega = 0$,

   $$H(e^{j0}) = 0$$

   At $\Omega = \pi$, $\sin(k\pi) = 0$ for integer k and therefore,

   $$H(e^{j\pi}) = 0$$

8.12 An FIR filter of type III is characterized as follows:

- The filter length M is an odd integer.
- The filter is symmetric about the noninteger midpoint $n = M/2$ in that its coefficients satisfy the condition

$$h[n] = h[M-n] \quad \text{for} \quad 0 \le n \le M$$

The frequency response of the filter is

$$H(e^{j\Omega}) = \sum_{n=0}^{M} h[n]\, e^{-jn\Omega}$$

which may be reformulated as follows

$$H(e^{j\Omega}) = \sum_{n=0}^{\frac{M-1}{2}} h[n]\, e^{-jn\Omega} + \sum_{n=\frac{M+1}{2}}^{M} h[n]\, e^{-jn\Omega}$$

$$= \sum_{n=0}^{\frac{M-1}{2}} h[n]\, e^{-jn\Omega} + \sum_{n=0}^{n=\frac{M-1}{2}} h[M-n]\, e^{-j(M-n)\Omega}$$

$$= \sum_{n=0}^{\frac{M-1}{2}} h[n] \left(e^{-jn\Omega} + e^{-j(M-n)\Omega} \right)$$

$$= e^{-jM\Omega/2} \sum_{n=0}^{\frac{M-1}{2}} h[n] \left(e^{j(\frac{M}{2}-n)\Omega} + e^{-j(\frac{M}{2}-n)\Omega} \right) \qquad (1)$$

Define

$$b[k] = 2h\left[\frac{M+1}{2} - k\right] \qquad \text{for } k = 1, 2, \ldots, \frac{M+1}{2}$$

and let

$$n = \frac{M+1}{2} - k$$

We may then rewrite Eq. (1) in the equivalent from:

$$H(e^{j\Omega}) = e^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} \frac{1}{2} b(k) \left(e^{j\Omega(k-\frac{1}{2})} + e^{-j\Omega(k-\frac{1}{2})} \right)$$

$$= e^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} b(k) \cos\left(\Omega (k - \tfrac{1}{2})\right) \qquad (2)$$

From Eq. (2) we may make the following observations on the frequency response of an FIR filter of type III:

1. The phase response of the filter is linear as exemplified by the exponential factor $e^{-jM\Omega/2}$.

2. At $\Omega = 0$,

$$H(e^{j0}) = \sum_{k=1}^{\frac{M+1}{2}} b(k)$$

which shows that there is no restriction on $H(e^{j0})$.

At $\Omega = \pi$,

$$H(e^{j\pi}) = e^{-jM\pi/2} \sum_{k=1}^{\frac{M+1}{2}} b(k) \cos\left(\pi (k - \tfrac{1}{2})\right)$$

$$= e^{-jM\pi/2} \sum_{k=1}^{\frac{M+1}{2}} b(k) \sin(\pi k)$$

which is zero since $\sin(\pi k) = 0$ for all integer values of k.

8.13 An FIR filter of type IV is characterized as follows:

- The filter length M is an odd integer.
- The filter is antisymmetric about the noninteger midpoint $n = \frac{M}{2}$ in that its coefficients satisfy the condition

$$h[n] = -h[M-n] \quad \text{for} \quad 0 \le n \le M$$

The frequency response of the filter is

$$H(e^{j\Omega}) = \sum_{n=0}^{M} h[n]\, e^{-jn\Omega}$$

which may be reformulated as follows:

$$H(e^{j\Omega}) = \sum_{n=0}^{\frac{M-1}{2}} h[n]\, e^{-jn\Omega} + \sum_{n=\frac{M+1}{2}}^{M} h[n]\, e^{-jn\Omega}$$

$$= \sum_{n=0}^{\frac{M-1}{2}} h[n]\, e^{-jn\Omega} + \sum_{n=0}^{\frac{M-1}{2}} h[M-n]\, e^{-j(M-n)\Omega}$$

$$= \sum_{n=0}^{\frac{M-1}{2}} h[n]\, e^{-jn\Omega} - \sum_{n=0}^{\frac{M-1}{2}} h[n]\, e^{-j(M-n)\Omega}$$

$$= e^{-jM\Omega/2} \sum_{n=0}^{\frac{M-1}{2}} h[n] \left(e^{-j(n-\frac{M}{2})\Omega} - e^{j(n-\frac{M}{2})\Omega} \right) \qquad (1)$$

Define

$$b[k] = 2\,h\left[\frac{M+1}{2} - k\right] \quad \text{for } k = 1, 2, \ldots, \frac{M+1}{2}$$

and let

$$k = \frac{M+1}{2} - n$$

We may then rewrite Eq. (1) in the equivalent form:

$$H(e^{j\Omega}) = e^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} h\left[\frac{M+1}{2} - k\right]\left(e^{j(k-\frac{1}{2})\Omega} - e^{-j(k-\frac{1}{2})\Omega}\right)$$

$$= j e^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} 2h\left[\frac{M+1}{2} - k\right] \sin\left(\left(k-\tfrac{1}{2}\right)\Omega\right)$$

$$= j e^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} b[k] \sin\left(\left(k-\tfrac{1}{2}\right)\Omega\right) \qquad (2)$$

From Eq. (2) we may make the following observations on the FIR filter of type IV:

1. The phase response of the filter includes a linear component exemplified by the exponential $e^{-jM\Omega/2}$.

2. At $\Omega = 0$,

$$H(e^{j0}) = 0$$

At $\Omega = \pi$,

$$H(e^{j\pi}) = j e^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} b[k] \sin\left(\left(k-\tfrac{1}{2}\right)\pi\right)$$

$$= j e^{-jM\Omega/2} \sum_{k=1}^{\frac{M+1}{2}} (-1)^{k+1} b[k]$$

which shows that $H(e^{j\pi})$ can assume an arbitrary value.

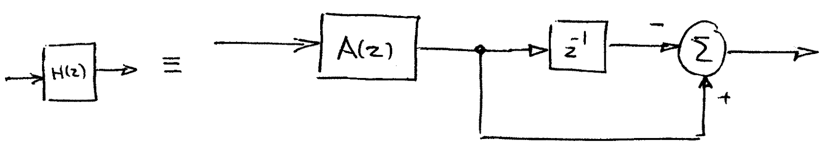
8.14 Let the transfer function of the FIR filter be defined by

$$H(z) = (1 - z^{-1})\,A(z) \qquad (1)$$

where $A(z)$ is an arbitrary polynomial in z^{-1}. The $H(z)$ of Eq. (1) has a zero at $z = 1$ as prescribed. Let the sequence $a[n]$ denote the inverse z-transform of $A(z)$. Expanding Eq. (1):

$$H(z) = A(z) - z^{-1}A(z) \qquad (2)$$

which may be represented by the block diagram:



From Eq. (2) we readily find that the impulse response of the filter is defined in terms of $a[n]$ as follows:

$$h[n] = a[n] - a[n-1]$$

8.15 Let the transfer function of the FIR filter be defined by

$$H'(e^{j\Omega}) = \sum_{n=-M/2}^{M/2} h_d[n]\, e^{-jn\Omega} \qquad (1)$$

For $H'(e^{j\Omega})$ to have zero phase we require the filter to be symmetric about $n=0$:

$$h_d[-n] = h_d[n] \qquad (2)$$

for then we may express $H'(e^{j\Omega})$ as

$$H'(e^{j\Omega}) = \sum_{n=-1}^{-M/2} h_d[n]\, e^{-jn\Omega} + h_d[0] + \sum_{n=1}^{M/2} h_d[n]\, e^{-jn\Omega}$$

$$= \sum_{n=1}^{M/2} h_d[-n]\, e^{jn\Omega} + h_d[0] + \sum_{n=1}^{M/2} h_d[n]\, e^{-jn\Omega}$$

$$= \sum_{n=1}^{M/2} h_d[n]\left(e^{jn\Omega} + e^{-jn\Omega}\right) + h_d[0]$$

$$= 2\sum_{n=1}^{M/2} h_d[n]\cos(n\Omega) + h_d[0]$$

Returning to Eq. (1) we may also write

$$H'(e^{j\Omega}) = e^{jM\Omega/2} \sum_{n=0}^{M} h_d[n-\tfrac{M}{2}]\, e^{-jn\Omega}$$

$$= e^{jM\Omega/2} \sum_{n=0}^{M} h_d[n]\, e^{-jn\Omega}$$

where, in the last line, we have invoked the symmetry condition of Eq. (2). Define

$$H(e^{j\Omega}) = e^{-jM\Omega/2} H'(e^{j\Omega})$$

$$= \sum_{n=0}^{M} h_d[n]\, e^{-jn\Omega}$$

which is the desired relation.

8.16 The FIR digital filter used as a discrete-time differentiator in Example 8.6 exhibits the following properties:

- The filter length M is an odd integer.
- The frequency response of the filter satisfies the conditions:

  1. At $\Omega = 0$,

     $H(e^{j0}) = 0$

  2. At $\Omega = \pi$,

     $H(e^{j\pi}) = 0$

These properties are basic properties of an FIR filter of type III discussed in Problem 8.11. We therefore immediately deduce that the FIR filter of Example 8.6 is antisymmetric about the noninteger point $n = M/2$.

8.17 According to Eqs. (8.64) and (8.65), the magnitude $r = |z|$ and phase $\theta = \arg\{z\}$ are defined by

$$r = \left(\frac{(1+\sigma)^2 + \omega^2}{(1-\sigma)^2 + \omega^2} \right)^{1/2} \qquad (1)$$

$$\theta = \tan^{-1}\left(\frac{\omega}{1+\sigma}\right) - \tan^{-1}\left(\frac{\omega}{1-\sigma}\right)$$

These two relations are based on $\qquad (2)$

$$z = r e^{j\theta} = \frac{1+s}{1-s}, \qquad s = \sigma + j\omega$$

For the more general case of a sampling rate $1/T$ for which we have

$$s = \frac{2}{T} \frac{z-1}{z+1}$$

or

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}$$

we may rewrite Eqs. (1) and (2) by replacing ω with $\frac{T}{2}\omega$ and σ with $\frac{T}{2}\sigma$, obtaining

$$r = \left(\frac{(\frac{2}{T} + \sigma)^2 + \omega^2}{(\frac{2}{T} - \sigma)^2 + \omega^2} \right)^{1/2}$$

$$\theta = \tan^{-1}\left(\frac{\omega}{\frac{2}{T} + \sigma}\right) - \tan^{-1}\left(\frac{\omega}{\frac{2}{T} - \sigma}\right)$$

8.18 For a digital IIR filter, the transfer function $H(z)$ may be expressed as

$$H(z) = \frac{N(z)}{D(z)}$$

where $N(z)$ and $D(z)$ are polynomials in z^{-1}. The filter is unstable if any pole of $H(z)$ or, equivalently, any zero of the denominator polynomial $D(z)$ lies outside the unit circle in the z-plane. According to the bilinear transform,

$$H(z) = H_a(s)\Big|_{s=\frac{z-1}{z+1}}$$

where $H_a(s)$ is the transfer function of an analog filter used as the basis for designing the digital IIR filter. The poles of $H(z)$ outside the unit circle in the z-plane correspond to certain poles of $H(s)$ in the right half of the s-plane. Conversely, the poles of $H_a(s)$ in the right half of the s-plane are mapped onto the outside of the unit circle in the z-plane. Now if any pole of $H_a(s)$ lies in the right-half plane, the analog filter is unstable. Hence if any such filter is used in the bilinear transform, the resulting digital filter is unstable.

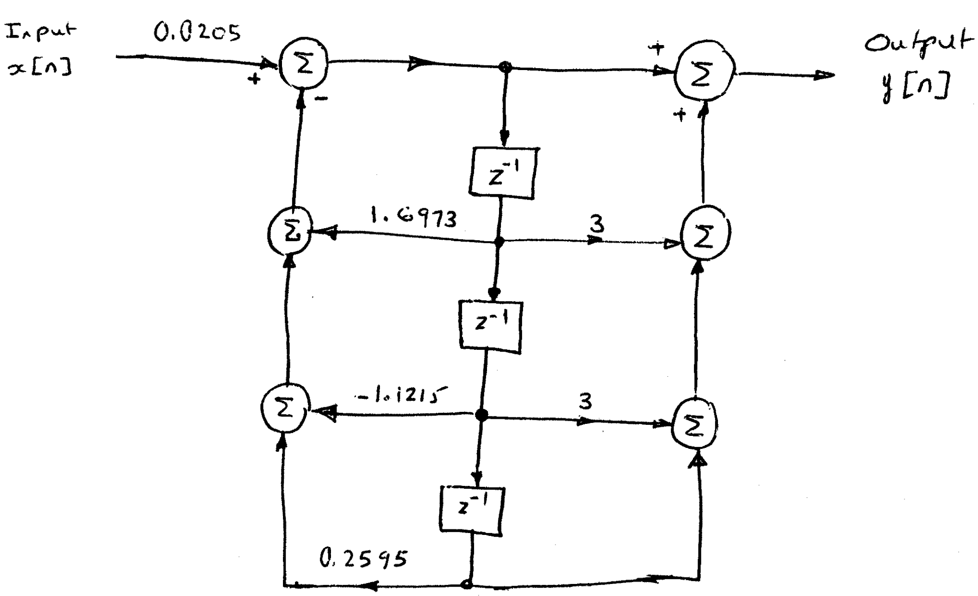
8.19 The transfer function of the digital IIR filter is

$$H(z) = \frac{0.0205\,(z+1)^3}{(z-0.4893)(z^2-1.208z+0.5304)}$$

Expanding the numerator and denominator polynomials of $H(z)$ in ascending powers of z^{-1}, we may write

$$H(z) = \frac{0.0205\,(1+3z^{-1}+3z^{-2}+z^{-3})}{1-1.6973z^{-1}+1.1215z^{-2}-0.2595z^{-3}}$$

Hence the filter may be implemented in direct II using the following configuration:



8.20 We are given an analog filter whose transfer function is defined by

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - d_k}$$

Recall the Laplace transform pair

$$e^{d_k t} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s - d_k}$$

It follows therefore that the impulse response of the analog filter is

$$h_a(t) = \sum_{k=1}^{N} A_k e^{d_k t} \qquad (1)$$

According to the method of impulse invariance, the impulse response of a digital filter derived from the analog filter of Eq. (1) is defined by

$$h[n] = T h_a(nT)$$

where T is the sampling period. Hence from Eq. (1) we find that

$$h[n] = \sum_{k=1}^{N} T A_k e^{n d_k T} \qquad (2)$$

Now recall the z-transform pair:

$$e^{n \alpha_k T} \xleftrightarrow{z} \frac{1}{1 - e^{\alpha_k T} z^{-1}}$$

Hence the transfer function of the digital filter is deduced from Eq. (2) to be

$$H(z) = \sum_{k=1}^{N} \frac{T A_k}{1 - e^{\alpha_k T} z^{-1}}$$

8.21 Consider a discrete-time system whose transfer function is denoted by $H(z)$. By definition,

$$H(z) = \frac{Y(z)}{X(z)}$$

where $Y(z)$ and $X(z)$ are the z-transforms of the output sequence $y[n]$ and input sequence $x[n]$. Let $H_{eq}(z)$ denote the z-transform of the equalizer connected in cascade with $H(z)$. Let $x'[n]$ denote the equalizer output in response to $y[n]$ as the input. Ideally,

$$x'[n] = x[n - n_0]$$

where n_0 is an integer delay. Hence

$$H_{eq}(z) = \frac{X'(z)}{Y(z)} = \frac{z^{-n_0} X(z)}{Y(z)} = \frac{z^{-n_0}}{H(z)}$$

Putting $z = e^{j\Omega}$:

$$H_{eq}(e^{j\Omega}) = \frac{e^{-jn_0\Omega}}{H(e^{j\Omega})}$$

8.22 The phase delay of an FIR filter of even length M and antisymmetric impulse response is linear with frequency Ω as shown by

$$\theta(\Omega) = -M\Omega$$

Hence such a filter used as an equalizer introduces a constant delay

$$\tau(\Omega) = -\frac{\partial \theta(\Omega)}{\partial \Omega} = M \text{ samples}$$

The implication of this result is that as we make the filter length M larger, the constant delay introduced by the equalizer is correspondingly increased.

8.23 We are given a continuous-time channel whose input-output relation is defined by

$$y(t) = K_1 x(t - t_{01}) + K_2 x(t - t_{02})$$

where the scaling factors K_1 and K_2 are constants and likewise for the transmission delays t_{01} and t_{02}. The transfer function of the channel is

$$H(s) = \frac{Y(s)}{X(s)}$$

$$= K_1 e^{-st_{01}} + K_2 e^{-st_{02}}$$

$$= K_1 e^{-st_{01}} \left(1 + \frac{K_2}{K_1} e^{-s(t_{02} - t_{01})}\right) \qquad (1)$$

where it is noted that $t_{02} > t_{01}$ and therefore the difference $t_{02} - t_{01}$ is positive. Ideally, we would like to have an equalizer whose transfer function $H_{eq}(s)$ satisfies the condition

$$H_{eq}(s)\, H(s) = K_1 e^{-st_{01}} \qquad (2)$$

The result of such an equalizer is to introduce an amplitude scaling by the factor K_1 and a constant transmission delay equal to t_{01}. From Eqs. (1) and (2) it therefore follows that

$$H_{eq}(s) = \frac{K_1 e^{-st_{01}}}{H(s)}$$

$$= \frac{1}{1 + \frac{K_2}{K_1} e^{-s(t_{02} - t_{01})}} \qquad (3)$$

We are given that $K_2 << K_1$ and $t_{02} > t_{01}$. It follows therefore that

$$\left| \frac{K_2}{K_1} e^{-j\omega(t_{02}-t_{01})} \right| << 1 \quad \text{for all } \omega >> 0$$

Thus putting $s = j\omega$ in Eq. (3) and invoking the approximation

$$\frac{1}{1+x} \simeq 1 - x \quad \text{for } |x| << 1$$

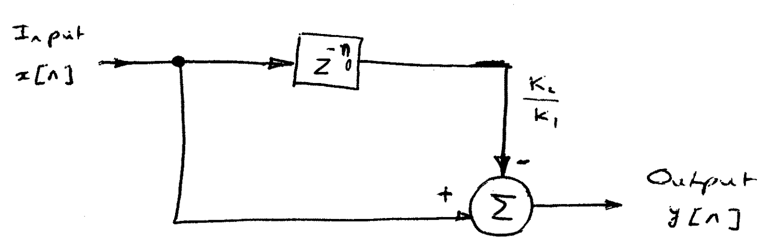
we may approximate Eq. (3) as

$$H_{eq}(j\omega) \simeq 1 - \frac{K_2}{K_1} e^{-j\omega(t_{02}-t_{01})} \qquad (4)$$

Let n_0 be the positive integer closest in value to the ratio $(t_{02}-t_{01})/T$, where T is the sampling interval. We may then use for the equalizer a digital FIR filter whose frequency response is defined by

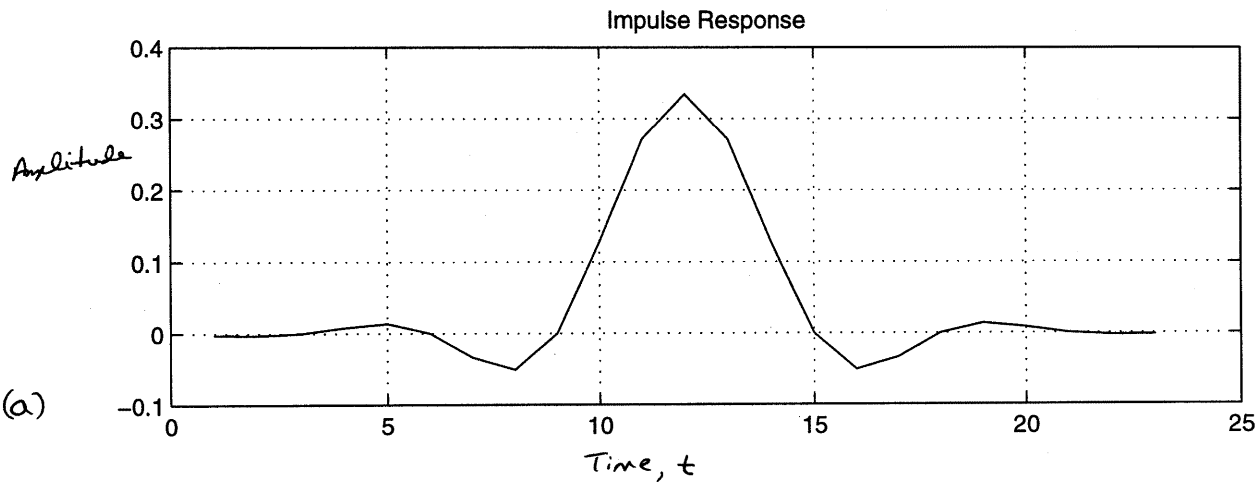
$$H_{eq}(j\Omega) \simeq 1 - \frac{K_2}{K_1} e^{-jn_0\Omega} \qquad (5)$$

which may be implemented as follows:

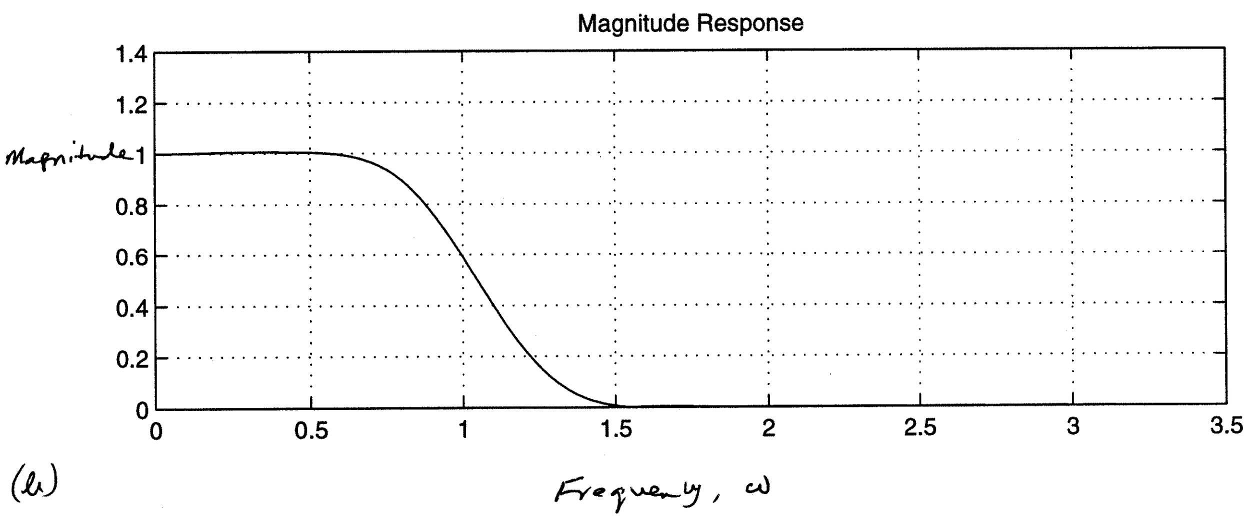


```
b=fir1(22,1/3,hamming(23));

subplot(2,1,1)
plot(b)
title('Impulse Response')
grid on

[H,w]=freqz(b,1,512,2*pi);
subplot(2,1,2)
plot(w,abs(H))
title('Magnitude Response')
grid on
```

8.24


Impulse Response
Amplitude
0.4
0.3
0.2
0.1
0
−0.1
0
5
10
15
20
25
Time, t

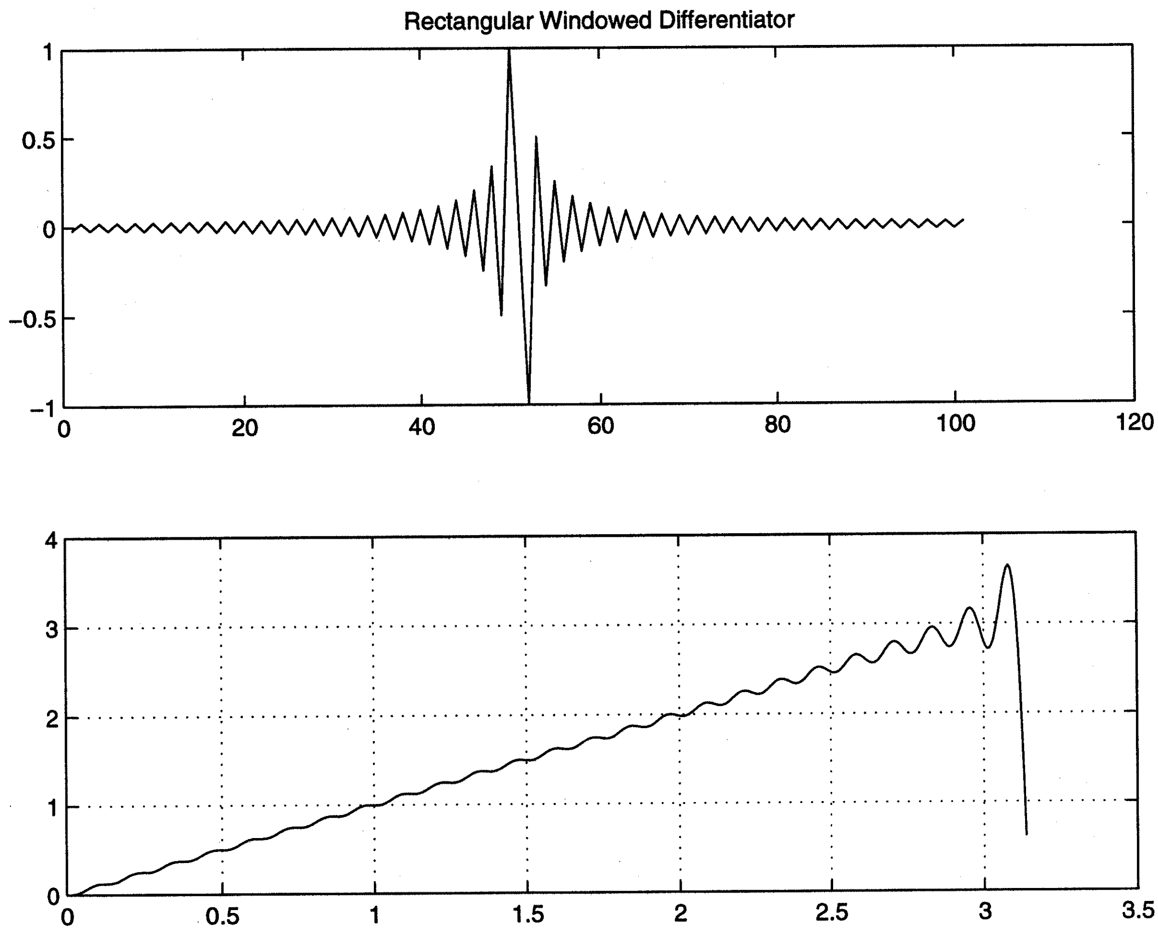
(a)


Magnitude Response
Magnitude
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
Frequency, ω

(b)

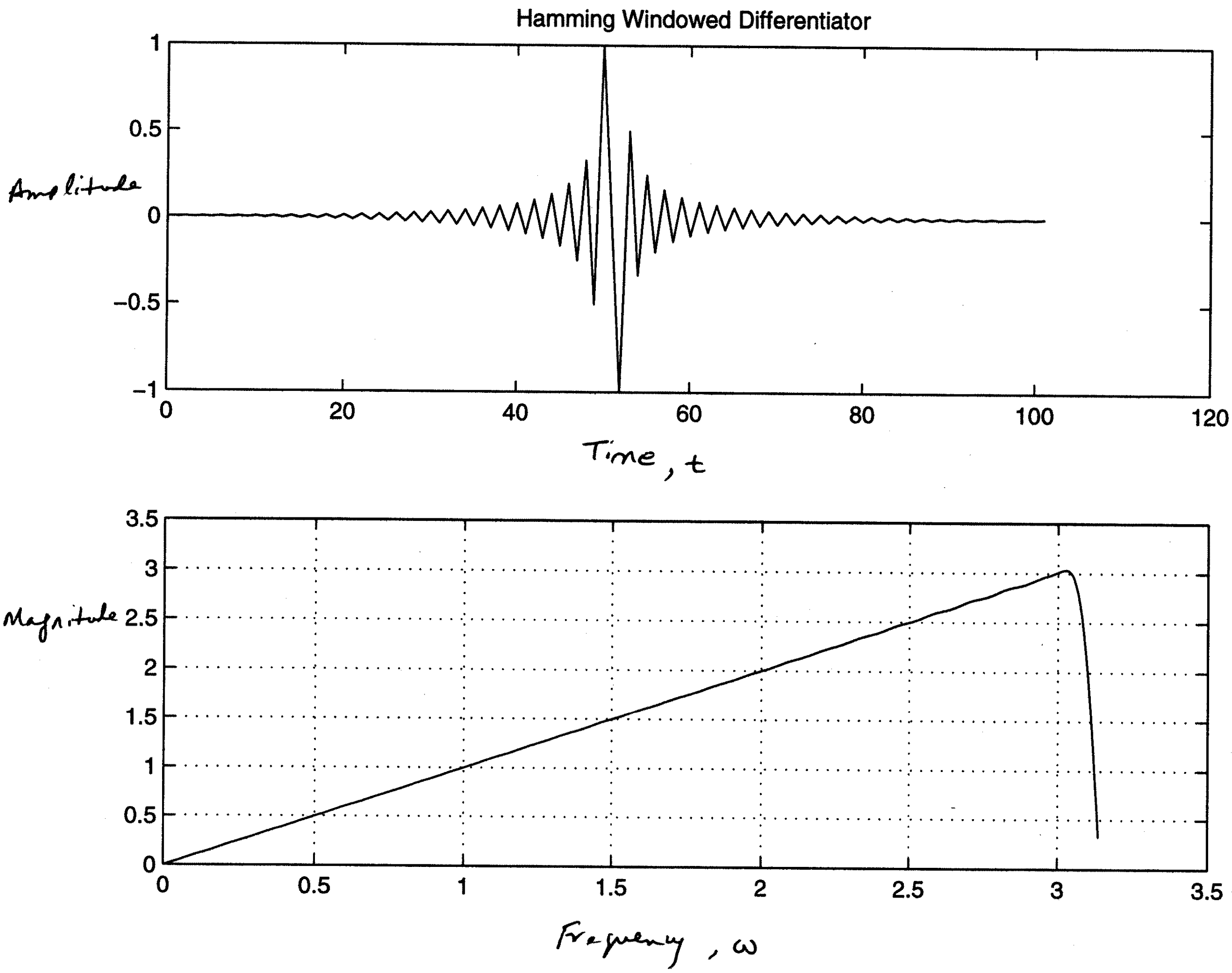
```
M=100;
n=0:M;   f=n-M/2;

a = cos(pi*f) ./ f;             % integration by parts as done
b = sin(pi*f) ./ (pi*f.^2);     % in equation 8.59

h=a-b;
k=isnan(h);  h(k)=0;     % get rid of NaN
h_rect=h;
h_hamm=h.*hamming(length(h))';

[H,w]=freqz(h_rect,1,512,2*pi);

subplot(2,1,1)
plot(h_rect)
title('Rectangular Windowed Differentiator')
subplot(2,1,2)
plot(w,abs(H));grid

figure
[H,w]=freqz(h_hamm,1,512,2*pi);
subplot(2,1,1)
plot(h_hamm)
title('Hamming Windowed Differentiator')
subplot(2,1,2)
plot(w,abs(H));grid
```

8.25(a)



(a) Rectangular window

8.25 (b)



(b) Hamming window

8.26 For a Butterworth low-pass filter of order N, the squared magnitude response is

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}} \qquad (1)$$

where ω_c is the cutoff frequency.

(a) We are given the following specifications:

(i) $\omega_c = 2\pi \times 800$ rad/s

(ii) At $\omega = 2\pi \times 1{,}500$ rad/s, we have

$$10 \log_{10} |H(j\omega)|^2 = -15 \text{ dB}$$

or, equivalently

$$|H(j\omega)|^2 = \frac{1}{31.6228}$$

Substituting these values in Eq. (1):

$$31.6228 = 1 + \left(\frac{2\pi \times 1{,}200}{2\pi \times 800}\right)^{2N}$$

$$= 1 + (1.5)^{2N}$$

Solving for the filter order N:

$$N = \frac{1}{2}\left(\log 30.6228 / \log 1.5\right)$$

$$= 4.2195$$

So we choose $N = 5$.

(b) The MATLAB code for designing the filter is given on the next page.

The page, following the code, lists the transfer function of the analog low-pass filter and the corresponding IIR low-pass filter.

(c) The frequency response of the filter is presented in Fig. 1.

```
omegaC=.2*pi;
N=5;
wc=tan(omegaC/2);

coeff=[1 3.2361 5.2361 5.2361 3.2361 1];

ds=coeff.*(wc.^[0:N]);
ns=wc^N;
[nz dz]=bilinear(ns,ds,.5);

[H,W]=freqz(nz,dz,512);
subplot(2,1,1); plot(W,abs(H))
xlabel('rad/s'); title('Magnitude Response')
grid on

phi=angle(H);
phi=(180/pi)*phi;
subplot(2,1,2)
plot(W,phi)
title('Phase Response'); xlabel('rad/s'); ylabel('degrees')
grid on

set(gcf,'name',['Low Pass: order=' num2str(N) ' wc='num2str(omegaC)])
```

```
>> [a b c d]=filterDesignLP(.2*pi,5);


H(s)  =  0.0036
         --------------------------------------------------------------------------------
         1.0000 s^5 +  1.0515 s^4 +  0.5528 s^3 +  0.1796 s^2 +  0.0361 s^1 +  0.0036 s^0

H(z)  =  0.0013 z^5 +  0.0064 z^4 +  0.0128 z^3 +  0.0128 z^2 +  0.0064 z^1 +  0.0013 z^0
         --------------------------------------------------------------------------------
         1.0000 z^5 + -2.9754 z^4 +  3.8060 z^3 + -2.5452 z^2 +  0.8811 z^1 + -0.1254 z^0
```

8.26

```
>> [a b c d]=filterDesignHP(.6,5);

H(s)  = 353.0579 s^5 +   0.0000 s^4 +   0.0000 s^3 +   0.0000 s^2 +   0.0000 s^1 +   0.0000 s^0
        ----------------------------------------------------------------------------------------
        353.0579 s^5 + 353.4261 s^4 + 176.8950 s^3 + 54.7200 s^2 + 10.4614 s^1 +  1.0000 s^0

H(z)  =  0.3718 z^5 + -1.8591 z^4 +  3.7181 z^3 + -3.7181 z^2 +  1.8591 z^1 + -0.3718 z^0
         --------------------------------------------------------------------------------
         1.0000 z^5 + -3.0660 z^4 +  4.0072 z^3 + -2.7279 z^2 +  0.9586 z^1 + -0.1382 z^0
```

8.27

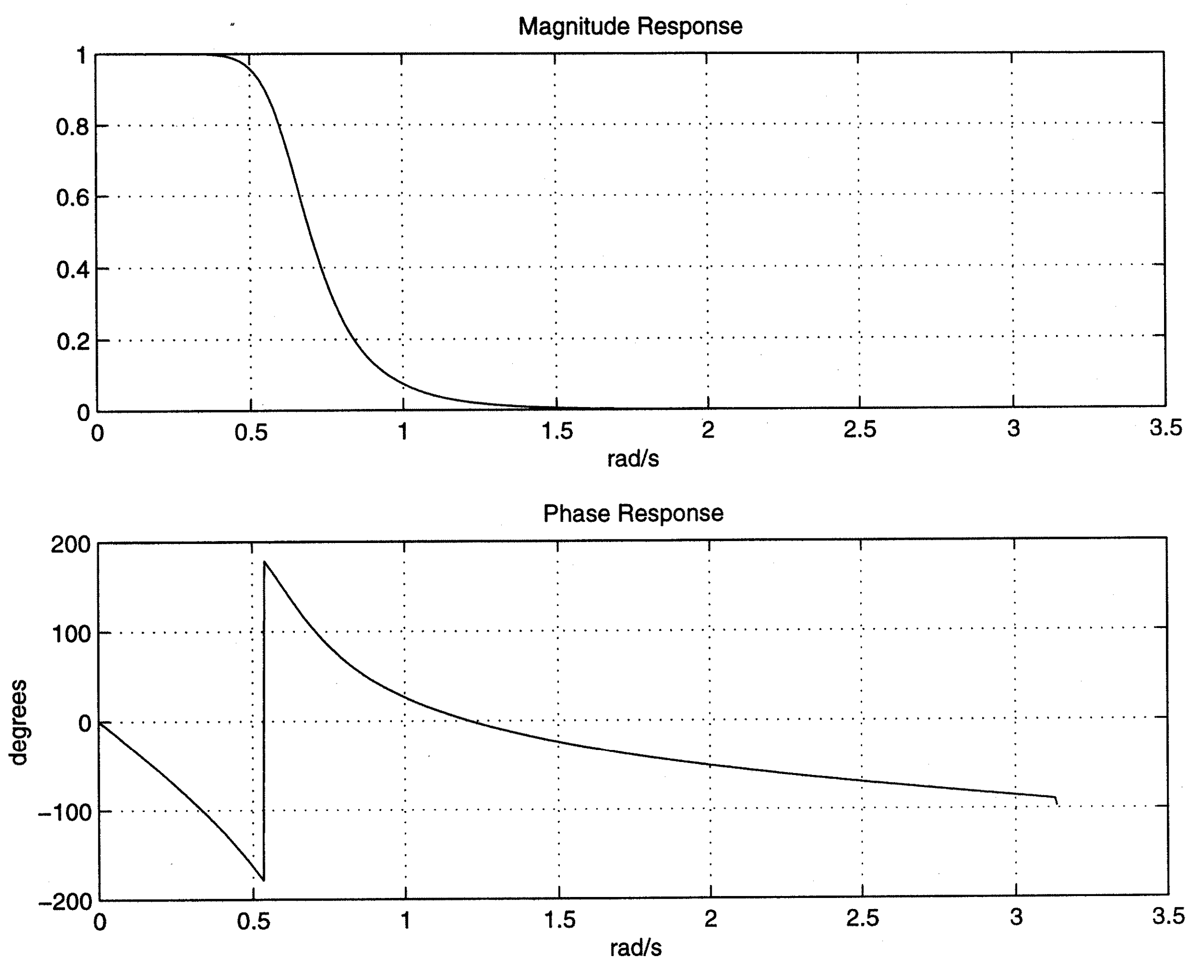


Figure 1

Frequency response of IIR low-pass filter

8.27 The transfer function $H(s)$ of the analog high-pass filter and the transfer function $H(z)$ of the IIR digital high-pass filter are listed on the same page where those pertaining to Problem 8.26 were given.

The MATLAB code for designing the IIR digital high-pass filter and Fig.1 displaying its frequency response are presented on the next two pages.

```
omegaC=.6;
N=5;
wc=tan(omegaC/2);

coeff=[1 3.2361 5.2361 5.2361 3.2361 1];

ds=fliplr(coeff./(wc.^[0:N]));
ns=[1/(wc^N) zeros(1,N)];
[nz dz]=bilinear(ns,ds,.5);

[H,W]=freqz(nz,dz,512);
subplot(2,1,1)
plot(W,abs(H))
xlabel('rad/s'); title('Magnitude Response')
grid on

phi=angle(H);
phi=(180/pi)*phi;
subplot(2,1,2)
plot(W,phi)
title('Phase Response'); xlabel('rad/s'); ylabel('degrees')
grid on

set(gcf,'name',['High Pass: order=' num2str(N) ' wc='num2str(omegaC)])
```

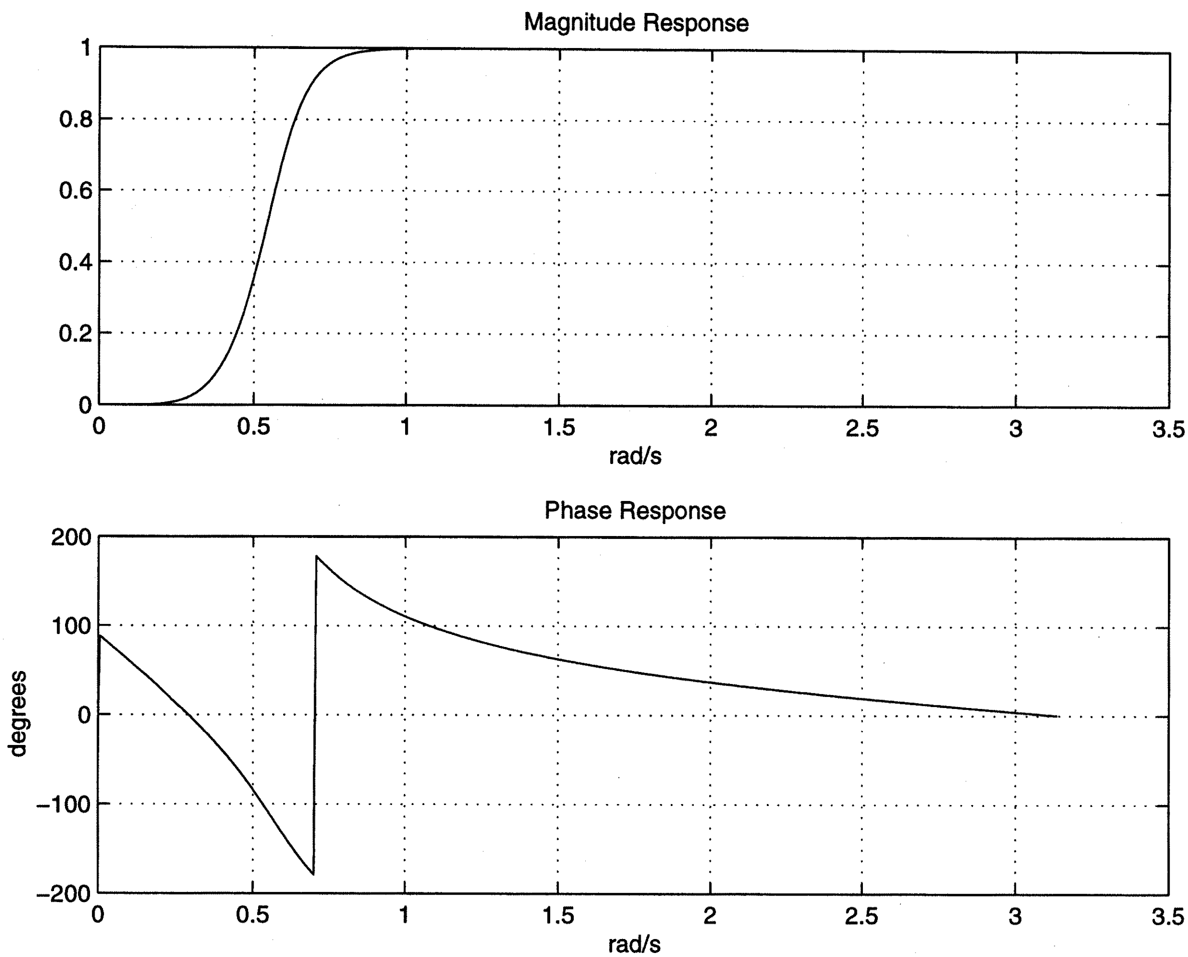


Figure 1

IIR high-pass filter

8.28 The frequency response of the channel is defined by

$$H_c(j\omega) = \frac{1}{\left(1 + \frac{j\omega}{\pi}\right)^2}$$

The requirement is to design an FIR filter for equalizing the channel.

Ideally, we may define the frequency response of the equalizer as

$$H_{eq}(j\omega) = \frac{e^{-j\omega t_0}}{H_c(j\omega)}$$

$$= \left(1 + \frac{j\omega}{\pi}\right)^2 e^{-j\omega t_0}$$

where t_0 is the transmission delay. The critical issue in designing the equalizer is how to compensate for the dispersive nature of $H_c(j\omega)$.

For the problem at hand, we first expand $(1 + j\omega/\pi)^2$ as follows:

$$\left(1 + \frac{j\omega}{\pi}\right)^2 = 1 + \frac{2}{\pi}(j\omega) - \frac{1}{\pi}(\omega^2)$$

There are three parts to this expression that merit individual considerations:

1. The unity term, which represents an ideal filter.
2. The $j\omega$ term, which represents differentiation.
3. The $(j\omega)^2$ term, which represents double differentiation.

In example 8.8 we showed how to deal with the first two terms. In this problem we extend the material described therein by considering the $(j\omega)^2$ term.

We are interested in the use of an FIR filter of length M (even) for the equalization. So with $-\pi \le \Omega \le \pi$ as the frequency band of interest, let

$$I(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Omega^2 e^{jk\Omega}\, d\Omega \qquad (1)$$

where $k = n - \frac{M}{2}$. In effect, $\frac{M}{2}$ samples represent the constant transmission delay through the filter.

Performing the integration by parts (twice) on Eq. (1), we may write

$$-\frac{1}{\pi^2} I(k) = -\frac{1}{k\pi}\sin(k\pi) - \frac{2}{(k\pi)^2}\cos(k\pi) + \frac{2}{(k\pi)^3}\sin(k\pi) \quad (2)$$

where we have included the multiplying factor $-1/\pi^2$ that is associated with Ω^2.

We may now describe the composition of the impulse response $h_{eq}[n]$ of the equalizer:

1. A sinc function corresponding to the unity term.
2. An antisymmetric FIR filter whose mid-point coefficient is zero, accounting for the $j\omega$ term.
3. A symmetric FIR filter accounting for the $(j\omega)^2$ term in accordance with Eq. (2).

The sum of these three contributions is finally weighted by the Hamming window.

A word of caution is in order when implementing the equalizer on a computer. In particular, we have to take special care in computing the mid-point coefficient of the FIR filter. By definition, we have

$$h_{eq}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{eq}(j\Omega)\, e^{jn\Omega}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-nM/2}}{H_c(j\Omega)}\, e^{jn\Omega}\, d\Omega$$

Putting $n = 0$:

$$h_{eq}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{H_c(j\Omega)}\, d\Omega$$

For the problem at hand,

$$h_{eq}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 + \frac{j\Omega}{\pi}\right)^2 d\Omega$$

$$= \frac{2}{3}$$

Hence, in software implementation of the FIR equalizer, the code will have to include a provision for ensuring that $h_{eq}[0] = 2/3$. The MATLAB code for this implementation is presented on the next page.

(Note: In Problem 8.28 the channel frequency response $H_c(j\omega)$ should be referred to as a second-order rational function.)

```
w=0:pi/511:pi;
den=sqrt(1+2*((w/pi).^2)+(w/pi).^4);
Hchan=1./den;
taps=95;M=taps-1;
n=0:M;  f=n-M/2;

%---------
% term 1
%---------
   hh1=fftshift(ifft(ones(taps,1))).*hamming(taps);

%---------
% term 2
%---------
   a=cos(pi*f)./f;   b=sin(pi*f)./(pi*f.^2);
   h=a-b;   k=isnan(h); h(k)=0;
   hh2=2*(hamming(taps)'.*h)/pi;
%---------
% term 3
%---------

   hh3a = -(1./(pi*f))        .*  sin(pi*f);
   hh3b = -(2./(pi*f).^2)     .*  cos(pi*f);
   hh3c =  (2./(pi*f).^3)     .*  sin(pi*f);


   hh3=hh3a+hh3b+hh3c;
   hh3=hamming(taps).*hh3';
   hh=hh1'+hh2+hh3';
   hh(48)=2/3;


[Heq,w]=freqz(hh,1,512,2*pi);

p = 0.7501;

Hcheq=(p*abs(Heq)).*Hchan';
plot(w,p*abs(Heq),'b--')
hold on
plot(w,abs(Hchan),'g-.')
plot(w,abs(Hcheq),'r-')
legend('Heq','Hchan','Hcheq',1)
hold off
```

Figure 1 on the next page presents performance evaluation of the equalizer. Specifically, three magnitude responses are included:

- The dashed-dotted curve corresponds to the channel.
- The dashed line curve corresponds to the equalizer.
- The solid curve corresponds to the cascade connection of the channel and the equalizer.

This figure demonstrates almost perfect equalization of the channel over a large fraction of the passband $0 \le |\Omega| \le \pi$, except for two minor modifications:

1. Amplitude scaling
2. Constant transmission delay.

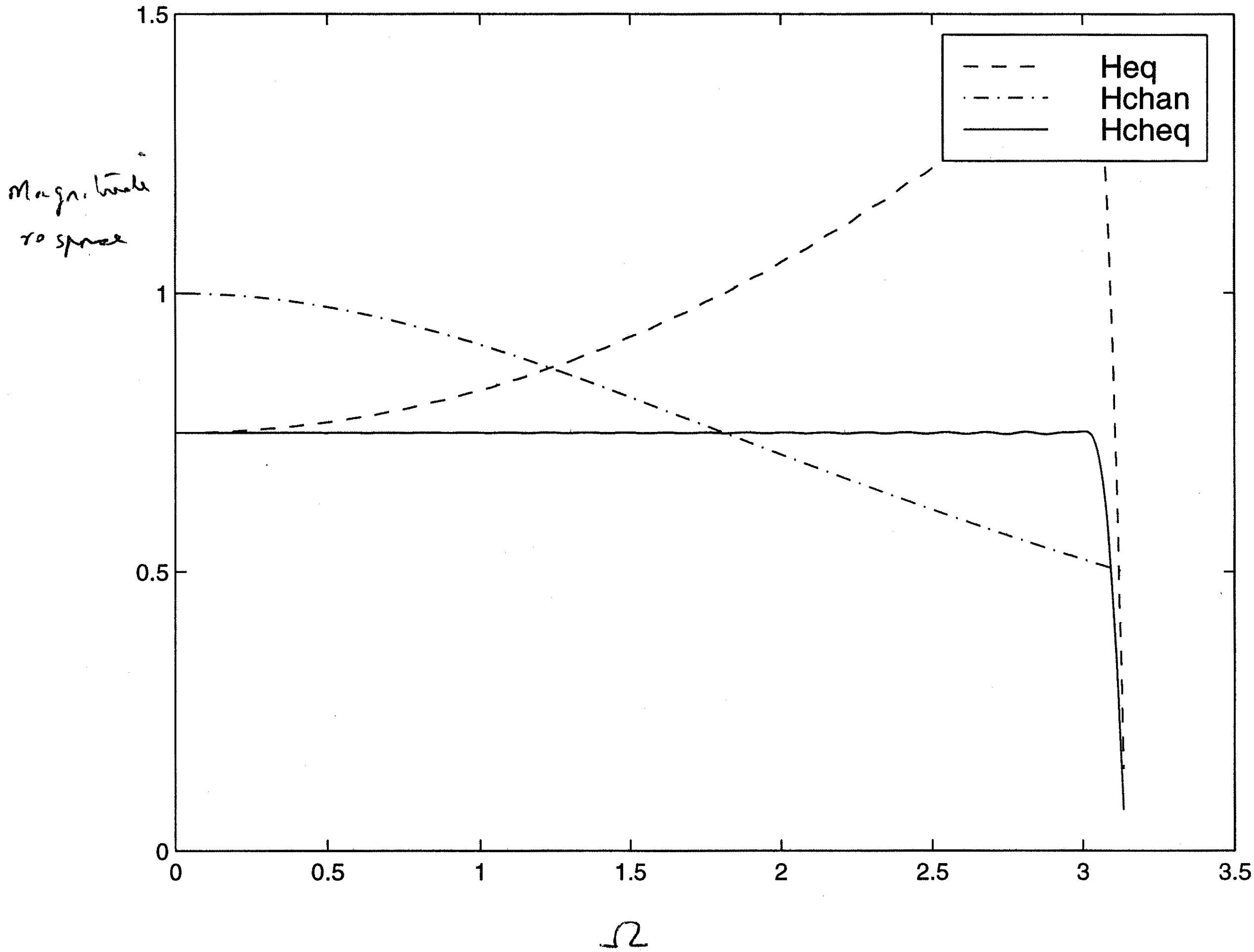


Figure 1