CHAPTER 3

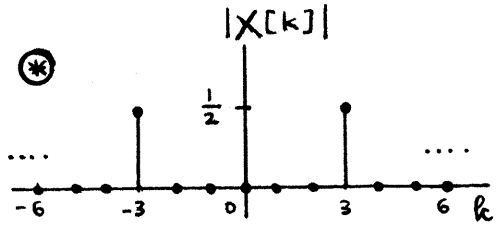
$\boxed{3.1}$

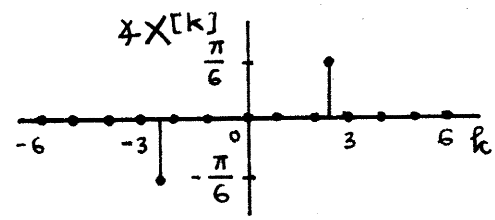
(a) $x[n] = \cos\left(\frac{6\pi}{13}n + \frac{\pi}{6}\right) = \dfrac{e^{j\frac{\pi}{6}} \cdot e^{j\frac{6\pi}{13}n} + e^{-j\frac{\pi}{6}} \cdot e^{-j\frac{6\pi}{13}n}}{2}$

$N = 13 \longrightarrow \Omega_0 = \frac{2\pi}{13}$ choose $n, k \in \{-6, -5, \ldots, 6\}$

$x[3] = \frac{1}{2} e^{j\frac{\pi}{6}}$; $x[-3] = \frac{1}{2} e^{-j\frac{\pi}{6}}$; $X[k] = 0,\ k \neq \pm 3$
$k \in \{-6, \ldots, 6\}$

⊛





(b) $x[n] = \sin\left(\frac{4\pi}{21}n\right) + \cos\left(\frac{10\pi}{21}n\right) + 1$

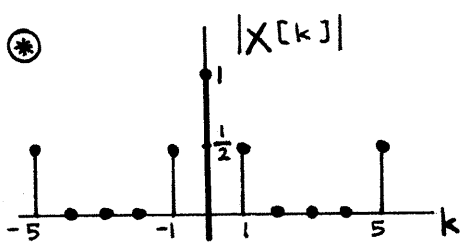
$= -\frac{j}{2}\left(e^{j\frac{4\pi}{21}n} - e^{-j\frac{4\pi}{21}n}\right) + \frac{1}{2}\left(e^{j\frac{10\pi}{21}n} + e^{-j\frac{10\pi}{21}n}\right) + 1$

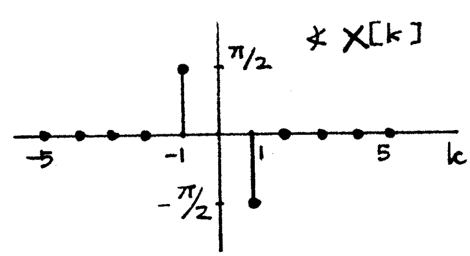
$\Omega_0 = \gcd\left(\frac{4\pi}{21}, \frac{10\pi}{21}\right) = \frac{2\pi}{21}$

$N = 21$ Choose $n, k \in \{-10, \ldots, 10\}$

$x[-5] = x[5] = \frac{1}{2}$
$x[-1] = \frac{1}{2} j$; $x[1] = -\frac{1}{2} j$
$x[0] = 1$
$k \in \{-10, \ldots, 10\}$

⊛





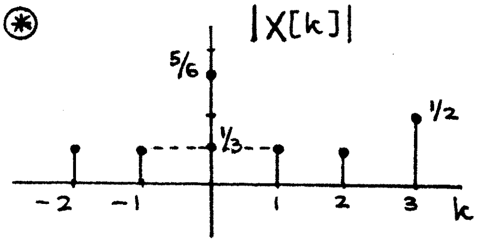
(c) $x[n] = \sum_{m=-\infty}^{\infty} \delta[n-2m] + \delta[n+3m] \qquad N=6 \rightarrow \Omega_0 = \frac{\pi}{3}$

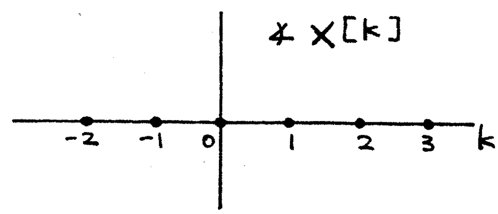
Choose $n, k \in \{-2, \ldots, 3\}$

$$X[k] = \frac{1}{6}\left[e^{jk\frac{\pi}{3}(2)} + 2 + e^{-jk\frac{\pi}{3}(2)} + e^{-jk\frac{\pi}{3}(3)}\right]$$

$$= \frac{1}{6}\left[2 + 2\cos\left(\frac{2\pi}{3}k\right) + (-1)^k\right] \qquad k \in \{-2, \ldots, 3\}$$

$\circledast$



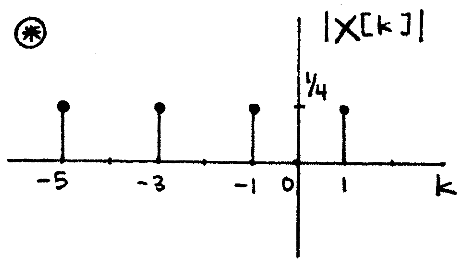


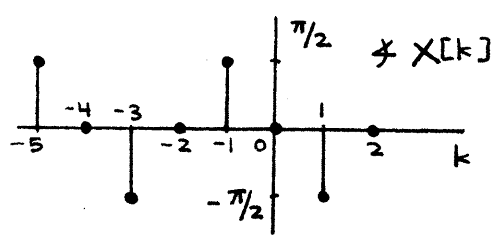
(d) $N = 8 \longrightarrow \Omega_0 = \frac{\pi}{4}$
choose $n, k \in \{-5, \ldots, 2\}$

$$X[k] = \frac{1}{8}\left(-e^{jk\frac{\pi}{4}(2)} + e^{-jk\frac{\pi}{4}(2)}\right)$$

$$= -\frac{j}{4}\sin\left(\frac{\pi}{2}k\right) \qquad , \; k \in \{-5, \ldots, 2\}$$

$\circledast$





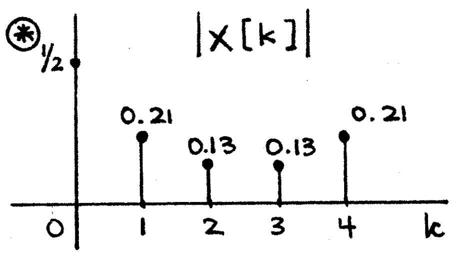
(e) $N = 5 \longrightarrow \Omega_0 = \frac{2\pi}{5}$
choose $n, k \in \{0, \ldots, 4\}$

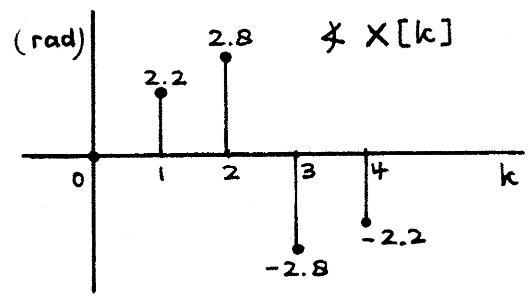
$$X[k] = \frac{1}{5}\left(e^{-j\frac{2\pi}{5}k} + 2e^{-j\frac{4\pi}{5}k} + 3e^{-j\frac{6\pi}{5}k} + 4e^{-j\frac{8\pi}{5}k}\right)\frac{1}{4}$$

$$X[k] = \frac{1}{20}\left(e^{-j\frac{2\pi}{5}k} + 2e^{-j\frac{4\pi}{5}k} + 3e^{-j\frac{6\pi}{5}k} + 4e^{-j\frac{8\pi}{5}k}\right)$$

$$k \in \{0, \dots, 4\}$$

⊛





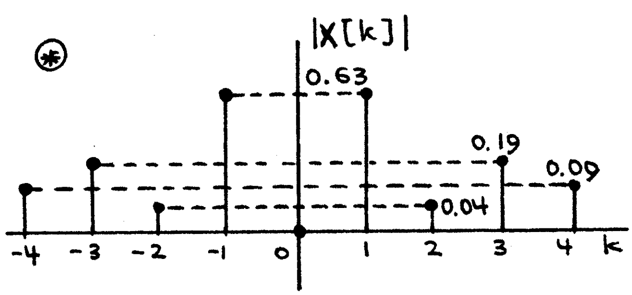
(f) $N = 9 \longrightarrow \Omega_0 = \frac{2\pi}{9}$

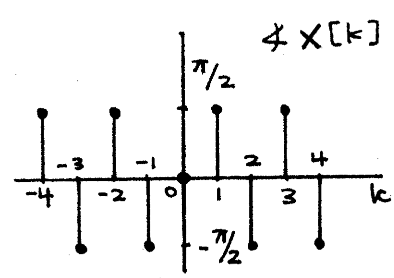
choose $n, k \in \{-4, \dots, 4\}$

$$X[k] = \frac{1}{9}(2j)\left(\sin\left(\frac{2\pi}{9}k\right) + \sin\left(\frac{4\pi}{9}k\right) + \sin\left(\frac{6\pi}{9}k\right) + \sin\left(\frac{8\pi}{9}k\right)\right)$$

$$k \in \{-4, \dots, 4\}$$

⊛





<u>Note</u> : ⊛ All drawings are only for one period time / frequency

$\boxed{3.2}$

(a) $x[k] = \cos\left(\frac{6\pi}{17}k\right)$; $N = 17 \longrightarrow \Omega_0 = \frac{2\pi}{17}$

choose $n, k \in \{-8, \dots, 8\}$

$$x[n] = \sum_{k=-8}^{8} \cos\left(\frac{6\pi}{17}k\right) e^{jkn\frac{2\pi}{17}} = \frac{1}{2}\sum_{k=-8}^{8} e^{jk\frac{2\pi}{17}(n+3)} + e^{jk\frac{2\pi}{17}(n-3)}$$

$$x[n] = \begin{cases} 0 & n \neq \pm 3 \\ \frac{17}{2} & n = \pm 3 \end{cases} \quad \text{since} \quad \sum_{k=-8}^{8} e^{jk\frac{2\pi}{17}(n+3)} = 0 \quad n \neq -3$$

$n \in \{-8, -7, \cdots 7, 8\}$

(b) $X[k] = \cos\left(\frac{10\pi}{21}k\right) + j\sin\left(\frac{4\pi}{21}k\right)$

$N = 21 \longrightarrow \Omega_0 = \frac{2\pi}{21}$

choose $n, k \in \{0, \ldots, 20\}$

$$x[n] = \frac{1}{2}\sum_{k=0}^{20}\left\{e^{j\frac{2\pi}{21}(n+5)k} + e^{j\frac{2\pi}{21}(n-5)k} + e^{j\frac{2\pi}{21}(n+2)k} - e^{j\frac{2\pi}{21}(n-2)k}\right\}$$

Now $\displaystyle\sum_{k=0}^{20} e^{j\frac{2\pi}{21}(n+n_0)k} = \frac{1 - e^{j2\pi(n+n_0)}}{1 - e^{j\frac{2\pi}{21}(n+n_0)}} \quad n \neq -n_0$

$= 0 \quad n \neq -n_0$

So $x[n] = \begin{cases} 21/2 & n = \pm 5, -2 \\ -21/2 & n = 2 \\ 0 & \text{otherwise} \end{cases} \quad n \in \{-10, -9, \ldots, 9, 10\}$

(c) $X[k] = \sum_{m=-\infty}^{\infty} \delta[k-2m] - 2\delta[k+3m]$

$N = 6 \longrightarrow \Omega_0 = \frac{\pi}{3}$, choose $n, k \in \{-2, \ldots, 3\}$

$x[n] = e^{-jn\frac{\pi}{3}(2)} - 1 + e^{jn\frac{\pi}{3}(2)} - 2e^{jn\frac{\pi}{3}(3)}$

$x[n] = 2\cos\left(\frac{2\pi}{3}n\right) - 1 - 2(-1)^n \quad , n \in \{-2, \ldots, 3\}$

(d) $N = 7 \longrightarrow \Omega_0 = \frac{2\pi}{7}$

choose $n, k \in \{-3, \ldots, 3\}$

$x[n] = j e^{-j\frac{6\pi}{7}n} - j e^{j\frac{6\pi}{7}n} = 2\sin\left(\frac{6\pi n}{7}\right)$

$n \in \{-3, \ldots, 3\}$

(e) $N = 7 \longrightarrow \Omega_0 = \frac{2\pi}{7}$

choose $n, k \in \{-3, \ldots, 3\}$

$x[n] = e^{-j\frac{2\pi}{7}n} - \frac{1}{2} + e^{j\frac{2\pi}{7}n} = 2\cos\left(\frac{2\pi n}{7}\right) - \frac{1}{2}$

$n \in \{-3, \ldots, 3\}$

(f) $N = 12 \longrightarrow \Omega_0 = \frac{\pi}{6}$

choose $n, k \in \{-5, \ldots, 6\}$

$X[k] = e^{-j\frac{\pi}{6}k}$

$x[n] = \sum_{k=-4}^{4} e^{j\frac{\pi}{6}(n-1)k} = \frac{\sin\left(\frac{3\pi}{4}(n-1)\right)}{\sin\left(\frac{\pi}{12}(n-1)\right)}$

$n \in \{-5, \ldots, 6\}$

$\boxed{3.3}$

(a) $x(t) = \sin(2\pi t) + \cos(3\pi t)$

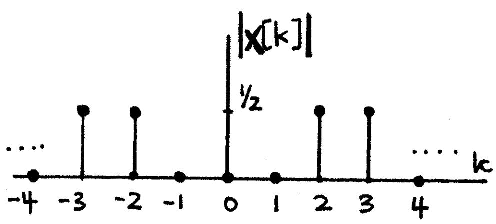
$$= \frac{e^{j2\pi t} - e^{-j2\pi t}}{2j} + \frac{e^{j3\pi t} + e^{-j3\pi t}}{2}$$

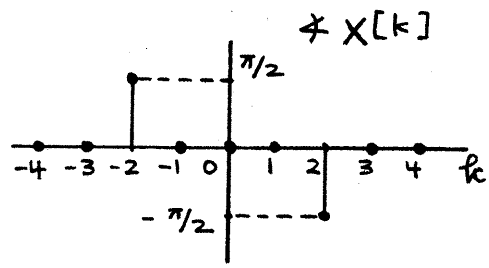
$$\omega_0 = \gcd(2\pi, 3\pi) = \pi$$

$$x[2] = \frac{1}{2j} \quad ; \quad x[-2] = -\frac{1}{2j}$$

$$x[3] = x[-3] = \frac{1}{2}$$

$$X[k] = 0 \quad , \quad k \neq \pm 2, \pm 3$$



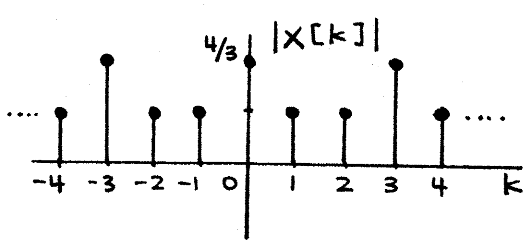


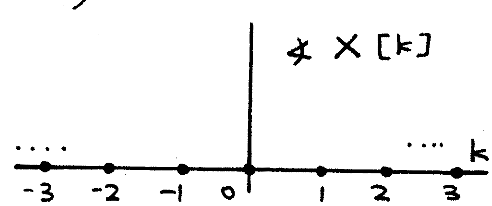
(b) $x(t) = \sum_{m=-\infty}^{\infty} \delta(t - \frac{1}{2}m) + \delta(t - \frac{3}{2}m)$

$$T = \frac{3}{2} \longrightarrow \omega_0 = \frac{4\pi}{3}$$

$$X[k] = \frac{2}{3}\left(e^{j\frac{2\pi}{3}k} + 2 + e^{-j\frac{2\pi}{3}k}\right)$$

$$= \frac{4}{3}\left(1 + \cos\left(\frac{2\pi}{3}k\right)\right)$$





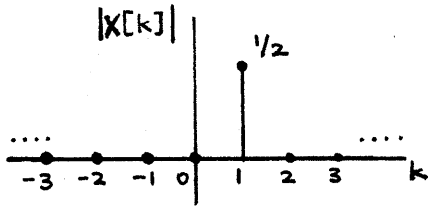
(c) $x(t) = \sum_{m=-\infty}^{\infty} e^{j\frac{2\pi}{3}m} \delta(t - 2m)$

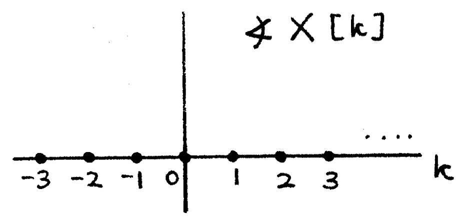
$$T = 6 \longrightarrow \omega_o = \frac{\;}{3}$$

$$X[k] = \frac{1}{6}\left(1 + e^{j\frac{2\pi}{3}} e^{-j\frac{2\pi}{3}k} + e^{j\frac{4\pi}{3}} e^{-j\frac{4\pi}{3}k}\right)$$

$$X[k] = \frac{1}{6}\left(1 + e^{j\frac{2\pi}{3}(1-k)} + e^{j\frac{4\pi}{3}(1-k)}\right)$$

$$X[k] = \frac{1}{2}\,\delta(k)$$





(d) $T = 2 \longrightarrow \omega_o = \pi$

$$X[k] = \frac{1}{2}\int_0^1 \frac{e^{j2\pi t} - e^{-j2\pi t}}{j2} . e^{-jk\pi t}\, dt$$

$\underline{k=0}$: $X[0] = \frac{1}{2}\int_0^1 \sin 2\pi t \,.\, dt = 0$

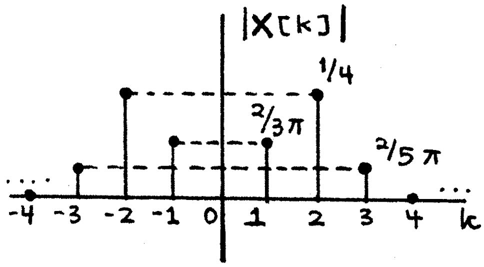
$\underline{k=2}$: $X[2] = \frac{1}{j.4}\int_0^1 1 - e^{-j4\pi t}\, dt = \frac{1}{j4}$

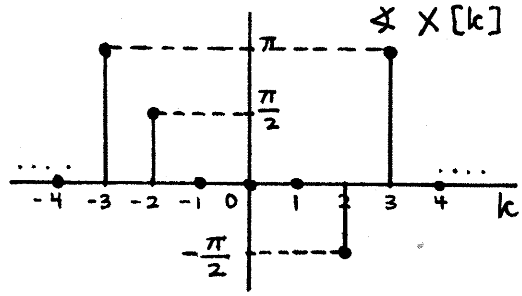
$\underline{k=-2}$: $X[-2] = \frac{1}{j.4}\int_0^1 e^{j4\pi} - 1\, dt = -\frac{1}{j4}$

$\underline{k \neq 0, \pm 2}$: $X[k] = \frac{1}{j4}\int_0^1 e^{j\pi(2-k)t} - e^{-j\pi(2+k)t}\, dt$

$$X[k] = \frac{1-(-1)^k}{\pi(4-k)^2}$$

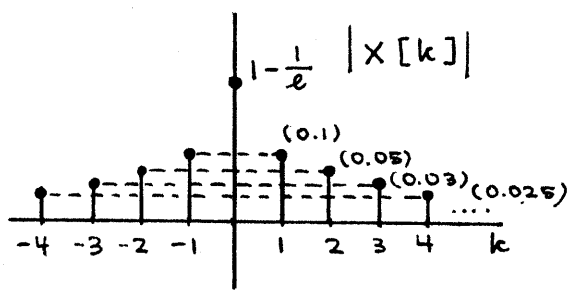
$$\therefore X[k] = \begin{cases} 0 & , \; k = 0 \\ \pm\frac{1}{j4} & , \; k = \pm 2 \\ \frac{1-(-1)^k}{\pi(4-k^2)} & , \; \text{otherwise} \end{cases}$$

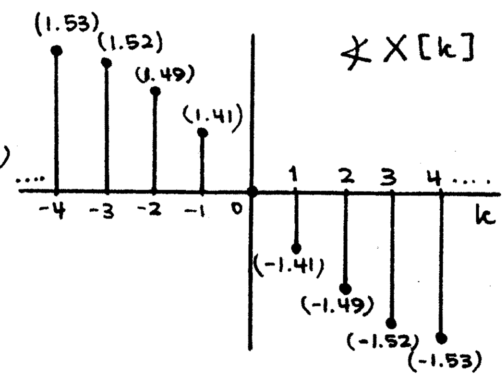




(e) $T = 1 \longrightarrow \omega_0 = 2\pi$

$$X[k] = \int_0^1 e^{-(1+j2\pi k)t}\, dt = \frac{1 - e^{-1}}{1 + j2\pi k}$$





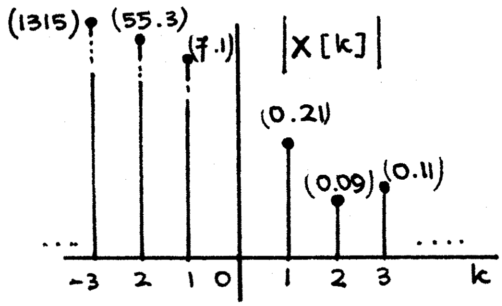
(f) $T = 3 \longrightarrow \omega_0 = \frac{2\pi}{3}$

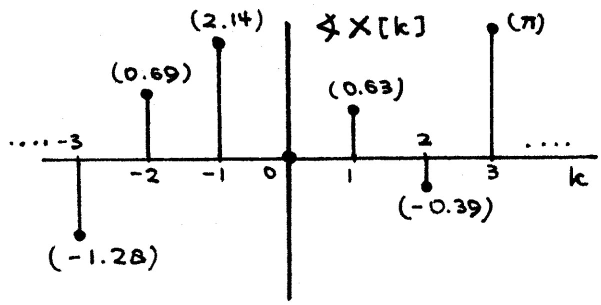
$$x[k] = \frac{1}{3}\left(\int_{-1}^{1} t\, e^{-j\frac{2\pi}{3}kt}\, dt + \int_{1}^{2} (3-2t)\, e^{-j\frac{2\pi}{3}kt}\, dt\right)$$

$\underline{k=0}$: $X[0] = \frac{1}{3}\left(\int_{-1}^{1} t\, dt + \int_{1}^{2} (3-2t)\, dt\right) = 0$

$\underline{k \neq 0}$, $$X[k] = \frac{\left(j2\sin a - 2a\cos a + 2a e^{-\frac{3}{2}a} \cos\frac{1}{2}a - j4e^{-\frac{3}{2}a}\sin\frac{1}{2}a\right)}{3a^2}$$

where $a = \frac{2\pi}{3}k$





$\boxed{3.4}$

(a) $X[k] = j\,\delta[k-1] - j\,\delta[k+1] + \delta[k-3] + \delta[k+3], \quad \omega_0 = \pi$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{jk\pi t} = j e^{j\pi t} - j e^{-j\pi t} + e^{j3\pi t} + e^{-j3\pi t}$$

$$x(t) = -2\sin(\pi t) + 2\cos(3\pi t)$$

(b) $X[k] = j\,\delta[k-1] - j\,\delta[k+1] + \delta[k-3] + \delta[k+3], \quad \omega_0 = 3\pi$

$$x(t) = j e^{j3\pi t} - j e^{-j3\pi t} + e^{j9\pi t} + e^{-j9\pi t}$$

$$x(t) = -2\sin(3\pi t) + 2\cos(9\pi t)$$

(c) $X[k] = \left(-\frac{1}{2}\right)^{|k|}, \quad \omega_0 = 1$

$$x(t) = \sum_{k=0}^{\infty} \left(-\tfrac{1}{2}\right)^{k} e^{jkt} + \sum_{k=-\infty}^{-1} \left(-\tfrac{1}{2}\right)^{-k} e^{jkt}$$

$$x(t) = \sum_{k=0}^{\infty} \left(-\tfrac{1}{2} e^{jt}\right)^{k} + \sum_{k=1}^{\infty} \left(-\tfrac{1}{2} e^{-jt}\right)^{k}$$

Since $\left|-\frac{1}{2} e^{\pm jt}\right| < 1$ always, the sum converges

$$x(t) = \frac{1}{1+\frac{1}{2}e^{jt}} - \frac{\frac{1}{2}e^{-jt}}{1-\frac{1}{2}e^{-jt}} = \frac{\frac{3}{4} - e^{-jt}}{\frac{3}{4} - j\sin t}$$

(d) $\omega_0 = \pi$

$$X[3] = e^{-j\frac{\pi}{4}} \quad , X[-3] = e^{j\frac{\pi}{4}}$$

$$X[4] = 2e^{j\frac{\pi}{8}} \quad , X[-4] = 2e^{-j\frac{\pi}{8}}$$

$$x(t) = e^{-j\frac{\pi}{4}} \cdot e^{j3\pi t} + e^{j\frac{\pi}{4}} \cdot e^{-j3\pi t} + 2e^{j\frac{\pi}{8}} \cdot e^{j4\pi t} + 2e^{-j\frac{\pi}{8}} e^{-j4\pi t}$$

$$x(t) = 2\cos\left(3\pi t - \frac{\pi}{4}\right) + 4\cos\left(4\pi t + \frac{\pi}{8}\right)$$

(e) $\omega_0 = 2\pi$

$$X[k] = e^{-j2\pi k} \qquad -4 \leq k \leq 4$$

$$x(t) = \sum_{k=-4}^{4} e^{j2\pi k(t-1)} = \frac{\sin(9\pi t)}{\sin(\pi t)}$$

(f) $\omega_0 = \pi$

$$X[k] = |k| \qquad -3 \leq k < 3$$

$$x(t) = 2\cos(\pi t) + 4\cos(2\pi t) + 6\cos(6\pi t)$$

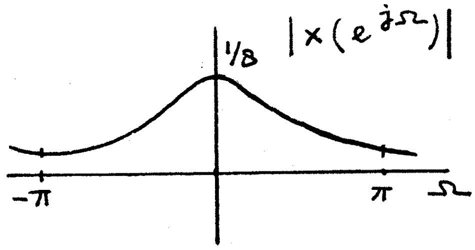
$\boxed{3.5}$

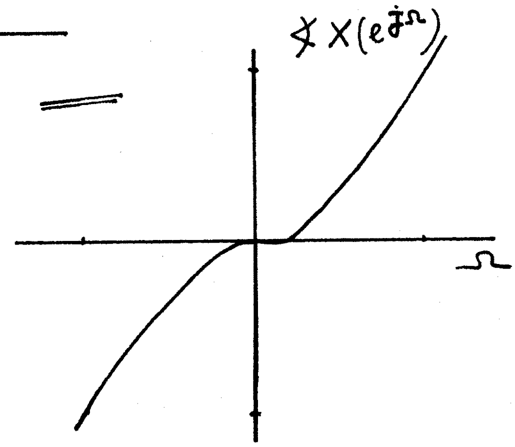
(a) $x[n] = \left(\frac{1}{2}\right)^n u[n-4]$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} = \sum_{n=4}^{\infty} \left(\tfrac{1}{2} e^{-j\Omega}\right)^n$$

$$, \left|\tfrac{1}{2} e^{-j\Omega}\right| < 1 \text{ always}$$

$$X(e^{j\Omega}) = \frac{\left(\frac{1}{2}e^{-j\Omega}\right)^4}{1-\frac{1}{2}e^{-j\Omega}}$$





⊛ only one period !

$$|X(e^{j\Omega})| = \frac{\left(\frac{1}{2}\right)^4}{\sqrt{\left(1-\frac{1}{2}\cos\Omega\right)^2+\left(\frac{1}{2}\sin\Omega\right)^2}}$$

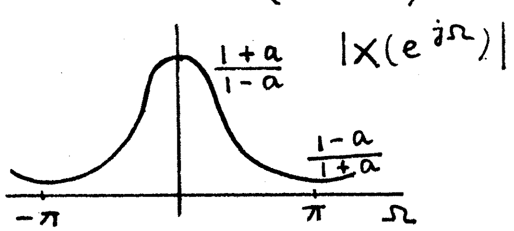
$$\measuredangle X(e^{j\Omega}) = \Omega - \tan^{-1}\left(\frac{\sin\Omega}{2-\cos\Omega}\right)$$

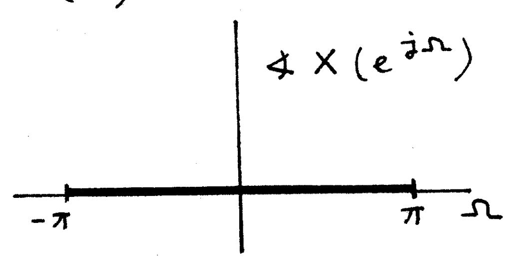
(b) $x[n] = a^{|n|} \qquad |a| < 1$

$$X(e^{j\Omega}) = \sum_{n=0}^{\infty}\left(a e^{-j\Omega}\right)^n + \sum_{n=-1}^{-\infty}\left(a e^{j\Omega}\right)^{-n}$$

$$X(e^{j\Omega}) = \frac{1}{1-ae^{-j\Omega}} + \frac{ae^{j\Omega}}{1-ae^{j\Omega}}$$

$$= \frac{1-a^2}{(1+a^2)-2a\cos(\Omega)}$$





$|X(e^{j\Omega})| = X(e^{j\Omega}) \qquad \measuredangle X(e^{j\Omega}) = 0$

⊛ only one period !

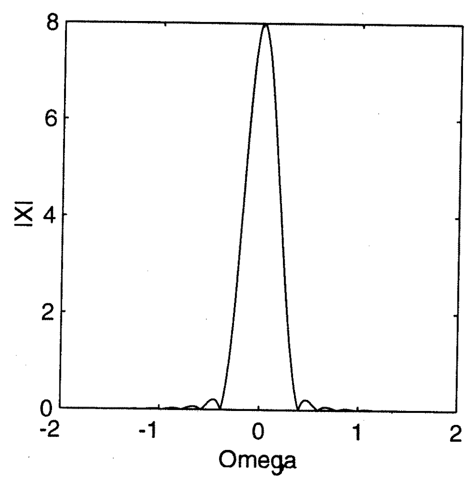
(c) $x[n] = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{N}\cdot n\right) \qquad |n| \leqslant N$

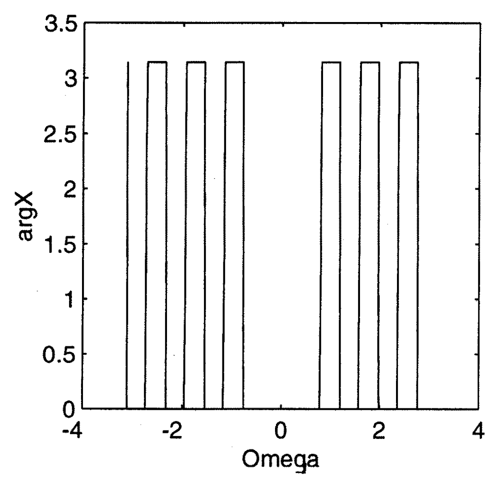
$$X(e^{j\Omega}) = \frac{1}{2}\sum_{n=N}^{N}\left(1 + \frac{e^{j\frac{\pi}{N}n} + e^{-j\frac{\pi}{N}n}}{2}\right)e^{-j\Omega n}$$

$$X(e^{j\Omega}) = \frac{1}{2}\frac{\sin\left(\left(N+\frac{1}{2}\right)\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)} + \frac{1}{4}\frac{\sin\left(\frac{\pi}{2N} - \Omega N - \frac{1}{2}\Omega\right)}{\sin\left(\frac{\pi}{2N} - \frac{1}{2}\Omega\right)}$$

$$- \frac{1}{4}\cdot\frac{\sin\left(\frac{\pi}{2N} + \Omega N + \frac{1}{2}\Omega\right)}{\sin\left(\frac{\pi}{2N} + \frac{1}{2}\Omega\right)}$$

Note : (*) Only one period ! Assuming $N=7$





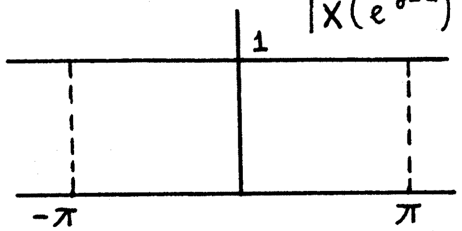
(d) $x[n] = \delta[6-3n]$

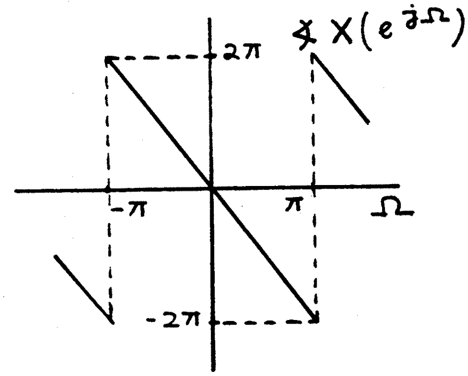
$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty}\delta[6-3n]e^{-j\Omega n} = e^{-j2\Omega}$$

$$|X(e^{j\Omega})| = 1$$

$$\measuredangle X(e^{j\Omega}) = -2\Omega$$

Note : Only one period!





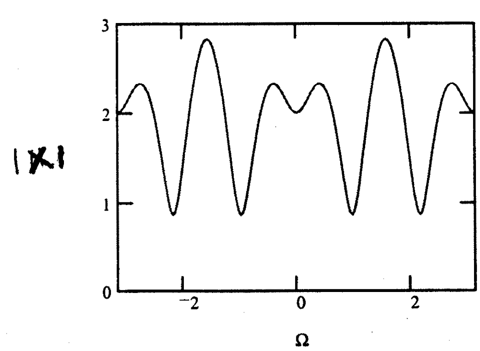
(e) $X(e^{j\Omega}) = e^{j3\Omega} + e^{j2\Omega} + e^{-j2\Omega} - e^{-j3\Omega}$

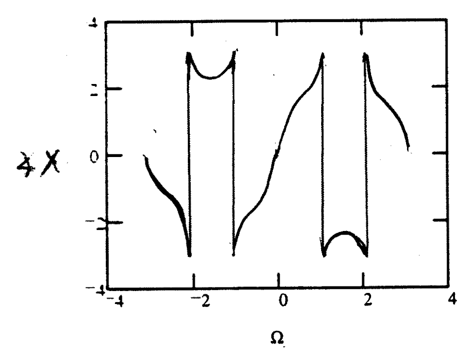
$$X(e^{j\Omega}) = 2\cos(2\Omega) + j\,2\sin(3\Omega)$$

$$|X(e^{j\Omega})| = 2\sqrt{\cos^2(2\Omega) + \sin^2(3\Omega)}$$

$$\measuredangle X(e^{j\Omega}) = +\tan^{-1}\left(\frac{\sin(3\Omega)}{\cos(2\Omega)}\right)$$

Note : only one period !



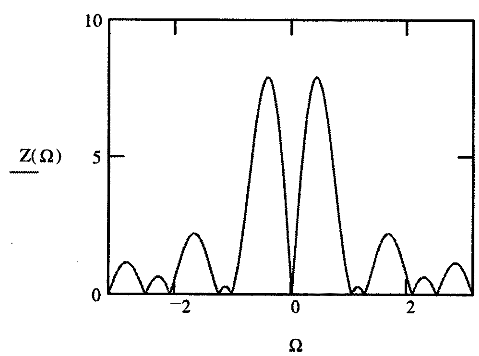


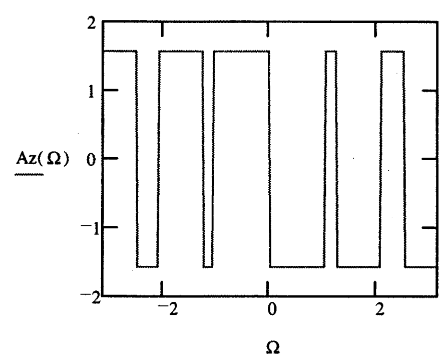
(f) $X(e^{j\Omega}) = -j2\big(\sin(\Omega) + \sin(2\Omega) + \sin(3\Omega) + \sin(4\Omega) + \sin(5\Omega)\big)$

$$|X(e^{j\Omega})| = 2\,\big|\sin(\Omega) + \sin(2\Omega) + \sin(3\Omega) + \sin(4\Omega) + \sin(5\Omega)\big|$$

$$\measuredangle X(e^{j\Omega}) = -\frac{\pi}{2}\,\mathrm{sgn}\left(\frac{X(e^{j\Omega})}{-j2}\right)$$

Note : Only one period !





$\boxed{3.6}$

(a) $X(e^{j\Omega}) = \cos(\Omega) + j\sin(\Omega) = e^{j\Omega}$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\,e^{j\Omega n}\,d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(n+1)}\,d\Omega$$

$n = -1 \quad : \quad x[-1] = 1$

$n \neq 1 \quad : \quad x[n] = 0$

$$\therefore\ x[n] = \underline{\underline{\delta[n+1]}}$$

(b) $X(e^{j\Omega}) = \sin(\Omega) + \cos\left(\frac{\Omega}{2}\right)$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{+j\Omega} - e^{-j\Omega}}{j2} + \frac{e^{j\frac{\Omega}{2}} + e^{-j\frac{\Omega}{2}}}{2}\right)e^{j\Omega n}\,d\Omega$$

, n discrete

$$= \frac{1}{2j}\delta[n+1] - \frac{1}{2j}\delta[n-1] + \frac{1}{2\pi}\frac{\cos\pi n}{(n+1/2)} - \frac{1}{2\pi}\frac{\cos\pi n}{n-1/2}$$

(c) $X(e^{j\Omega}) = e^{-j4\Omega} \qquad \frac{\pi}{2} < |\Omega| < \pi$

$$x[n] = \frac{1}{2\pi}\left(\int_{\pi}^{-\frac{\pi}{2}} e^{j\Omega(n-4)}\,d\Omega + \int_{\frac{\pi}{2}}^{\pi} e^{j\Omega(n-4)}\,d\Omega\right)$$

$$x[n] = \frac{-\sin\left(\frac{\pi}{2}n\right)}{\pi(n-4)} \quad , \underline{n \neq 4}$$

$$\underline{n=4} \quad , x[4] = \frac{1}{2\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{1}{2}$$

$$\therefore X(e^{j\Omega}) = \begin{cases} \dfrac{\sin\left(\frac{\pi}{2}n\right)}{\pi(4-n)} & , n \neq 4 \\ \dfrac{1}{2} & , n = 4 \end{cases}$$

(d) $x[n] = \frac{1}{2\pi}\left(\int_{-\pi}^{0} e^{(jn+1)\Omega}\,d\Omega + \int_{0}^{\pi} e^{(jn-1)\Omega}\,d\Omega\right)$

$$x[n] = \frac{[1+(-1)^n]}{\pi(n^2+1)}$$

(e) $x[n] = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -\sin(\Omega) e^{j2\Omega} e^{jn\Omega}\,d\Omega + \int_{0}^{\pi} \sin(\Omega) \cdot e^{j2\Omega} e^{jn\Omega}\,d\Omega\right)$

$$x[n] = \frac{1}{2\pi}\left(-\int_{-\pi}^{0} \frac{e^{j(n+3)\Omega} - e^{j(n+1)\Omega}}{j2}\,d\Omega + \int_{0}^{\pi} \frac{e^{j(n+3)\Omega} - e^{j(n+1)\Omega}}{j2}\,d\Omega\right)$$

$\underline{n=-1}$: $x[-1] = \frac{1}{2\pi}\left(-\int_{-\pi}^{0} \frac{e^{j2\Omega}-1}{j2}\, d\Omega + \int_{0}^{\pi} \frac{e^{j2\Omega}-1}{j2}\, d\Omega\right) = 0$

$\underline{n=-3}$: $x[-3] = \frac{1}{2\pi}\left(-\int_{-\pi}^{0} \frac{1-e^{-j2\Omega}}{j2}\, d\Omega + \int_{0}^{\pi} \frac{1-e^{-j2\Omega}}{j2}\, d\Omega\right) = 0$

$\underline{n \neq -1,-3}$: $x[n] = \left(1-(-1)^{n+1}\right)\left(\frac{-1}{\pi(n+1)(n+3)}\right)$

(f) $x[n] = \frac{1}{2\pi}\left(\int_{-\frac{\pi}{2}}^{0} e^{j\Omega n}\, d\Omega - \int_{0}^{\pi/2} e^{j\Omega n}\, d\Omega\right)$

$\underline{n=0}$: $x[0] = 0$

$\underline{n \neq 0}$: $x[n] = \frac{2}{jn}\left(1 - \cos\left(\frac{\pi}{2} n\right)\right)$

$\boxed{3.7}$

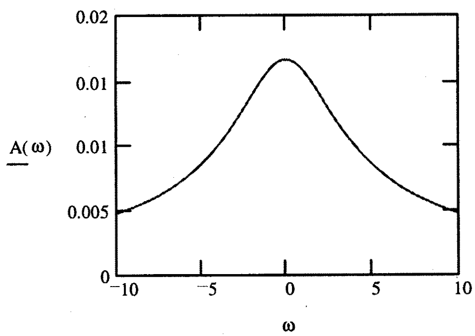
(a) $x(t) = e^{-3t}\, u(t-1)$

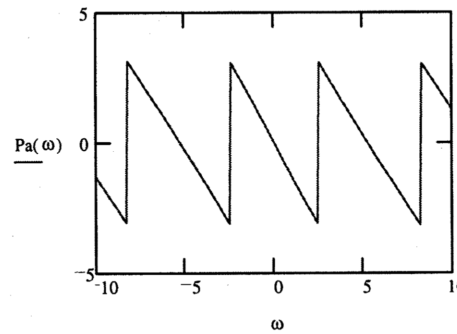
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{1}^{\infty} e^{-(j\omega+3)t}\, dt$$

$$X(j\omega) = \frac{e^{-(j\omega+3)}}{j\omega+3}$$

$$|X(j\omega)| = \frac{e^{-3}}{\sqrt{\omega^2+9}}$$

$$\measuredangle X(j\omega) = -\omega - \tan^{-1}\left(\frac{\omega}{3}\right)$$



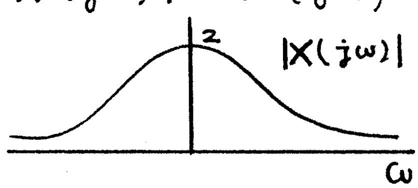


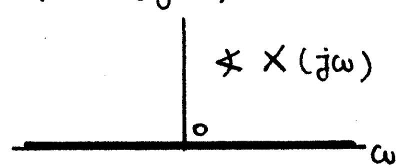
(b) $x(t) = e^{-|t|}$

$$X(j\omega) = \int_{-\infty}^{0} e^{(1-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(1+j\omega)t}\, dt$$

$$X(j\omega) = \frac{2}{\omega^2 + 1}$$

$$|X(j\omega)| = X(j\omega) \qquad \measuredangle X(j\omega) = 0$$





(c) $x(t) = t\, e^{-2t} u(t)$

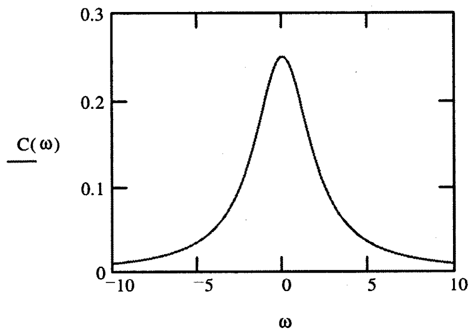
$$X(j\omega) = \int_{0}^{\infty} t\, e^{-(2+j\omega)t}\, dt$$

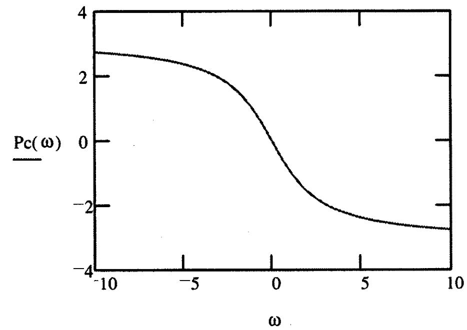
doing integral by parts

$$X(j\omega) = \frac{1}{(2+j\omega)^2}$$

$$|x(j\omega)| = \frac{1}{\sqrt{(4-\omega^2)^2 + (4\omega)^2}}$$

$$\measuredangle X(j\omega) = -\tan^{-1}\left(\frac{4\omega}{4-\omega^2}\right)$$





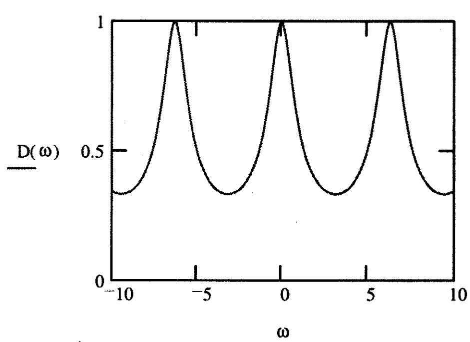
(d) $x(t) = \sum_{m=0}^{\infty} a^m \delta(t-m) \qquad |a| < 1$

$$X(j\omega) = \int_{-\infty}^{\infty} \left(\sum_{m=0}^{\infty} a^m \delta(t-m) \right) e^{-j\omega t}\, dt$$

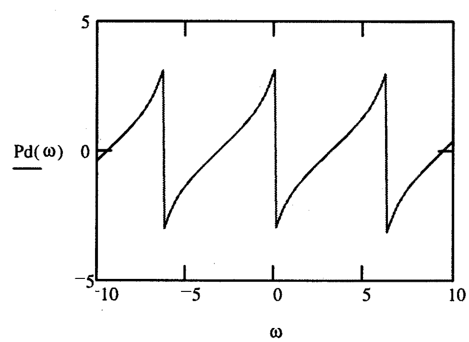
$$X(j\omega) = \sum_{m=0}^{\infty} \left(a e^{-j\omega} \right)^m$$

$$= \frac{1}{1 - a.e^{-j\omega}}$$

$$|X(j\omega)| = \frac{1}{\sqrt{(1-a\cos\omega)^2 + (a\sin\omega)^2}}$$



$$\measuredangle X(j\omega) = -\tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega} \right)$$

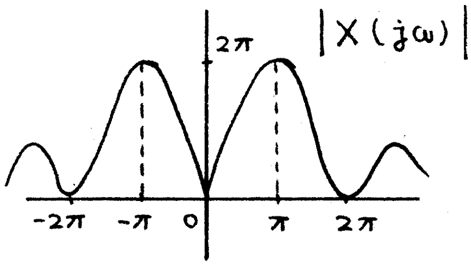


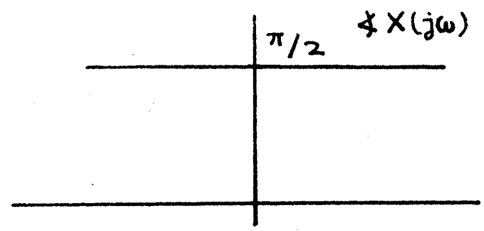
(e) $X(j\omega) = \int_{1}^{0} e^{-j\omega t}\,dt - \int_{0}^{1} e^{-j\omega t}\,dt$

$X(0) = 0$

$\omega \neq 0 \; : \; X(j\omega) = \dfrac{2(\cos(\omega)-1)}{j\omega}$

$|X(j\omega)| = \dfrac{2(1-\cos(\omega))}{\omega}$ $\qquad\qquad$ $\measuredangle X(j\omega) = \dfrac{\pi}{2}$



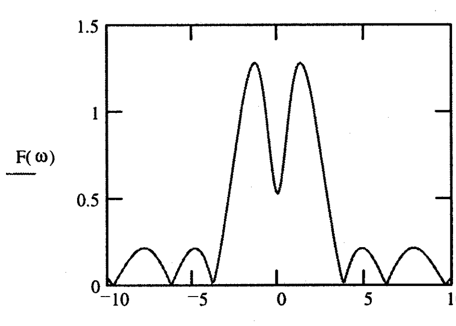


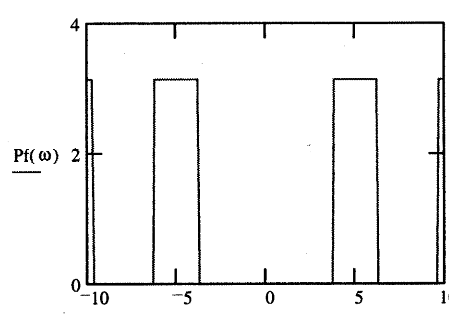
(f) $X(j\omega) = \int_{-1}^{0} e^{(1-j\omega)t}\,dt + \int_{0}^{1} e^{-(1+j\omega)t}\,dt$

$X(j\omega) = \dfrac{2}{\omega^2+1}\left(1 + 2e^{-1}(\omega\sin(\omega) - \cos(\omega))\right)$

$|X(j\omega)| = \dfrac{2}{\omega^2+1}\left|1 + 2e^{-1}(\omega\sin(\omega) - \cos(\omega))\right|$

$\measuredangle X(j\omega) = \begin{cases} 0 & , \operatorname{sgn}(X(j\omega)) = 1 \\ \pi & , \operatorname{sgn}(X(j\omega)) = -1 \end{cases}$





$\boxed{3.8}$

(a) $$X(j\omega) = \begin{cases} \cos(\omega) & , |\omega| < \frac{\pi}{2} \\ 0 & , \text{otherwise} \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{e^{j\omega} + e^{-j\omega}}{2} e^{j\omega t} \, d\omega$$

$$= \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} \left(e^{j\omega(t+1)} + e^{j\omega(t-1)} \right) d\omega$$

$\underline{t=1}$: $x(1) = \frac{1}{4\pi} \int_{\pi/2}^{\pi/2} e^{j2\omega} + 1 \, d\omega = \frac{1}{4}$

$\underline{t=-1}$: $x(-1) = x(1) = \frac{1}{4}$

$\underline{t \neq \pm 1}$: $x(t) = \frac{1}{2\pi(t+1)} \frac{e^{j\frac{\pi}{2}(t+1)} - e^{-j\pi/2(t+1)}}{2j}$

$+ \frac{1}{2\pi(t-1)} \frac{e^{j\pi/2(t-1)} - e^{-j\pi/2(t-1)}}{2j} \qquad t \neq \pm 1$

$$\therefore x(t) = \begin{cases} \frac{1}{4} & , t = \pm 1 \\ \frac{\sin(\frac{\pi}{2}(t+1))}{2\pi(t+1)} + \frac{\sin(\frac{\pi}{2}(t-1))}{2\pi(t-1)} & , \text{otherwise} \end{cases}$$

(b) $X(j\omega) = e^{-2\omega} u(\omega)$

$$x(t) = \frac{1}{2\pi} \int_0^\infty e^{-2\omega} e^{j\omega t} \, d\omega = \frac{1}{2\pi} \int_0^\infty e^{(jt-2)\omega} \, d\omega$$

$$x(t) = \frac{1}{2\pi(2-jt)}$$

(c) $X(j\omega) = e^{-|\omega|}$

$$x(t) = \frac{1}{2\pi} \left(\int_{-\infty}^0 e^{\omega} e^{j\omega t} \, d\omega + \int_0^\infty e^{-\omega} e^{j\omega t} \, d\omega \right)$$

$$x(t) = \frac{2}{t^2+1}$$

(d) $x(t) = \frac{1}{2\pi}\left(\int_{-2}^{2} e^{-j2\omega} e^{j\omega t}\, d\omega\right) = \frac{1}{2\pi}\int_{-2}^{2} e^{j\omega(t-2)}\, d\omega$

$\underline{t=2}$: $x(2) = \frac{2}{\pi}$

$\underline{t \neq 2}$: $x(t) = \frac{\sin(2(t-2))}{\pi(t-2)}$

(e) $x(t) = \frac{1}{2\pi}\int_{-3}^{3} \frac{2}{3}\, \omega\, e^{j\omega t}\, d\omega$

$x(t) = \frac{2}{j\pi t}\cos(3t) - \frac{2}{j3\pi t^2}\sin(3t)$, $\underline{t \neq 0}$

$\underline{t=0}$ = $x(0) = 0$

$$\therefore x(t) = \begin{cases} 0 & , t=0 \\ \frac{2}{j\pi t}\cos(3t) - \frac{2}{j3\pi t^2}\sin(3t) & , t \neq 0 \end{cases}$$

(f) $x(t) = \frac{j}{2\pi}\left(\int_{-2}^{0} e^{j\omega t}\, d\omega - \int_{0}^{2} e^{j\omega t}\, d\omega\right)$

$x(t) = \frac{1-\cos(2t)}{\pi t}$, $\underline{t \neq 0}$

$\underline{t=0}$: $x(0) = 0$

$$\therefore x(t) = \begin{cases} 0 & , t=0 \\ \frac{1-\cos(2t)}{\pi t} & , t \neq 0 \end{cases}$$

$\boxed{3.9}$ Note :

C	= continuous	P	= periodic
D	= discrete	A	= aperiodic

(a) $x(t) = e^{-3t} \cos(\pi t)\, u(t) \longrightarrow$ C, A : Fourier Transform

$$X(j\omega) = \int_0^\infty \frac{e^{j\pi t} + e^{-j\pi t}}{2} e^{-3t} \cdot e^{-j\omega t}\, dt$$

$$= \frac{1}{2} \int_0^\infty e^{(-3+j(\pi-\omega))t} + e^{(-3-j(\pi+\omega))t}\, dt$$

$$X(j\omega) = \frac{1}{3 + j(\omega - \pi)} + \frac{1}{3 + j(\omega+\pi)}$$

(b) $x[n] = \begin{cases} \cos\left(\frac{\pi}{5}n\right) + j \sin\left(\frac{\pi}{5}n\right) & , \; |n| < 10 \\ 0 & , \; \text{otherwise} \end{cases}$

D, A : discrete time Fourier Transform

$$X(e^{j\Omega}) = \sum_{n=-10}^{10} e^{j\left(\frac{\pi}{5} - \Omega\right)n}$$

$$X(e^{j\Omega}) = e^{-j\left(\frac{\pi}{5} - \Omega\right)10} \cdot \frac{1 - e^{j\left(\frac{\pi}{5} - \Omega\right)21}}{1 - e^{j\left(\frac{\pi}{5} - \Omega\right)}}$$

$$X(e^{j\Omega}) = \frac{\sin\left(\frac{21}{2}\left(\frac{\pi}{5} - \Omega\right)\right)}{\sin\left(\frac{1}{2}\left(\frac{\pi}{5} - \Omega\right)\right)}$$

(c) $x[n] \rightarrow$ D, P : discrete time Fourier Transform

$N = 7 \longrightarrow \Omega_0 = \frac{2\pi}{7}$ choose $n, k \in \{0, \ldots, 6\}$

$$X[k] = \frac{1}{7}\left(1 + e^{-j\frac{6\pi}{7}k} - e^{-j\frac{8\pi}{7}k}\right), k \in \{0, \ldots, 6\}$$

(d) $x(t) = e^{1+t}\, u(-t+2) \longrightarrow C, A$: Fourier Transform

$$X(j\omega) = \int_{-\infty}^{2} e \cdot e^{(1-j\omega)t}\, dt$$

$$X(j\omega) = \frac{e^{(3-j2\omega)}}{1-j\omega}$$

(e) $x(t) = |\sin(2\pi t)| \longrightarrow C, P$: Fourier Series

$$T = \frac{1}{2}, \quad \omega_0 = 4\pi$$

$$X[k] = 2\int_{0}^{1/2} \frac{e^{j2\pi t} - e^{-j2\pi t}}{j2} \cdot e^{-j4\pi k t}\, dt$$

$$X[k] = -j\int_{0}^{1/2} e^{j2\pi(1-2k)t} - e^{-j2\pi(1+2k)t}\, dt$$

$$X[k] = \frac{1 - e^{j\pi(1-2k)}}{2\pi(1-2k)} + \frac{1 - e^{-j\pi(1+2k)}}{2\pi(1+2k)}$$

(f) $x[n] \longrightarrow D, A$: discrete time Fourier Transform

$$X(e^{j\Omega}) = -j2\left(\frac{1}{4}\right)\left(\sin(\Omega) + 2\sin(2\Omega) + 3\sin(3\Omega) + 4\sin(4\Omega)\right)$$

$$X(e^{j\Omega}) = \frac{-j}{2}\left(\sin(\Omega) + 2\sin(2\Omega) + 3\sin(3\Omega) + 4\sin(4\Omega)\right)$$

(g) $x(t) \longrightarrow C, P$: Fourier Series

$$T = 4 \longrightarrow \omega_0 = \frac{\pi}{2}$$

$$X[k] = \frac{1}{4}\left(\int_0^2 e^{-j\frac{\pi}{2}kt}\,dt + 3\int_2^3 e^{-j\frac{\pi}{2}kt}\,dt\right)$$

$\underline{k=0}$: $X[0] = \frac{1}{4}(5) = \frac{5}{4}$

$\underline{k\neq 0}$: $X[k] = \frac{1}{4}\,\dfrac{2(-1)^k + 1 - 3e^{-j\frac{3\pi}{2}k}}{j\frac{\pi}{2}k}$

$$X[k] = \frac{2(-1)^k + 1 - 3e^{-j\frac{3\pi}{2}k}}{j\frac{\pi}{2}k}$$

$$\therefore X[k] = \begin{cases} \frac{5}{4} & , k=0 \\ \dfrac{2(-1)^k + 1 - 3e^{-j\frac{3\pi}{2}k}}{j\frac{\pi}{2}k} & , k\neq 0 \end{cases}$$

$\boxed{3.10}$

(a) $X[k] = \begin{cases} e^{-jk\pi} & , |k| < 10 \\ 0 & , \text{otherwise} \end{cases}$ $\qquad T=1 \rightarrow \omega_0 = 2\pi$

DA $\longleftrightarrow$ PC : Fourier Series

$$x(t) = \sum_{k=-10}^{10} e^{-jk\pi} \cdot e^{j2\pi kt}$$

$$x(t) = \sum_{k=-10}^{10} \left(e^{j\pi(2t-1)}\right)^k$$

$$x(t) = \frac{\sin\left(\frac{21}{2}\pi(2t-1)\right)}{\sin\left(\frac{\pi}{2}(2t-1)\right)}$$

$$x(t) = \frac{\cos(21\pi t)}{\cos(\pi t)}$$

(b) $X[k]$: DP $\longleftrightarrow$ PD : discrete time Fourier Series

$N = 5 \longrightarrow \Omega_0 = \frac{2\pi}{5}$ choose : $n, k \in \{-2, \dots, 2\}$

$$x[n] = j2\left(\sin\frac{2\pi}{5}n + \sin\frac{4\pi}{5}n\right)$$

(c) $X(j\omega) = \begin{cases} \cos\left(\frac{\omega}{2}\right) + j\sin\left(\frac{\omega}{2}\right) = e^{j\frac{\omega}{2}} & , |\omega| < \pi \\ 0 & , \text{otherwise} \end{cases}$

CA $\longleftrightarrow$ AC : Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\frac{\omega}{2}} \cdot e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(t+\frac{1}{2})}\, d\omega$$

$$x(t) = \frac{\cos(\pi t)}{\pi(t+\frac{1}{2})} \quad , \quad t \neq -\frac{1}{2}$$

$$t = -\frac{1}{2} \quad : \quad x\left(-\frac{1}{2}\right) = 1$$

$$\therefore x(t) = \begin{cases} 1 & , t = -\frac{1}{2} \\ \frac{\cos(\pi t)}{\pi(t+\frac{1}{2})} & , t \neq -\frac{1}{2} \end{cases}$$

(d) $X(j\omega) \longrightarrow$ Fourier Transform

$$x(t) = \frac{1}{2\pi}\left(\int_{-1}^{0} -e^{\omega} e^{j\omega t}\, d\omega + \int_{0}^{1} e^{-\omega} e^{j\omega t}\, d\omega\right)$$

$$x(t) = \frac{1}{2\pi}\left(-\int_{-1}^{0} e^{(1+jt)\omega}\, d\omega + \int_{0}^{1} e^{(jt-1)\omega}\, d\omega\right)$$

$$x(t) = \frac{j2t}{1+t^2}\left(1 - e^{-(t+1)}\right)$$

(e) $X(e^{j\Omega}) \longrightarrow$ discrete time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Omega}{\pi} e^{j\Omega n}\, d\Omega$$

$$x[n] = \frac{j}{n\pi}(-1)^{n+1} \quad , \quad \underline{n \neq 0}$$

$\underline{n = 0}$: $x[0] = 0$

$$\therefore x[n] = \begin{cases} 0 & , n = 0 \\ \frac{j}{n\pi}(-1)^{n+1} & , n \neq 0 \end{cases}$$

(f) $X[k]$: DA $\longleftrightarrow$ PC : Fourier Series

$$\omega_0 = \frac{2\pi}{T} = \pi$$

$$x(t) = 2\left(\sin(3\pi t) + \sin(4\pi t) + \sin(5\pi t) + \sin(6\pi t)\right)$$

(g) $X(e^{j\Omega}) = |\sin(\Omega)| \longrightarrow$ discrete time Fourier Transform

$$x[n] = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -\sin\Omega\, e^{j\Omega n}\, d\Omega + \int_{0}^{\pi} \sin\Omega . e^{j\Omega n}\, d\Omega\right)$$

$$x[n] = \frac{1}{\pi}\,\frac{((-1)^{n+1} - 1)}{n^2 - 1} \quad , \quad n \neq \pm 1$$

$\underline{n = 1}$: $x[-1] = 0$

$\underline{n = -1}$: $x[-1] = 0$

$$\therefore x[n] = \begin{cases} 0 & , n = \pm 1 \\ \frac{1}{\pi} \frac{(-1)^{n+1} - 1}{n^2 - 1} & , n \neq \pm 1 \end{cases}$$

$\boxed{3.11}$

(a) $$X(j\omega) = \frac{5j\omega + 12}{(j\omega)^2 + 5j\omega + 6}$$

$$X(j\omega) = \frac{2}{j\omega + 2} + \frac{3}{j\omega + 3}$$

$$\therefore x(t) = \left(2e^{-2t} + 3e^{-3t}\right) u(t)$$

(b) $$X(j\omega) = \frac{4}{(j\omega)^2 + 4j\omega + 3}$$

$$X(j\omega) = \frac{2}{j\omega + 1} - \frac{3}{j\omega + 3}$$

$$\therefore x(t) = 2\left(e^{-t} - e^{-3t}\right) u(t)$$

(c) $$X(j\omega) = \frac{-j\omega}{(j\omega)^2 + 3j\omega + 2}$$

$$X(j\omega) = \frac{-2}{j\omega + 2} + \frac{1}{j\omega + 1}$$

$$\therefore x(t) = \left(e^{-t} - 2e^{-2t}\right) u(t)$$

(d) $$X(j\omega) = \frac{-(j\omega)^2 - 4j\omega - 6}{\left((j\omega)^2 + 3j\omega + 2\right)(j\omega + 4)}$$

$$X(j\omega) = \frac{-1}{j\omega + 1} + \frac{1}{j\omega + 2} + \frac{-1}{j\omega + 4}$$

$$\therefore x(t) = \left(e^{-t} + e^{-2t} - e^{-4t}\right) u(t)$$

(e) $$X(j\omega) = \frac{2(j\omega)^2 + 12(j\omega) + 14}{(j\omega)^2 + 6j\omega + 5}$$

$$X(j\omega) = 2 + \frac{4}{(j\omega)^2 + 6j\omega + 5}$$

$$X(j\omega) = 2 + \frac{-1}{j\omega + 5} + \frac{1}{j\omega + 1}$$

$$\therefore x(t) = 2\delta(t) + (e^{-t} - e^{-5t})\,u(t)$$

(f) $$X(j\omega) = \frac{2j\omega + 1}{(j\omega + 2)^2}$$

$$X(j\omega) = \frac{2}{(j\omega + 2)} + \frac{-3}{(j\omega + 2)^2}$$

$$\therefore x(t) = (2 - 3t)\,e^{-2t}\,u(t)$$

$\boxed{3.12}$

(a) $$X(e^{j\Omega}) = \frac{3 - \frac{1}{4}e^{-j\Omega}}{-\frac{1}{16}e^{-j2\Omega} + 1}$$

$$X(e^{j\Omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\Omega}} + \frac{2}{1 + \frac{1}{4}e^{-j\Omega}}$$

$$\therefore x[n] = \left(\left(\frac{1}{4}\right)^n + 2\left(-\frac{1}{4}\right)^n\right) u[n]$$

(b) $$X(e^{j\Omega}) = \frac{3 - \frac{5}{4}e^{-j\Omega}}{\frac{1}{8}e^{-j2\Omega} - \frac{3}{4}e^{-j\Omega} + 1}$$

$$X(e^{j\Omega}) = \frac{2}{1-\frac{1}{4}e^{-j\Omega}} + \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

$$\therefore\ x[n] = \left(2\left(\tfrac{1}{4}\right)^n + \left(\tfrac{1}{2}\right)^n\right) u[n]$$

(c) $$X(e^{j\Omega}) = \frac{6}{e^{-j2\Omega} - 5e^{-j\Omega} + 6}$$

$$X(e^{j\Omega}) = \frac{-6}{e^{-j\Omega} - 2} + \frac{6}{e^{-j\Omega} - 3}$$

$$X(e^{j\Omega}) = \frac{3}{1-\frac{1}{2}e^{-j\Omega}} - \frac{2}{1-\frac{1}{3}e^{-j\Omega}}$$

$$\therefore\ x[n] = \left(3\left(\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{3}\right)^n\right) u[n]$$

(d) $$X(e^{j\Omega}) = \frac{6 - 2e^{-j\Omega} + \frac{1}{2}e^{-j2\Omega}}{\left(-\frac{1}{4}e^{-j2\Omega} + 1\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)}$$

$$X(e^{j\Omega}) = \frac{4}{1-\frac{1}{2}e^{-j\Omega}} + \frac{4}{1+\frac{1}{2}e^{-j\Omega}} + \frac{-2}{1-\frac{1}{4}e^{-j\Omega}}$$

$$\therefore\ x[n] = \left(4\left(\tfrac{1}{2}\right)^n + 4\left(-\tfrac{1}{2}\right)^n - 2\left(\tfrac{1}{4}\right)^n\right) u[n]$$

(e) $$X(e^{j\Omega}) = \frac{6 - \frac{2}{3}e^{-j\Omega} - \frac{1}{6}e^{-j2\Omega}}{-\frac{1}{6}e^{-j2\Omega} + \frac{1}{6}e^{-j\Omega} + 1}$$

$$X(e^{j\Omega}) = 1 + \frac{4}{1+\frac{1}{2}e^{-j\Omega}} + \frac{1}{1-\frac{1}{3}e^{j\Omega}}$$

$$\therefore\ x[n] = \delta[n] + \left(4\left(-\tfrac{1}{2}\right)^n + \left(\tfrac{1}{3}\right)^n\right) u[n]$$

$\boxed{3.13}$

(a) $x(t) = \sin(\pi t)\, e^{-2t}\, u(t)$

$$x(t) = \frac{e^{-2t} \cdot e^{j\pi t}\, u(t)}{j.2} - \frac{e^{-2t} \cdot e^{-j\pi t} u(t)}{j2}$$

$$e^{-2t} u(t) \xleftrightarrow{FT} \frac{1}{2 + j\omega}$$

$$e^{j\gamma t} s(t) \xleftrightarrow{FT} S(j(\omega - \gamma))$$

$$\therefore\ X(j\omega) = \frac{1}{j2}\left(\frac{1}{2 + j(\omega - \pi)} - \frac{1}{2 + j(\omega - \pi)}\right)$$

(b) $x(t) = e^{-3|t-2|}$

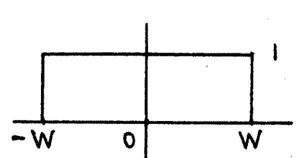
$$e^{-3|t|} \xleftrightarrow{FT} \frac{6}{9 + \omega^2}$$

$$s(t-2) \longleftrightarrow e^{-j2\omega} S(j\omega)$$

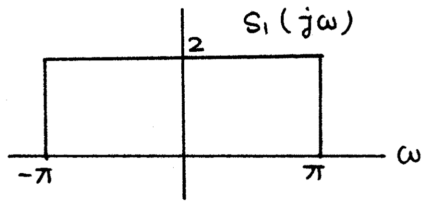
$$\therefore\ X(j\omega) = \frac{6e^{-j2\omega}}{\omega^2 + 9}$$

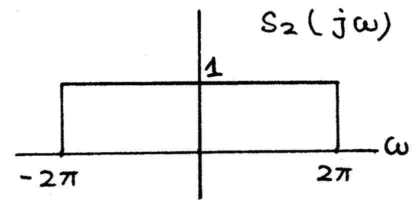
(c) $x(t) = \left[\dfrac{2\sin(\pi t)}{\pi t}\right]\left[\dfrac{\sin(2\pi t)}{\pi t}\right]$

$$\frac{\sin(wt)}{\pi t} \xleftrightarrow{FT}$$



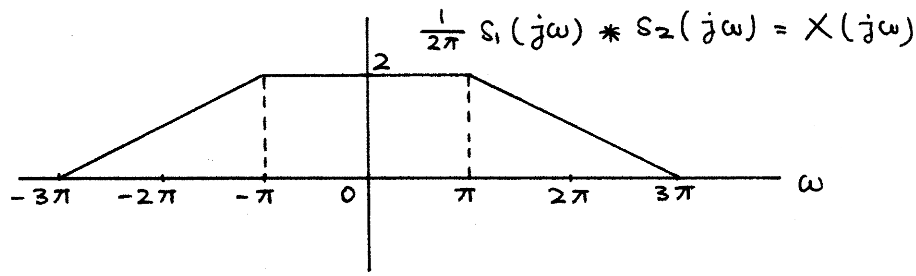
$$s_1(t) \cdot s_2(t) \xleftrightarrow{FT} \frac{1}{2\pi} S_1(j\omega) * S_2(j\omega)$$





$$\therefore X(j\omega) = \frac{1}{2\pi} S_1(j\omega) * S_2(j\omega)$$

$$\therefore X(j\omega) = \begin{cases} 3 - \dfrac{|\omega|}{\pi}, & \pi \leq |\omega| \leq 3\pi \\ 2, & |\omega| \leq \pi \\ 0, & \text{otherwise} \end{cases}$$



(d) $x(t) = \dfrac{d}{dt}\left(t e^{-2t} \sin(t)\, u(t)\right)$

$$x(t) = \frac{d}{dt}\left(t e^{-2t} u(t) \frac{e^{jt} - e^{-jt}}{j2}\right)$$

$$t e^{-2t} u(t) \longleftrightarrow \frac{1}{(2 + j\omega)^2}$$

$$e^{jt} s(t) \longleftrightarrow S(j(\omega - 1))$$

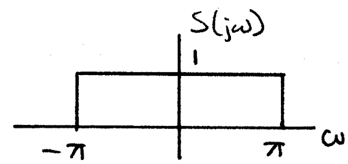
$$\frac{d}{dt} s(t) \longleftrightarrow j\omega S(j\omega)$$

$$X(j\omega) = j\omega\left(\frac{1}{j2}\right)\left(\frac{1}{(2 + j(\omega-1))^2} - \frac{1}{(2 + j(\omega+1))^2}\right)$$

$$\therefore X(j\omega) = \frac{\omega}{2}\left(\frac{1}{(2+j(\omega-1))^2} - \frac{1}{(2+j(\omega+1))^2}\right)$$

(e) $$x(t) = \int_{-\infty}^{t} \frac{\sin(\pi\tau)}{\pi\tau}\, d\tau$$

$$\frac{\sin(\pi t)}{\pi t} \longleftrightarrow$$



$$\int_{-\infty}^{t} s(\tau)\, d\tau \longleftrightarrow \frac{S(j\omega)}{j\omega} + \pi S(j0)\, \delta(\omega)$$

$$\therefore X(j\omega) = \begin{cases} \pi & , \ \omega = 0 \\ \dfrac{1}{j\omega} & , \ |\omega| < \pi \ , \ \omega \neq 0 \\ 0 & , \ \text{otherwise} \end{cases}$$

(f) $$x(t) = e^{-2t+1} u\left(\frac{t-4}{2}\right) = e^{-7} \cdot e^{-4\left(\frac{t}{2}-2\right)} u\left(\frac{t}{2}-2\right)$$

$$e^{-4t} u(t) \longleftrightarrow \frac{1}{4+j\omega}$$

$$s(t-2) \longleftrightarrow S(j\omega)\, e^{-j2\omega}$$

$$s\left(\frac{1}{2}t\right) \longleftrightarrow 2\, S(j(2\omega))$$

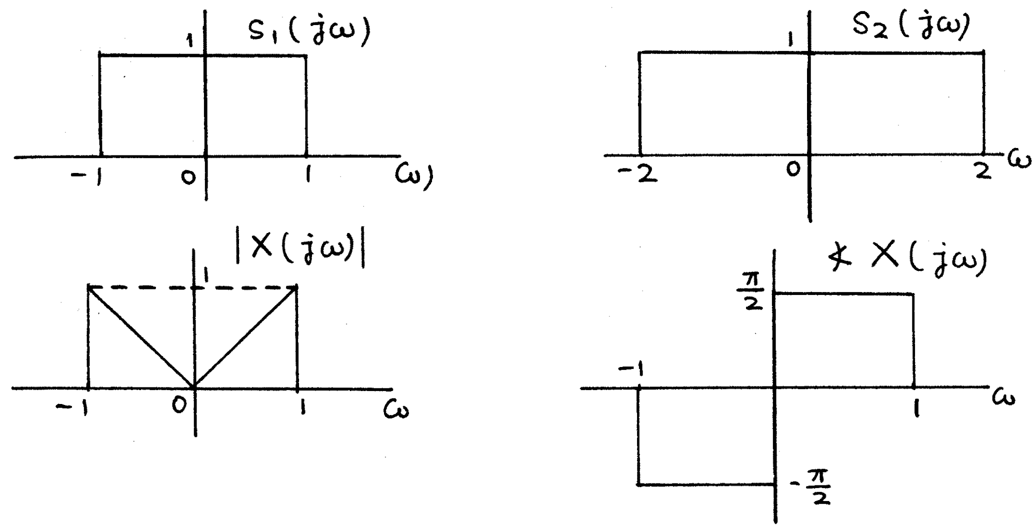
$$\therefore X(j\omega) = e^{-7} \frac{2}{4+j2\omega}\, e^{-j2\omega} = \frac{e^{-(7+j2\omega)}}{2+j\omega}$$

(g) $$x(t) = \frac{d}{dt}\left[\left(\frac{\sin(t)}{\pi t}\right) * \left(\frac{\sin(2t)}{\pi t}\right)\right]$$

$$s_1(t) * s_2(t) \longleftrightarrow S_1(j\omega) \cdot S_2(j\omega)$$

$$\frac{d}{dt} s(t) \longleftrightarrow j\omega\, S(j\omega)$$

$$\therefore X(j\omega) = \begin{cases} j\omega & , |\omega| < 1 \\ 0 & , \text{otherwise} \end{cases}$$



$\boxed{3.14}$

(a) $$X(j\omega) = \frac{j\omega}{(2+j\omega)^2}$$

$$\frac{1}{(2+j\omega)^2} \xleftrightarrow{FT} te^{-2t}\,u(t)$$

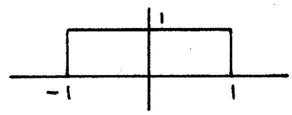
$$j\omega\, S(j\omega) \longleftrightarrow \frac{d}{dt}\, s(t)$$

$$\therefore x(t) = \frac{d}{dt}\left(te^{-2t}u(t)\right)$$

$$x(t) = (-2te^{-2t} + e^{-2t})\,u(t)$$

$$x(t) = (1-2t)e^{-2t}\,u(t)$$

(b) $$X(j\omega) = \frac{4\sin(2\omega-2)}{2\omega-2} + \frac{4\sin(2\omega+2)}{2\omega+2}$$

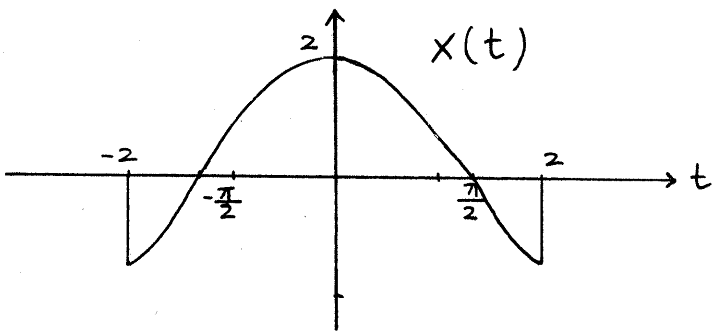
$$\frac{2\sin\omega}{\omega} \longleftrightarrow$$  $= \text{rect}(t)$

$$S(j2\omega) \longleftrightarrow \frac{1}{2}\, s\left(\frac{t}{2}\right)$$

$$S(j\omega-2) \longleftrightarrow e^{j2t}\, s(t)$$

$$\therefore\ x(t) = e^{j2t}\,\text{rect}\left(\frac{t}{2}\right) + e^{-j2t}\,\text{rect}\left(\frac{t}{2}\right)$$

$$x(t) = 2\cos(2t)\,.\,\text{rect}\left(\frac{t}{2}\right)$$



(c) $$X(j\omega) = \frac{1}{j\omega(j\omega+1)} + 2\pi\,\delta(\omega)$$

$$\frac{1}{j\omega+1} \longleftrightarrow e^{-t}\,u(t) \quad \text{and} \quad \pi\,\delta(\omega) \longleftrightarrow \frac{1}{2}$$

integration property

$$\therefore\ x(t) = \left(\int_{-\infty}^{t} e^{-\tau}\,u(\tau)\,d\tau\right) + \frac{1}{2}$$

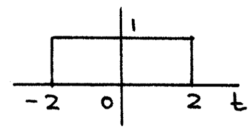
$$x(t) = \left(1 - e^{-t}\right)u(t) + \frac{1}{2}$$

(d) $$X(j\omega) = \frac{d}{d\omega}\left[4\cos(3\omega)\,\frac{\sin(2\omega)}{\omega}\right]$$

$$\frac{d}{d\omega} S(\omega) \longleftrightarrow -jt\,x(t)$$

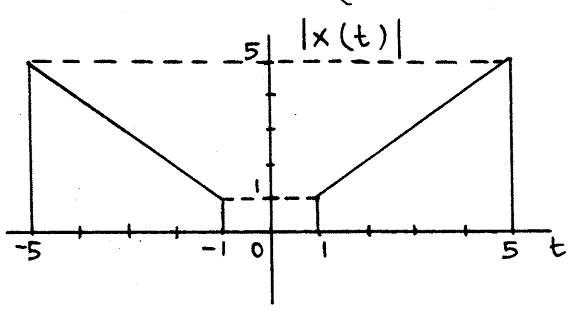
$$\cos(3\omega)\, S_2(j\omega) \longleftrightarrow \tfrac{1}{2} S_2(t+3) + \tfrac{1}{2} S_2(t-3)$$

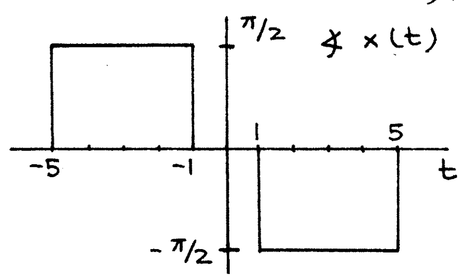
$$\frac{2\sin(2\omega)}{\omega} \longleftrightarrow \operatorname{rect}(2t) =$$



$$2\cos(3\omega)\,\frac{2\sin 2\omega}{\omega} \longleftrightarrow \operatorname{rect}(2(t-3)) + \operatorname{rect}(2(t+3))$$

$$\therefore\ x(t) = -jt\left(\operatorname{rect}(2(t-3)) + \operatorname{rect}(2(t+3))\right)$$





(e) $$X(j\omega) = \frac{2\sin(\omega)}{\omega(j\omega+1)}$$

$$\frac{2\sin(\omega)}{\omega} \longleftrightarrow \operatorname{rect}(t)$$

$$\frac{1}{j\omega+1} \longleftrightarrow e^{-t}u(t)$$

$$S_1(j\omega)\cdot S_2(j\omega) \longleftrightarrow s_1(t) * s_2(t)$$

$$\therefore\ x(t) = e^{-t}u(t) * \operatorname{rect}(t) = \begin{cases} 0 & , t<-1 \\ 1-e^{-(t+1)} & , -1\leq t<1 \\ 2e^{-t}\sinh(1) & , t\geq 1 \end{cases}$$

(f) $X(j\omega) = \text{Im}\left\{ e^{-j3\omega} \dfrac{1}{j\omega+2} \right\}$

$$e^{-j3\omega} \longleftrightarrow \delta(t-3)$$

$$\frac{1}{j\omega+2} \longleftrightarrow e^{-2t}\, u(t)$$

$$\text{Im}\,(S(j\omega)) \longleftrightarrow \text{odd}\,(s(t)) = \frac{s(t)-s(-t)}{2}$$

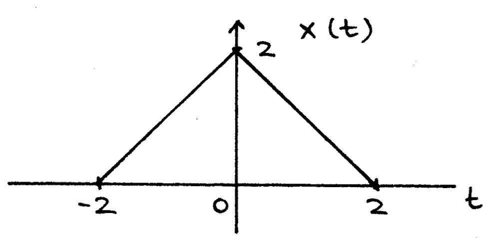
$$\therefore\ x(t) = \frac{e^{-2(t-3)}\, u(t-3) - e^{2(t+3)}\, u(-t-3)}{2}$$

(g) $X(j\omega) = \dfrac{4\sin^2(\omega)}{\omega^2}$

$$\frac{2\sin(\omega)}{\omega} \longleftrightarrow \text{rect}\,(t)$$

$$[S_1(j\omega)]^2 \longleftrightarrow s_1(t) * s_1(t)$$

$$\therefore\ x(t) = \begin{cases} 2-|t| & , |t| \leqslant -2 \\ 0 & , \text{otherwise} \end{cases}$$



$\boxed{3.15}$

(a) $x[n] = \left(\frac{1}{2}\right)^n u[n-2]$

$$x[n] = \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{n-2} u[n-2]$$

$$\left(\tfrac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

$$s[n-2] \longleftrightarrow e^{-j2\Omega} S'(e^{j\Omega})$$

$$\therefore X(e^{j\Omega}) = \frac{1}{4} e^{-j2\Omega} \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

(b) $x[n] = (n-2)(u[n-5] - u[n-6])$
$x[n] = (n-2)\,\delta[n-5]$
$x[n] = 3 \cdot \delta[n-5]$

$$\therefore X(e^{j\Omega}) = e^{-j5\Omega}$$

(c) $x[n] = \sin\left(\frac{\pi}{4}n\right)\left(\frac{1}{4}\right)^n u[n-1]$

$$x[n] = \frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{j2}\left(\tfrac{1}{4}\right)\left(\tfrac{1}{4}\right)^{n-1} u[n-1]$$

$$x[n] = \frac{1}{j8}\left(\tfrac{1}{4}\right)^{n-1} u[n-1]\left(e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}\right)$$

$$\left(\tfrac{1}{4}\right)^{n-1} u[n-1] \longleftrightarrow \frac{1}{1-\frac{1}{4}e^{-j\Omega}} = \frac{1}{e^{j\Omega}-\frac{1}{4}}$$

$$\therefore X(e^{j\Omega}) = \frac{1}{j8}\left(\frac{1}{e^{j(\Omega-\frac{\pi}{4})}-\frac{1}{4}} - \frac{1}{e^{j(\Omega+\frac{\pi}{4})}-\frac{1}{4}}\right)$$

(d) $x[n] = \left[\dfrac{\sin\left(\frac{\pi}{4}n\right)}{\pi n}\right] * \left[\dfrac{\sin\left(\frac{\pi}{4}(n-2)\right)}{\pi(n-2)}\right]$

$$s_1[n] * s_2[n] \longleftrightarrow S_1(e^{j\Omega}) \cdot S_2(e^{j\Omega})$$

$$\frac{\sin\left(\frac{\pi}{4}n\right)}{\pi n} \longleftrightarrow \begin{cases} 1, & |\Omega| < \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\Omega| \leq \pi \end{cases}$$

$$S[n-2] \longleftrightarrow e^{-j2\Omega}\, S(e^{j\Omega})$$

$$\therefore X(e^{j\Omega}) = \begin{cases} e^{-j2\Omega} & , |\Omega| < \frac{\pi}{4} \\ 0 & , \frac{\pi}{4} < |\Omega| \leq \pi \end{cases}$$

(e) $x[n] = \left[\dfrac{\sin(\frac{\pi}{2} n)}{\pi n} \right]^2$

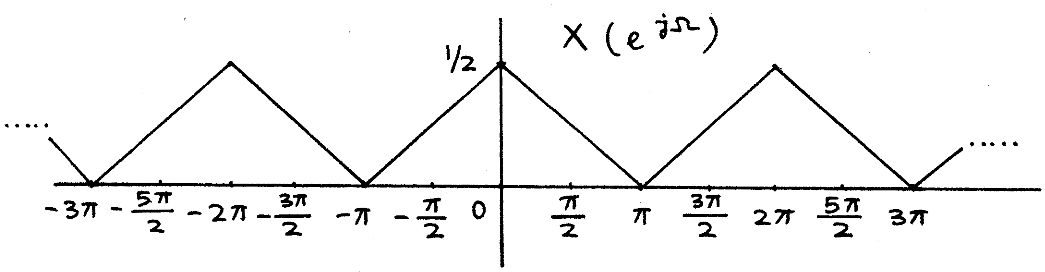
$$\frac{\sin(\frac{\pi}{2}\pi)}{\pi n} \longleftrightarrow \begin{cases} 1 & , |\Omega| < \pi/2 \\ 0 & , \pi/2 < |\Omega| \leq \pi , \end{cases}$$

periodic

$$(S_1[n])^2 \longleftrightarrow \frac{1}{2\pi} S_1'(e^{j\Omega}) * S_1(e^{j\Omega})$$

periodic convolution

$$\therefore X(e^{j\Omega}) = \frac{1}{2} - \frac{1}{2\pi}|\Omega| \quad , \; 2\pi \text{ periodic}$$



$\boxed{3.16}$

(a) $X(e^{j\Omega}) = \cos(2\Omega) + 1$

$$X(e^{j\Omega}) = \frac{e^{j2\Omega} + e^{-j2\Omega}}{2} + 1$$

$$\therefore x[n] = \frac{1}{2}\left(\delta[n+2] + \delta[n-2] + 2\,\delta[n] \right)$$

(b) $X(e^{j\Omega}) = \left[\frac{\sin(\frac{15}{2}\Omega)}{\sin(\frac{\Omega}{2})}\right] \circledast \left[e^{-j3\Omega} \frac{\sin(\frac{7}{2}\Omega)}{\sin(\frac{\Omega}{2})}\right]$

$$S_1(e^{j\Omega}) * S_2(e^{j\Omega}) \longleftrightarrow 2\pi s_1[n] \cdot s_2[n]$$

$$\frac{\sin(\frac{15}{2}\Omega)}{\sin(\frac{\Omega}{2})} \longleftrightarrow \begin{cases} 1 & , |n| \leq 7 \\ 0 & , \text{otherwise} \end{cases}$$

$$e^{-j3\Omega} \cdot \frac{\sin(\frac{7}{2}\Omega)}{\sin(\frac{\Omega}{2})} \longleftrightarrow \begin{cases} 1 & , |n-3| \leq 3 \\ 0 & , \text{otherwise} \end{cases}$$

$$\therefore \quad x[n] = \begin{cases} 2\pi & , |n-3| \leq 3 \\ 0 & , \text{otherwise} \end{cases}$$

(c) $X(e^{j\Omega}) = \cos(\Omega) \left[\frac{\sin(\frac{3}{2}\Omega)}{\sin(\frac{\Omega}{2})}\right]$

$$X(e^{j\Omega}) = \frac{1}{2}\left(e^{j\Omega} + e^{-j\Omega}\right) \cdot \frac{\sin(\frac{3}{2}\Omega)}{\sin(\frac{\Omega}{2})}$$

$$\frac{\sin(\frac{3}{2}\Omega)}{\sin(\frac{\Omega}{2})} \longleftrightarrow \begin{cases} 1 & , |n| \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\therefore \quad x[n] \begin{cases} \frac{1}{2} & , |n| \leq 2, n \neq 0 \\ 0 & , \overset{n=0}{\text{otherwise}} \end{cases}$$

(d) $X(e^{j\Omega}) = \begin{cases} 1 & , \frac{\pi}{4} < |\Omega| < \frac{3\pi}{4} \\ 0 & , \text{otherwise} \end{cases}$ for $|\Omega| < \pi$

$$X(e^{j\Omega}) = \begin{cases} 1 & , |\Omega - \frac{\pi}{2}| < \frac{\pi}{4} \\ 0 & , \text{otherwise} \end{cases}$$

$$\therefore \quad x[n] = \frac{1}{\pi(n-\frac{\pi}{2})} \sin\left(\frac{\pi}{4}\left(n - \frac{\pi}{2}\right)\right)$$

(e) $$X(e^{j\Omega}) = e^{-j(4\Omega+\frac{\pi}{2})} \frac{d}{d\Omega}\left[\frac{2}{1+\frac{1}{4}e^{-j(\Omega-\frac{\pi}{8})}} + \frac{2}{1+\frac{1}{4}e^{-j(\Omega+\frac{\pi}{8})}}\right]$$

$$-j.e^{-j4\Omega} \longleftrightarrow -j.\delta[n-4]$$

$$\frac{2}{1+\frac{1}{4}e^{-j.\Omega}} \longleftrightarrow 2\left(-\frac{1}{4}\right)^n . u[n]$$

$$\frac{d}{d\Omega} S(e^{j\Omega}) \longleftrightarrow -jn\, s[n]$$

$$\therefore x[n] = -jn\left(2\left(-\tfrac{1}{4}\right)^n u[n] e^{j\frac{\pi}{8}n} + 2\left(-\tfrac{1}{4}\right)^n u[n] e^{-j\frac{\pi}{8}n}\right)$$

$$* (-j).\delta[n-4]$$

$$x[n] = -4\cos\left(\frac{\pi}{8}(n-4)\right)\left(-\frac{1}{4}\right)^{n-4}(n-4)\; u[n-4]$$

$\boxed{3.17}$

(a) $y(t) = x(t-1)$

$$\therefore Y(j\omega) = e^{-j\omega} X(j\omega) = e^{-j\omega}\frac{2\sin(\omega)}{\omega}$$

(b) $y(t) = \cos(\pi t).x(t)$

$$y(t) = \frac{e^{j\pi t} + e^{-j\pi t}}{2} \cdot x(t)$$

$$\therefore Y(j\omega) = \frac{1}{2}X(j(\omega-\pi)) + \frac{1}{2}X(j(\omega+\pi))$$

$$Y(j\omega) = \frac{\sin(\omega-\pi)}{\omega-\pi} + \frac{\sin(\omega+\pi)}{\omega+\pi}$$

$$Y(j\omega) = -\left(\frac{\sin(\omega)}{\omega-\pi} + \frac{\sin(\omega)}{\omega+\pi}\right)$$

(c) $y(t) = x(2t+1) - x(2t-1)$

$$\therefore Y(j\omega) = e^{j\omega} \frac{1}{2} X\left(j\frac{\omega}{2}\right) - e^{-j\omega} \frac{1}{2} X\left(j\frac{\omega}{2}\right)$$

$$Y(j\omega) = \frac{\sin(\omega)}{\omega} \; j2\sin(\omega)$$

$$Y(j\omega) = j2\frac{\sin^2(\omega)}{\omega}$$

(d) $y(t) = x(\;t-1) + 2x(t-3) - x(2t-9)$

$$\therefore Y(j\omega) = e^{-j\omega} \; X(j\;\omega) + 2e^{-j3\omega} X(j\omega) - e^{-j9\omega} \cdot \frac{1}{2} X\left(j\frac{\omega}{2}\right)$$

$$Y(j\omega) = e^{-j\omega}\frac{2\sin(\;\omega)}{\omega} + e^{-j3\omega}\frac{4\sin(\omega)}{\omega} - e^{-j9\omega}\frac{2\sin\left(\frac{\omega}{2}\right)}{\omega}$$

(e) $y(t) = x(t)t$

$$-jt\,x(t) \longleftrightarrow \frac{d}{d\omega} X(j\omega)$$

$$\therefore Y(j\omega) = j\frac{d}{d\omega}\left(\frac{2\sin(\omega)}{\omega}\right)$$

$$Y(j\omega) = j\left(\frac{2\cos(\omega)}{\omega} - \frac{2\sin(\omega)}{\omega^2}\right)$$

(f) $y(t) = (2t-3)\,x(t)$

$y(t) = 2t.x(t) - 3x(t)$

From (e) : $t\,x(t) \longleftrightarrow j\left(\frac{2\cos(\omega)}{\omega} - \frac{2\sin(\omega)}{\omega^2}\right)$

$$\therefore Y(\omega) = j4\left(\frac{\cos(\omega)}{\omega} - \frac{2\sin(\omega)}{\omega^2}\right) - \frac{6\sin(\omega)}{\omega^2}$$

(g) From (c) , $y(t) = \int_{-\infty}^{t} y_c\left(\frac{\tau}{2}\right) d\tau$

$$\therefore Y(j\omega) = j4 \frac{\sin^2(2\omega)}{2\omega} \cdot \frac{1}{j\omega} + \pi(0)\,\delta(\omega)$$

$$Y(j\omega) = \frac{2\sin^2(2\omega)}{\omega^2}$$

(h) $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

$$\therefore Y(j\omega) = \frac{2\sin(\omega)}{\omega} \frac{1}{j\omega} + \pi(2)\,\delta(\omega)$$

$$Y(j\omega) = \frac{2\sin(\omega)}{j\omega^2} + 2\pi\,\delta(\omega)$$

(i) $y(t) = \frac{d}{dt}\left(x(t)\right)$

$$\therefore X(j\omega) = j\omega \cdot \frac{2\sin(\omega)}{\omega}$$

$$X(j\omega) = j2\sin(\omega)$$

$\boxed{3.18}$

(a) $Y(e^{j\Omega}) = e^{j3\Omega} X(e^{j\Omega}) \longleftrightarrow x[n+3]$

$$\therefore y[n] = (n+3)\left(-\frac{1}{2}\right)^{n+3} u[n+3]$$

(b) $Y(e^{j\Omega}) = \text{Real}\{X(e^{j\Omega})\} \longleftrightarrow \text{Even}\{x[n]\}$

$$\therefore y[n] = \frac{x[n] + x[-n]}{2} = \frac{n(-\frac{1}{2})^n u[n] + (-n)(-\frac{1}{2})^{-n} u[-n]}{2}$$

$$y[n] = \frac{1}{2} n\left(\left(-\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{2}\right)^{-n} u[-n]\right)$$

(c) $Y(e^{j\Omega}) = \frac{d}{d\Omega} X(e^{j\Omega}) \longleftrightarrow -jn\,x[n]$

$\therefore y[n] = -jn^2 \left(-\frac{1}{2}\right)^n u[n]$

(d) $Y(e^{j\Omega}) = X(e^{j\Omega}) \circledast X(e^{j(\Omega-\pi)}) \longleftrightarrow 2\pi\, x[n] \cdot x[n]e^{j\pi n}$

$\therefore y[n] = 2\pi n^2 \left(\frac{1}{4}\right)^n e^{j\pi n} u[n]$

(e) $Y(e^{j\Omega}) = \frac{d}{d\Omega} e^{j2\Omega} X(e^{j\Omega}) \longleftrightarrow -jn\left(x[n+2]\right)$

$\therefore y[n] = -jn^2 \left(-\frac{1}{2}\right)^{n+2} u[n+2]$

(f) $Y(e^{j\Omega}) = X(e^{j\Omega}) + X(e^{-j\Omega}) \longleftrightarrow x[n] + x[-n]\left(\frac{1}{2}\right)$

$\therefore y[n] = n\left(-\frac{1}{2}\right)^n u[n] - n\left(-\frac{1}{2}\right)^{-n} u[-n]$

(g) $Y(e^{j\Omega}) = \frac{d}{d\Omega}\left\{e^{-j2\Omega}\left[X(e^{j(\Omega+\frac{\pi}{4})}) - X(e^{j(\Omega-\frac{\pi}{4})})\right]\right\}$

$\longleftrightarrow -jn\left(x[n-2].e^{-j\frac{\pi}{4}(n-2)} - x[n-2].e^{j\frac{\pi}{4}(n-2)}\right)$

$\therefore y[n] = -jn^2\left(-\frac{1}{2}\right)^{n-2} e^{-j\frac{\pi}{4}n}(-j) + jn^2\left(-\frac{1}{2}\right)^{n-2}$

$.\, e^{j\frac{\pi}{4}n}(j)$

$y[n] = -n^2\left(-\frac{1}{2}\right)^{n-2} 2\cos\left(\frac{\pi}{4}n\right)$

$\boxed{3.19}$

(a) $y(t) = x(2t) \xleftrightarrow{2\omega_0} X[k]$

$\therefore Y[k] = -k2^{-|k|}$ $\qquad \omega_0' = 2\pi$

(b) $y(t) = \frac{d}{dt}x(t) \xleftrightarrow{\omega_0} jk\omega_0 X[k]$

$\therefore\ y[k] = -jk^2\pi\, 2^{-|k|}$ $\qquad \omega_0' = \pi$

(c) $y(t) = x(t - \frac{1}{4}) \xleftrightarrow{\omega_0} e^{-jk\omega_0 \frac{1}{4}}\, X[k]$

$\therefore\ y[k] = -\frac{1}{4}\, e^{-jk\pi}\, k 2^{-|k|}$

$y[k] = \frac{1}{4}(-1)^{k+1}\, k . 2^{-|k|}$ $\qquad \omega_0' = \pi$

(d) $y(t) = \text{Real}\{x(t)\} \xleftrightarrow{\omega_0} \text{even}\{X[k]\}$

$\therefore\ y[k] = \dfrac{X[k] + X[-k]}{2}$

$y[k] = \dfrac{-k2^{-|k|} + k2^{-|k|}}{2} = 0$

$y[k] = 0$

(e) $y(t) = \cos(2\pi t)\, x(t)$

$y(t) = \dfrac{e^{j2\pi t} + e^{-j2\pi t}}{2}\ x(t)$

$\xleftrightarrow{\omega_0} \frac{1}{2}\left(X[k - \frac{2\pi}{\omega_0}] + X[k + \frac{2\pi}{\omega_0}]\right)$

$\therefore\ y[k] = \frac{1}{2}\left(-(k-2)2^{-|k-2|} - (k+2)2^{-|k+2|}\right)$

(f) $y(t) = x(t) * x(t - \frac{1}{2}) \xleftrightarrow{\omega_0} X[k] . X[k] e^{-jk\omega_0 \frac{1}{2}}$

$\therefore\ y[k] = k^2\, 4^{-|k|}\, e^{-j\frac{1}{2}\pi k}$

$\boxed{3.20}$

(a) $y[k] = X[k-5] + X[k+5] \xleftrightarrow{\overset{\frac{\pi}{10}}{\downarrow}\ \Omega_0} [e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}]\, x[n]$

$$\therefore\ y[n] = 2\cos\left(\frac{\pi}{2}n\right)\frac{\sin\left(\frac{11\pi n}{20}\right)}{\sin\left(\frac{\pi}{20}n\right)}$$

(b) $$y[k] = \cos\left(k\frac{\pi}{5}\right).X[k] \overset{\Omega_0}{\longleftrightarrow} \frac{1}{2}\left(X[n-2]+X[n+2]\right)$$

$$\therefore\ y[n] = \frac{1}{2}\left(\frac{\sin\left(\frac{11\pi}{20}(n-2)\right)}{\sin\left(\frac{\pi}{20}(n-2)\right)} + \frac{\sin\left(\frac{11\pi}{20}(n+2)\right)}{\sin\left(\frac{\pi}{20}(n-2)\right)}\right)$$

(c) $$y[k] = X[k] * X[k] \overset{\Omega_0}{\longleftrightarrow} (x[n])^2$$

$$\therefore\ y[n] = \frac{\sin^2\left(\frac{11\pi}{20}n\right)}{\sin^2\left(\frac{\pi n}{20}\right)}$$

(d) $$y[k] = \text{Real}\{X[k]\} \overset{\Omega_0}{\longleftrightarrow} \text{Even}(x[n])$$

$$\therefore\ y[n] = \frac{x[n]+x[-n]}{2}$$

$$y[n] = \frac{\sin\left(\frac{11\pi}{20}n\right)}{\sin\left(\frac{\pi}{20}n\right)}$$

$\boxed{3.21}$
(a) $$\int_{-\infty}^{\infty}\frac{d\Omega}{\left|1-\frac{1}{4}e^{-j\Omega}\right|^2} \xleftrightarrow{\text{DTFT}} = 2\pi\sum_{n=-\infty}^{\infty}\left|\left(\frac{1}{4}\right)^n u[n]\right|^2$$

$$= 2\pi\sum_{n=0}^{\infty}\left(\frac{1}{16}\right)^n = \frac{32\pi}{15}$$

(b) $$\sum_{k=0}^{\infty}\frac{\sin^2\left(k\frac{\pi}{4}\right)}{k^2} \xleftrightarrow{\text{FS}} = \pi^2\frac{\omega_0}{2\pi}\int_{-\frac{\pi}{4\omega_0}}^{\frac{\pi}{4\omega_0}}(1)^2\,dt$$

$$= \frac{\omega_0}{2\pi} \cdot \frac{\pi}{2\omega_0} \pi^2$$

$$= \frac{\pi^2}{4}$$

(c) $\int_{-\infty}^{\infty} \frac{4\,d\omega}{(\omega^2+1)^2} \xleftrightarrow{FT} = 2\pi \int_{-\infty}^{\infty} e^{-2|t|}\,dt$

$$= 2\pi(2) \int_{0}^{\infty} e^{-2t}\,dt$$

$$= 2\pi$$

(d) $\sum_{k=0}^{19} \frac{\sin^2\left(\frac{11\pi}{20}k\right)}{\sin^2\left(\frac{\pi}{20}k\right)}$ looking at $\frac{\sin\left(\frac{11\pi}{20}k\right)}{N\left(\sin\left(\frac{\pi}{20}k\right)\right)}$

$\big\downarrow$ DTFS

we have : $\Omega_0 = \frac{\pi}{10}$, $M = 5$

So : $= \frac{N}{N} \cdot \sum_{n=-5}^{5} (1)^2$

$$= 10$$

(e) $\int_{-\infty}^{\infty} \frac{\sin^2(\pi t)}{\pi t}\,dt \xleftrightarrow{FT} = \pi \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega$

$$= \pi$$

$\boxed{3.22}$

(a) $x(t) \xleftrightarrow{FT} e^{-2\omega} u(\omega)$

$$e^{2t} u(-t) \longleftrightarrow \frac{1}{2 - j\omega}$$

$$\Rightarrow \frac{1}{2 - jt} \longleftrightarrow 2\pi e^{-2\omega} u(\omega)$$

$$\therefore x(t) = \frac{1}{2\pi} \cdot \frac{1}{2 - jt}$$

(b) $\frac{1}{1+t^2} \overset{FT}{\longleftrightarrow} X(j\omega)$

$e^{-|t|} \longleftrightarrow \frac{2}{1+\omega^2}$

$\Rightarrow \frac{2}{1+t^2} \longleftrightarrow 2\pi e^{-|\omega|}$

$\therefore X(j\omega) = \pi e^{-|\omega|}$

(c) $\frac{\sin\left(\frac{11\pi}{20} n\right)}{\sin\left(\frac{\pi}{20} n\right)} \xleftrightarrow{DTFS;\ \frac{\pi}{10}} X[k]$ $\qquad \Omega_0 = \frac{\pi}{10} \downarrow N = 20$

$$\begin{cases} 1 & |n| \leq 5 \\ 0 & 5 < |n| < 10 \end{cases} \longleftrightarrow \frac{\sin\left(\frac{11\pi}{20} k\right)}{20 \sin\left(\frac{\pi}{20} k\right)}$$

$$\Rightarrow \frac{\sin\left(\frac{11\pi}{20}\right)}{20 \sin\left(\frac{\pi}{20} n\right)} \longleftrightarrow \frac{1}{20} \begin{cases} 1 & |k| \leq 5 \\ 0 & 5 < |k| < 10 \end{cases}$$

$$\therefore X[k] = \begin{cases} \frac{1}{20} & |k| \leq 5 \\ 0 & 5 < |k| < 10 \end{cases}$$

$$X[k + i.20] = X[k]$$

$$k = \text{integers}$$

$\boxed{3.23}$

(a) $\int_{-\infty}^{\infty} X(j\omega)\, d\omega = 2\pi\, x(0)$
$= 2\pi\,(1)$
$= 2\pi$

(b) $\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega = 2\pi \int_{-\infty}^{\infty} |x(t)|^2\, dt$

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-1}^{0} (t+1)^2 dt + \int_{0}^{3} (1-t)^2 dt + \int_{2}^{3} (t-3)^2 dt$$

$$= \frac{4}{3}$$

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = \frac{8\pi}{3}$$

(c) $$\int_{-\infty}^{\infty} X(j\omega) e^{j2\omega} \, d\omega = 2\pi \, x(2)$$

$$= -2\pi$$

(d) $\text{Arg}\,[X(j\omega)] = \ldots\ldots$

$x(t)$ is actually an odd and real signal shifted by 1 to the right (delay of 1) , i.e : $x(t) = x_0(t-1)$

Thus , $X(j\omega) = \underbrace{X_0(j\omega)}_{\text{imaginary}} e^{-j\omega} = |X_0(j\omega)| \, e^{-j(\omega - \frac{\pi}{2})}$

$$\therefore \text{Arg}\,\{X(j\omega)\} = \frac{\pi}{2} - \omega$$

(e) $$X(j0) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \Big|_{\omega=0}$$

$$= \int_{-\infty}^{\infty} x(t) \, dt$$

$$= 0$$

$\boxed{3.24}$

(a) $$x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} \, d\Omega \Big|_{n=0}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) \, d\Omega$$

$$= \frac{1}{2\pi} \cdot \pi = \frac{1}{2}$$

(b) $\arg\{x[n]\} = \ldots$

$$X(e^{j\Omega}) = X_e(e^{j(\Omega+\frac{\pi}{4})}) \quad , \; X_e = \text{even}$$

$$\Rightarrow x[n] = x\;[n] . e^{-j\frac{\pi}{4}n}$$

$$\therefore \arg\{x[n]\} = -\frac{\pi}{4}n$$

(c) $$\sum_{n=-\infty}^{\infty} x[n] = X(e^{j0}) = 1$$

(d) $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\Omega})\right|^2 d\Omega$$

$$= 2\left(\frac{1}{2\pi}\right)\int_{-\pi/4}^{\pi/4}\left(1+\frac{4}{\pi}\Omega\right)^2 d\Omega$$

$$= \frac{2}{3}$$

(e) $$\sum_{n=-\infty}^{\infty} x[n] . e^{j\frac{\pi}{4}n} = X(e^{-j\frac{\pi}{4}}) = 0$$

$\boxed{3.25}$

(a) (i) $$X[k] = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jk\omega_0 t}\, dt$$

If $x(t)$ is real-valued : $x(t) = x^*(t)$

$$X^*[k] = \frac{1}{T}\int_{\langle T\rangle} x^*(t)\, e^{jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-j(-k)\omega_0 t}\, dt$$

$$\therefore X^*[k] = X[-k]$$

(ii) Further, if $x(t)$ even : $x(-t) = x(t)$

$$X[-k] = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{jk\omega_0 t}\, dt$$

$$X[-k] = \frac{1}{T} \int_{\langle T \rangle} x(-t)\, e^{jk\omega_0 t}\, dt$$

$$X[-k] = \frac{1}{T} \int_{\langle T \rangle} x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau$$

$\because \tau = -t$, $d\tau = -dt$

flip order of $\int$

$$X[-k] = X[k]$$
$$= X^*[k]$$

$\therefore$ The $X[k]$ is real valued or $\mathrm{Im}\{X[k]\} = 0$

(b) $$x[n-n_0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega(n-n_0)}\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underline{X(e^{j\Omega}) . e^{j\Omega n_0}} . e^{j\Omega n}\, d\Omega$$

$$\therefore x[n-n_0] \xleftrightarrow{DTFT} X(e^{j\Omega})\, e^{-j\Omega n_0}$$

(c) $$e^{jk_0\Omega_0 n}\, x[n] = \sum_{k=\langle N \rangle} x[n]\, e^{jk_0\Omega_0 n} . e^{jk\Omega_0 n}$$

$$= \sum_{k=\langle N \rangle} x[n]\, e^{j(k-k_0)\Omega_0 n}$$

$$= X[k-k_0]$$

$$\therefore e^{jk_0\Omega_0 n}\, x[n] \xleftrightarrow{DTFS} X[k-k_0]$$

(d) Let $\mathcal{F}\{ax(t)+by(t)\} = S(j\omega)$

$$S(j\omega) = \int_{-\infty}^{\infty} (ax(t)+by(t))\, e^{-j\omega t}\, dt$$

$$= a\int_{-\infty}^{\infty} x(t).\, e^{-j\omega t}\, dt + b\int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= a\, X(j\omega) + b\, Y(j\omega)$$

$$\therefore\ ax(t) + by(t) \xleftrightarrow{FT} aX(j\omega) + bY(j\omega)$$

(e) Let $\mathcal{F}\{x[n] * y[n]\} = C(e^{j\Omega})$

$$C(e^{j\Omega}) = \sum_{n=-\infty}^{\infty}\ \sum_{\ell=-\infty}^{\infty} x[\ell].\, y[n-\ell].\, e^{-j\Omega n}$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell] \underbrace{\sum_{n=-\infty}^{\infty} y[n-\ell]\, e^{-j\Omega(n-\ell)}\, e^{-j\Omega\ell}}$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\, e^{-j\Omega\ell}\ .\ Y(e^{j\Omega})$$

$$= X(e^{j\Omega})\, Y(e^{j\Omega})$$

$$\therefore\ x[n] * y[n] \xleftrightarrow{DTFT} X(e^{j\Omega})\, Y(e^{j\Omega})$$

(f) Let $\mathcal{F}\{x[n].\, y[n]\} = M(e^{j\Omega})$

$$M(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n].y[n].\, e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Gamma})\, e^{j\Gamma n}\, d\Gamma$$

$$M(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y[n].\, \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Gamma})\, e^{j\Gamma n}\, d\Gamma\, e^{-j\Omega n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Gamma}) \sum_{n=-\infty}^{\infty} y[n]\, e^{j\Gamma n}\, e^{-j\Omega n}\, d\Gamma$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Gamma}) \underbrace{\sum_{n=-\infty}^{\infty} y[n]\, e^{-j(\Omega-\Gamma)n}}\, d\Gamma$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Gamma}) \,.\, Y(e^{j(\Omega-\Gamma)})\, d\Gamma$$

$$= \frac{1}{2\pi} X(e^{j\Omega}) * Y(e^{j\Omega})$$

$$\therefore x[n]\,.\,y[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi} X(e^{j\Omega}) * Y(e^{j\Omega})$$

(g) Let $\mathcal{F}\{x[n] \circledast y[n]\} = C[k]$ $\qquad N = \frac{2\pi}{\Omega_0}$

$$C[k] = \frac{1}{N}\sum_{n=\langle N\rangle}\left(\sum_{\ell=\langle N\rangle} x[\ell]\,.\,y[n-\ell]\right) e^{-jk\Omega_0 n}$$

$$= \sum_{\ell=\langle N\rangle} x[\ell]\,.\,\underbrace{\sum_{n-\ell=\langle N\rangle}\frac{1}{N}\,.\,y[n-\ell]\, e^{-jk\Omega_0(n-\ell)}}\,.\,e^{-jk\Omega_0 \ell}$$

$$= \underbrace{N\,.\,\frac{1}{N}\sum_{\ell=\langle N\rangle} x[\ell]\, e^{-jk\Omega_0 \ell}}\,.\; Y[k]$$

$$= N\,.\quad X[k]\quad .\,Y[k]$$

$$\therefore x[n] \circledast y[n] \xleftrightarrow{\text{DTFS}} N\,.\,X[k]\,.\,Y[k]$$

(h) Let $\mathcal{F}\{x(t)\,.\,y(t)\} = M[k]$

$$M[k] = \frac{1}{T}\int_{\langle T\rangle} x(t)\,.\,y(t)\,.\,e^{-jk\omega_0 t}\, dt$$

$$x(t) = \sum_{\ell=-\infty}^{\infty} X[\ell] \cdot e^{j\ell\omega_0 t}$$

$$M[k] = \frac{1}{T} \int_{\langle T \rangle} \sum_{\ell=-\infty}^{\infty} X[\ell] e^{j\ell\omega_0 t} \; y(t) \cdot e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \sum_{\ell=-\infty}^{\infty} X[\ell] \int_{\langle T \rangle} y(t) e^{-j(k-\ell)\omega_0 t} dt$$

$$= \sum_{\ell=-\infty}^{\infty} X[\ell] \; Y[k-\ell]$$

$$= X[k] * Y[k-\ell]$$

$$\therefore x(t) \cdot y(t) \longleftrightarrow X[k] * Y[k-\ell]$$

(i) $$\frac{1}{T} \int_{\langle T \rangle} |x(t)|^2 dt = \frac{1}{T} \int_{\langle T \rangle} x(t) \sum_{k=-\infty}^{\infty} X^*[k] e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X^*[k] \underbrace{\int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt}_{X[k]}$$

$$= \sum_{k=-\infty}^{\infty} X^*[k] \cdot X[k]$$

$$= \sum_{k=-\infty}^{\infty} |X[k]|^2$$

$\boxed{3.26}$

(a) $x(t) = e^{\frac{-t^2}{2}}$, $X(j\omega) = e^{\frac{-\omega^2}{2}}$

$$T_d = \left[\frac{\int_{-\infty}^{\infty} t^2 e^{-t^2} dt}{\int_{-\infty}^{\infty} e^{-t^2} dt} \right]^{1/2}$$

$$Td = \left[\frac{\left(\frac{1}{\sqrt{2}}\right)^3 \sqrt{2\pi}}{\left(\frac{1}{\sqrt{2}}\right) \sqrt{2\pi}} \right]^{1/2}$$

$$Td = \frac{1}{\sqrt{2}}$$

$$Bw = \left[\frac{\int_{-\infty}^{\infty} \omega^2 e^{-\omega^2} d\omega}{\int_{-\infty}^{\infty} e^{-\omega^2} d\omega} \right]^{\frac{1}{2}}$$

$$Bw = \frac{1}{\sqrt{2}}$$

$$\therefore \ Td \ Bw = \frac{1}{2}$$

(b) $x(t) = e^{\frac{-t^2}{2a^2}}$

$$f\left(\frac{t}{a}\right) \xleftrightarrow{FT} aF(j\omega a)$$

so : $X(j\omega) = a e^{-\frac{\omega^2 a^2}{2}}$

$$Td = \left[\frac{\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{a^2}} dt}{\int_{-\infty}^{\infty} e^{-\frac{t^2}{a^2}} dt} \right]^{\frac{1}{2}}$$

$$Td = \left[\frac{\left(\frac{a}{\sqrt{2}}\right)^3 \sqrt{2\pi}}{\left(\frac{a}{\sqrt{2}}\right) \sqrt{2\pi}} \right]^{\frac{1}{2}}$$

$$Td = \frac{a}{\sqrt{2}}$$

$$Bw = \left[\frac{\int_{-\infty}^{\infty} \omega^2 e^{-\omega^2 a^2} d\omega}{\int_{-\infty}^{\infty} e^{-\omega^2 a^2} d\omega} \right]^{\frac{1}{2}}$$

$$Bw = \left[\frac{\left(\frac{1}{\sqrt{2}a}\right)^3 \sqrt{2\pi}}{\left(\frac{1}{\sqrt{2}a}\right) \sqrt{2\pi}} \right]^{\frac{1}{2}}$$

$$Bw = \frac{1}{\sqrt{2}a}$$

$$\therefore \; Td.Bw = \frac{1}{2}$$

If a increases :

(1) Td increases

(2) Bw decreases

(3) Td . Bw stays the same

$\boxed{3.27}$

$$x(t) = \begin{cases} 1 & , \; |t| < T \\ 0 & , \; \text{otherwise} \end{cases}$$

$$Td.\,Bw \geq 1/2 \; , \quad \text{so} \quad Bw \geq \frac{1}{2Td}$$

(a) $x(t)$

$$Td = \left[\frac{\int_{-T}^{T} t^2 \, dt}{\int_{-T}^{T} dt} \right]^{\frac{1}{2}} = \left[\frac{\frac{2}{3} T^3}{2T} \right]^{\frac{1}{2}} = \frac{T}{\sqrt{3}}$$

$$Bw \geq \frac{\sqrt{3}}{2T}$$

(b) $$x(t) * x(t) = \begin{cases} 2T - |t| & , |t| < 2T \\ 0 & , \text{otherwise} \end{cases}$$

$$\int_{-2T}^{2T} t^2 (2T - |t|)^2 dt = 2 \int_0^{2T} 4T^2t^2 - 4Tt^3 + t^4 dt$$

$$= \frac{32}{15} T^5$$

$$\int_{-2T}^{T} (2T - |t|^2)^2 dt = 2 \int_0^{2T} 4T^2 - 4Tt + t^2 dt$$

$$= \frac{16}{3} T^3$$

$$Td = \left[\frac{\frac{32}{15} T^5}{\frac{16}{3} T^3} \right]^{\frac{1}{2}}$$

$$Td = \sqrt{\frac{2}{5}}\, T$$

$$Bw \geq \frac{\sqrt{5}}{2\sqrt{2}\, T}$$

$\boxed{3.28}$ $$Td = \left[\frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \right]^{\frac{1}{2}}$$

$$Bw = \left[\frac{\int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega} \right]^{\frac{1}{2}}$$

$$T_d^{(s)} = \left[\frac{\int_{-\infty}^{\infty} t^2 |x(at)|^2 \, dt}{\int_{-\infty}^{\infty} |x(at)|^2 \, dt} \right]^{\frac{1}{2}}$$

$\underline{\underline{u = at}}$

$$\left[\frac{\frac{1}{a^3} \int_{-\infty}^{\infty} u^2 |x(u)|^2 \, du}{\frac{1}{a} \int_{-\infty}^{\infty} |x(u)|^2 \, du} \right]^{\frac{1}{2}} = \frac{1}{a} \cdot T_d$$

$$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$B_w^{(s)} = \left[\frac{\frac{1}{a^2} \int_{-\infty}^{\infty} \omega^2 . \left|X\left(\frac{j\omega}{a}\right)\right|^2 d\omega}{\frac{1}{a^2} \int_{-\infty}^{\infty} \left|X\left(\frac{j\omega}{a}\right)\right|^2 d\omega} \right]^{\frac{1}{2}}$$

$\underline{\underline{v = \frac{\omega}{a}}}$

$$\left[\frac{a^3 \int_{-\infty}^{\infty} v^2 |X(jv)|^2 \, dv}{a \int_{-\infty}^{\infty} |X(jv)|^2 \, dv} \right]^{\frac{1}{2}} = a . B_w$$

$$\therefore \; T_d^{(s)} . B_w^{(s)} = T_d . B_w$$

Time - Bandwidth product is invariant to scaling

$\boxed{3.29}$ $$x(t) = \sum_{k=-\infty}^{\infty} X[k] \, e^{jk\omega_0 t}$$

(a) (i) $$x(t) = \sum_{k=-\infty}^{\infty} X[k] \left(\cos(k\omega_0 t) + j \sin(k\omega_0 t) \right)$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k].\cos(k.\omega_0.t) + j\,X[k].\sin(k.\omega_0.t)$$

$x(t)$ real : $X[-k] = X^*[k]$

$\cos(-x) = \cos(x)$

$\sin(-x) = -\sin(x)$

Thus : $x(t) = X(0) + \sum_{k=1}^{\infty} (X[k] + X^*[k])\cos(k\omega_0 t)$

$+\, j\,(X[k] - X^*[k])\sin(k\omega_0 t)$

$$x(t) = X(0) + \sum_{k=1}^{\infty} 2.Re\{X[k]\}\cos(k.\omega_0 t) - 2\,Im\{X[k]\}$$

$\sin(k.\omega_0 t)$

compare with :

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k.\cos(k.\omega_0.t) + b_k.\sin(k.\omega_0.t)$$

(ii) $a_0 = X(0)$

$a_k = 2\,Re\{X[k]\}$ $\quad b_k = -2\,Im\{X[k]\}$

since :

$X[k] + X^*[k] = a_k$; $X[k] - X^*[k] = -j\,b_k$

$X[0] = a_0$

$$X[k] = \frac{a_k - j\,b_k}{2}$$

(iii) $a_0 = X(0) = \frac{1}{T}\int_{\langle T\rangle} x(t)\,e^{-j(0)\omega_0 t}\,dt$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t)\,dt$$

$$a_k = \frac{2}{T} Re \left\{ \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} \, dt \right\}$$

$$a_k = \frac{2}{T} \int_{\langle T \rangle} x(t) \, Re \left\{ e^{-jk\omega_0 . t} \right\} dt$$

$$a_k = \frac{2}{T} \int_{\langle T \rangle} x(t) \cos(k . \omega_0 . t) \, dt$$

$$b_k = -\frac{2}{T} \int_{\langle T \rangle} x(t) . Im \left\{ e^{-jk\omega_0 . t} \right\} dt$$

$$= \frac{2}{T} \int_{\langle T \rangle} x(t) . \sin(k \omega_0 t) \, dt$$

(iv) If $x(t)$ is even : $X[k] = X^*[k]$

Thus : $X[k]$ is real valued $\longrightarrow Im\{X[k]\} = 0$

$$\therefore b_k = -2(0)$$

$$b_k = 0$$

If $x(t)$ is odd : $X[k] = -X^*[k]$

Thus : $X[k]$ is imaginary $\longrightarrow Re\{X[k]\} = 0$

$$\therefore a_k = 2(0)$$

$$a_k = 0$$

(b) From (a) : $x(t) = a_0 + \sum_{k=1}^{\infty} a_k . \cos(k . \omega_0 . t)$

$$+ \, b_k \sin(k . \omega_0 . t)$$

(i) Let : $a_k = \cos(\theta_k) . \sqrt{a_k^2 + b_k^2}$

$$b_k = -\sin(\theta_k) . \sqrt{a_k^2 + b_k^2}$$

$$\cos(\theta_k)\cdot\cos(k.\omega_0.t) - \sin(\theta_k)\cdot\sin(k.\omega_0.t)$$
$$= \cos(k.\omega_0.t + \theta_k)$$

Thus : $x(t) = a_0 + \sum_{k=1}^{\infty} \sqrt{a_k^2 + b_k^2}\ \cos(k.\omega_0.t + \theta_k)$

compare with :
$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cdot \cos(k.\omega_0.t + \theta_k)$$

(ii) $c_k = \sqrt{a_k^2 + b_k^2}$
$$= 2\sqrt{(\text{Re}\{X[k]\})^2 + (\text{Im}\{X[k]\})^2}$$

$$\theta_k = -\tan^{-1}\frac{b_k}{a_k}$$
$$= -\tan^{-1}\left(\frac{\text{Im}\{X[k]\}}{\text{Re}\{X[k]\}}\right)$$

(iii) $c_k = \sqrt{a_k^2 + b_k^2}$

$$\theta_k = -\tan^{-1}\frac{b_k}{a_k}$$

$\boxed{3.30}$ $x(t) = -x\left(t - \frac{T}{2}\right)$ halfwave symmetry

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$X[k] = \frac{1}{T}\left\{\int_0^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt + \int_{-T/2}^{0} x(t) e^{-jk\omega_0 t}\, dt\right\}$$

$$X[k] = \frac{1}{T}\left\{\int_0^{T/2} x(t) e^{-jk\omega_0 t}\, dt + \int_0^{T/2} -x(\tau) e^{-jk\omega_0 t}\cdot e^{jk\pi}\, dt\right\}$$

$$X[k] = \frac{1}{T}\left\{ \int_0^{T/2} x(t)\, e^{-jk\omega_0 t} \left(1-(-1)^k\right) dt \right\}$$

when k even $(=0, \pm 2, \pm 4, \ldots)$, the integrand vanishes,

Thus :

$$X[k] = \begin{cases} \frac{1}{T} \int_0^{T/2} 2x(t)\, e^{-jk\omega_0 t}\, dt & , k \text{ odd} \\ 0 & , k \text{ even} \end{cases}$$

$\boxed{3.31}$

(a) $x[n] = \sum_{k=0}^{N-1} X[k]\, e^{jk\Omega_0 n}$, $n = 0, 1, \ldots, N-1$

equivalent to :

$x[0] = X[0] \cdot e^{j(0)\Omega_0(0)} + \ldots + X[N-1]\, e^{j(N-1)\Omega_0(0)}$

$x[1] = X[0] \cdot e^{j(0)\Omega_0(1)} + \ldots + X[N-1]\, e^{j(N-1)\Omega_0(1)}$

$\vdots$

$x[N-1] = X[0] \cdot e^{j(0)\Omega_0(N-1)} + \ldots + X[N-1]\, e^{j(N-1)\Omega_0(N-1)}$

can be written :

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} e^{j(0)\Omega_0(0)} & \cdots & e^{j(N-1)\Omega_0(0)} \\ e^{j(0)\Omega_0(1)} & \cdots & e^{j(N-1)\Omega_0(1)} \\ \vdots & & \\ e^{j(0)\Omega_0(N-1)} & \cdots & e^{j(N-1)\Omega_0(N-1)} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

which is : $\overline{x} = V\, \overline{X}$, V is $N \times N$ matrix

V is defined as : $\boxed{v_{r,c} = e^{j(c)\Omega_o(r)}}$

where r (= row) $= 0, 1, \ldots, N-1$
c (= column) $= 0, 1, \ldots, N-1$

(b) $$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\Omega_o n}, \quad k = 0, 1, \ldots, N-1$$

$$N.\,X[0] = x[0].\,e^{-j(0)\Omega_o(0)} + \ldots + x[N-1]\,e^{-j(0)\Omega(N-1)}$$

$$N.\,X[1] = x[0].\,e^{-j(1)\Omega_o(0)} + \ldots + x[N-1]\,e^{-j(1)\Omega_o(N-1)}$$

$$\vdots$$

$$N.\,X[N-1] = x[0].\,e^{-j(N-1)\Omega_o(0)} + \ldots + x[N-1]\,e^{-j(N-1)\Omega_o(N-1)}$$

can be written :

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} e^{-j(0)\Omega_o(0)}/N & \cdots\cdots & e^{-j(0)\Omega_o(N-1)}/N \\ e^{-j(1)\Omega_o(0)}/N & \cdots\cdots & e^{-j(1)\Omega_o(N-1)}/N \\ \vdots & & \\ e^{-j(N-1)\Omega_o(0)}/N & \cdots\cdots & e^{-j(N-1)\Omega_o(N-1)}/N \end{bmatrix}$$

which is : $\overline{X} = W\,\overline{x}$, W is $N \times N$ matrix

W is defined as : $\boxed{W_{r,c} = \frac{1}{N}\, e^{-j(r)\Omega_o(c)}}$

where r, c are defined as (a)

(c) W . V is defined as :

$$WV_{r,c} = \sum_{k=0}^{N-1} e^{j(k)\Omega_o(r)}\,.\,e^{-j(k)\Omega_o(c)}\,\frac{1}{N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} e^{jk\Omega_o(r-c)}$$

$$\sum_{k=0}^{N-1} e^{jk\Omega_o(r-c)} \begin{cases} N & , r = c \\ \dfrac{1-e^{jk\frac{2\pi}{N}(r-c)(N)}}{1-e^{j\frac{2\pi}{N}(r-c)}} = 0 & , r \neq c \end{cases}$$

$$\therefore wv_{r,c} = \begin{cases} 1 & , r = c \\ 0 & , r \neq c \end{cases}$$

which is an <u>identity</u> matrix

$$\therefore \mathbf{W.V} = \underline{\underline{\mathbf{I}}}$$

$\boxed{3.32}$ $\hat{x}_J(t) \sum_{k=-J}^{J} X[k]\, e^{jk\omega_o t}$

$$MSE_J = \frac{1}{T}\int_{\langle T\rangle} \left|x(t) - \hat{x}_J(t)\right|^2 dt$$

(a) $\left|x(t) - \hat{x}_J(t)\right|^2 = (x(t) - \hat{x}_J(t))(x(t)^* - \hat{x}_J(t)^*)$

$$= \left|x(t)\right|^2 - x(t).\hat{x}_J^*(t) - \hat{x}_J(t)\, x^*(t) + \hat{x}_J(t)\, \hat{x}_J^*(t)$$

$$\hat{x}_J^*(t) = \sum_{k=-J}^{J} X[k]^*\, e^{-jk\omega_o t}$$

Thus :

$$MSE = \frac{1}{T}\int_{\langle T\rangle} \left|x(t)\right|^2 dt - \sum_{k=-J}^{J} X^*[k]\left(\frac{1}{T}\int_{\langle T\rangle} x(t) e^{-jk\omega_o t} dt\right)$$

$$-\sum_{k=-J}^{J} X[k]\left(\frac{1}{T}\int_{\langle T\rangle} x^*(t).e^{jk\omega_0 t}\,dt\right)$$

$$+\sum_{m=-J}^{J}\sum_{k=-J}^{J} X^*[k].X[m]\left(\frac{1}{T}\int_{\langle T\rangle} e^{-jk\omega_0 t}.e^{jm\omega_0 t}\,dt\right)$$

(b) $\gamma[k] = \frac{1}{T}\int_{\langle T\rangle} x(t).e^{-jk\omega_0 t}\,dt$

$$MSE_J = \frac{1}{T}\int_{\langle T\rangle} |x(t)|^2 dt - \sum_{k=-J}^{J} X^*[k]\,\gamma[k] - \sum_{k=-J}^{J} X[k].\gamma^*[k]$$

$$+\sum_{k=-J}^{J}\sum_{m=-J}^{J} X^*[k].X[m]\left(\frac{1}{T}\int_{\langle T\rangle} e^{-j(k-m)\omega_0 t}\,dt\right)$$

$$\int_{\langle T\rangle} e^{-j(k-m)\omega_0 t}\,dt = \begin{cases} T, & k=m \\ 0, & k\neq m \end{cases} = T.\delta_{mk}$$

And : $\sum_{k=-J}^{J}\sum_{m=-J}^{J} X^*[k].X[m].\delta_{mk} = \sum_{k=-J}^{J} X^*[k].X[k]$

$$= \sum_{k=-J}^{J} |X[k]|^2$$

Thus :

$$MSE_J = \frac{1}{T}\int_{\langle T\rangle} |x(t)|^2 dt - \sum_{k=-J}^{J} X^*[k]\gamma[k] - \sum_{k=-J}^{J} X[k]\,\gamma^*[k]$$

$$+\sum_{k=-J}^{J} |X[k]|^2$$

(c) $-\sum_{k=-J}^{J} X^*[k]\,\gamma[k] - \sum_{k=-J}^{J} X[k]\,\gamma^*[k]$

$$= \sum_{k=-J}^{J} \left|X[k]-\gamma[k]\right|^2 - \sum_{k=-J}^{J} \left|X[k]\right|^2 - \sum_{k=-J}^{J} \left|\gamma[k]\right|^2$$

Thus, we can re-write (b) as :

$$MSE_J = \frac{1}{T}\int_{\langle T\rangle} |x(t)|^2\,dt + \sum_{k=-J}^{J} \left|X[k]-\gamma[k]\right|^2 - \sum_{k=-J}^{J} \left|\gamma[k]\right|^2$$

(d) Minimize MSE_J.

MSE_J is minimized when the sum at the middle vanishes :

$$\therefore\ X[k] = \gamma[k]$$

(e) $MSE_J\ \text{min} = \frac{1}{T}\int_{\langle T\rangle} |x(t)|^2\,dt - \sum_{k=-J}^{J} |X[k]|^2$

As J increases :

$|X[k]|^2$ is always greater than zero, so the sum :

$$\sum_{k=-J}^{J} |X[k]|^2 > 0 .$$

So is $\frac{1}{T}\int_{\langle T\rangle} |x(t)|^2\,dt$

As J increases, $\sum_{k=-J}^{J} |X[k]|^2$ increases and

<u>MSE_J min decreases</u>

In fact, it is expected as $J \to \infty$, $MSE_J \min \to 0$

$\boxed{3.33}$

$$MSE = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left| x(t) - \sum_{k=1}^{N} c_k \cdot \phi_k(t) \right|^2 dt$$

(a)
$$\left| x(t) - \sum_{k=1}^{N} c_k \cdot \phi_k(t) \right|^2 = |x(t)|^2 - x(t) \sum_{k=0}^{N-1} c_k^* \phi_k^*(t) - x^*(t) \sum_{k=0}^{N-1} c_k \cdot \phi_k(t) + \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} c_k \cdot c_m^* \cdot \phi_k(t) \cdot \phi_m^*(t)$$

Thus :

$$MSE = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt - \sum_{k=0}^{N-1} c_k^* \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x(t) \cdot \phi_k^*(t) dt \right) - \sum_{k=0}^{N-1} c_k \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x^*(t) \phi_k(t)\, dt \right) + \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} c_k \cdot c_m^* \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \phi_k(t) \phi_m^*(t) dt \right)$$

Let : $\gamma[k] = \frac{1}{f_k} \int_{t_1}^{t_2} x(t) \cdot \phi_k^*(t)\, dt$ and use the orthogonality

$$MSE = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt - \frac{f_k}{t_2 - t_1} \sum_{k=0}^{N-1} c_k^* \gamma[k] - \frac{f_k}{t_2 - t_1} \sum_{k=0}^{N-1} c_k \cdot \gamma^*[k] + \sum_{k=0}^{N-1} \frac{f_k}{t_2 - t_1} |c_k|^2$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt + \frac{f_k}{t_2 - t_1} \sum_{k=0}^{N-1} |c_k - \gamma[k]|^2$$

$$- \frac{f_k}{t_2 - t_1} \sum_{k=0}^{N-1} |\gamma[k]|^2$$

Analogous to 3.29, $c_k = \gamma[k] = \frac{1}{f_k} \int_{t_1}^{t_2} x(t) . \phi_k^*(t) \, dt$

(b) $$MSE_{min} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt - \frac{f_k}{t_2 - t_1} \sum_{k=0}^{N-1} |c_k|^2$$

$$= \frac{1}{t_2 - t_1} \left[\int_{t_1}^{t_2} |x(t)|^2 dt - \sum_{k=0}^{N-1} f_k |c_k|^2 \right]$$

$MSE_{min} = 0$, when :

$$\int_{t_1}^{t_2} |x(t)|^2 dt = \sum_{k=0}^{N-1} f_k |c_k|^2$$

(c) We can see that the orthogonality relation for the Walsh function is as follows :

$$\int_0^1 \phi_k(t) . \phi_e^*(t) \, dt = \begin{cases} 1 & , k = e \\ 0 & , k \neq e \end{cases} \qquad \text{so: } f_k = 1$$

(i) $$x(t) = \begin{cases} 2 & , \frac{1}{2} < t < \frac{3}{4} \\ 0 & , 0 < t < \frac{1}{2} , \frac{3}{4} < t < 1 \end{cases}$$

$$c_0 = \int_{1/2}^{3/4} 2 \, dt = \frac{1}{2}$$

$$c_1 = \int_{1/2}^{3/4} -2 \, dt = -\frac{1}{2}$$

$$c_2 = \int_{1/2}^{3/4} -2 \, dt = -\frac{1}{2}$$

$$c_3 = 2\left(\frac{1}{4}\right) = \frac{1}{2}$$

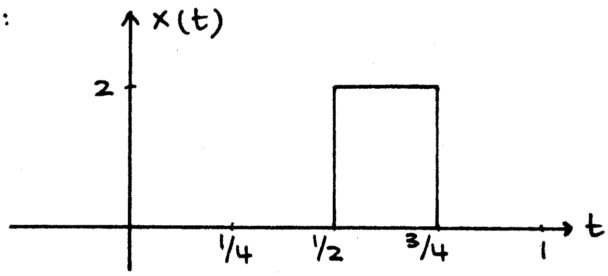
$$c_4 = 2(0) = 0$$

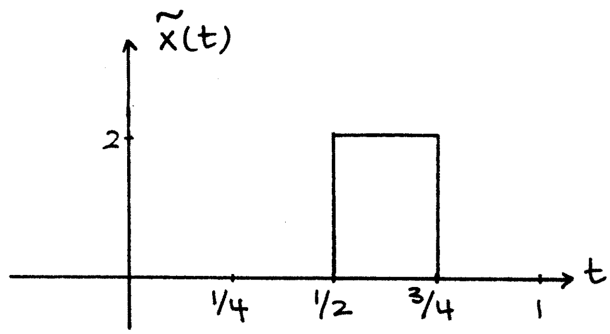
$$c_5 = 2(0) = 0$$

$$\Rightarrow \tilde{x}(t) = \frac{1}{2}\left(\phi_0(t) - \phi_1(t) - \phi_2(t) + \phi_3(t)\right)$$

where $x(t) \simeq \tilde{x}(t)$

Sketch :





From the sketch, we can see that $x(t) = \tilde{x}(t)$

MSE $= 0$

(ii) $x(t) = \sin(2\pi t) \qquad 0 \leq t \leq 1$

$$C_0 = \int_0^1 \sin(2\pi t)\, dt = 0$$

$$C_1 = \int_0^{1/2} \sin(2\pi t)\, dt - \int_{1/2}^{1} \sin(2\pi t)\, dt = \frac{2}{\pi}$$

$$C_2 = \int_0^1 \phi_2(t) \cdot \sin(2\pi t)\, dt = \frac{1}{\pi}$$

$C_3 = 0$ by inspection

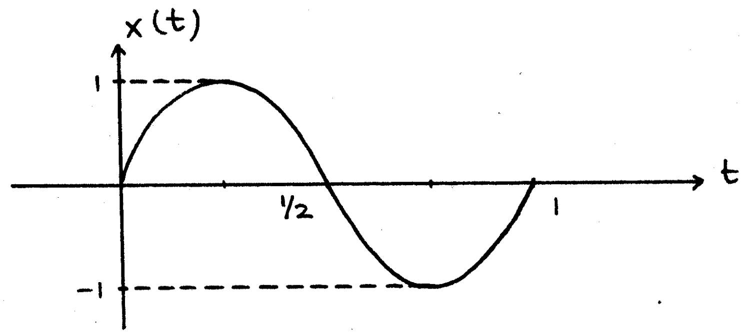
$C_4 = 0$ by inspection

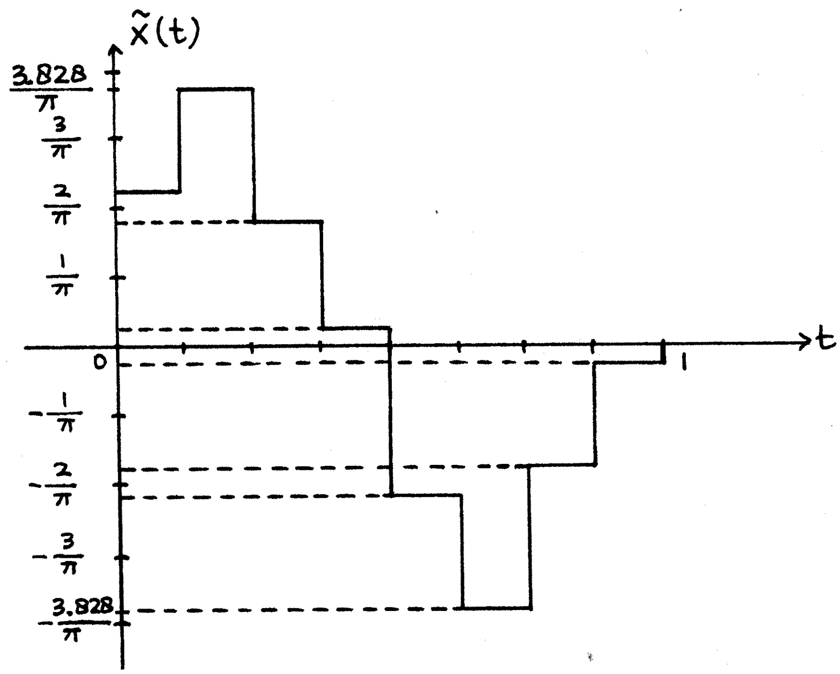
$$c_5 = 2\left[\int_0^{1/8} \sin(2\pi t)\,dt - \int_{1/8}^{3/8} \sin(2\pi t)\,dt + \int_{3/8}^{1/2} \sin(2\pi t)dt \right.$$

$$= \frac{2}{\pi}\left(1-\sqrt{2}\right) \simeq \frac{1}{\pi}(-0.83)$$

$$\therefore\ \tilde{x}(t) = \frac{1}{\pi}\left(2\phi_1(t) + \phi_2(t) + 2(\sqrt{2}-1)\,\phi_5(t)\right)$$

<u>Sketch</u> :





$$\text{MSE} = \int_0^1 \left|\sin(2\pi t) - \tilde{x}(t)\right|^2 dt$$

$$= 2\left(\int_0^{1/8} \left|\sin(2\pi t) - \frac{2.17}{\pi}\right|^2 dt + \int_{1/8}^{2/8} \left|\sin(2\pi t)\frac{3.83}{\pi}\right|^2 dt \right.$$

$$+\int_{2/8}^{3/8}\left|\sin(2\pi t)-\frac{1.83}{\pi}\right|^2 dt+\int_{3/8}^{1/2}\left|\sin(2\pi t)-\frac{0.17}{\pi}\right|^2 dt\Big)$$

$$\cong 0.1265$$

(d) $\phi_k(t) = \frac{2k-1}{k}\, t\, \phi_{k-1}(t) - \frac{k-1}{k}\phi_{k-2}(t)$

$\phi_0(t) = 1 \quad , \phi_1(t) = t$

$\phi_2(t) = \frac{3}{2}\, t\,(t) - \frac{1}{2}(1) = \frac{1}{2}(3t^2-1)$

$\phi_3(t) = \frac{5}{3}\, t\, \frac{1}{2}(3t^2-1) - \frac{2}{3}(t) = \frac{1}{2}(5t^3-3t)$

$\phi_4(t) = \frac{7}{4}\, t\, \frac{1}{2}(5t^3-3t) - \frac{3}{4}\cdot\frac{1}{2}(3t^2-1)$

$\quad = \frac{1}{8}(35t^4 - 30t^2 + 3)$

$\phi_5(t) = \frac{9}{5}\, t\, \frac{1}{8}(35t^4-30t^2+3) - \frac{4}{5}\cdot\frac{1}{2}(5t^3-3t)$

$\quad = \frac{1}{40}(315t^5 - 350t^3 + 75t)$

The orthogonality relation for Legendre polynomial is :

$$\int_{-1}^{1}\phi_k(t)\,\phi_\ell^*(t)\,dt = \delta_{k\ell}\frac{2}{2k+1}$$

so : $f_k = \frac{2}{2k+1}$

(i) $x(t) = \begin{cases} 2 & , 0 < t < \frac{1}{2} \\ 0 & , -1 < t < 0\ ,\ \frac{1}{2} < t < 1 \end{cases}$

$$C_0 = \frac{1}{2}\int_0^{1/2} 2(1)\,dt = \frac{1}{2}$$

$$C_1 = \frac{3}{2}\int_0^{1/2} 2t\,dt = \frac{3}{8}$$

$$C_2 = \frac{5}{2}\int_0^{1/2} 3t^2 - 1\,dt = -\frac{15}{16}$$

$$C_3 = \frac{7}{2}\int_0^{1/2} 5t^2 - 3t\,dt = -1.039$$

$$C_4 = \frac{9}{2}\int_0^{1/2} \frac{1}{4}(35t^4 - 30t^2 + 3)\,dt = 0.527$$

$$C_5 = \frac{11}{2}\int_0^{1/2} \frac{1}{20}(315t^5 - 350t^3 + 75t)\,dt = 1.300$$

$$\tilde{x}(t) = \sum_{k=0}^{5} c_k \cdot \phi_k(t)$$

$$\text{MSE} = \frac{1}{2}\left[\int_0^{1/2} 4\,dt - \sum_{k=0}^{5} \frac{2}{2k+1}|c_k|^2\right]$$

$$= 0.1886$$

(ii) $x(t) = \sin(\pi t)$, $-1 \leqslant t \leqslant 1$

$$C_0 = \frac{1}{2}\int_{-1}^{1} \sin(\pi t)\,dt = 0$$

$$C_1 = \frac{3}{2}\int_{-1}^{1} t\sin(\pi t)\,dt = 0.955$$

$$C_2 = \frac{5}{2}\int_{-1}^{1} \frac{1}{2}(3t^2 - 1)\sin(\pi t)\,dt = 0$$

$$C_3 = \frac{7}{2}\int_{-1}^{1} \frac{1}{2}(5t^2 - 3t)\sin(\pi t)\,dt = -1.158$$

$$C_4 = \frac{9}{2}\int_{-1}^{1} \frac{1}{8}(35t^4 - 30t^2 + 3)\sin(\pi t)\,dt = 0$$

$$c_5 = \frac{11}{2}\int_{-1}^{1} \frac{1}{40}\left(315t^5 - 350t^3 + 75t\right)\sin(\pi t)\,dt$$

$$c_5 = 0.213$$

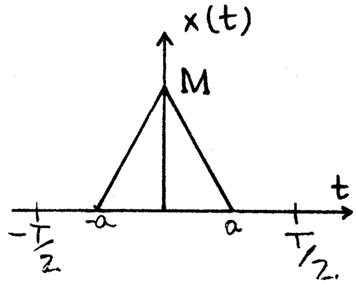
$$\tilde{x}(t) = \sum_{k=0}^{5} c_k \cdot \phi_k(t)$$

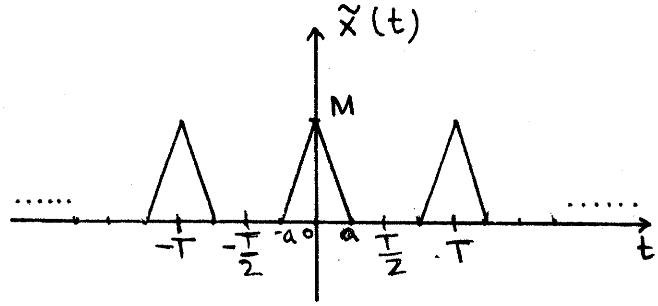
$$MSE = \frac{1}{2}\left[\int_{-1}^{1}(\sin(\pi t))^2\,dt - \sum_{k=0}^{5}\frac{2}{2k+1}|c_k|^2\right]$$

$$MSE = 3 \times 10^{-4}$$

$\boxed{3.34}$

(a)





(b) $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{jk\omega_0 t} \qquad \ldots(1)$$

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\,dt \qquad \ldots(2)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\,dt \qquad \ldots(3)$$

Using : $x(t) = \lim_{T\to\infty} \tilde{x}(t)$

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-j.k.\omega_0.t}\,dt$$

As $T \rightarrow \infty$, $X[k] = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$

$$X[k] = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot e^{-jk\omega_0 t} dt$$

Compare with (3) ; we can see that

$$X[k] = \frac{1}{T} \cdot X(jk\omega_0)$$

(c) $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) \cdot e^{jk\omega_0 t}$

$T = \frac{2\pi}{\omega_0}$

$\therefore \tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \cdot \omega_0$

(d) $x(t) = \lim_{T \to \infty} \tilde{x}(t)$

$= \lim_{T \to \infty} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$

let $\omega \approx k\omega_0$

as $T \rightarrow \infty$; $\omega_0 \rightarrow 0$; $k \rightarrow \infty$; $\omega \rightarrow \infty$

$\Longrightarrow x(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{\omega=-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} \omega_0$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \cdot d\omega$$

$\boxed{3.35}$

$$X[k] = \frac{1}{N} \frac{\sin\left(k \cdot \frac{\pi}{N}(2M+1)\right)}{\sin\left(k \cdot \frac{\pi}{N}\right)}$$

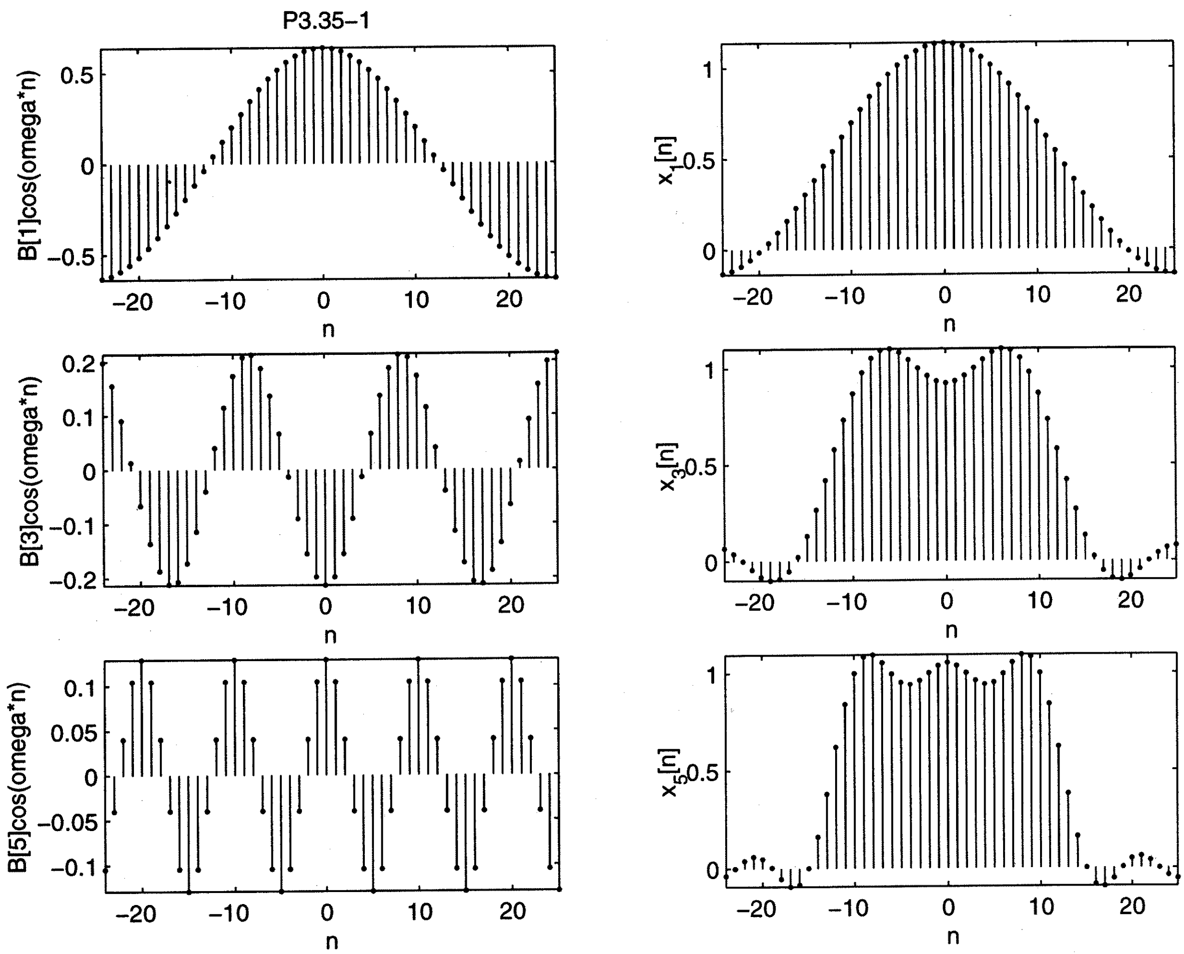
$$B[k] = \begin{cases} X[k] & , k = 0, \frac{N}{2} \\ 2X[k] & , k = 1, 2, \ldots \frac{N}{2} - 1 \end{cases}$$

$N = 50 \quad , \quad J = 1, 3, 5, 23, 25$

$$\hat{x}_j[n] = \sum_{k=0}^{J} B[k] \cos(k \cdot \Omega_0 \cdot n)$$

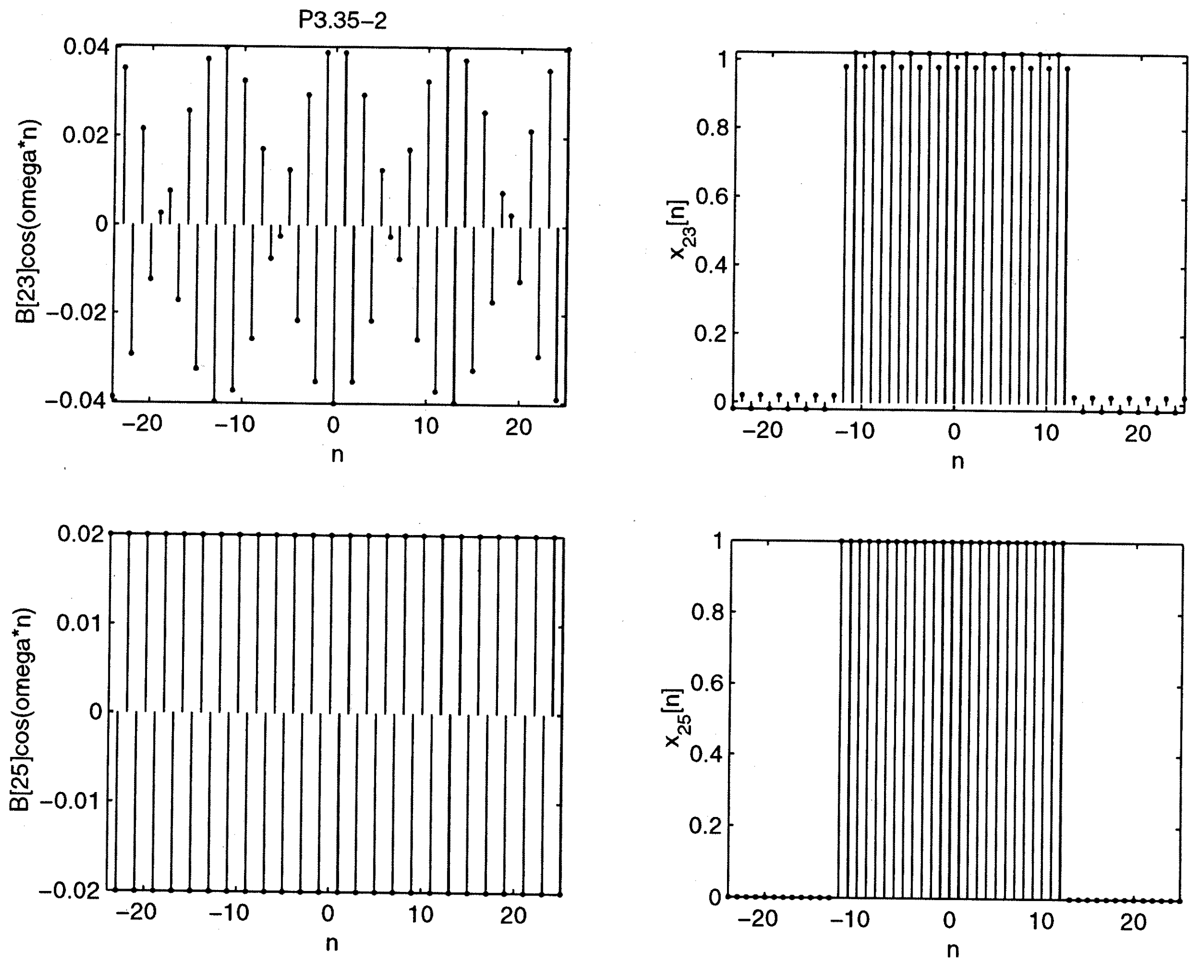
P 3.35

—Plot 1 of 6—



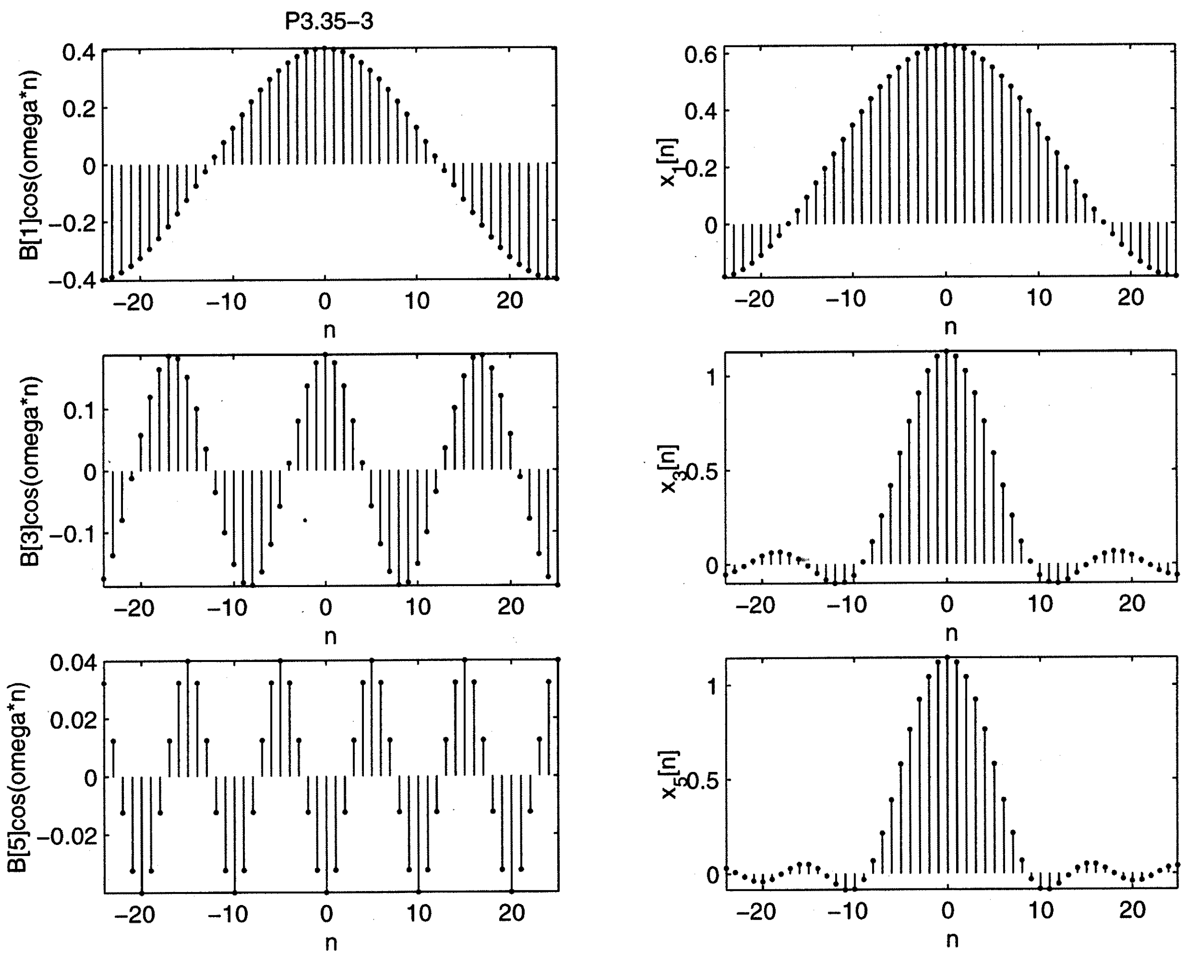
P 3.35

– Plot 2 of 6 –



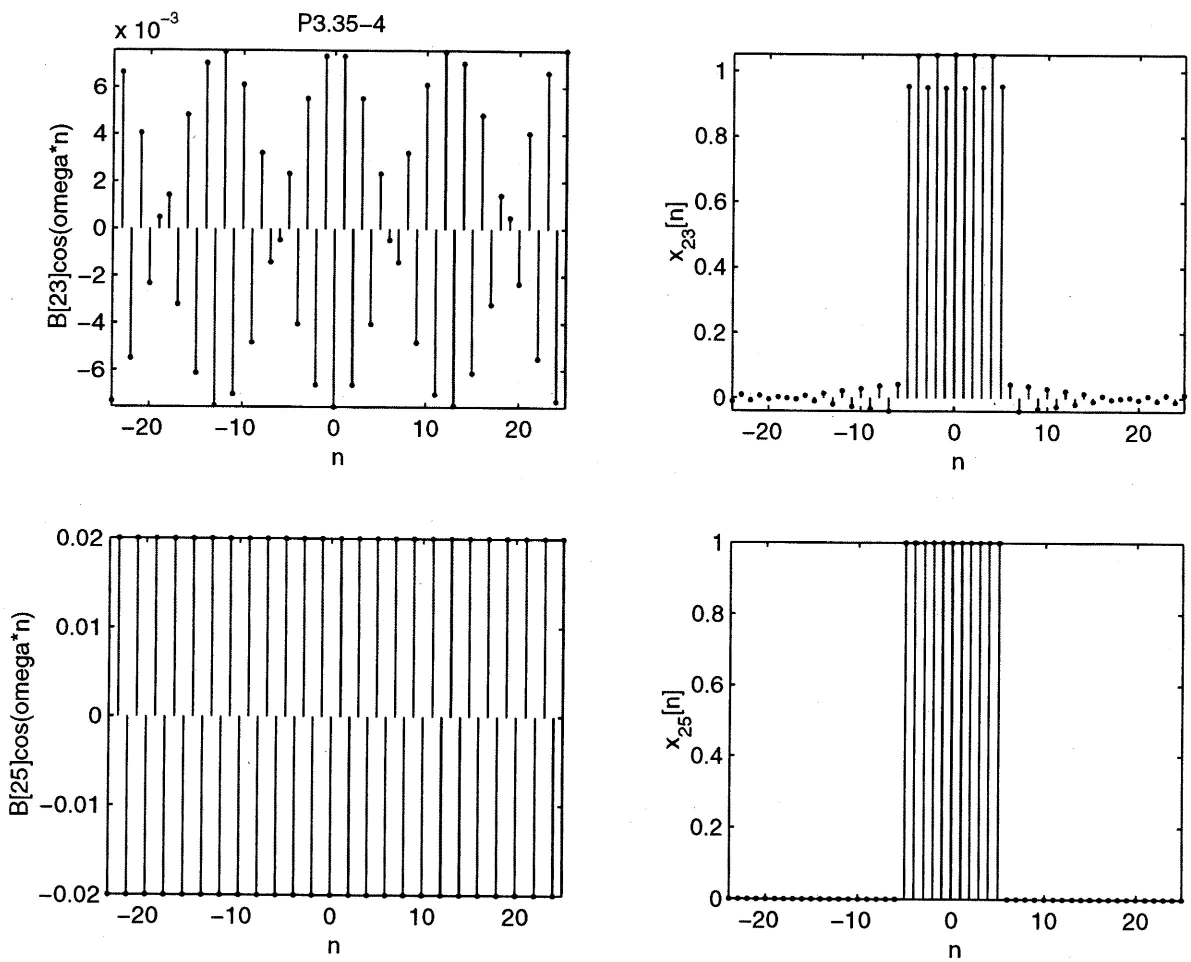
P 3.35

- Plot 3 of 6 -



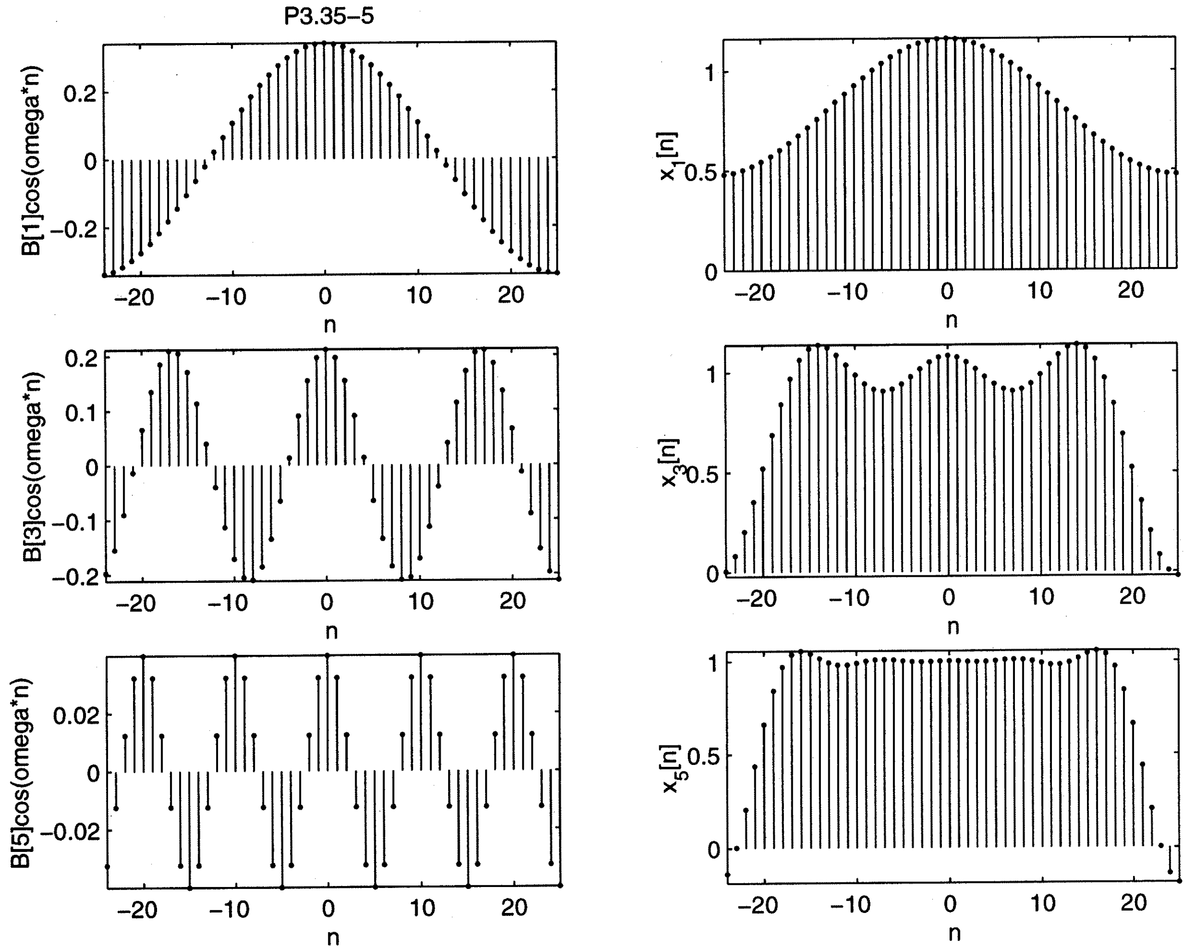
P 3.35

- Plot 4 of 6 -



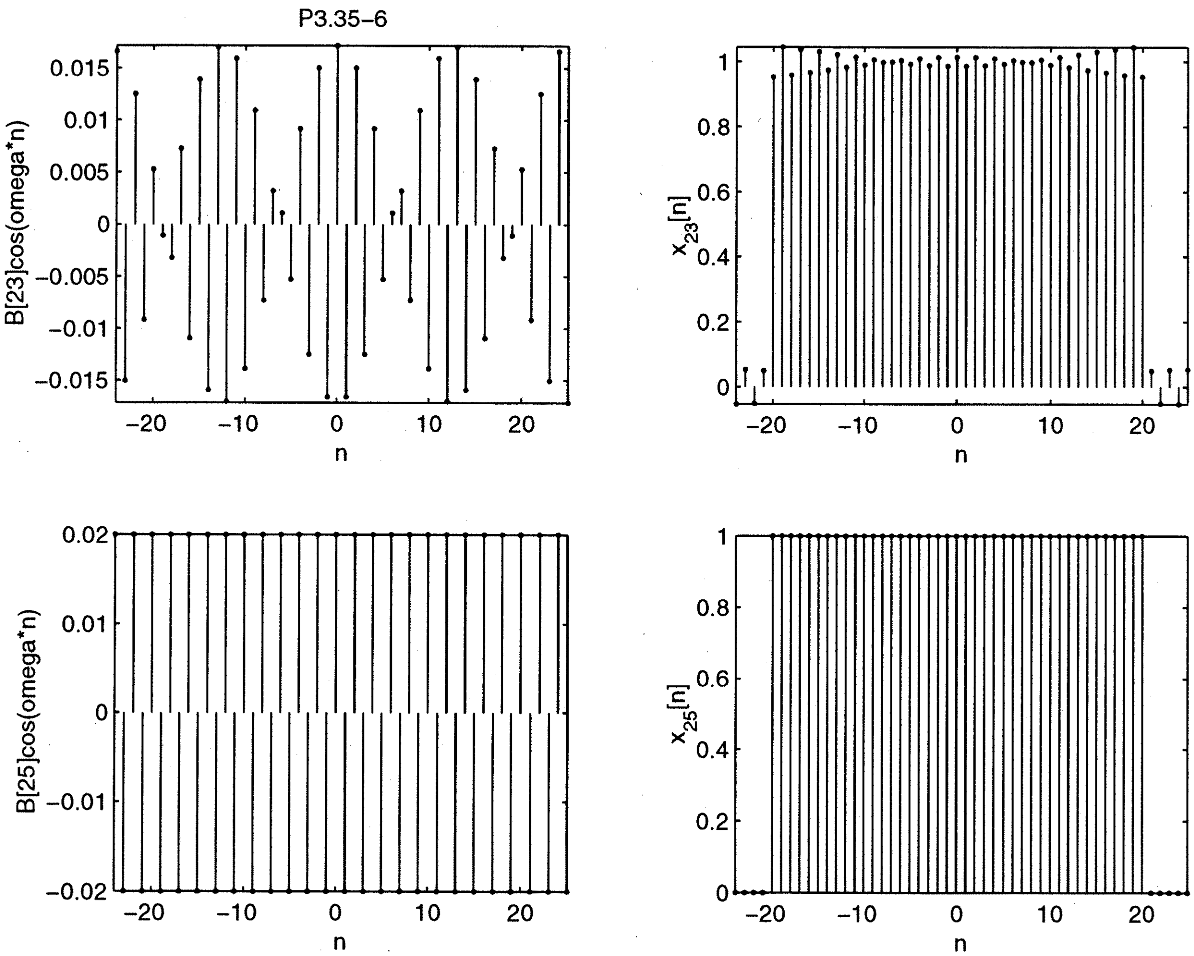
P 3.35

— Plot 5 of 6 —



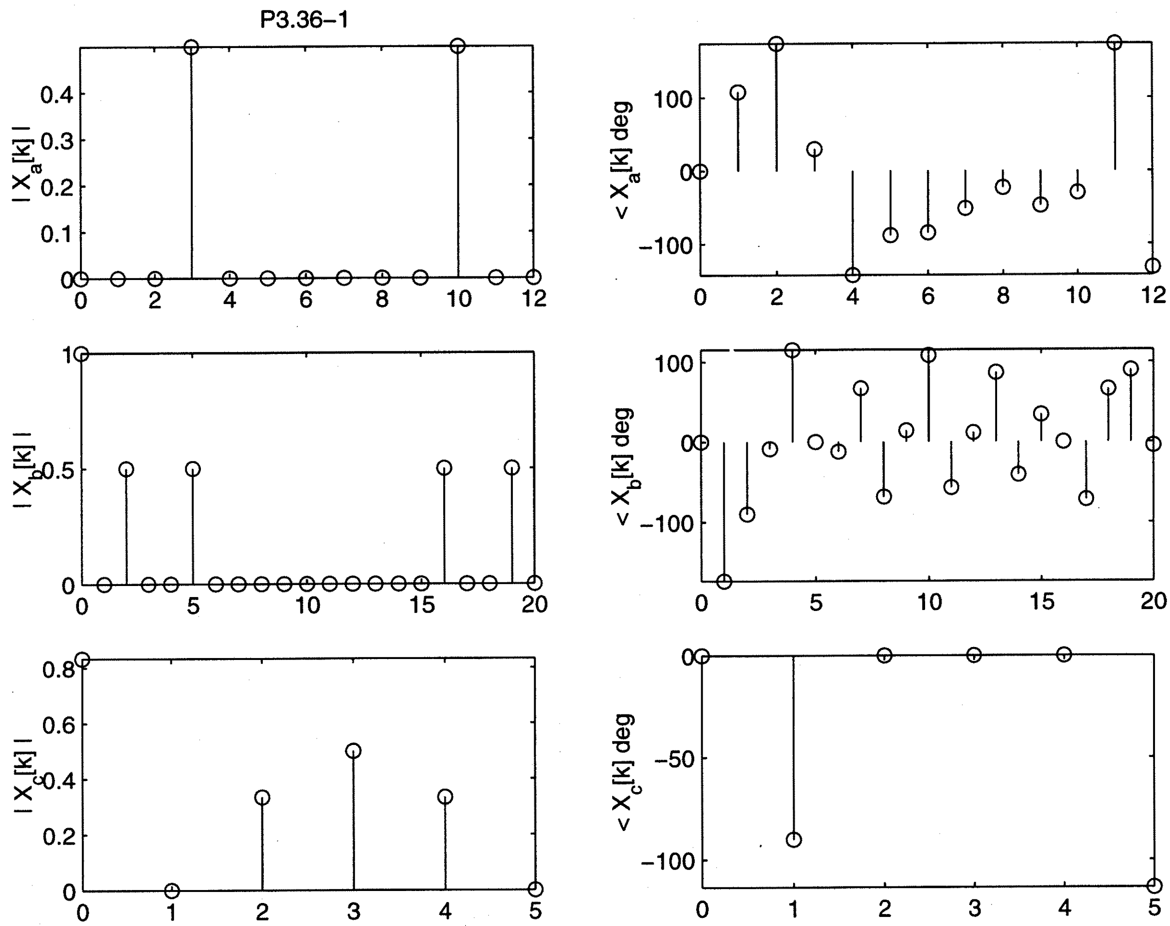
P 3.35

– Plot 6 of 6 –



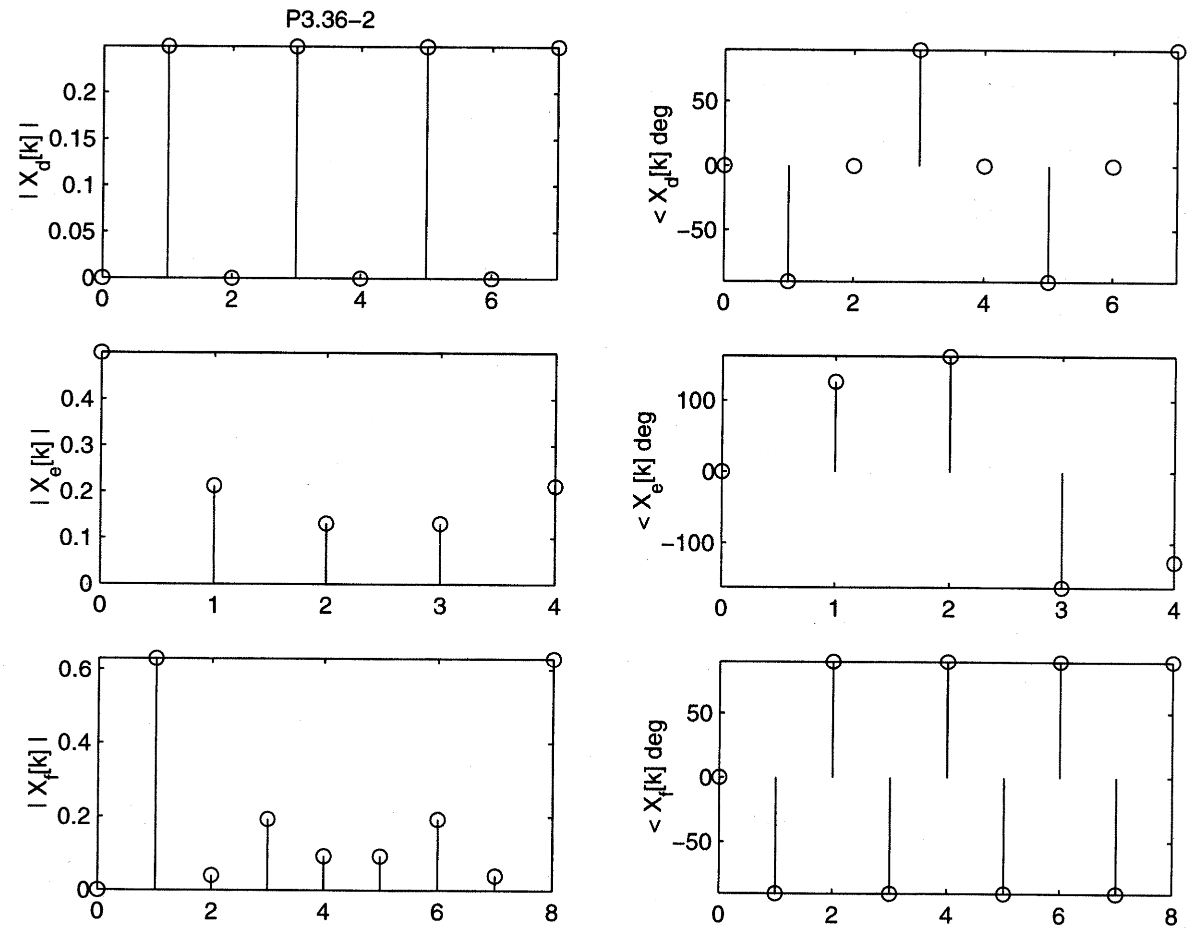
P 3.36

— Plot 1 of 2 —



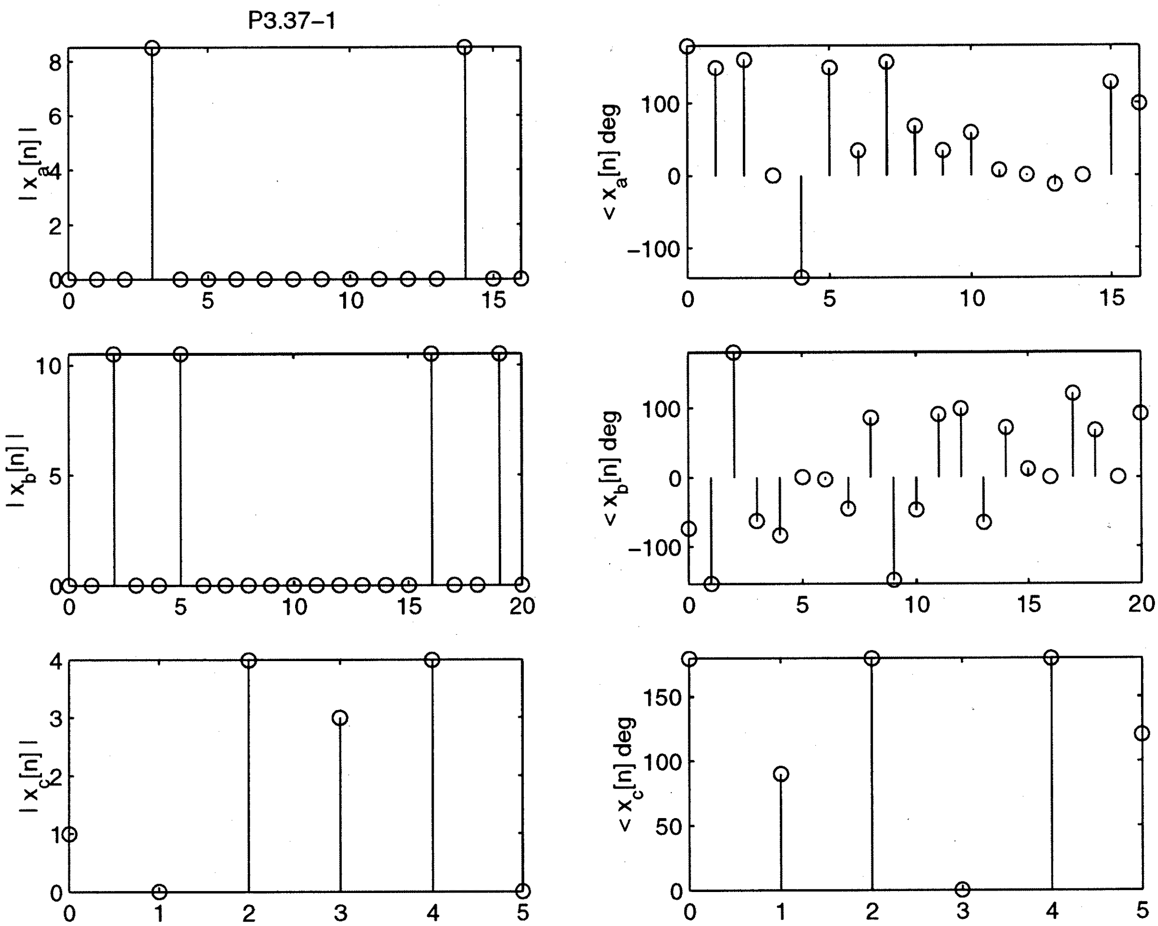
P 3.36

- Plot 2 of 2 -



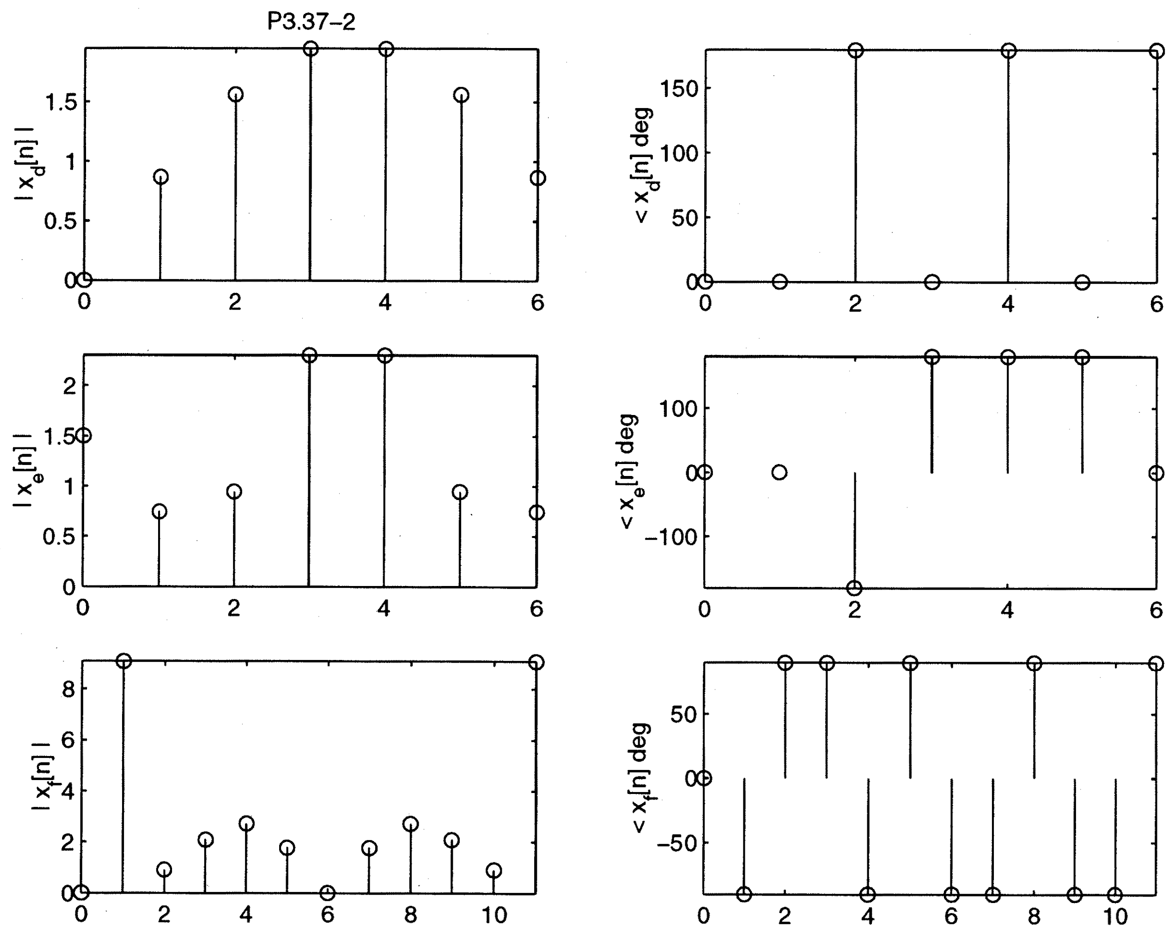
P 3.37

- Plot 1 of 2 -



P 3.37

- Plot 2 of 2 -



$\boxed{3.38}$ To compute the time signals from FS coefficients using ifft (.), we look at the expressions :

$$\text{FS} \quad : \hat{x}(t) = \sum_{k=-\infty}^{\infty} \bar{X}[k]\, e^{j\frac{2\pi}{T}kt}$$

$$\text{DTFS} : x[n] = \sum_{\langle N \rangle} X[k]\, e^{j\frac{2\pi}{N}kn}$$

Using a finite number of FS coefficients (ones that are significantly big), a discrete approximation can be done as follow :

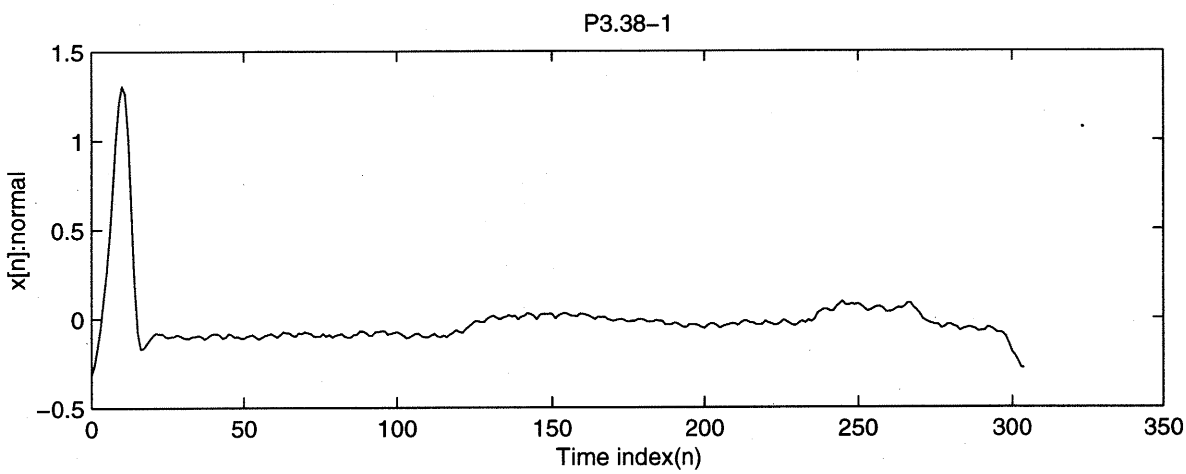
$$\text{approximation of FS} : \hat{X}[n_t] \cong \sum_{\langle N_t \rangle} \hat{X}[k] e^{j\frac{2\pi}{N_T}kn_t}$$

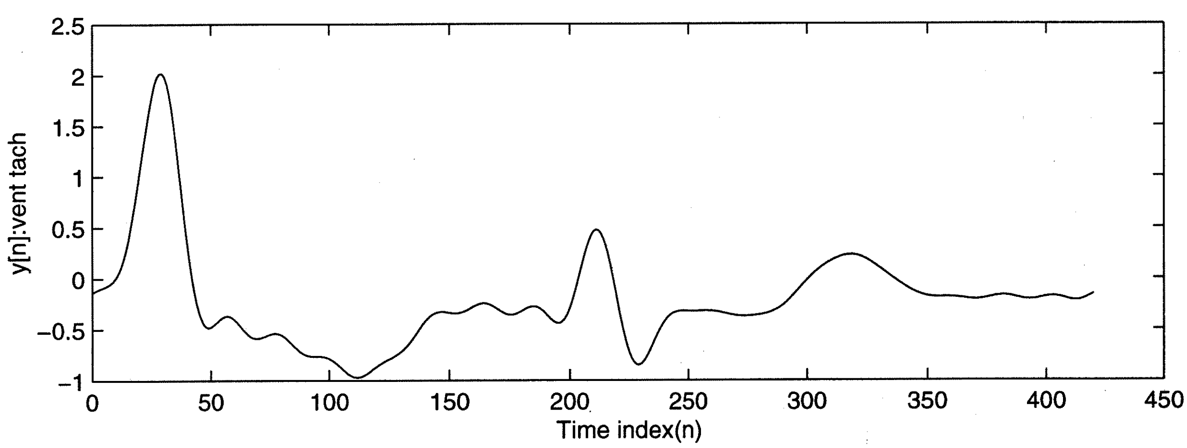
n_t is the sampled time vector and can be determined from :

$$\frac{t}{T} = \frac{n_t}{N_T} \quad \text{or} \quad T = \underset{\substack{\uparrow \\ \text{stepsize}}}{\Delta} . N_T$$

P 3.38

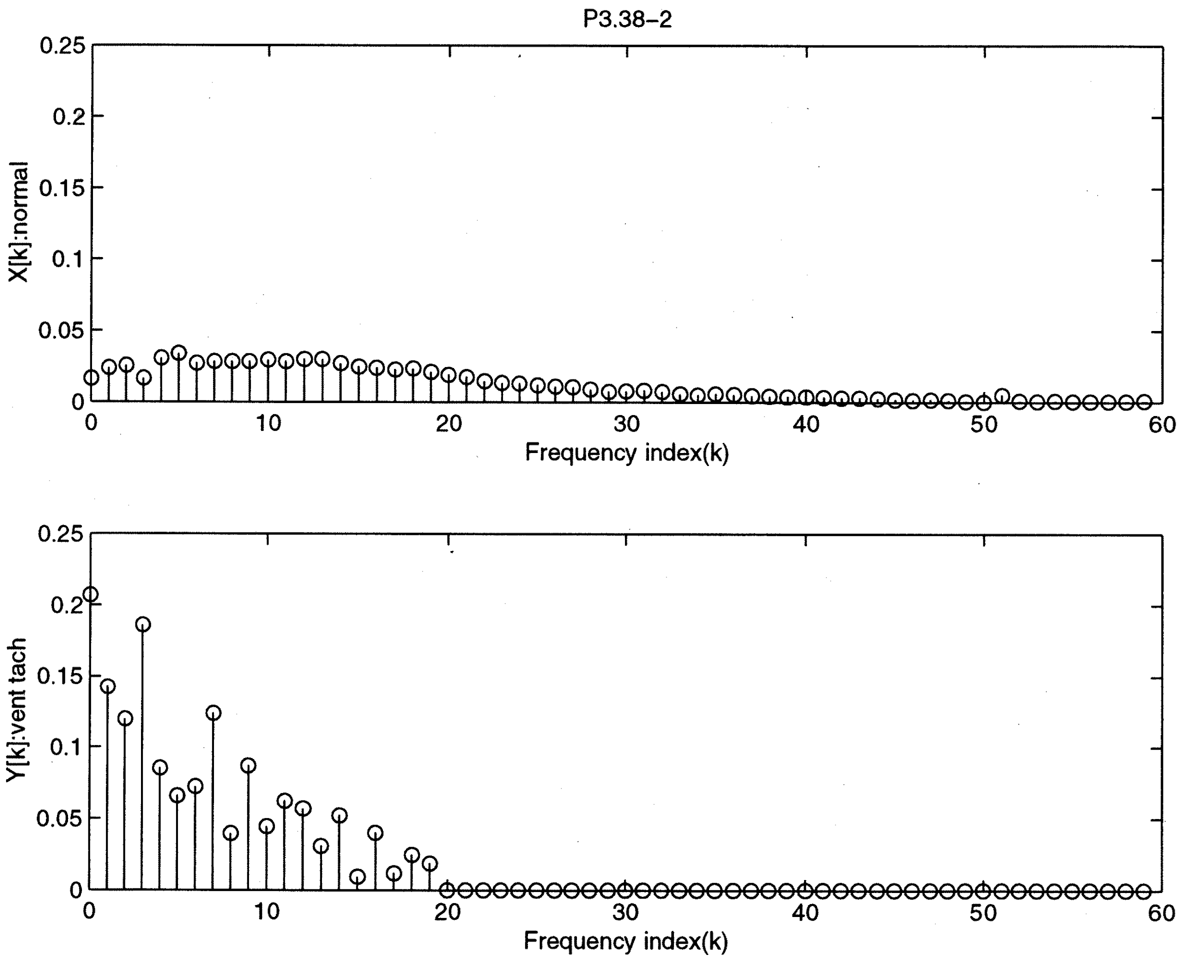
- Plot 1 of 2 -





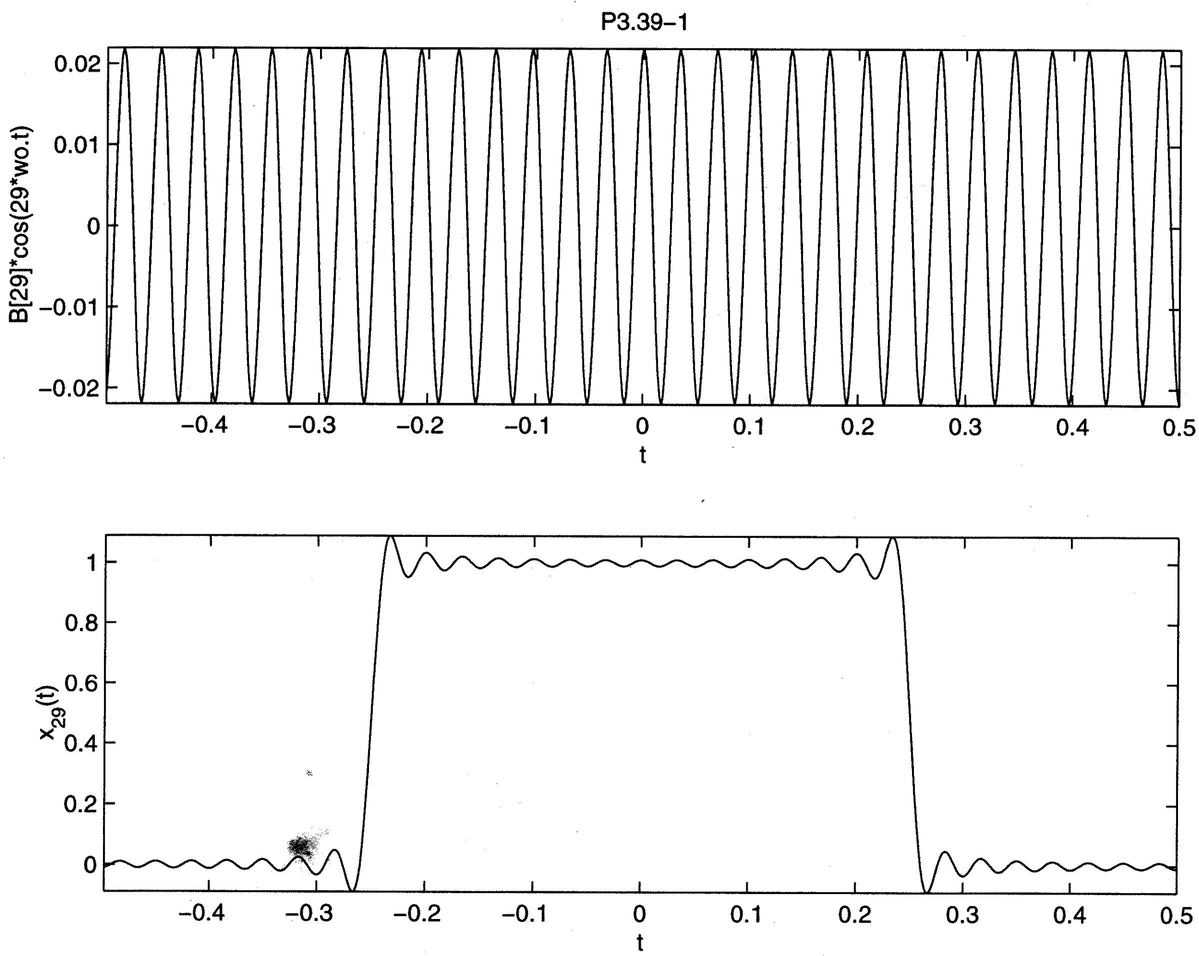
P 3.38

– Plot 2 of 2 –



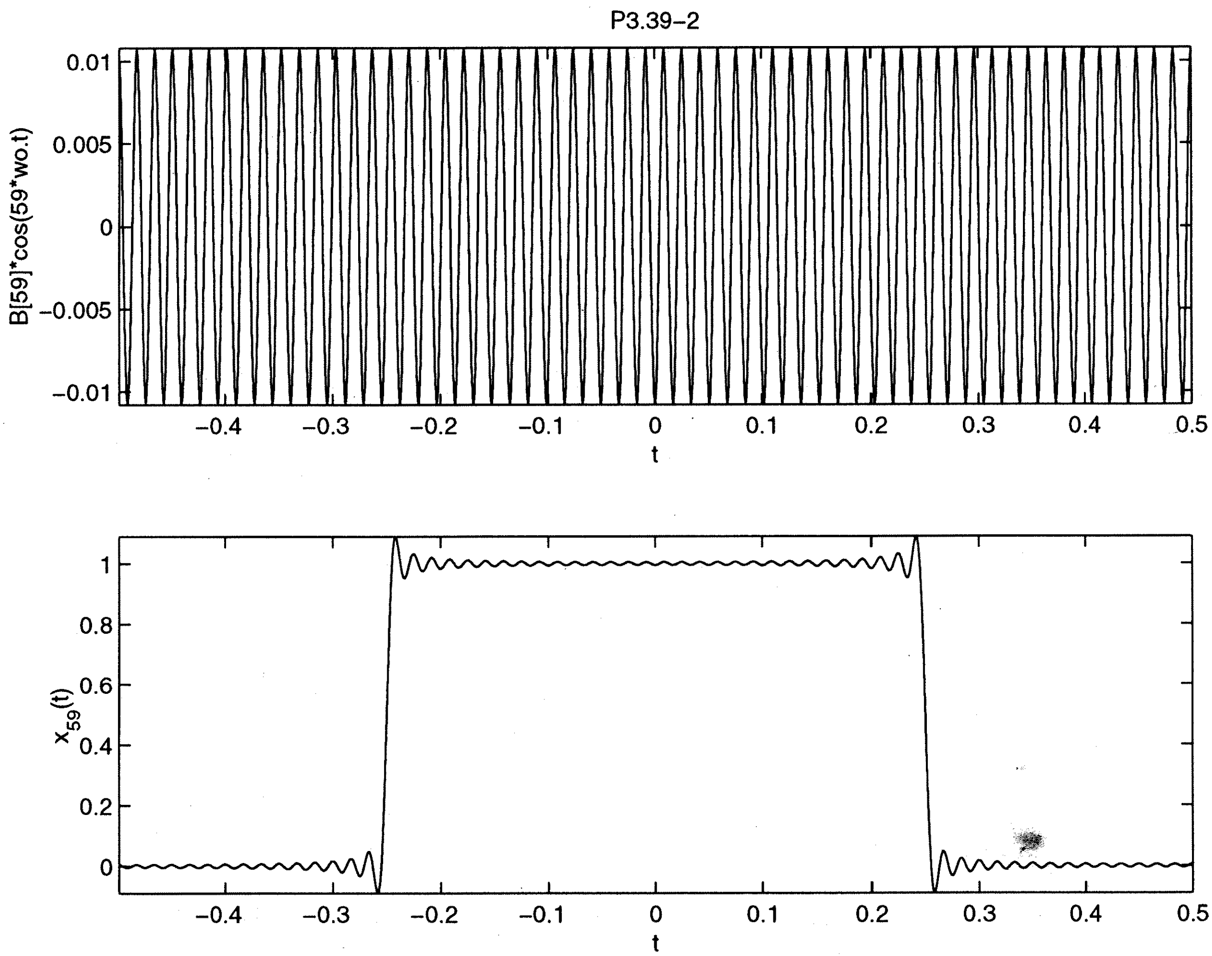
P 3.39

- Plot 1 of 3 -



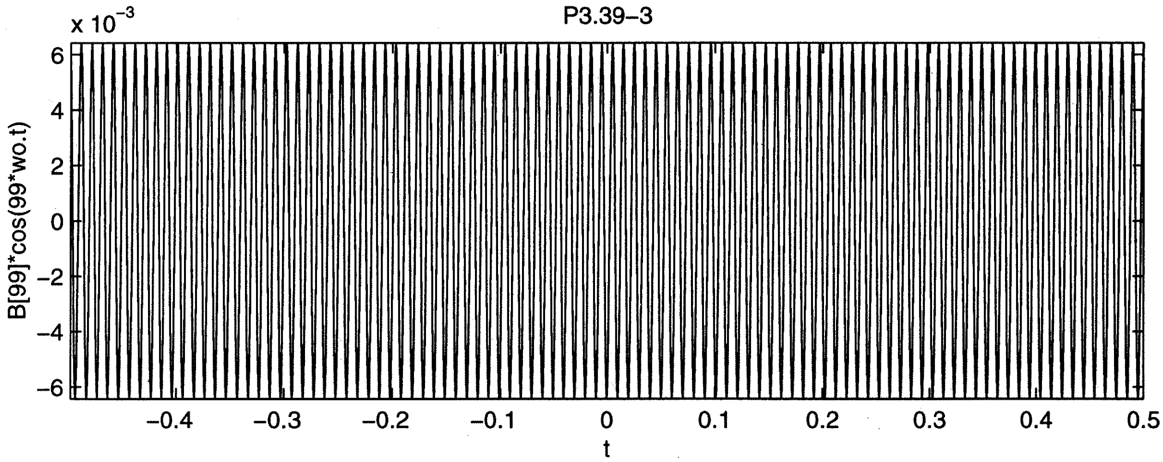
P 3.39

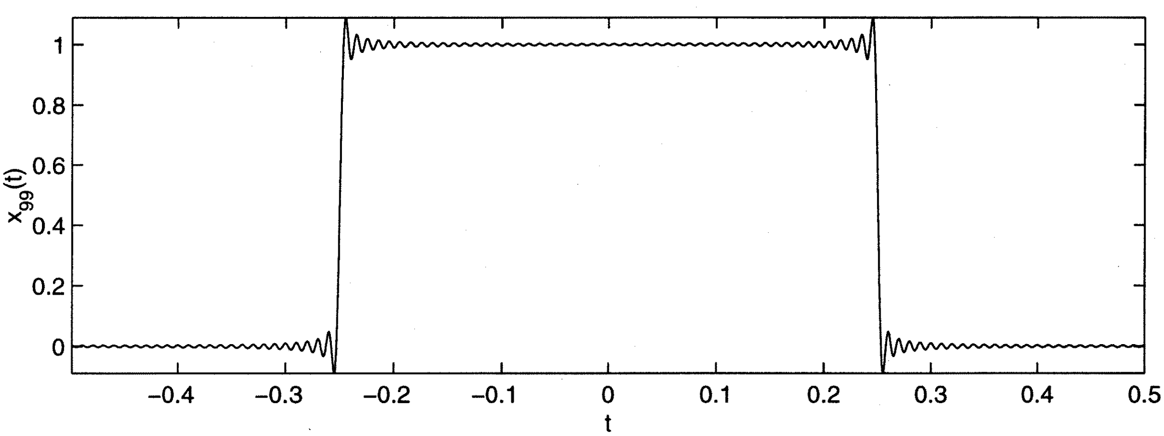
– Plot 2 of 3 –



P 3.39

- Plot 3 of 3 -





$\boxed{3.39}$ Using function max (.), the peak overshoot can be found

J	Peak
29	1.0891
59	1.0894
99	1.0895

$\boxed{3.40}$

(a) $X[k] = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-j\frac{2\pi}{T}kt}\, dt \quad , \quad T = 1$

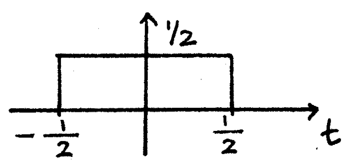
From example 3.26 with $T = 1$, $x(t) = y(t+\frac{1}{4}) - \frac{1}{2}$

we have $x(t-t_0) \overset{FS;\omega_0}{\longleftrightarrow} e^{-jk\omega_0 t_0} X[k]$

and $Y[k] = \begin{cases} \frac{\pi}{\omega_0} & , k = 0 \\ \frac{4\sin(\frac{k\pi}{2})}{jk^2\pi\omega_0} & , k \neq 0 \end{cases}$ where $\omega_0 = \frac{2\pi}{T} = 2\pi$

$\Rightarrow Y[k] = \begin{cases} \frac{1}{2} & , k = 0 \\ \frac{4\sin(\frac{k\pi}{2})}{jk^2 2\pi^2} & , k \neq 0 \end{cases}$

Also



$\overset{FS;2\pi}{\longleftrightarrow} \delta[k]$

$\Rightarrow X[k] = \begin{cases} -\frac{1}{2} & , k = 0 \\ \frac{4\sin(\frac{k\pi}{2})}{k^2 2\pi^2} e^{j\frac{\pi}{2}(k-1)} & , k \neq 0 \end{cases}$

(b) $x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{j\omega_0 kt}$

since $x(t)$ is real and even, $X[k] = X[-k]$

$x(t) = \sum_{k=1}^{\infty} X[k]\, e^{j\omega_0 kt} + \sum_{k=-\infty}^{-1} X[k]\, e^{j\omega_0 kt} + X[0]$

$= \sum_{k=1}^{\infty} X[k]\, e^{j\omega_0 kt} + \sum_{k=1}^{\infty} \underbrace{X[-k]}_{=X[k]}\, e^{-j\omega_0 kt} + X[0]$

$$= \sum_{k=1}^{\infty} X[k] \left(e^{j\omega_0 kt} + e^{-j\omega_0 kt}\right) + X[0]$$

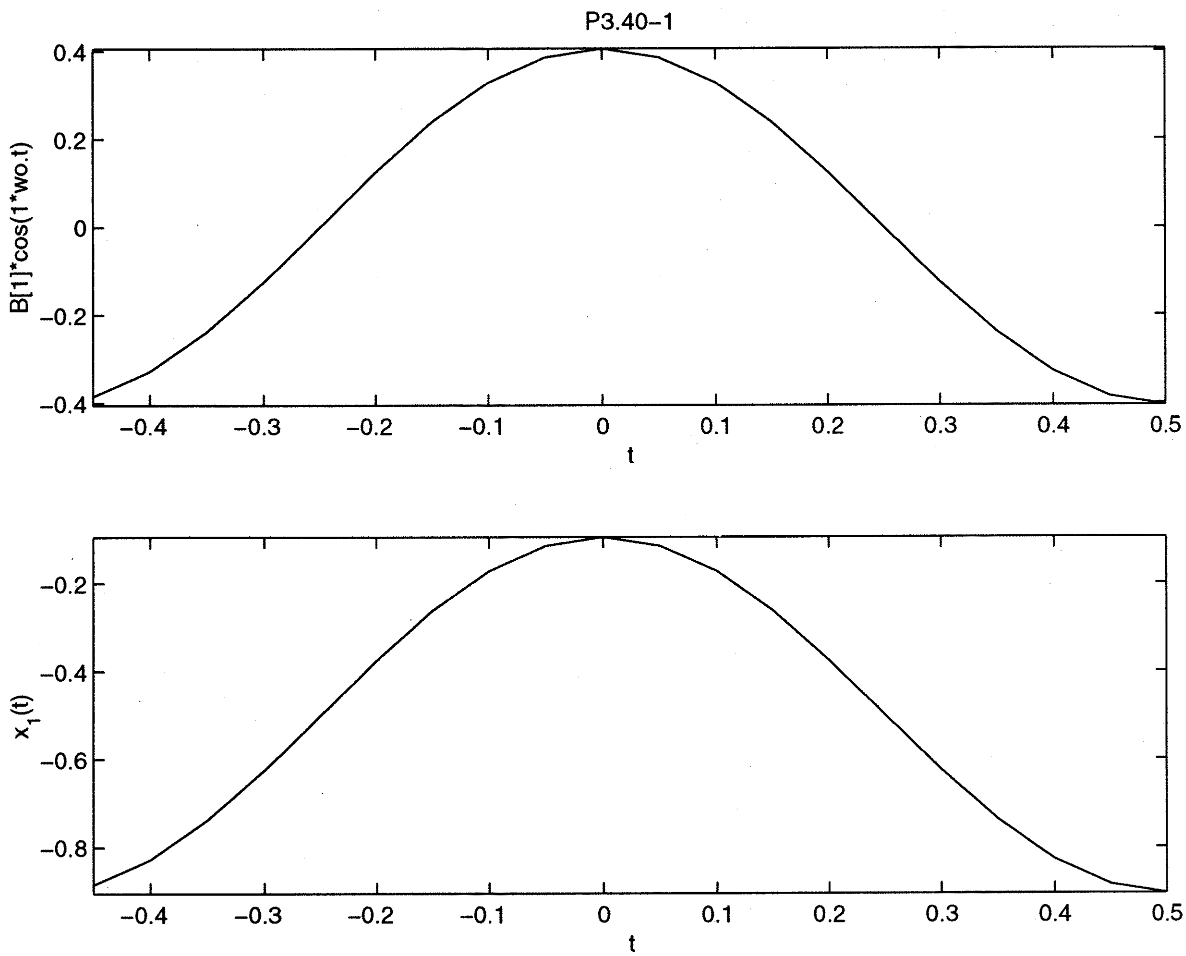
$$= \sum_{k=0}^{\infty} B[k] \cos(k\omega_0 t)$$

where $B[k] = \begin{cases} X[0] & , k=0 \\ 2X[k] & , k \neq 0 \end{cases}$

(c) $\hat{x_j}(t) \sum_{k=0}^{j} B[k] \cos(k\omega_0 t)$

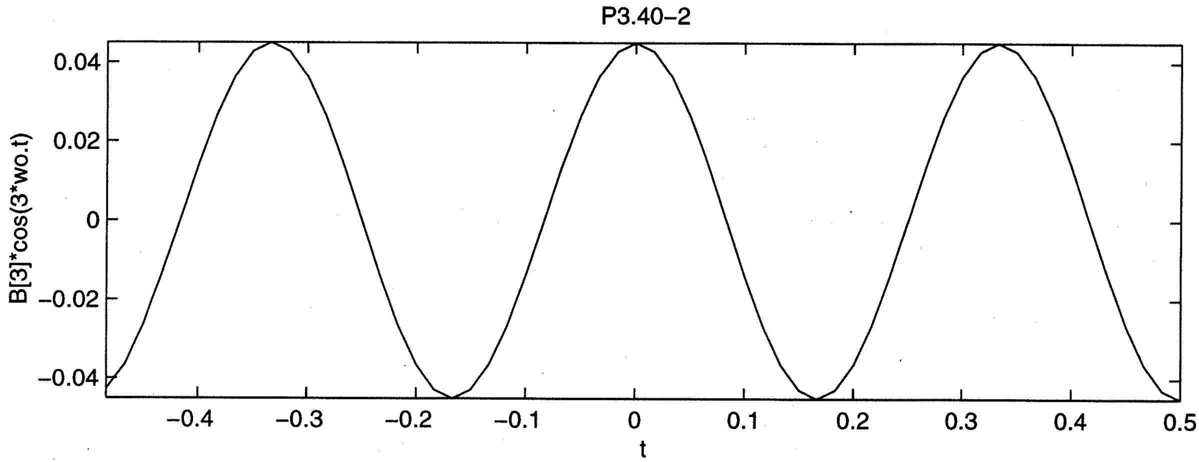
P 3.40

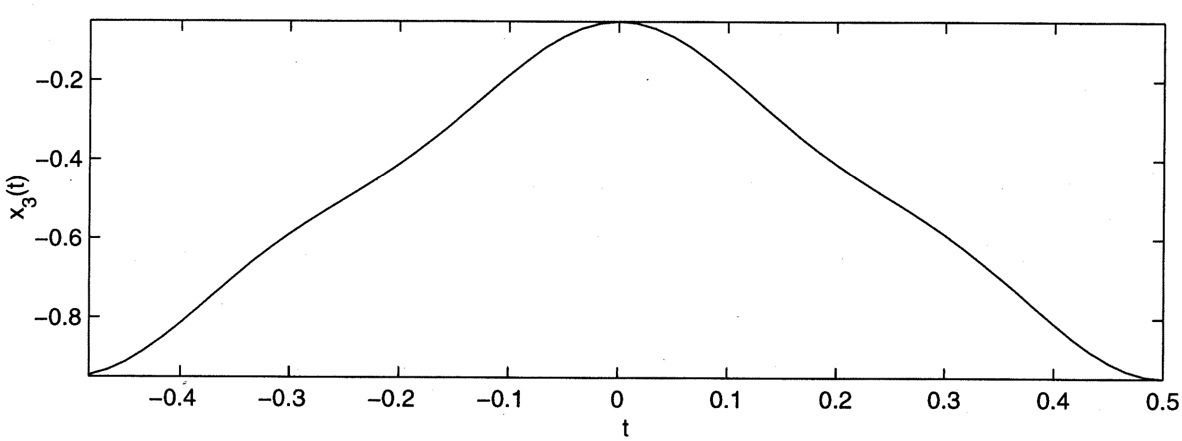
- Plot 1 of 5 -



P 3.40

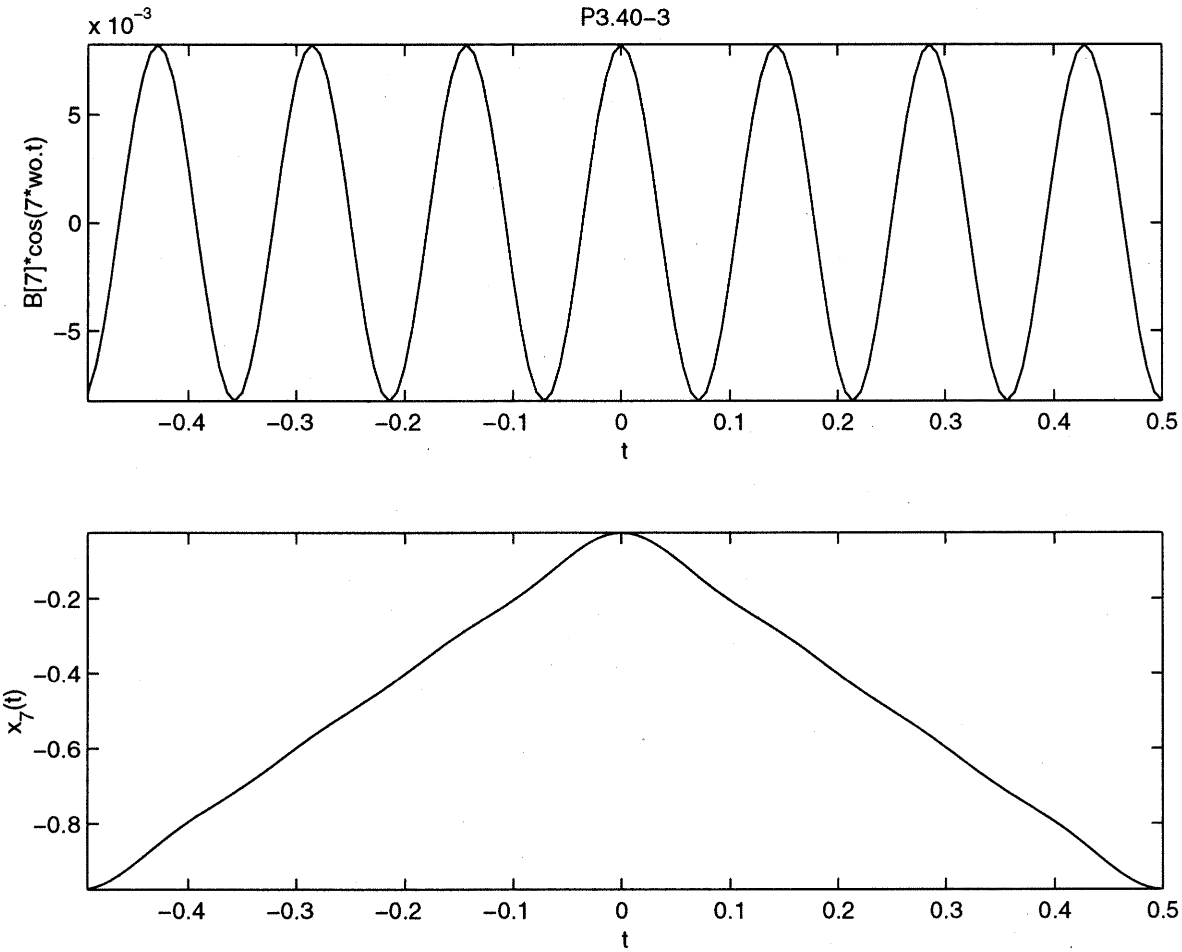
– Plot 2 of 5 –





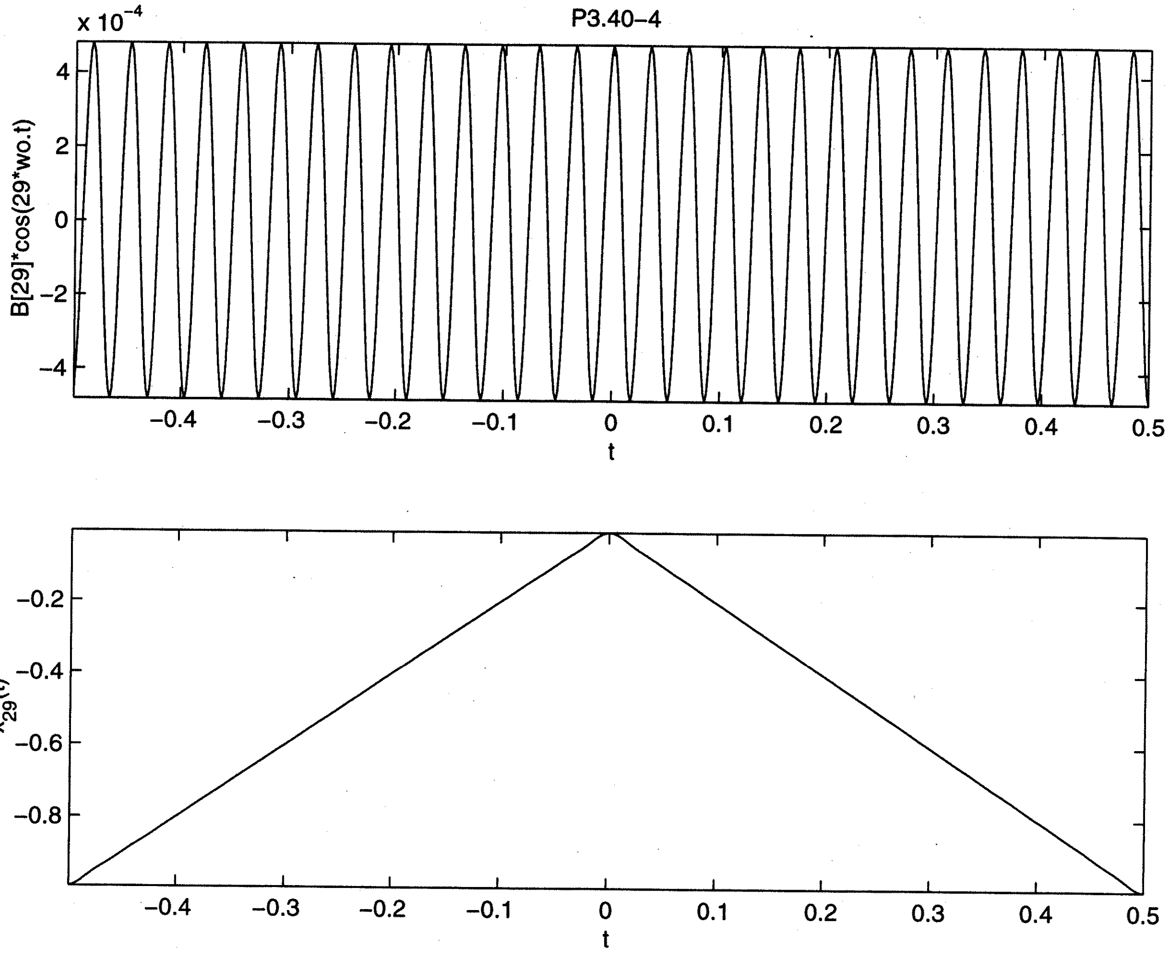
P 3.40

— Plot 3 of 5 —



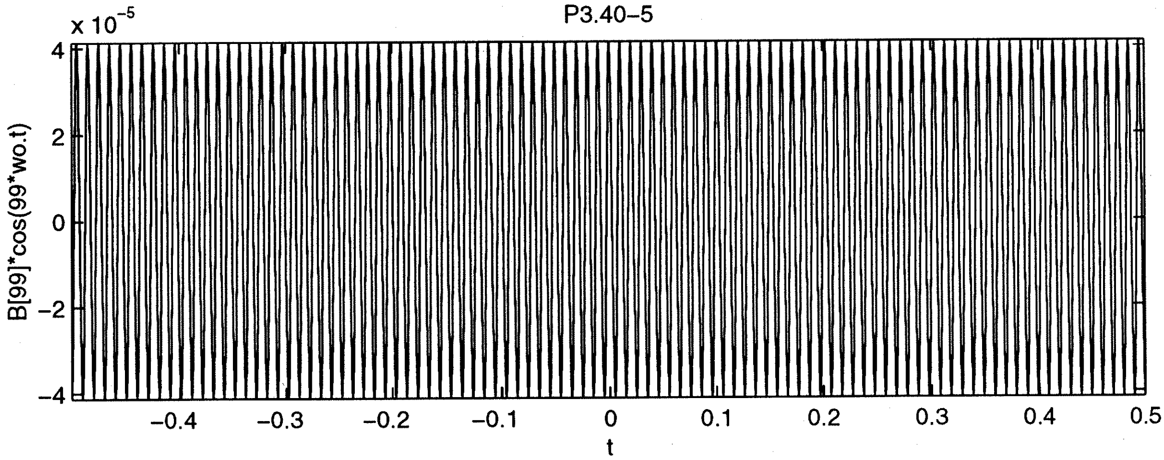
P 3.40

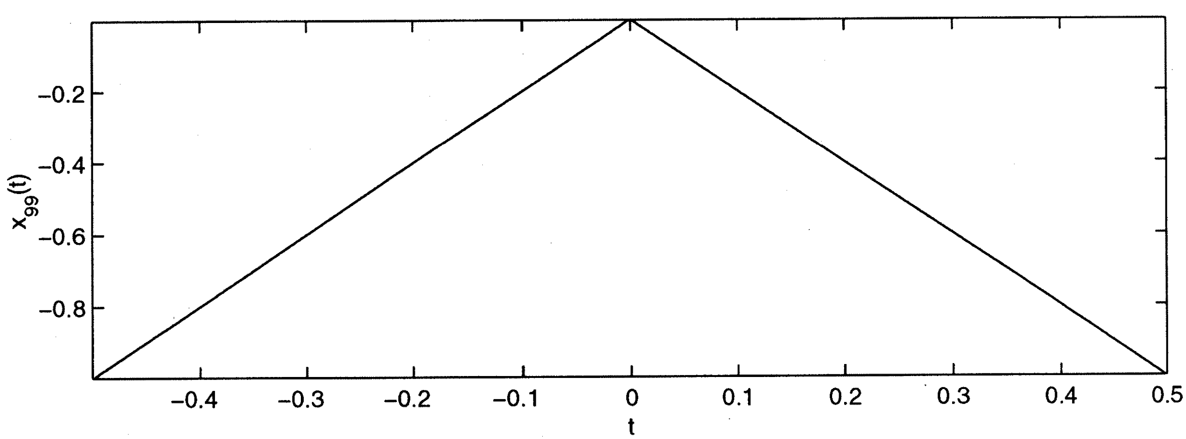
- Plot 4 of 5 -



P 3.40

– Plot 5 of 5 –





$\boxed{3.41}$

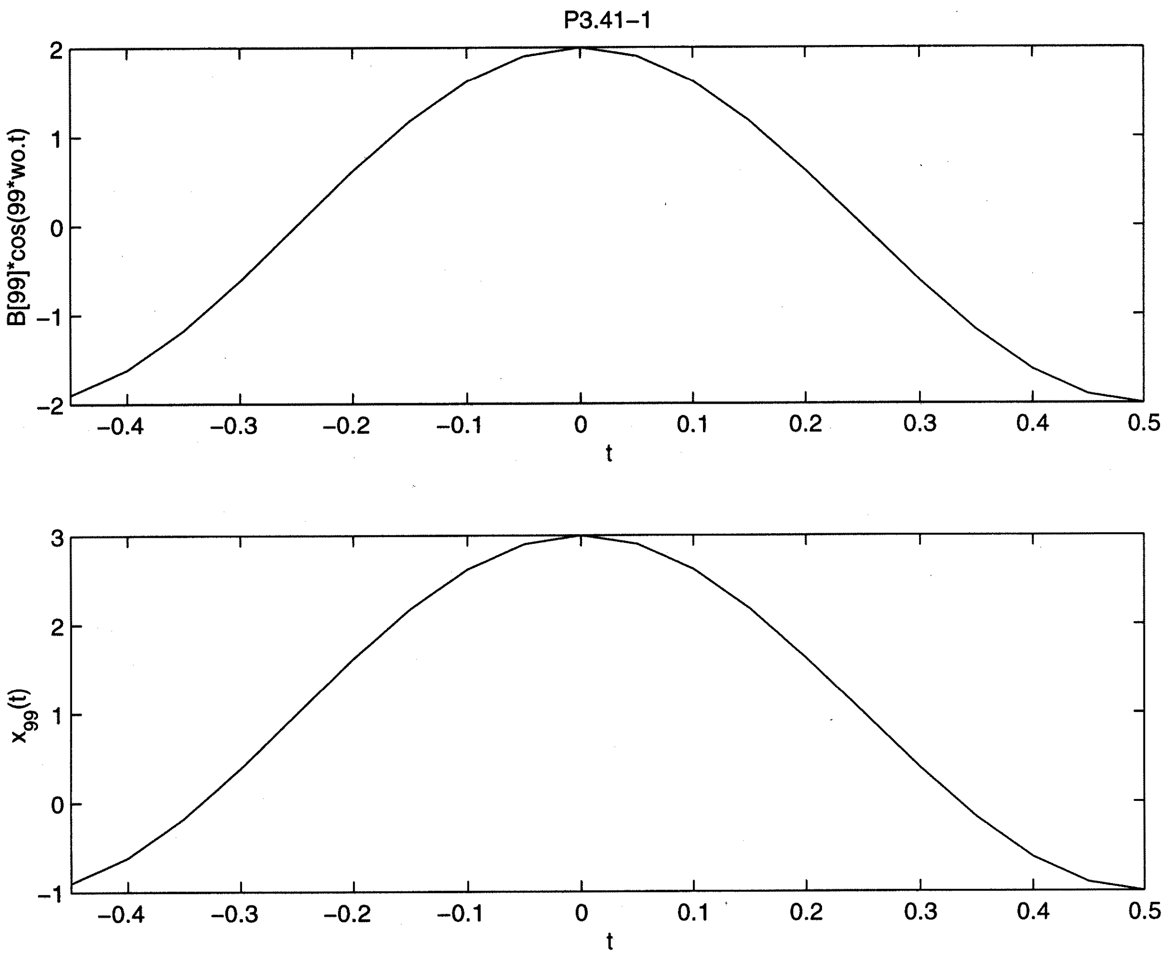
(a) $X[k] = \frac{1}{1}\int_0^T \delta(t)\, e^{-j2\pi kt}\, dt = 1$, all k

(b) same deal

$$B[k] = \begin{cases} 1 & , k = 0 \\ 2 & , k \neq 0 \end{cases}$$

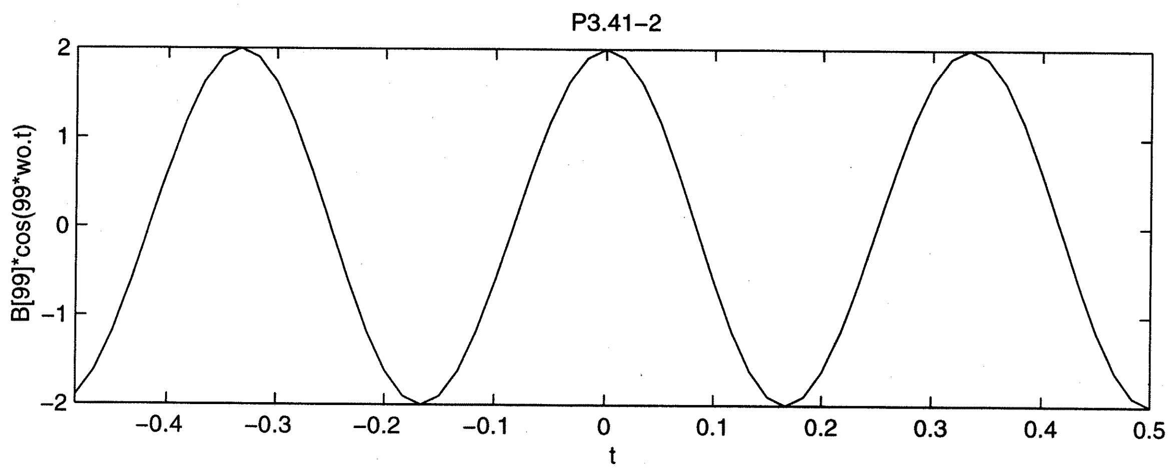
P 3.41

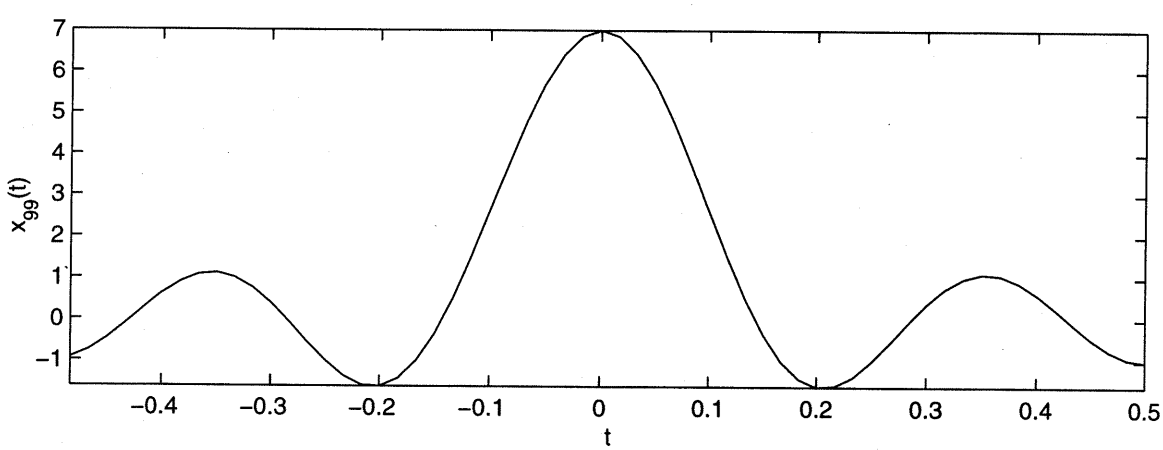
– Plot 1 of 5 –



P 3.41

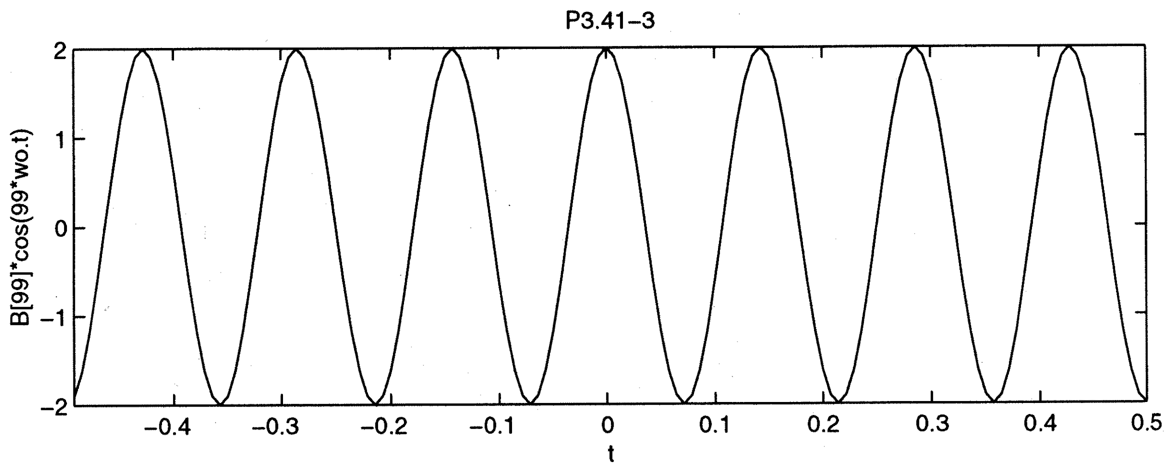
- Plot 2 of 5 -

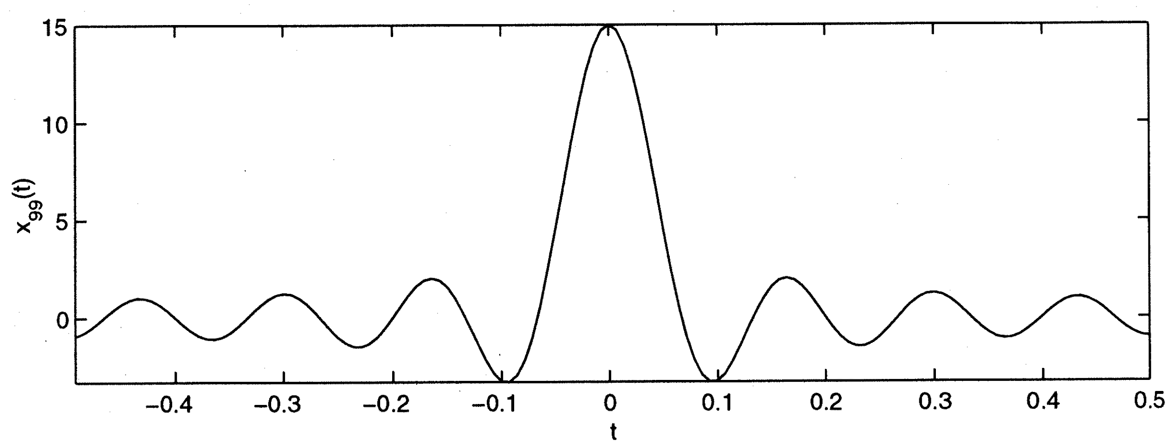




P 3.41

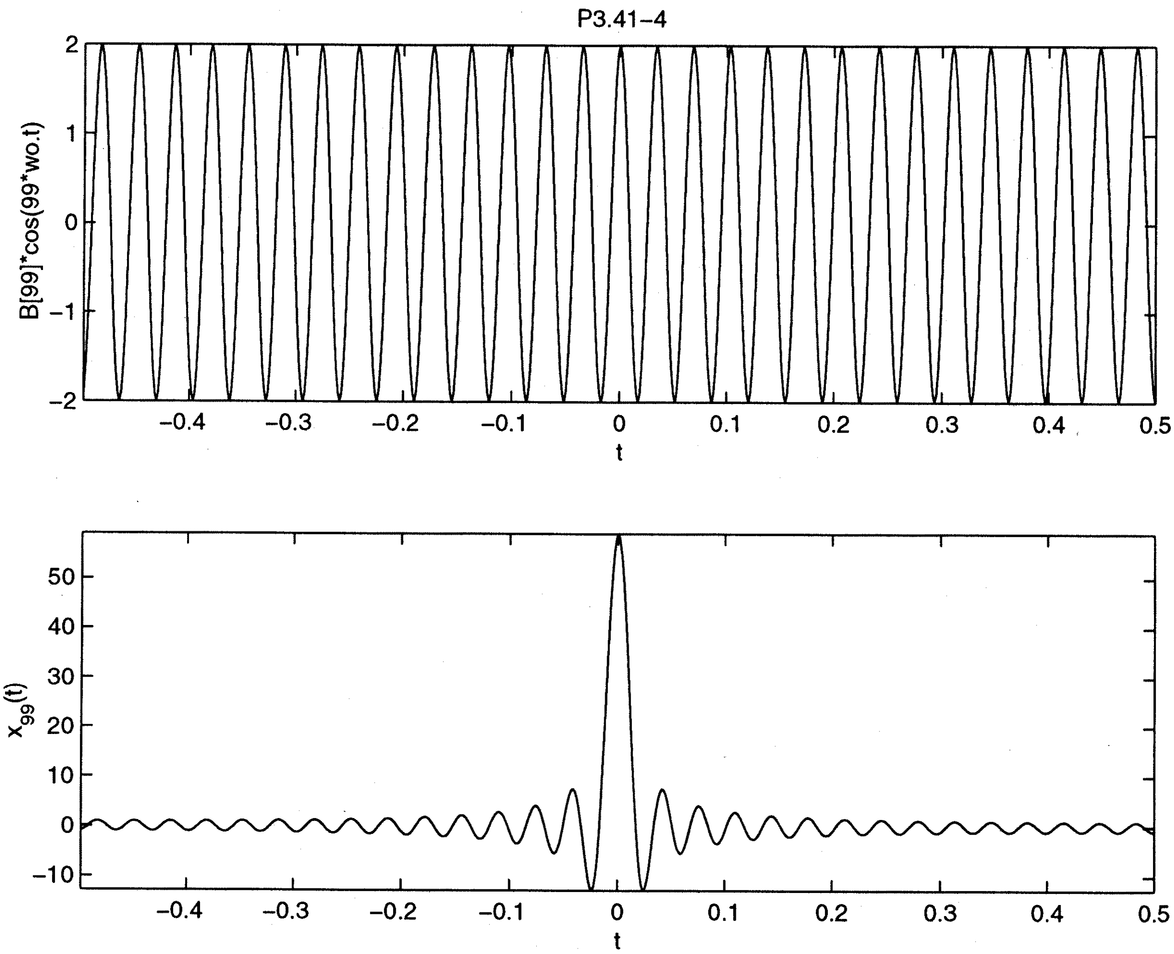
- Plot 3 of 5 -





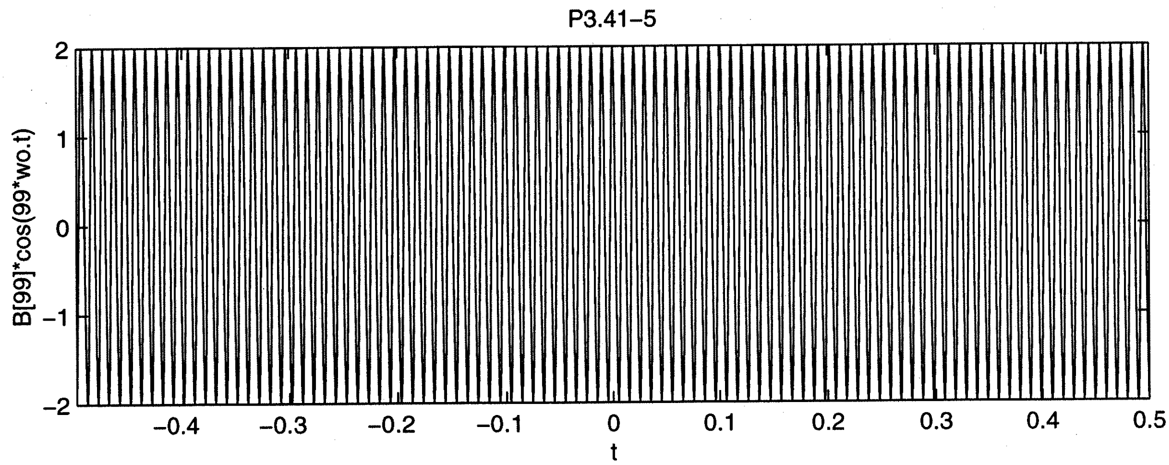
P 3.41

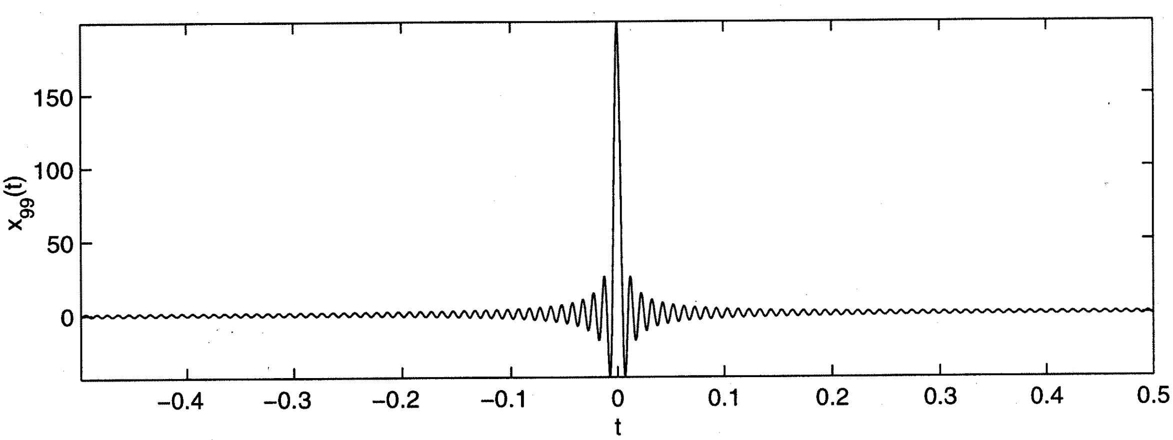
– Plot 4 of 5 –



P 3.41

— Plot 5 of 5 —



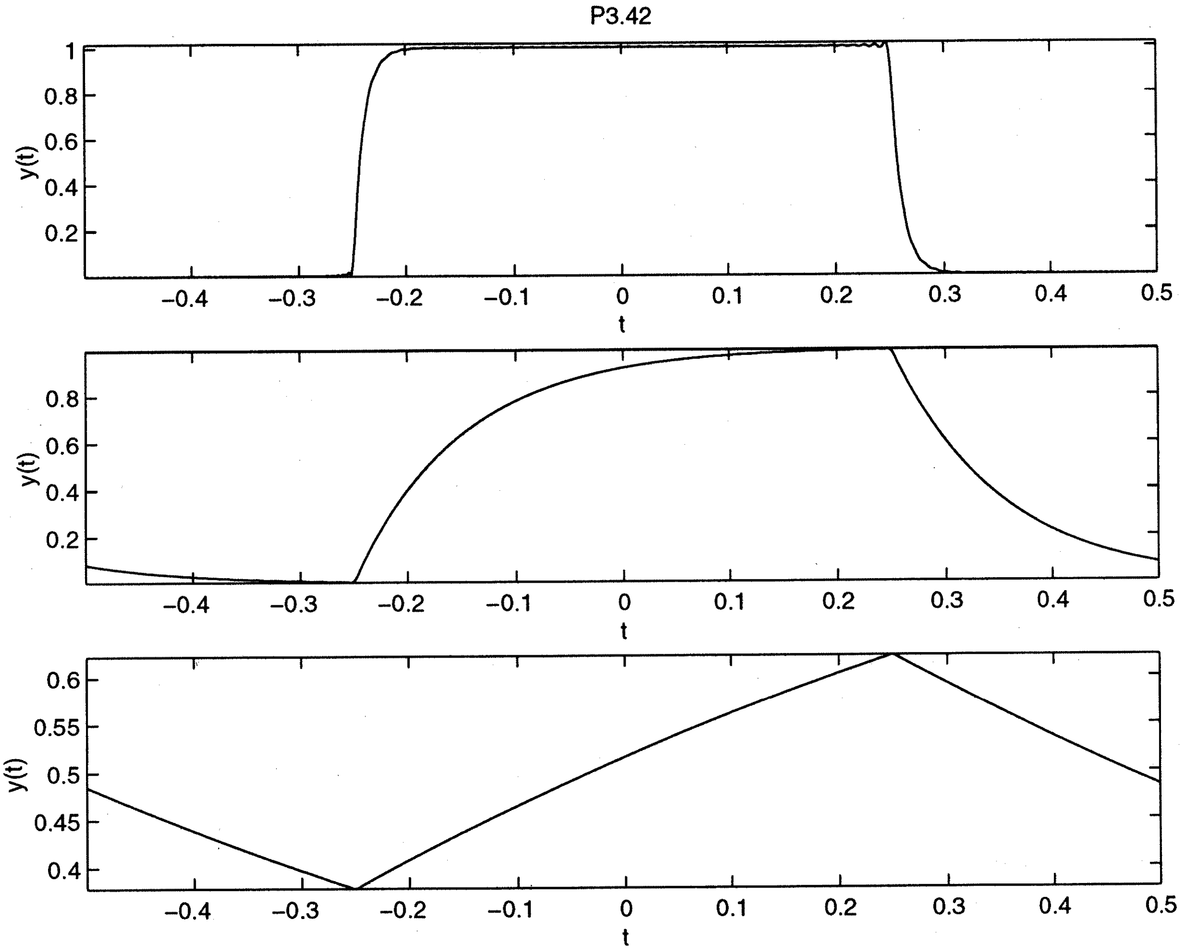


$\boxed{3.42}$ $$Y[k] = \frac{\frac{1}{RC}}{j2\pi k + \frac{1}{RC}} \cdot \frac{\sin(k\frac{\pi}{2})}{k\pi}$$

$$y(t) \approx \sum_{k=-100}^{100} Y[k]\, e^{j2\pi k t}$$

P 3.42

– Plot 1 of 1 –



$\boxed{3.43}$ It can be seen that as M increases :

1) The ripple is decreased
2) Transition Rolls of faster

Observe that $h[n] = \frac{1}{\pi n} \sin\left(\frac{\pi n}{2}\right)$ is symmetric,

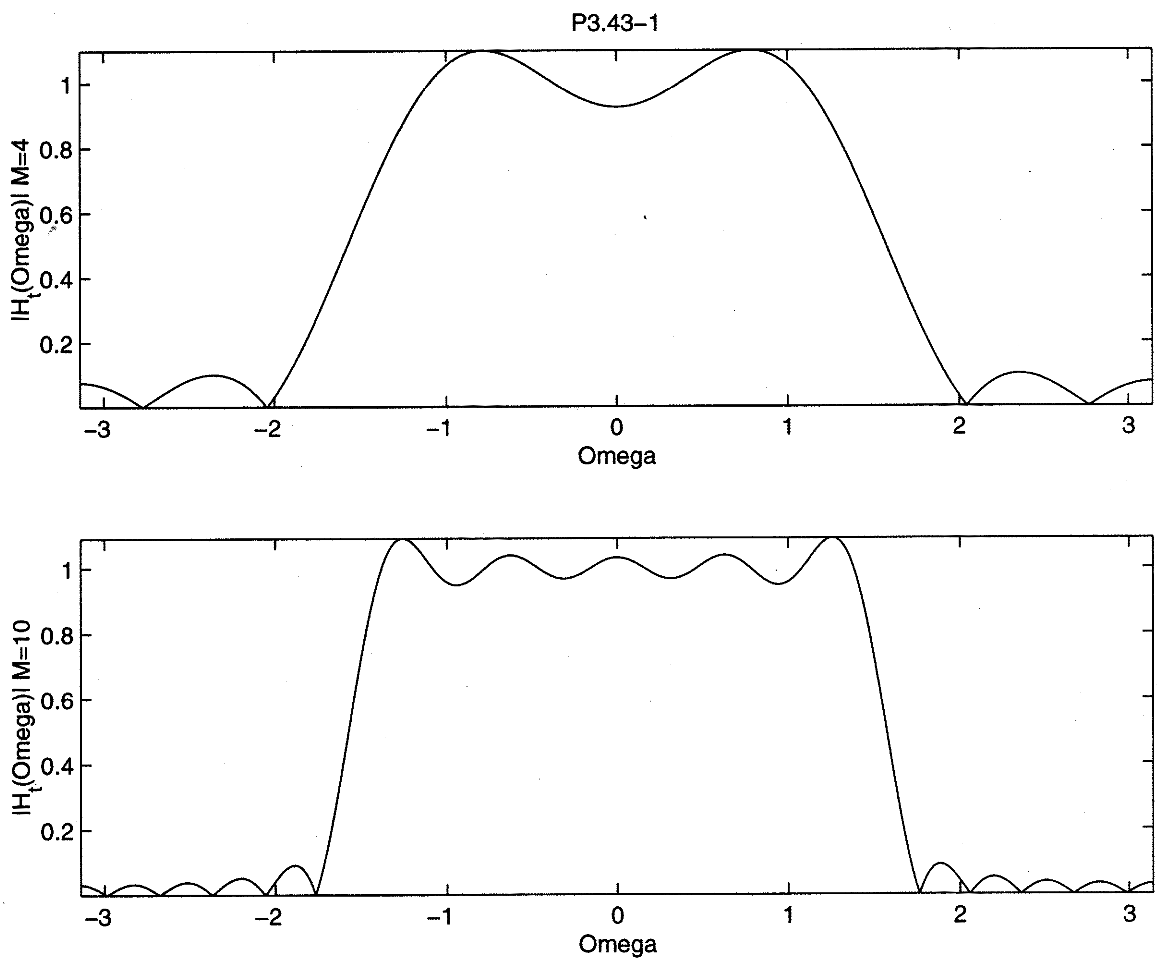
hence $H_t(e^{j\Omega}) = \sum_{n=1}^{M} h[n] e^{-j\Omega n} + \sum_{n=-M}^{-1} h[n] e^{-j\Omega n} + h[0]$

$H_t(e^{j\Omega}) = \sum_{n=1}^{M} 2h[n] \cos \Omega n + h[0] \cos \Omega(0)$

hence the approximation is more accurate as M increases

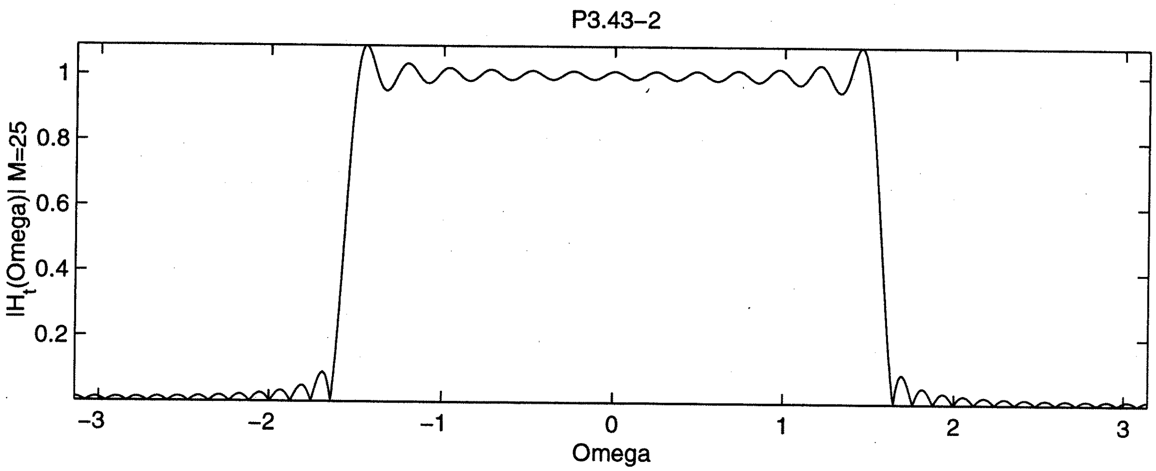
P 3.43

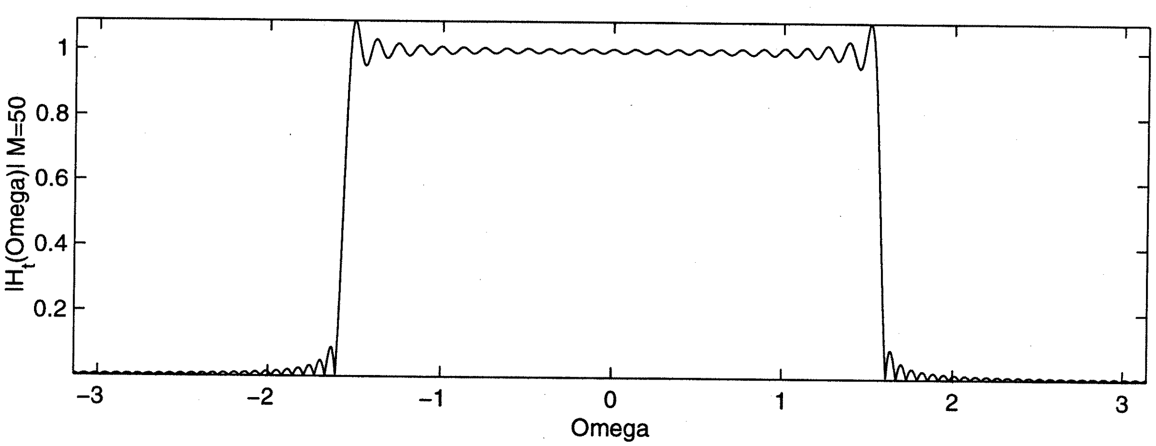
- Plot 1 of 2 -



P 3.43

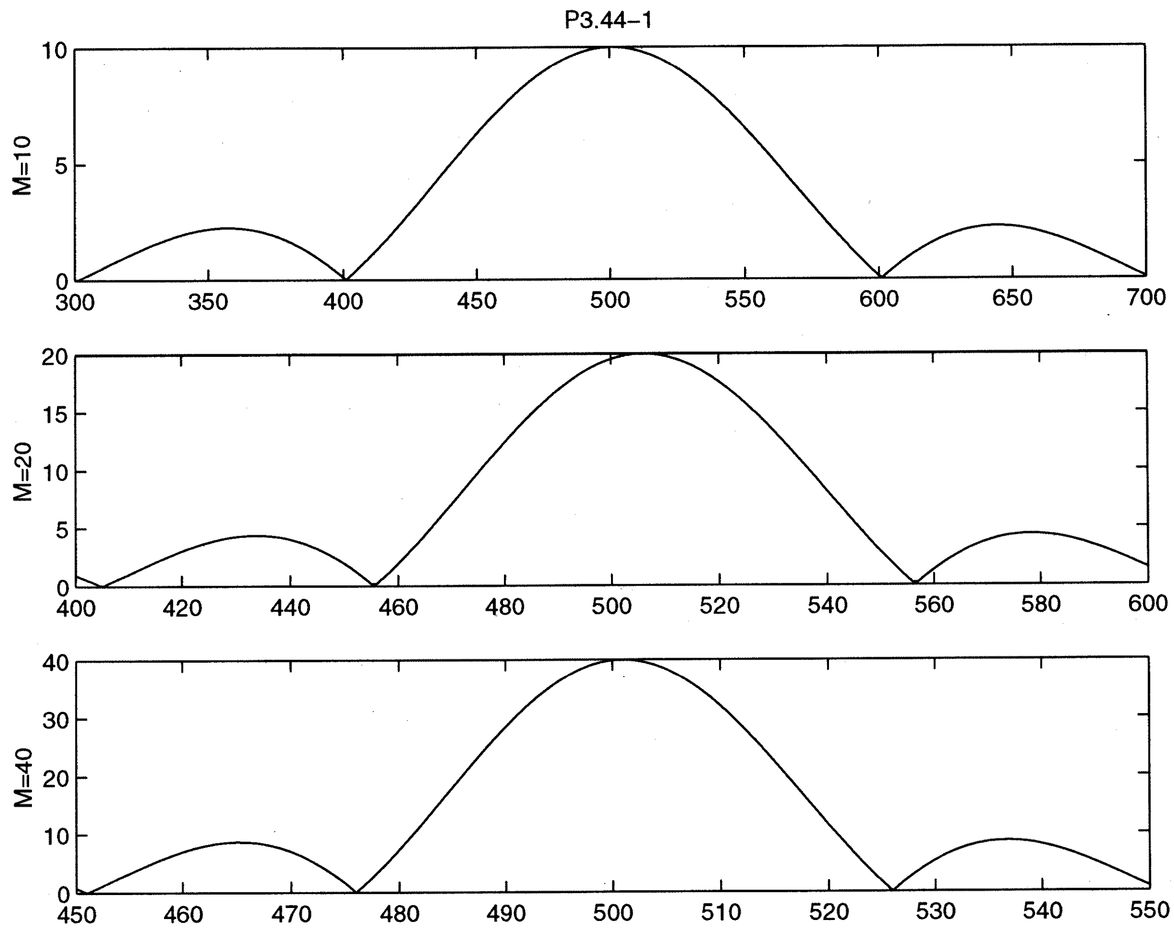
- Plot 2 of 2 -





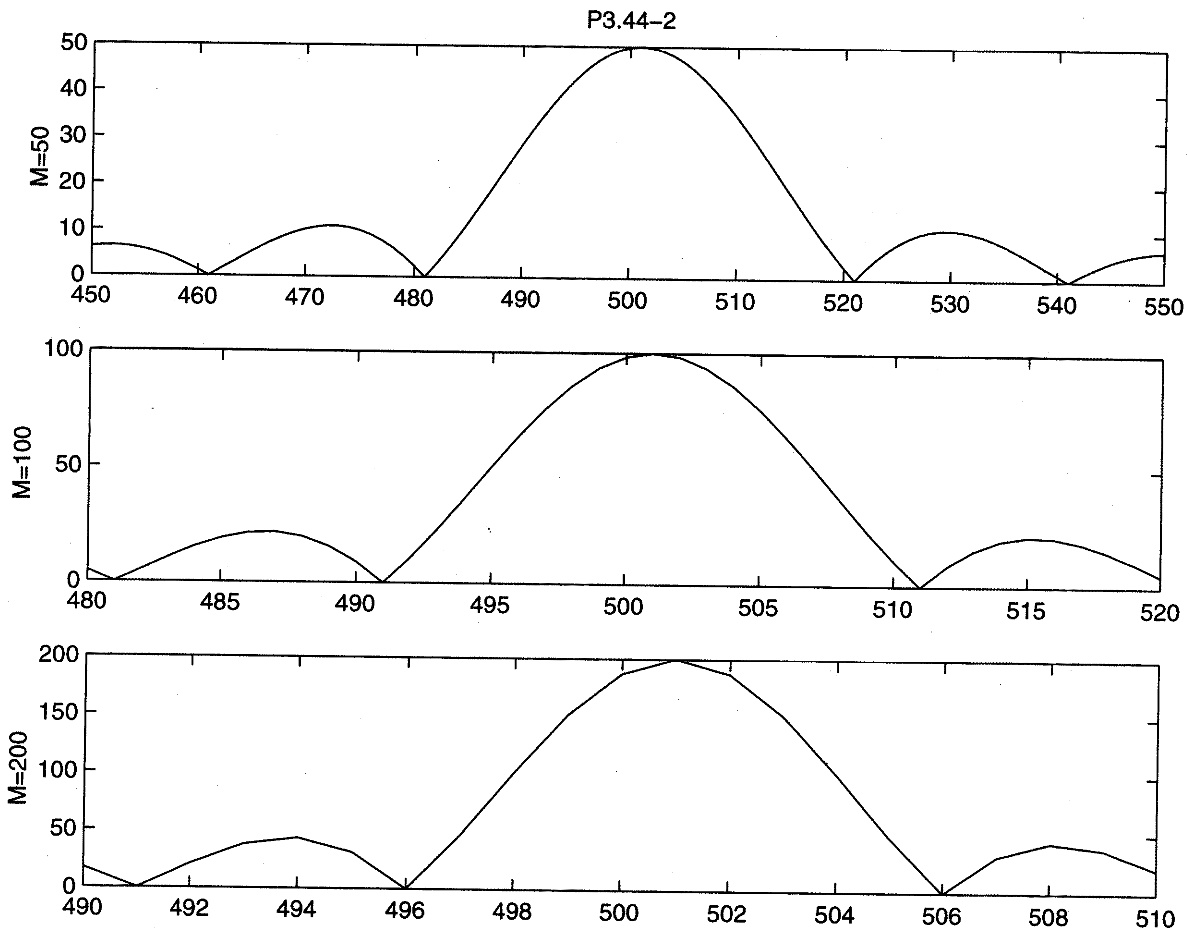
P 3.44

- Plot 1 of 2



P 3.44

- Plot 2 of 2 -



M	BW
10	201
20	101
40	51
50	41
100	21
200	11

P 3.45

- Plot 1 of 1 -