CHAPTER 2

$\boxed{2.1}$ Here, we are to use the fact that :

$$\delta[n-k] * h[n] = h[n-k]$$
$$(a x_1[n] + b x_2[n]) * h[n] = a x_1[n] * h[n] + b x_2[n] * h[n]$$

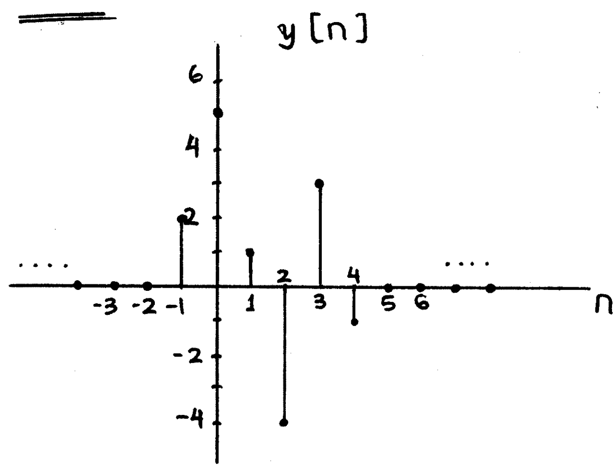
$$h[n] = \delta[n+1] + 3\delta[n] + 2\delta[n-1] - \delta[n-2] + \delta[n-3]$$

(a) $x[n] = 2\delta[n] - \delta[n-1]$

$y[n] = x[n] * h[n] = 2h[n] - h[n-1]$

$$2h[n] = 2\delta[n+1] + 6\delta[n] + 4\delta[n-1] - 2\delta[n-2] + 2\delta[n-3]$$
$$h[n-1] = \qquad \delta[n] + 3\delta[n-1] + 2\delta[n-2] - \delta[n-3] + \delta[n-4] \; -$$

$$\underline{\underline{y[n] = 2\delta[n+1] + 5\delta[n] + \delta[n-1] - 4\delta[n-2] + 3\delta[n-3] - \delta[n-4]}}$$

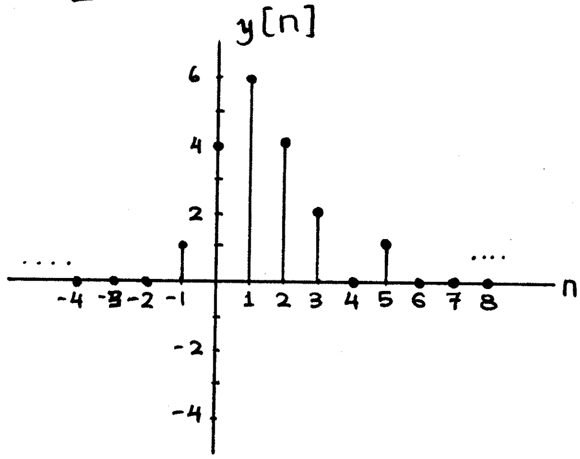


(b) $x[n] = u[n] - u[n-3] = \delta[n] + \delta[n-1] + \delta[n-2]$

$y[n] = x[n] * h[n] = h[n] + h[n-1] + h[n-2]$

$$h[n] = \delta[n+1] + 3\delta[n] + 2\delta[n-1] - \delta[n-2] + \delta[n-3]$$
$$h[n-1] = \qquad \delta[n] + 3\delta[n-1] + 2\delta[n-2] - \delta[n-3] + \delta[n-4]$$
$$h[n-2] = \qquad \delta[n-1] + 3\delta[n-2] + 2\delta[n-3] - \delta[n-4] + \delta[n-5] \; +$$

$$y[n] = \delta[n+1] + 4\,\delta[n] + 6\,\delta[n-1] + 4\,\delta[n-2] + 2\,\delta[n-3] + \delta[n-5]$$

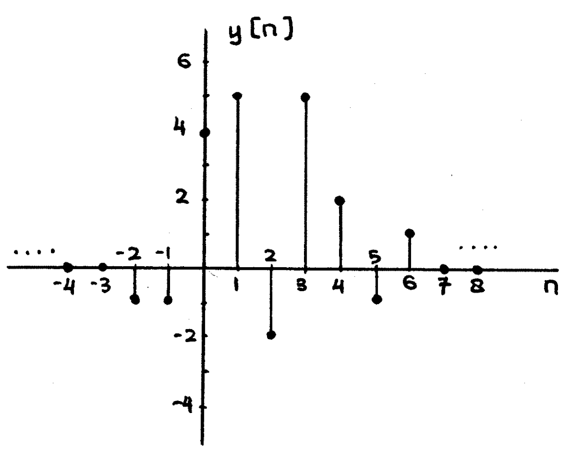


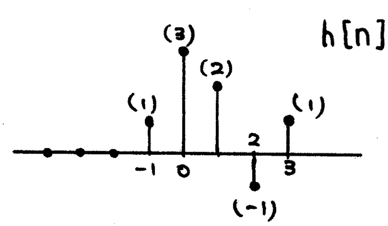
(c) $x[n] = -\delta[n+1] + 2\,\delta[n] + \delta[n-3]$

$y[n] = x[n] * h[n] = -h[n+1] + 2\,h[n] + h[n-3]$

$$\begin{aligned}
-h[n+1] &= -\delta[n+2] - 3\,\delta[n+1] - 2\,\delta[n] + \delta[n-1] - \delta[n-2] \\
2h[n] &= 2\,\delta[n+1] + 6\,\delta[n] + 4\,\delta[n-1] - 2\,\delta[n-2] + 2\,\delta[n-3] \\
h[n-3] &= \delta[n-2] + 3\,\delta[n-3] + 2\,\delta[n-4] - \delta[n-5] + \delta[n-6] \quad +
\end{aligned}$$

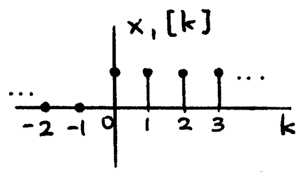
$$y[n] = -\delta[n+2] - \delta[n+1] + 4\,\delta[n] + 5\,\delta[n-1] - 2\,\delta[n-2] + 5\,\delta[n-3] + 2\,\delta[n-4] - \delta[n-5] + \delta[n-6]$$

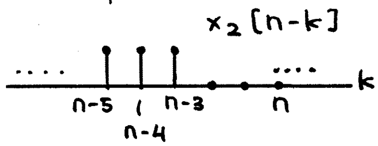




$\boxed{2.2.}$

(a) $y[n] = \underbrace{u[n]}_{x_1[n]} * \underbrace{u[n-3]}_{x_2[n]}$





$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k]$$

For $\underline{n-3<0 \text{ or } n<3}$: $y[n]=0$

$\underline{n=3}$: $y[3] = 1(1) = 1$

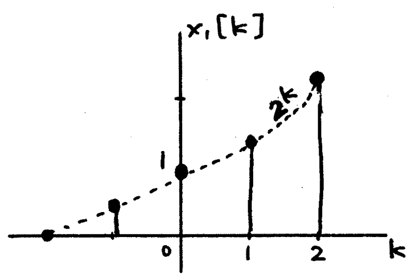
$\underline{n=4}$: $y[4] = 1(1)+1(1) = 2$

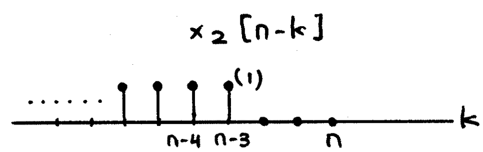
$\vdots$

$\underline{n=p}$: $y[p] = (p-2)$

$$\therefore\ y[n] = (n-2)\,u[n-2]$$

(b) $y[n] = \underbrace{2^n u[-n+2]}_{x_1[n]} * \underbrace{u[n-3]}_{x_2[n]}$





for $\underline{n-3 \leq 2 \text{ or } n \leq 5}$: $y[n] = \sum_{k=-\infty}^{n-3} 2^k$

substitution $e = -(k+3)$ gives $y[n] = \sum_{e=-n}^{\infty} 2^{-e-3}$

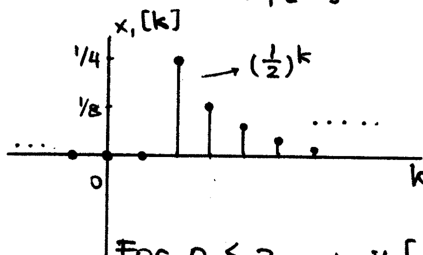
$$y[n] = \frac{1}{8} \sum_{e=-n}^{\infty} \left(\frac{1}{2}\right)^e \qquad \because \text{(Appendix A.3)}$$

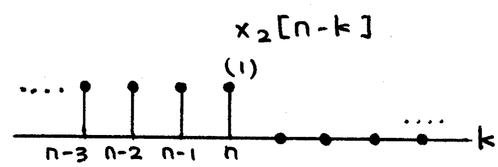
$$= \frac{1}{8}\,\frac{\left(\frac{1}{2}\right)^{-n}}{1-\frac{1}{2}} = \frac{1}{4}(2)^n$$

$\underline{n > 5}$: $y[n] = \sum_{k=\infty}^{2} 2^k = y[5] = \underline{\underline{8}}$

$$\therefore \ y[n] = \begin{cases} \frac{1}{4}(2)^n & n = -\infty, \ldots\ldots, 4, 5 \\ 8 & n = 6, 7, \ldots\ldots, +\infty \end{cases}$$

(c) $y[n] = \underbrace{(\tfrac{1}{2})^n u[n-2]}_{x_1[n]} * \underbrace{u[n]}_{x_2[n]}$





For $\underline{n < 2}$: $y[n] = 0$

For $\underline{n \geqslant 2}$: $y[n] = \sum_{k=2}^{n} (\tfrac{1}{2})^k$ (Appendix A.3)

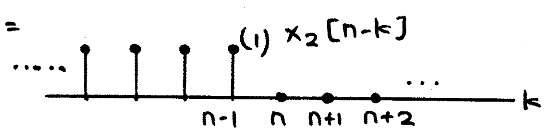
$$= \frac{1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} - (1 + \tfrac{1}{2}) = \tfrac{1}{2} - \left(\tfrac{1}{2}\right)^n$$

$$\therefore y[n] = \left\{\tfrac{1}{2} - (\tfrac{1}{2})^n\right\} u[n-2]$$

(d) $y[n] = \underbrace{\cos(\tfrac{\pi}{2} n)\, u[n]}_{x_1[n]} * \underbrace{u[n-1]}_{x_2[n]}$

$x_1[k] = \cos(\tfrac{\pi}{2} k)\, u[k]$

$x_2[n-k] =$



$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k]$$

For $n - 1 < 0$ or $\underline{n < 1}$: $y[n] = 0$

For $n-1 \geqslant 0$ or $\underline{n \geqslant 1}$:

$$y[n] = \sum_{k=0}^{n-1} \cos\left(\frac{\pi}{2}k\right) = \begin{cases} 1 & n = 4v+1,\ 4v+2 \\ 0 & n = 4v,\ 4v+3 \end{cases}$$

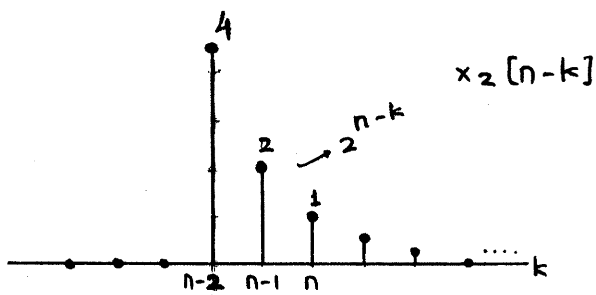
v = integer

$$\therefore\ y[n] = u[n-1] \cdot f[n], \text{where } f[n] = \begin{cases} 1 & n = 4v+1,\ 4v+2 \\ 0 & n = 4v,\ 4v+3 \end{cases}$$

(e) $y[n] = \cos\left(\frac{\pi}{2}n\right) * 2^n u[-n+2]$

$x_1[k] = \cos\left(\frac{\pi}{2}k\right)$

$x_2[n-k] =$



$$y[n] = \sum_{k=n-2}^{\infty} \cos\left(\frac{\pi}{2}k\right) . 2^{n-k}$$

$$\cos\left(\frac{\pi}{2}k\right) = \begin{cases} 1 & k = \ldots, 0, 4, 8, \ldots \\ -1 & k = \ldots, 2, 6, 10, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Thus, the only contributions are from the even-indexed terms.

Consider :

$\underline{n = \text{even}}$: $y[n] = \sum_{k=n-2}^{\infty} (-1)^{\frac{k}{2}}\, 2^{n-k}$

alternate sum $(n-2, n, n+2, \ldots)$

substitution $e = \frac{k}{2}$ (to account only even terms)

$$y[n] = 2^n \sum_{e=\frac{n-2}{2}}^{\infty} (-1)^e \left(\tfrac{1}{2}\right)^{2e} = 2^n \sum_{e=\frac{n-2}{2}}^{\infty} \left(-\tfrac{1}{4}\right)^e$$

$$\text{(App.3)} = 2^n \, \frac{\left(-\frac{1}{4}\right)^{\frac{n-2}{2}}}{1+\frac{1}{4}} = 2^n \left(\tfrac{4}{5}\right)(-4)\left(-\tfrac{1}{2}\right)^n$$

$$= \frac{16}{5}(-1)^{n+1}$$

$\underline{n = \text{odd}}$: $y[n] = \sum_{k=n-1}^{\infty} (-1)^{\frac{k}{2}} \, 2^{n-k}$ alternate sum $(n-1; n+1, n+3, \ldots)$

Analogous to above, yield :

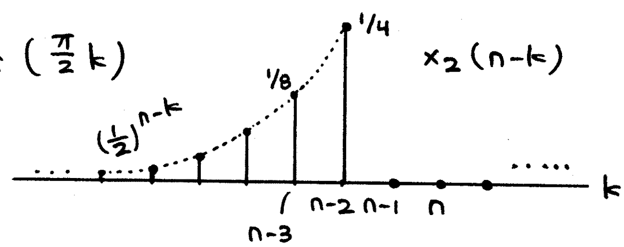
$$y[n] = 2^n \sum_{e=\frac{n-1}{2}}^{\infty} \left(-\tfrac{1}{4}\right)^e = 2^n \left(\tfrac{4}{5}\right)\left(-\tfrac{1}{4}\right)^{\frac{n-1}{2}}$$

$$= \frac{8}{5}(-1)^{n+1}$$

(f) $y[n] = \cos\left(\frac{\pi}{2} n\right) * \left(\frac{1}{2}\right)^n u[n-2]$

$x_1[k] = \cos\left(\frac{\pi}{2} k\right)$

$x_2[n-k] =$



$$y[n] = \sum_{k=-\infty}^{n-2} \cos\left(\tfrac{\pi}{2} k\right) . \left(\tfrac{1}{2}\right)^{n-k}$$

substitution $p = -k$ yields :

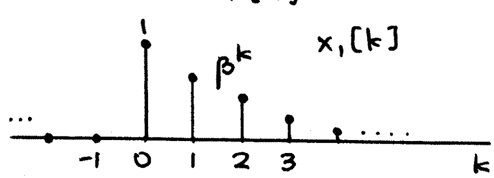
$$y[n] = \sum_{p=-(n-2)}^{\infty} \cos\left(\tfrac{\pi}{2} p\right) . \left(\tfrac{1}{2}\right)^{n+p} \quad \because \cos(-x) = \cos(x)$$

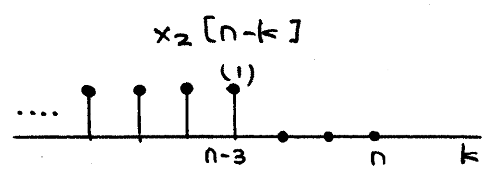
Then, using the same technique as (e) gives us :

$$y[n] = \begin{cases} \displaystyle\sum_{p=-(n-2)}^{\infty} (-1)^{\frac{p}{2}} \left(\tfrac{1}{2}\right)^{n+p} & n \text{ even} \\ \displaystyle\sum_{p=-(n-3)}^{\infty} (-1)^{\frac{p}{2}} \left(\tfrac{1}{2}\right)^{n+p} & \end{cases}$$

$$y[n] \begin{cases} \left(\tfrac{1}{2}\right)^n \dfrac{\left(-\frac{1}{4}\right)^{-\frac{(n-2)}{2}}}{\frac{5}{4}} = \frac{1}{5}(-1)^n & n \text{ even} \\ \left(\tfrac{1}{2}\right)^n \dfrac{\left(-\frac{1}{4}\right)^{-\left(\frac{n-3}{2}\right)}}{\frac{5}{4}} = \frac{1}{10}(-1)^{n+1} & n \text{ odd} \end{cases}$$

(g) $y[n] = \underbrace{\beta^n u[n]}_{x_1[n]} * \underbrace{u[n-3]}_{x_2[n]}$, $|\beta| < 1$



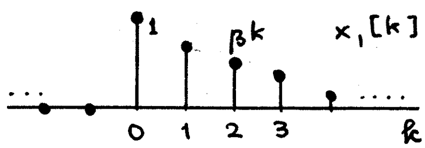


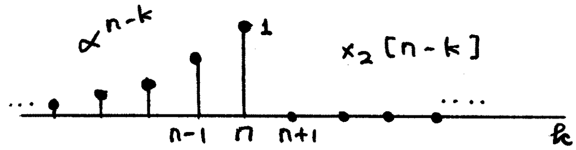
For $\underline{n-3 < 0 \text{ or } n < 3}$: $y[n] = 0$

$\underline{n \geq 3}$: $y[n] = \displaystyle\sum_{k=0}^{n-3} \beta^k = \frac{1-\beta^{n-2}}{1-\beta}$

$$\therefore\ y[n] = \left(\frac{1-\beta^{n-2}}{1-\beta}\right) u[n-3] \quad , |\beta| < 1$$

(h) $y[n] = \underbrace{\beta^n u[n]}_{x_1[n]} * \underbrace{\alpha^n u[n]}_{x_2[n]} \qquad |\beta| < 1 \ , \ |\alpha| < 1$





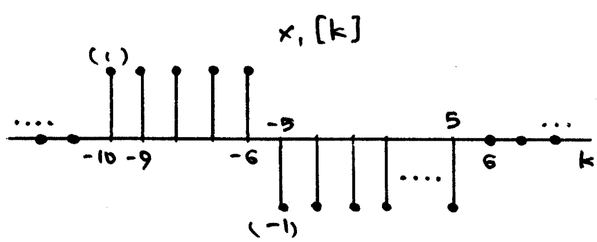
For $\underline{n < 0}$: $y[n] = 0$

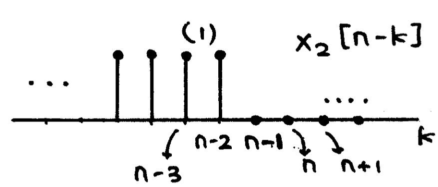
$\underline{n \geq 0}$: $y[n] = \sum_{k=0}^{n} \beta^k \alpha^{n-k} = \alpha^n \sum_{k=0}^{n} \left(\frac{\beta}{\alpha}\right)^k$

$$= \begin{cases} \alpha^n \cdot \dfrac{1 - \left(\frac{\beta}{\alpha}\right)^{n+1}}{1 - \frac{\beta}{\alpha}} & \alpha \neq \beta \\ \alpha^n (n+1) & \alpha = \beta \end{cases}$$

$$\therefore \ y[n] = \begin{cases} \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \cdot u[n] & \alpha \neq \beta \\ \alpha^n (n+1) \, u[n] & \alpha = \beta \end{cases}$$

(i) $y[n] = \left(\underbrace{u[n+10] - 2u[n+5] + u[n-6]}_{x_1[n]}\right) * \underbrace{u[n-2]}_{x_2[n]}$





For $\underline{n-2 < -10 \text{ or } n < -8}$: $y[n] = 0$

$-10 \le n-2 < -5$ or $\underline{-8 \le n < -3}$: $y[n] = n+9$

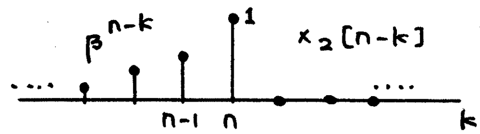
$-5 \le n-2 < 6$ or $\underline{-3 \le n < 8}$: $y[n] = 5+(-n-4) = 1-n$

$n-2 \geq 6$ or $\underline{n \geq 8}$: $y[n] = y[7] = -6$

$$\therefore\ y[n] = (n+9)\big(u[n+8]-u[n+3]\big) + (1-n)\big(u[n+3] - u[n-8]\big) + (-6)\big(u[n-8]\big)$$

$$y[n] = (n+9)\,u[n+8] - 2(n+4)\,u[n+3] + (n-7)\,u[n-8]$$

(j) $y[n] = \underbrace{\big(u[n+10] - 2u[n+5] + u[n-6]\big)}_{x_1[n] \Rightarrow \text{ see } (i)} * \underbrace{\beta^n u[n]}_{x_2[n]}, \ |\beta| < 1$



For $\underline{n < -10}$: $y[n] = 0$

$\underline{-10 \le n \le -6}$: $y[n] = \sum_{k=-10}^{n} \beta^{n-k}$

$$= \beta^n \left(\frac{1-\left(\frac{1}{\beta}\right)^{n+1}}{1-\frac{1}{\beta}} + \beta \cdot \frac{1-\beta^{10}}{1-\beta} \right)$$

$$= \frac{1-\beta^{11+n}}{1-\beta}$$

$\underline{-5 \le n \le 5}$: $y[n] = \sum_{k=-10}^{-6} \beta^{n-k} - \sum_{k=-5}^{n} \beta^{n-k}$

$$y[n] = \beta^n \left(\frac{2\beta^6 - \beta^{11} - \beta^{-n}}{1-\beta} \right)$$

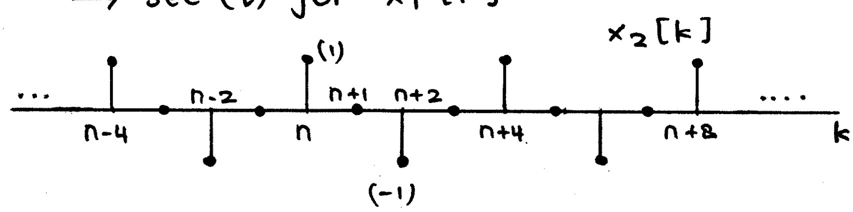
$$n \geq 6 : y[n] = \sum_{k=-10}^{-6} \beta^{n-k} - \sum_{k=-5}^{5} \beta^{n-k} = y[5] \frac{\beta^n}{\beta^5}$$

$$y[n] = \beta^n \left(\frac{2\beta^6 - \beta^{11} - \beta^{-5}}{1-\beta} \right)$$

$$\therefore y[n] = \begin{cases} 0 & n = -\infty, \dots, -12, -11 \\ \dfrac{1-\beta^{n+11}}{1-\beta} & n = -10, -9, \dots, -6 \\ \beta^n \left(\dfrac{2\beta^6 - \beta^{11} - \beta^{-n}}{1-\beta} \right) & n = -5, -4, \dots, 5 \\ \beta^n \left(\dfrac{2\beta^6 - \beta^{11} - \beta^{-5}}{1-\beta} \right) & n = 6, 7, \dots, \infty \end{cases}$$

(k) $y[n] = \left(u[n+10] - 2u[n+5] + u[n-6] \right) * \underbrace{\cos\left(\tfrac{\pi}{2} n\right)}_{x_2[n]}$

$\Longrightarrow$ see (i) for $x_1[k]$



There are 4 different cases :

(i) $\underline{n = 4v}$ $\quad v$ is any integer

$$y[n] = (1)[-1 + 0 + 1 + 0 + -1] + (-1)[0 + 1 + 0 + -1 + 0 + 1 + 0 + -1 + 0 + 1 + 0]$$
$$= -1 - 1 = -2$$

(ii) $\underline{n = 4v + 2}$:

$$y[n] = (1)[1+0+-1+0+1] + (-1)[0+-1+0+1+0+-1+0+1+0+-1+0]$$
$$= 1 + 1 = 2$$

(iii) $\underline{n = 4v + 3}$:

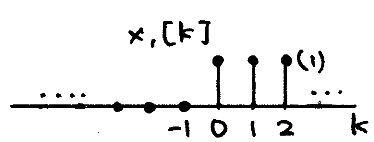
$$y[n] = (1)[0+-1+0+1+0] + (-1)[-1+0+1+0+-1+0+1+0+-1+0+1]$$
$$= 0 + 0 = 0$$

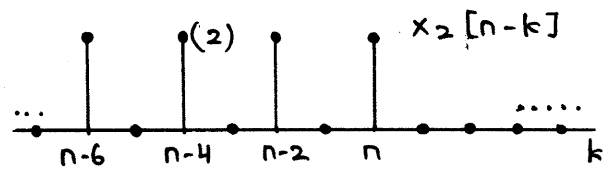
(iv) $\underline{n = 4v + 1}$:

$$y[n] = 0$$

$$\therefore y[n] = \begin{cases} -2 & n = 4v \\ 2 & n = 4v+2 \\ 0 & \text{otherwise} \end{cases} \qquad v = \text{integer}$$

(ℓ) $$y[n] = \underbrace{u[n]}_{x_1[n]} * \underbrace{\sum_{p=0}^{\infty} \delta[n-2p]}_{x_2[n]}$$



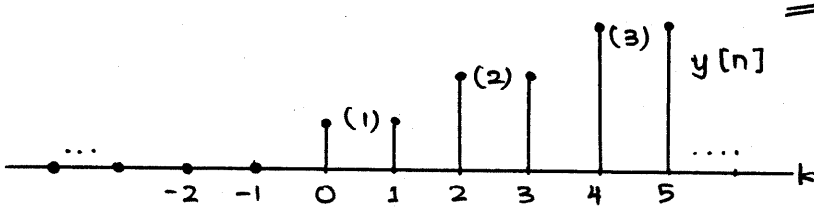


For $\underline{n < 0}$: $y[n] = 0$

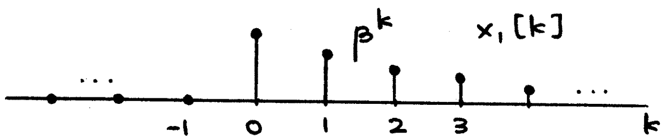
$\underline{n \geq 0}$: $n = \text{even } (0, 2, 4, \ldots) : y[n] = \frac{n}{2} + 1$

$n = \text{odd } (1, 3, 5, \ldots) : y[n] = \frac{n+1}{2}$

$$\therefore\ y[n] = u[n]\,.\,f[n] \text{ where } f[n] = \begin{cases} \frac{n}{2}+1 & , n = \text{even} \\ \frac{n+1}{2} & , n = \text{odd} \end{cases}$$



(m) $$y[n] = \underbrace{\beta^n u[n]}_{x_1[n]} * \underbrace{\sum_{p=0}^{\infty} \delta[n-2p]}_{x_2[n]} \qquad |\beta| < 1$$



see (e) for $x_2[n-k]$

For $\underline{n < 0}$: $y[n] = 0$

$\underline{n \geq 0}$: n = even $(0, 2, 4, \ldots)$:

$$y[n] = \sum_{e=0}^{n/2} \beta^{2e} = \frac{1-\beta^{2(\frac{n}{2}+1)}}{1-\beta^2} = \frac{1-\beta^{(n+2)}}{1-\beta^2}$$

n = odd $(1, 3, 5, \ldots)$:

$$y[n] = \sum_{e=0}^{\frac{n-1}{2}} \beta^{2e-1} = \frac{1}{\beta} \cdot \frac{1-\beta^{2(\frac{n-1}{2}+1)}}{1-\beta^2}$$

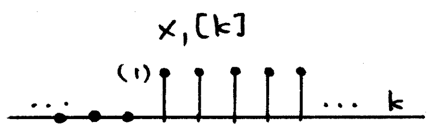
$$y[n] = \frac{1}{\beta} \cdot \frac{1-\beta^{(n+1)}}{1-\beta^2}$$

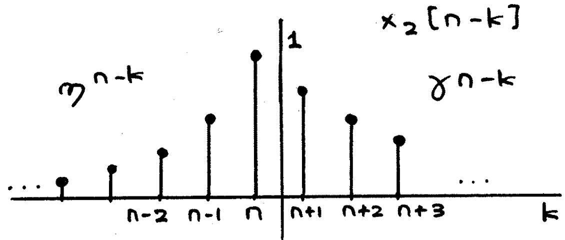
$\therefore\ y[n] = u[n]\,.\,g[n]$ where :

$$g[n] = \begin{cases} \frac{1-\beta^{(n+2)}}{1-\beta^2} & , n = \text{even} \\ \frac{1}{\beta} \cdot \frac{1-\beta^{(n+1)}}{1-\beta^2} & , n = \text{odd} \end{cases}$$

(n) $y[n] = \underbrace{u[n-2]}_{x_1[n]} * \underbrace{h[n]}_{x_2[n]}$ where $h[n] = \begin{cases} \gamma^n & n<0 \quad |\gamma|>1 \\ \eta^n & n \geq 0 \quad |\eta|<1 \end{cases}$

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k] \cdot x_2[n-k]$$





For $\underline{n \leq 2}$: $y[n] = \sum_{k=2}^{\infty} \gamma^{n-k}$

$$= \gamma^n \cdot \frac{(1/\gamma)^2}{1-\frac{1}{\gamma}} = \frac{\gamma^{n-1}}{\gamma-1}$$

$\underline{n > 2}$: $y[n] = \sum_{k=2}^{n} \eta^{n-k} + \sum_{k=n+1}^{\infty} \gamma^{n-k}$

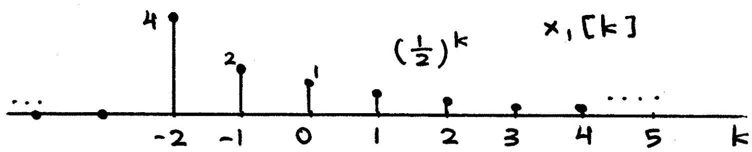
$$= \eta^n \left(\frac{1}{\eta^n} \cdot \frac{1-\eta^{n-1}}{1-\eta} \right) + \gamma^n \left(\frac{1}{\gamma^n} \frac{\left(\frac{1}{\gamma}\right)^{n+1}}{1-\frac{1}{\gamma}} \right)$$

$$y[n] = \frac{1-\eta^{n-1}}{1-\eta} + \frac{1}{\gamma-1}$$

$$\therefore y[n] = \begin{cases} \dfrac{\gamma^{n-1}}{\gamma-1} & n = -\infty, \ldots, 0, 1 \\ \dfrac{1}{\gamma-1} - \dfrac{1-\eta^{n-1}}{1-\eta} & n = 2, 3, \ldots, \infty \end{cases}$$

(o) $y[n] = \underbrace{\left(\frac{1}{2}\right)^n u[n+2]}_{x_1[n]} * \underbrace{h[n]}_{x_2[n]}$

$$y[n] = \sum_{n=-\infty}^{\infty} x_1[k] \cdot x_2[n-k]$$



See (n) for $x_2[k]$

For $n < -2$: $y[n] = \sum_{k=-2}^{\infty} (\frac{1}{2})^k . \gamma^{n-k} = \gamma^n . \sum_{k=-2}^{\infty} (\frac{1}{2\gamma})^k$

$$= \gamma^n . \frac{(\frac{1}{2\gamma})^{-2}}{1 - \frac{1}{2\gamma}} = \frac{8(\gamma)^{n+3}}{2\gamma - 1}$$

For $n \geqslant -2$: $y[n] = \sum_{k=-2}^{n} (\frac{1}{2})^k \eta^{n-k} + \sum_{k=n+1}^{\infty} (\frac{1}{2})^k \gamma^{n-k}$

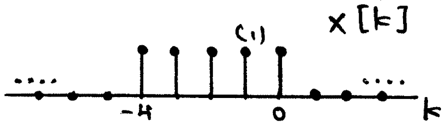
$$y[n] = \eta^n . \sum_{k=-2}^{n} (\frac{1}{2\eta})^k + \gamma^n \sum_{k=n+1}^{\infty} (\frac{1}{2\gamma})^k$$

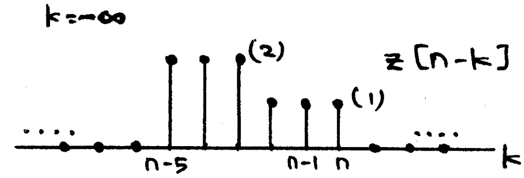
$$y[n] = \eta^n (2\eta)^2 . \frac{1 - (\frac{1}{2\eta})^{n-1}}{1 - \frac{1}{2\eta}} + \gamma^n \frac{(\frac{1}{2\gamma})^{n+1}}{1 - \frac{1}{2\gamma}}$$

$$= 8(\eta)^{n+3} \frac{(2\eta)^{1-n} - 1}{1 - 2\eta} + (\frac{1}{2})^n \frac{1}{2\gamma - 1}$$

$$\therefore y[n] = \begin{cases} \dfrac{8(\gamma)^{n+3}}{2\gamma - 1} & , \quad n = -\infty, \ldots, -4, -3 \\[2ex] 8(\eta)^{n+3} \dfrac{(2\eta)^{1-n} - 1}{1 - 2\eta} + (\frac{1}{2})^n \dfrac{1}{2\gamma - 1} & , \quad n = -2, -1, \ldots, \infty \end{cases}$$

$\boxed{2.3.}$ (a) $a[n] = x[n] * z[n] = \sum_{k=-\infty}^{\infty} x[k] . z[n-k]$





For $n+4 < 0$ or $\underline{n < -4}$: $a[n] = 0$

$0 \leq n+4 < 3$ or $\underline{-4 \leq n < -1}$; $a[n] = n+5$

$n \geq -1$ and $n < 1$ or $-1 \leq n < 1$: $a[n] = 3 + 2(n+2)$

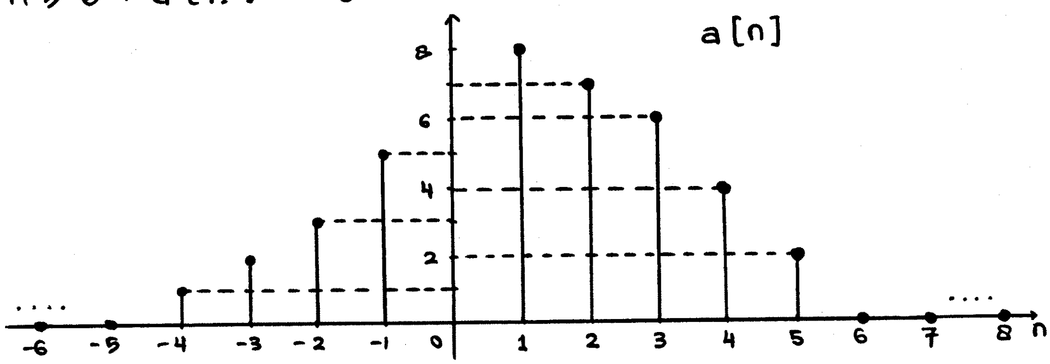
$= 2n + 7$

$n = 1$: $a[n] = 2(1) + 3(2) = 8$

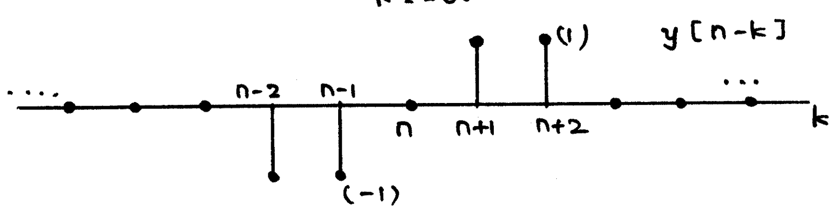
$n = 2$: $a[n] = 1(1) + 3(2) = 7$

$3 \leq n < 6$: $a[n] = 2(6-n)$

$n \geq 6$: $a[n] = 0$



(b) $a[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot y[n-k]$



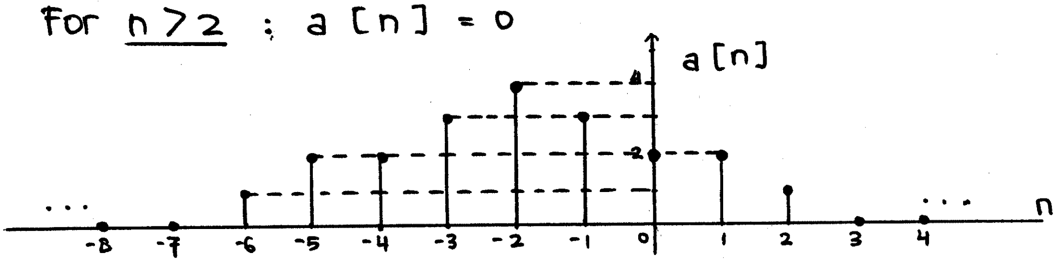
For $n+4 < -2$ or $\underline{n < -6}$: $a[n] = 0$

$a[-6] = 1$; $a[-5] = 2$; $a[-4] = 2$

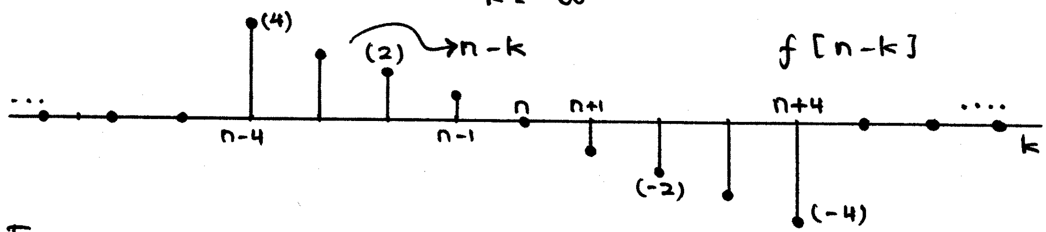
$a[-3] = 3$; $a[-2] = 4$; $a[-1] = 3$

$a[0] = 2$; $a[1] = 2$; $a[2] = 1$

For $\underline{n > 2}$: $a[n] = 0$



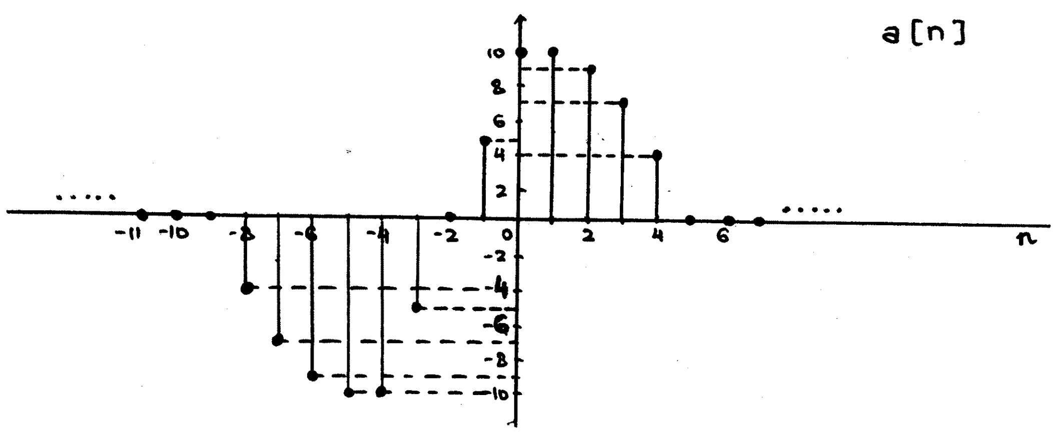
(c) $a[n] = x[n] * f[n] = \sum_{k=-\infty}^{\infty} x[k]\, f[n-k]$



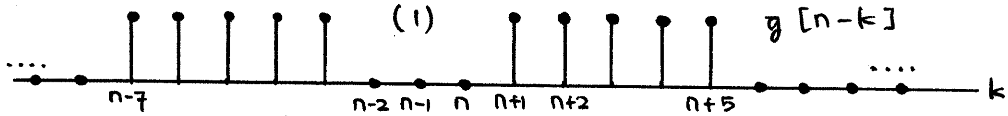
For $n+4 < -4$ or $\underline{n < -8}$: $a[n] = 0$

$a[-8] = -4$; $a[-7] = -4-3 = -7$; $a[-6] = -7-2 = -9$;
$a[-5] = -9-1 = -10$; $a[-4] = -10-0 = -10$;
$a[-3] = -3-2-1-0+1 = -5$; $a[-2] = -2-1-0+1+2 = 0$;
$a[-1] = -1-0+1+2+3 = 5$; $a[0] = 0+1+2+3+4 = 10$;
$a[1] = 1+2+3+4+0 = 10$; $a[2] = 2+3+4 = 9$;
$a[3] = 3+4 = 7$; $a[4] = 4$;

For $n-4 > 0$ or $\underline{n > 4}$: $a[n] = 0$

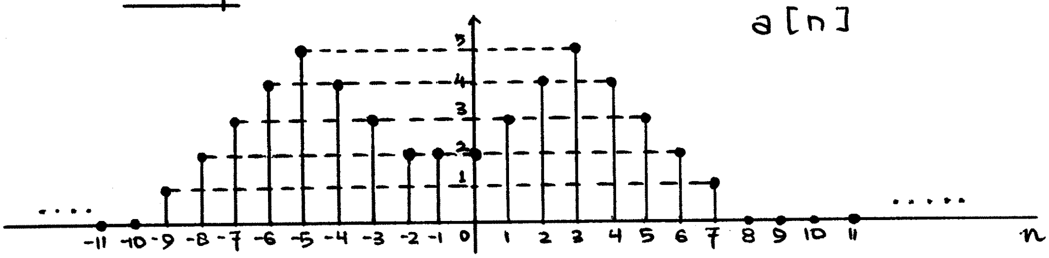


(d) $a[n] = x[n] * g[n] = \sum_{k=-\infty}^{\infty} x[k]\, g[n-k]$

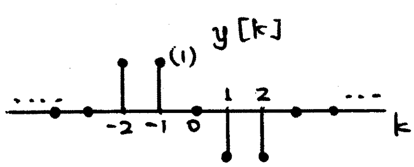


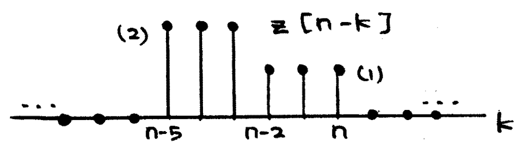
For $n+5<-4$ or $\underline{n<-9}$: $a[n]=0$

$a[-9]=1$; $a[-8]=2$; $a[-7]=3$; $a[-6]=4$;
$a[-5]=5$; $a[-4]=4$; $a[-3]=3$; $a[-2]=2$;
$a[-1]=2$; $a[0]=2$; $a[1]=3$; $a[2]=4$;
$a[3]=5$; $a[4]=4$; $a[5]=3$; $a[6]=2$;
$a[7]=1$
For $\underline{n>7}$: $a[n]=0$



(e) $a[n]=y[n]*z[n]=\sum_{k=-\infty}^{\infty} y[k]\,z[n-k]$

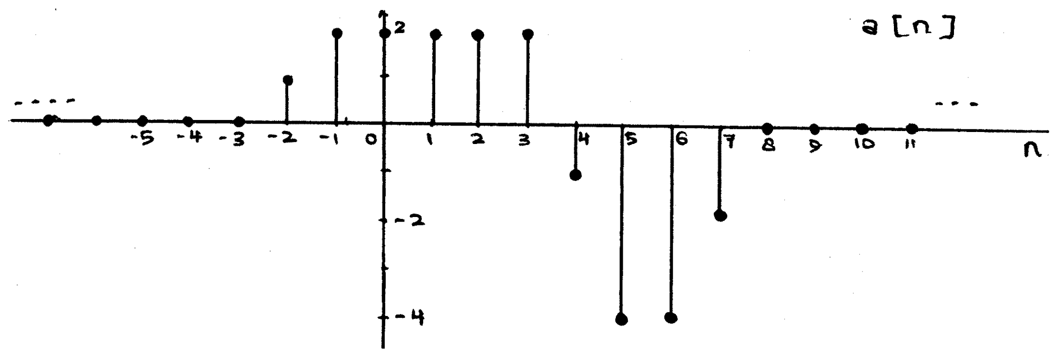




For $\underline{n<-2}$: $a[n]=0$

$a[-2]=1$; $a[-1]=2$; $a[0]=2$; $a[1]=2(1)+1-1$
$=2$;
$a[2]=2(2)-2(1)=2$; $a[3]=2(-1)+2(2)=2$;
$a[4]=-1+2-2=-1$; $a[5]=2(-2)=-4$;
$a[6]=2(-2)=-4$; $a[7]=-2$;

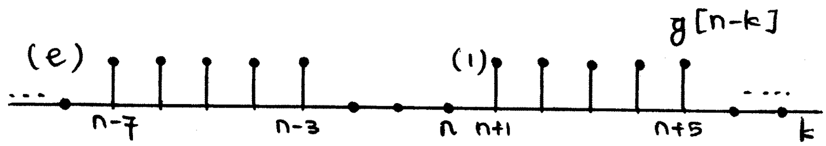
For $\underline{n>7}$; $a[n]=0$;



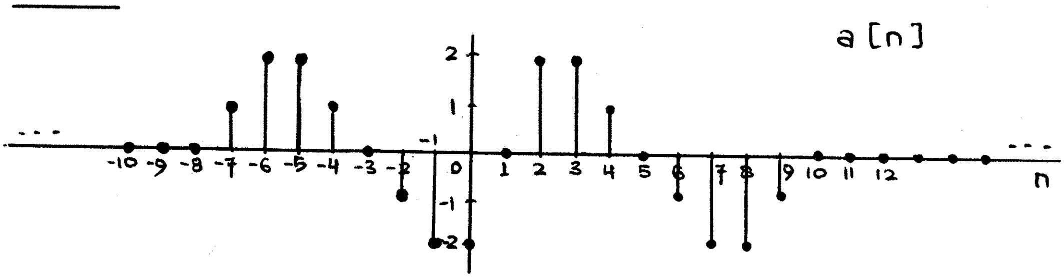
(f) $a[n] = y[n] * g[n] = \sum_{k=-\infty}^{\infty} y[k]\, g[n-k]$

For $n+5 < -2$ or $\underline{n < -7}$: $a[n] = 0$

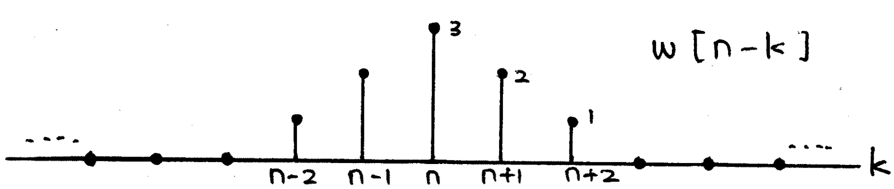
$\Rightarrow y[k]$ see (e)



$a[-7] = 1$; $a[-6] = 2$; $a[-5] = 2$; $a[-4] = 1$; $a[-3] = 0$;
$a[-2] = -1$; $a[-1] = -2$; $a[0] = -2$; $a[1] = -1+1 = 0$;
$a[2] = 2(1) = 2$; $a[3] = 2(1) = 2$; $a[4] = 2(1)-1 = 1$;
$a[5] = 0$; $a[6] = 2(-1)+1 = -1$; $a[7] = -2$;
$a[8] = -2$; $a[9] = -1$;
$\underline{n > 9}$: $a[n] = 0$

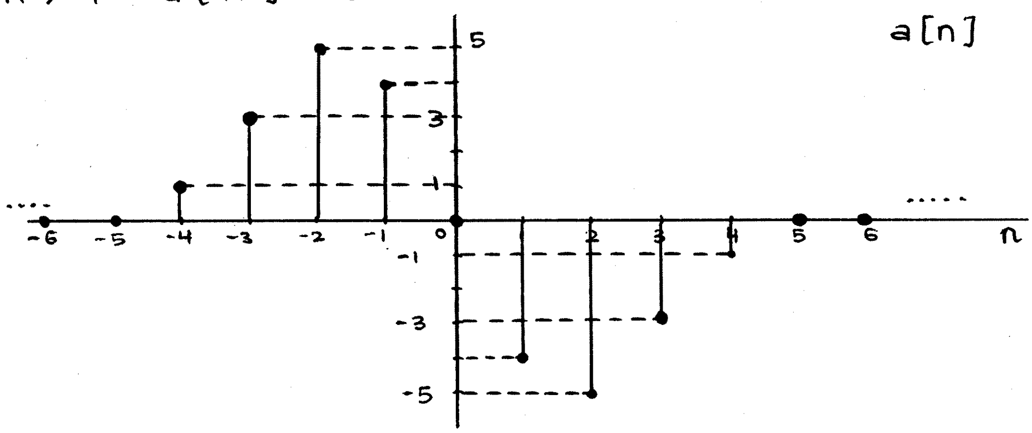


(g) $a[n] = y[n] * w[n] = \sum_{k=-\infty}^{\infty} y[k]\, w[n-k]$

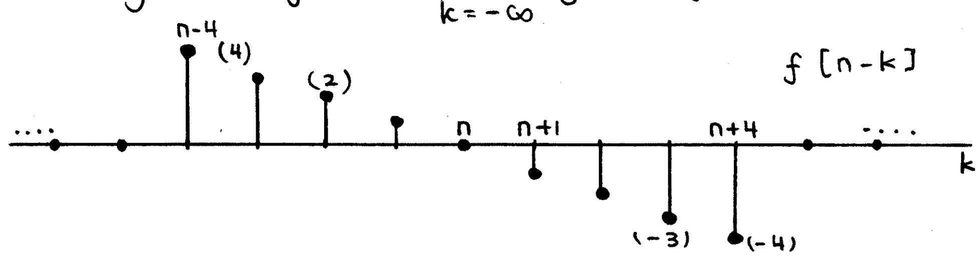


For $n+2 < -2$ or $n < -4$: $a[n] = 0$

$a[-4] = 1$; $a[-3] = 1+2 = 3$; $a[-2] = 2+3 = 5$;
$a[-1] = -1+3+2 = 4$; $a[0] = (-1+1)(2+1) = 0$;
$a[1] = 1-3-2 = -4$; $a[2] = -3-2 = -5$;
$a[3] = -2-1 = -3$; $a[4] = -1$;
$n > 4$: $a[n] = 0$

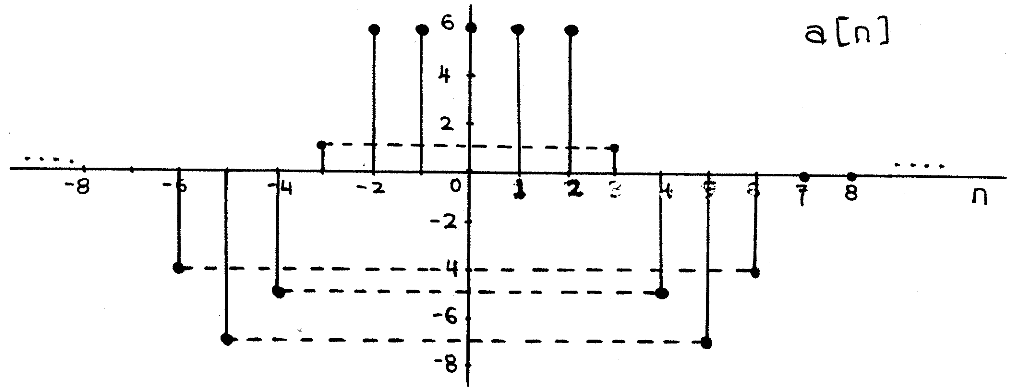


(h) $a[n] = y[n] * f[n] = \sum_{k=-\infty}^{\infty} y[k]\, f[n-k]$

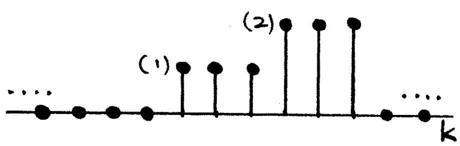


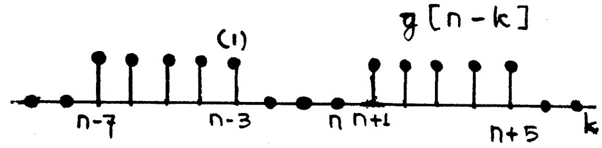
For $n+4 < -2$ or $n < -6$: $a[n] = 0$

$a[-6] = -4$; $a[-5] = -4-3 = -7$; $a[-4] = -3-2 = -5$;
$a[-3] = -2-1+4 = 1$; $a[-2] = -1+3+4 = 6$;
$a[-1] = 1+2+3 = 6$; $a[0] = 2(1+2) = 6$;
$a[1] = 1+2+3 = 6$; $a[2] = -1+3+4 = 6$;
$a[3] = -3+4 = 1$; $a[4] = -2-3 = -5$; $a[5] = -3-4 = -7$;
$a[6] = -4$;
$\underline{n > 6}$: $a[n] = 0$



(i) $a[n] = z[n] * g[n] = \sum_{k=-\infty}^{\infty} z[k]\, g[n-k]$

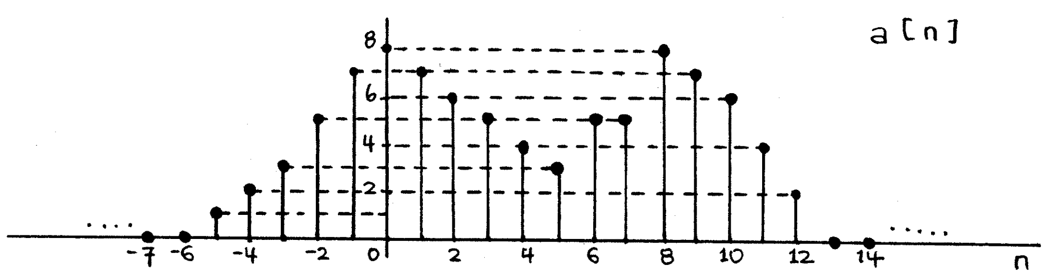




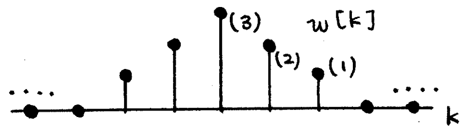
For $n+5 < 0$ or $\underline{n < -5}$: $a[n] = 0$

$a[-5] = 1$; $a[-4] = 2$; $a[-3] = 3$; $a[-2] = 3+2 = 5$;
$a[-1] = 3 + 4 = 7$; $a[0] = 2 + 2(3) = 8$; $a[1] = 1 + 2(3) = 7$;
$a[2] = 2(3) = 6$; $a[3] = 1 + 2(2) = 5$; $a[4] = 2 + 2(1) = 4$;
$a[5] = 3(1) = 3$; $a[6] = 2 + 3(1) = 5$; $a[7] = 3(1) + 2 = 5$;
$a[8] = 3(2) + 2 = 8$; $a[9] = 3(2) + 1 = 7$; $a(10) = 3(2) = 6$;
$a[11] = 2(2) = 4$; $a[12] = 2$

$\underline{n > 12}$: $a[n] = 0$



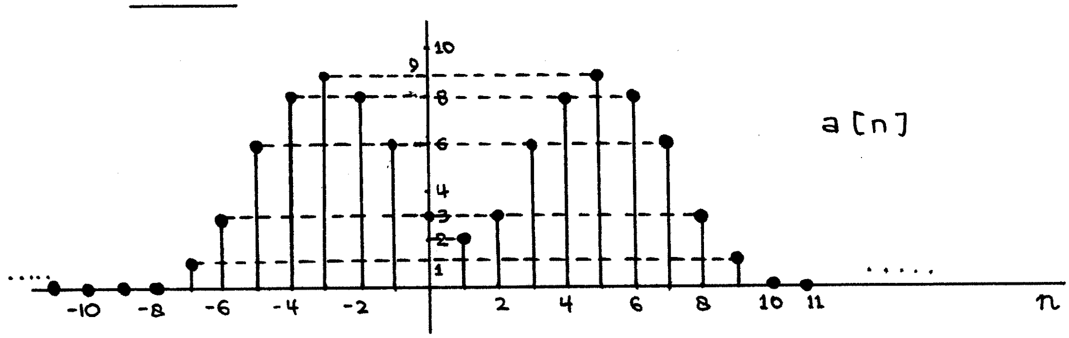
(j) $a[n] = w[n] * g[n] = \sum_{k=-\infty}^{\infty} w[k] \cdot g[n-k]$



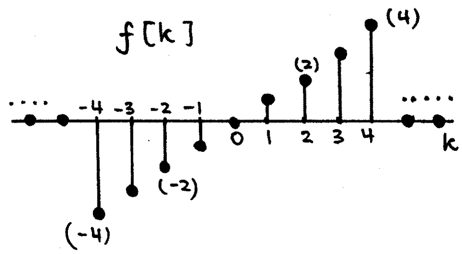
See (i) for $g[n-k]$

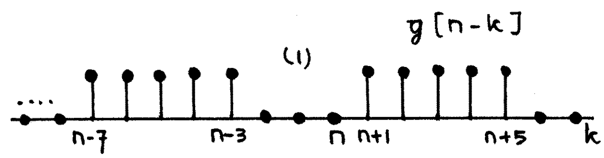
For $n+5<-2$ or $\underline{n<-7}$: $a[n]=0$

$a[-7]=1$; $a[-6]=1+2=3$; $a[-5]=1+2+3=6$;
$a[-4]=6+2=8$; $a[-3]=8+1=9$; $a[-2]=1+2+3+2$
$=8$; $a[-1]=3+2+1=6$; $a[0]=1+2=3$; $a[1]=1+1=2$;
$a[2]=2+1=3$; $a[3]=3+2+1=6$; $a[4]=6+2=8$;
$a[5]=1+2+3+2+1=9$; $a[6]=6+2=8$;
$a[7]=1+2+3=6$; $a[8]=1+2=3$; $a[9]=1$;
$\underline{n>9}$: $a[n]=0$



(k) $a[n]=f[n]*g[n]=\sum_{k=-\infty}^{\infty} f[k].g[n-k]$



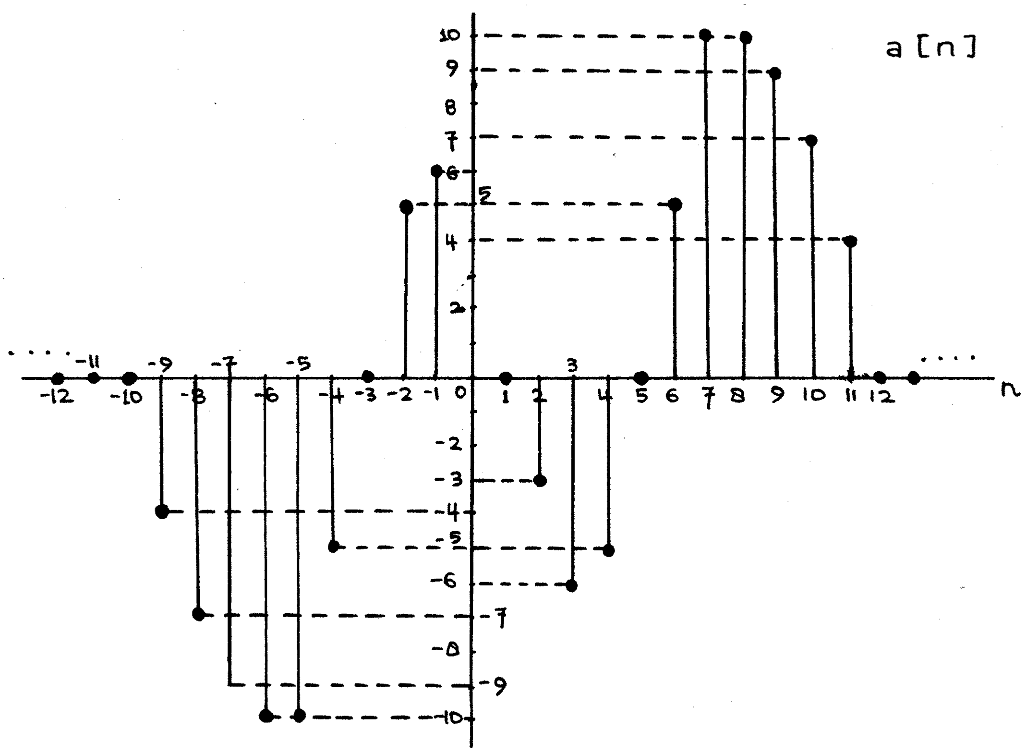


For $n+5<-4$ or $\underline{n<-9}$: $a[n]=0$

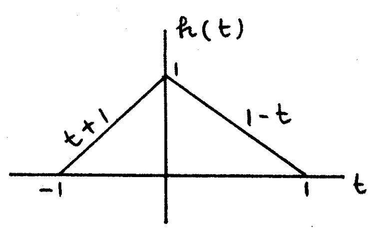
$a[-9]=-4$; $a[-8]=-4-3=-7$; $a[-7]=-7-2=-9$;
$a[-6]=-9-1=-10$; $a[-5]=-10$; $a[-4]=-3-2-1$
$+1=-5$; $a[-3]=0$; $a[-2]=-1+1+2+3=5$;

$a[-1] = -4+1+2+3+4 = 6$; $a[0] = 1+2+3+4-3-4 = 3$; $a[1] = 0$; $a[2] = 3+4-1-2-3-4 = -3$; $a[3] = 4-1-2-3-4 = -6$; $a[4] = 1+0-1-2-3 = -5$; $a[5] = 2+1+0-1-2 = 0$; $a[6] = 3+2+1-1 = 5$; $a[7] = 4+3+2+1 = 10$; $a[8] = 4+3+2+1 = 10$; $a[9] = 4+3+2 = 9$; $a[10] = 4+3 = 7$; $a[11] = 4$

$\underline{n > 11}$: $a[n] = 0$



$\boxed{2.4}$



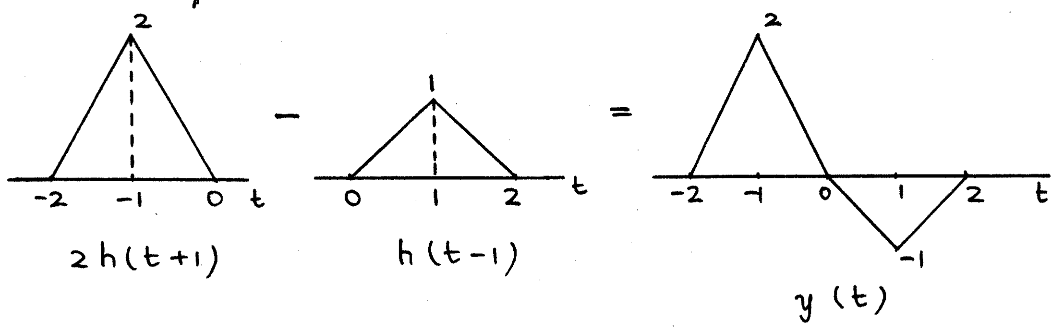
Use the facts that

$\delta(t-d) * h(t) = h(t-d)$

and the linearity property

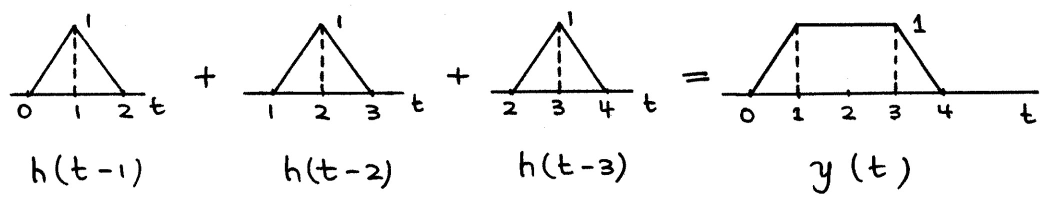
(a) $x(t) = 2\delta(t+1) - \delta(t-1) \rightarrow x(t) * h(t) \overset{\text{linearity}}{=} 2h(t+1) - h(t-1)$

Graphically :



(b) $x(t) = \delta(t-1) + \delta(t-2) + \delta(t-3) \rightarrow x(t) * h(t) \overset{\text{linearity}}{=} h(t-1) + h(t-2) + h(t-3)$

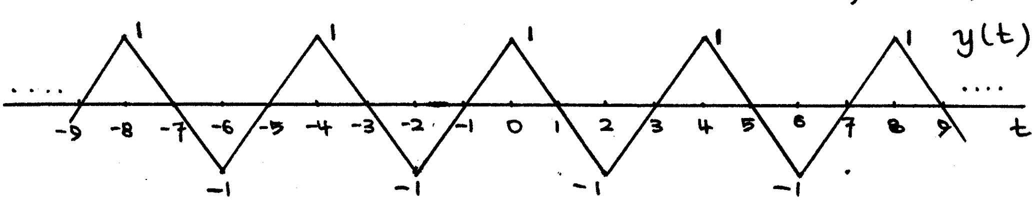
Graphically :



(c) $x(t) = \sum_{p=-\infty}^{\infty} (-1)^p \, \delta(t-2p)$

By linearity : $y(t) = x(t) * h(t) = \sum_{p=-\infty}^{\infty} (-1)^p \, h(t-2p)$

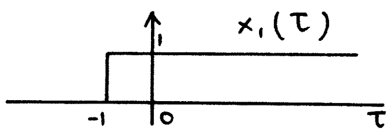
$$= \ldots + h(t+8) - h(t+6) + h(t+4) - h(t+2) + h(t) - h(t-2) + h(t-4) - h(t-6) + h(t-8) + \ldots$$

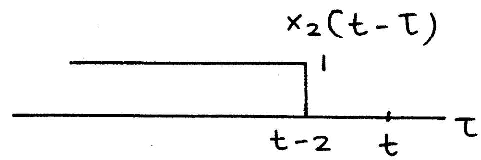


$\boxed{2.5}$ $$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

(a) $$y(t) = \underbrace{u(t+1)}_{x_1(t)} * \underbrace{u(t-2)}_{x_2(t)}$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$





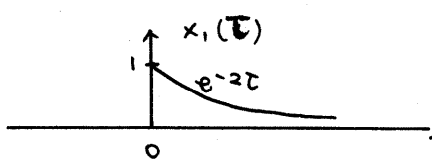
For $t-2 < -1$ or $\underline{t < 1}$: $y(t) = 0$

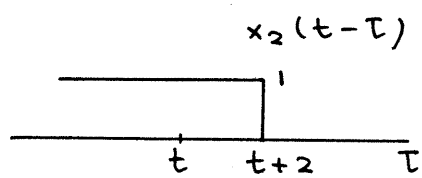
$\underline{t \geq 1}$: $y(t) = \int_{-1}^{t-2} (1)(1)\, d\tau = t-1$

$\therefore\ y(t) = (t-1)\, u(t-1)$

(b) $$y(t) = \underbrace{e^{-2t} u(t)}_{x_1(t)} * \underbrace{u(t+2)}_{x_2(t)}$$

$$= x_1(t) * x_2(t)$$





For $t+2 \geq 0$ or $\underline{t \geq -2}$: $y(t) = \int_{0}^{t+2} (1)\, e^{-2\tau} d\tau$

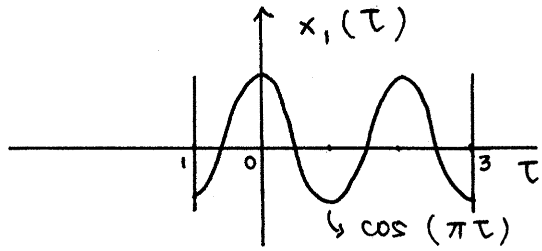
$$= \frac{1}{2}\left(1 - e^{-2(t+2)}\right)$$

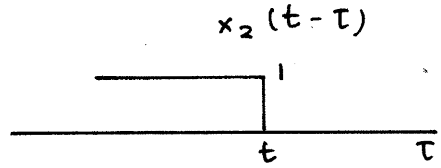
$\underline{t < -2}$: $y(t) = 0$

$\therefore\ y(t) = \frac{1}{2}\left(1 - e^{-2(t+2)}\right) u(t+2)$

(c) $$y(t) = \cos\underbrace{(\pi t)\big(u(t+1) - u(t+3)\big)}_{x_1(t)} * \underbrace{u(t)}_{x_2(t)}$$

$$= x_1(t) * x_2(t)$$



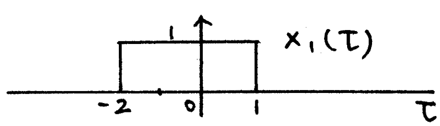


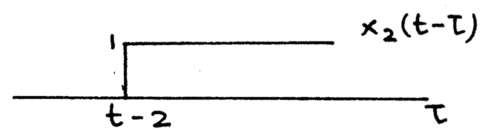
For $\underline{t < -1}$: $y(t) = 0$

$\underline{-1 \leq t < 3}$: $y(t) = \int_{-1}^{t} \cos \pi\tau \, d\tau = \frac{1}{\pi} \sin \pi t$

$\therefore \; y(t) = \frac{1}{\pi} \sin(\pi t) \, . \, u(t)$

(d) $y(t) = \big(\underbrace{u(t+2) - u(t-1)}_{x_1(t)}\big) * \underbrace{u(-t+2)}_{x_2(t)} = x_1(t) * x_2(t)$





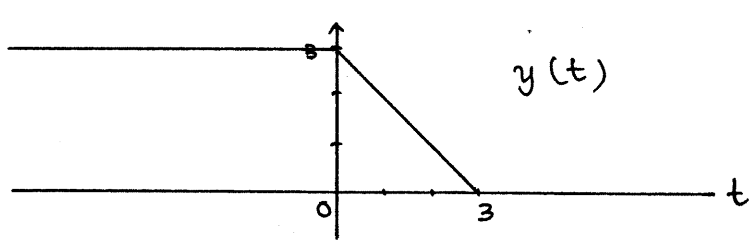
For $t - 2 < -2$ or $\underline{t < 0}$: $y(t) = 3 \times 1 = 3$

$t \geq 0$ but $t - 2 < 1$ or $\underline{0 \leq t < 3}$:

$y(t) = \int_{t-2}^{t} (1)(1) \, d\tau = 3 - t$

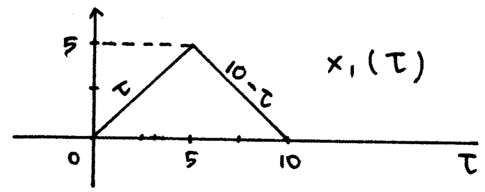
$\underline{t \geq 3}$: $y(t) = 0$

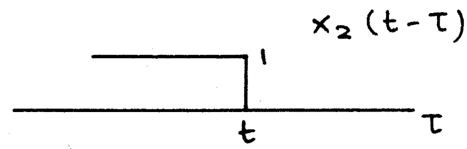
$$\therefore \; y(t) = \begin{cases} 3 & t < 0 \\ 3 - t & 0 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$



(e) $y(t) = [t u(t) + (10-2t)u(t-5) - (10-t)u(t-10)] * u(t)$

$= x_1(t) * x_2(t)$





For $t < 0$: $y(t) = 0$

$0 \leq t < 5$: $y(t) = \int_0^t \tau(1)\,d\tau = \frac{1}{2}t^2$

$5 \leq t < 10$: $y(t) = \int_0^5 \tau\,d\tau + \int_5^t (10-\tau)\,d\tau$

$= \frac{25}{2} + 10(t-5) - \frac{1}{2}(t^2 - 25)$

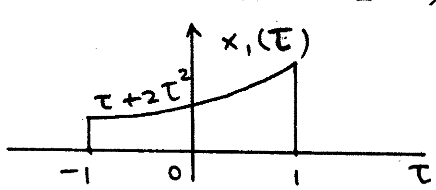
$= -\frac{1}{2}t^2 + 10t - 25$

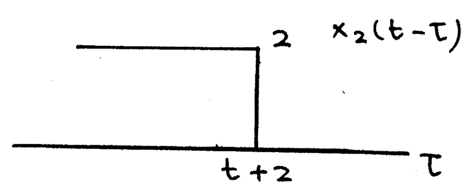
$t \geq 10$: $y(t) = \frac{1}{2} \times 10 \times 5 = 25$

$$\therefore\ y(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}t^2 & 0 \leq t < 5 \\ -\frac{1}{2}t^2 + 10t - 25 & 5 \leq t < 10 \\ 25 & t \geq 10 \end{cases}$$

(f) $y(t) = (t + 2t^2)\big(u(t+1) - u(t-1)\big) * 2u(t+2)$

$= x_1(t) * x_2(t)$





For $t+2 < -1$ or $\underline{t < -3}$: $y(t) = 0$

$\underline{-3 \leq t < -1}$: $y(t) = 2\int_{-1}^{t+2} \tau + 2\tau^2\,d\tau$

$$= (t+2)^2 - 1 + \frac{4}{3}\left((t+2)^3 + 1\right)$$

$$= \frac{4}{3}(t+2)^3 + (t+2)^2 + \frac{1}{3}$$

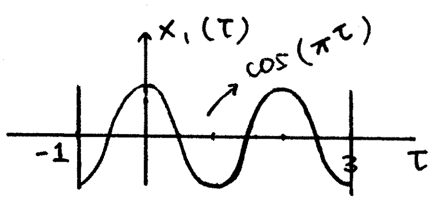
For $t \geq -1$: $y(t) = 2\int_{-1}^{1} \tau + 2\tau^2 \, d\tau$

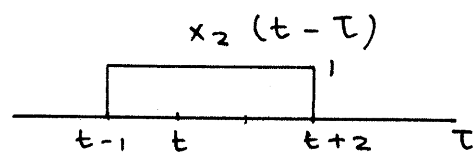
$$= \frac{4}{3} + 1 + \frac{1}{3} = \frac{8}{3}$$

$$\therefore y(t) = \begin{cases} 0 & , \quad t < -3 \\ \frac{4}{3}(t+2)^3 + (t+2)^2 + \frac{1}{3} & , \quad -3 \leq t < -1 \\ \frac{8}{3} & , \quad t \geq -1 \end{cases}$$

(g) $y(t) = \cos(\pi t)\left(u(t+1) - u(t-3)\right) * \left(u(t+2) - u(t-1)\right)$

$= x_1(t) * x_2(t)$





For $t+2 < -1$ or $\underline{t < -3}$: $y(t) = 0$

$t \geq -3$ but $t-1 < -1$ or $\underline{-3 \leq t < 0}$:

$$y(t) = \int_{-1}^{t+2} \cos(\pi\tau)\, d\tau = \frac{1}{\pi}\sin(\pi t)$$

$t \geq 0$ but $t+2 < 3$ or $\underline{0 \leq t < 1}$:

$$y(t) = \int_{t-1}^{t+2} \cos(\pi\tau)\, d\tau = \frac{2}{\pi}\sin(\pi t)$$

$t \geq 1$ but $t-1 < 3$ or $\underline{1 \leq t < 4}$:

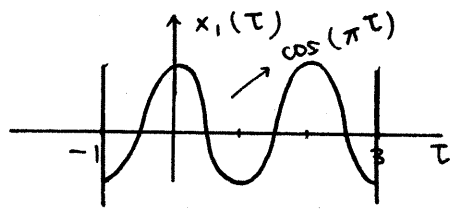
$$y(t) = \int_{t-1}^{3} \cos(\pi\tau)\, d\tau = \frac{1}{\pi}\sin(\pi t)$$

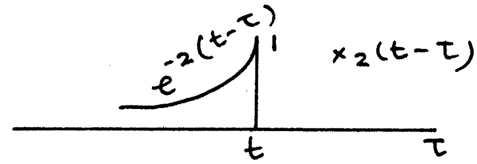
$\underline{t \geq 4}$: $y(t) = 0$

$$\therefore \quad y(t) = \begin{cases} 0 & , \ t < -3 \\ \frac{1}{\pi}\sin(\pi t) & , \ -3 \leq t < 0 \\ \frac{2}{\pi}\sin(\pi t) & , \ 0 \leq t < 1 \\ \frac{1}{\pi}\sin(\pi t) & , \ 1 \leq t < 4 \\ 0 & , \ t \geq 4 \end{cases}$$

(h) $y(t) = \cos(\pi t)\left(u(t+1) - u(t+3)\right) * e^{-2t}\, u(t)$

$= x_1(t) * x_2(t)$





For $t < -1$: $y(t) = 0$

$-1 \leq t < 3$: $y(t) = \int_{-1}^{t} e^{-2(t-\tau)} \cos(\pi\tau)\, d\tau$

$$= e^{-2t} \int_{-1}^{t} e^{2\tau} \cos(\pi\tau)\, d\tau$$

(Appendix A4) $y(t) = e^{-2t} \frac{e^{2t}}{n^2+4}\left[2 + 2\cos(\pi t) + \pi\sin(\pi t)\right]$

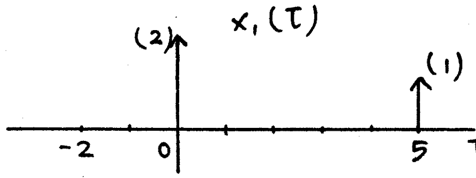
$$y(t) = \frac{1}{\pi^2+4}\left(2 + 2\cos(\pi t) + \pi\sin(\pi t)\right)$$

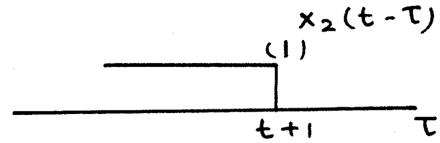
$t \geq 3$: $y(t) = \int_{-1}^{3} e^{-2(t-\tau)} \cos(\pi\tau)\, d\tau$

$$= \frac{1}{\pi^2+4}(0) = 0$$

$$\therefore y(t) = \begin{cases} 0 & , \ t < -1 \\ \frac{1}{\pi^2+4}\left(2 + 2\cos(\pi t) + \pi\sin(\pi t)\right) & , \ -1 \leq t < 3 \\ 0 & , \ t \geq 3 \end{cases}$$

(i) $y(t) = (2\delta(t) + \delta(t-5)) * u(t+1) = x_1(t) * x_2(t)$





For $t+1 < 0$ or $\underline{t < -1}$: $y(t) = 0$

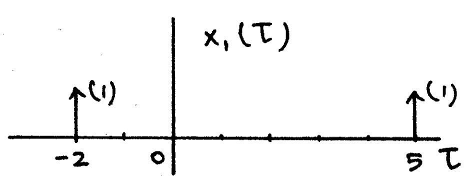
$t \geq -1$ but $t+1 < 5$ or $\underline{-3 \leq t < 4}$:

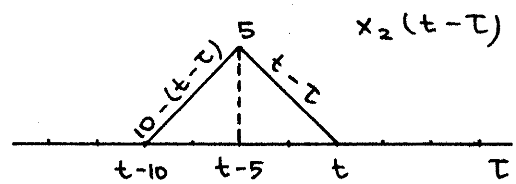
$$y(t) = \int_{-\infty}^{t+1} 2\delta(\tau)\, d\tau = 2$$

$t \geq 4$: $$y(t) = \int_{-\infty}^{t+1} 2\delta(\tau) + \delta(\tau-5)\, d\tau = 3$$

$$\therefore y(t) = \begin{cases} 0 & , \quad t < -1 \\ 2 & , \quad -3 \leq t < 4 \\ 3 & , \quad t \geq 4 \end{cases}$$

(j) $y(t) = (\delta(t+2) + \delta(t-5)) * (t\,u(t) + (10-2t)u(t-5)$
$- (10-t)\,u(t-10))$
$= x_1(t) * x_2(t)$





For $\underline{t < -2}$: $y(t) = 0$

$\underline{-2 \leq t < 3}$: $$y(t) = \int_{t-10}^{t} \delta(\tau+2).(t-\tau)\, d\tau = 2+t$$

$\underline{3 \leq t < 5}$: $$y(t) = \int_{t-10}^{t} \delta(\tau+2)[10-(t-\tau)]\, d\tau$$

$$= 8 - t$$

$\underline{5 \leq t < 8}$: $$y(t) = \int_{t-10}^{t} \delta(\tau+2)(10-(t-\tau)) + \delta(\tau-5)$$
$$(t-\tau)\, d\tau$$

$$y(t) = 8 - t + t - 5 = 3$$

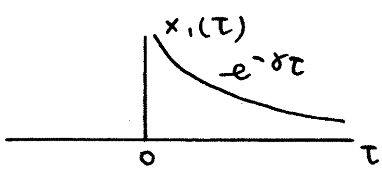
$\underline{8 \leq t < 10}$: $y(t) = \int_{t-10}^{t} \delta(\tau-5)(t-\tau) = t-5$

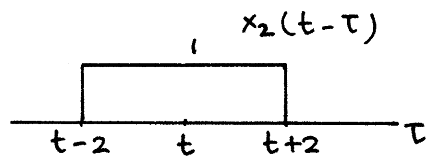
$\underline{10 \leq t < 15}$: $y(t) = \int_{t-10}^{t} \delta(\tau-5)\big(10-(t-\tau)\big) = 15-t$

$\underline{t \geq 15}$: $y(t) = 0$

$$\therefore y(t) = \begin{cases} 0 & , \quad t < -2 \\ 2+t & , \quad -2 \leq t < 3 \\ 8-t & , \quad 3 \leq t < 5 \\ 3 & , \quad 5 \leq t < 8 \\ t-5 & , \quad 8 \leq t < 10 \\ 15-t & , \quad 10 \leq t < 15 \\ 0 & , \quad t \geq 15 \end{cases}$$

(k) $y(t) = e^{-\gamma t} u(t) * \big(u(t+2) - u(t-2)\big) = x_1(t) * x_2(t)$





For $t+2 < 0$ or $\underline{t < -2}$: $y(t) = 0$

$t \geq -2$ but $t-2 > 0$ or $\underline{-2 \leq t < 2}$:

$$y(t) = \int_{0}^{t+2} e^{-\gamma\tau} d\tau = \frac{1}{\gamma}\left(1 - e^{-\gamma(t+2)}\right)$$

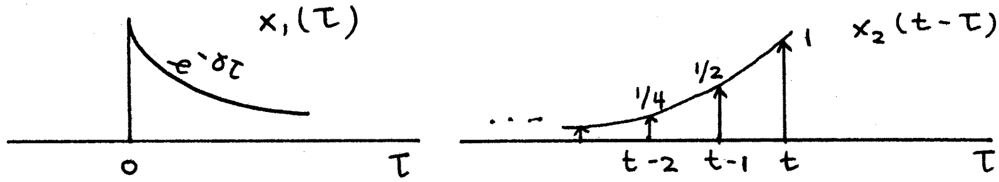
$t \geq 2$: $y(t) = \int_{t-2}^{t+2} e^{-\gamma\tau} d\tau$

$$= \frac{1}{\gamma}\left(e^{-\gamma(t-2)} - e^{-\gamma(t+2)}\right)$$

$$= \frac{2}{\gamma} \sinh(2\gamma)$$

$$\therefore y(t) = \begin{cases} 0 & , \quad t < -2 \\ \frac{1}{\gamma}\left(1 - e^{-\gamma(t+2)}\right) & , \quad -2 \leq t < 2 \\ \frac{2}{\gamma} \sinh(2\gamma) & , \quad t \geq 2 \end{cases}$$

(ℓ) $y(t) = e^{-\gamma t} u(t) * \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p) = x_1(t) * x_2(t)$



For $t < 0$: $y(t) = 0$

$t \geq 0$: $y(t) = \int_0^t e^{-\gamma\tau} \cdot \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p-\tau)\,d\tau$

Notice that the sum starts at $p=0$, but ends at biggest $\tilde{p}$ where $t - \tilde{p} \geq 0$ or $\tilde{p} = \lfloor t \rfloor$

⊛ Note : $f(x) = \lfloor x \rfloor$ is a floor function
e.g : $\lfloor 2 \rfloor = 2$; $\lfloor 2.8 \rfloor = 2$; $\lfloor 3.1 \rfloor = 3$,

So : $y(t) = \sum_{p=0}^{\lfloor t \rfloor} \left(\frac{1}{2}\right)^p \int_0^t e^{-\gamma\tau} \delta(t-p-\tau)\,d\tau$

since Σ and $\int$ are exchangeable in this case

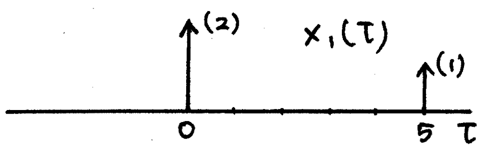
$$y(t) = \sum_{p=0}^{\lfloor t \rfloor} \left(\frac{1}{2}\right)^p e^{-\gamma(t-p)}$$

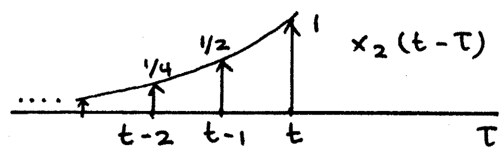
$$= e^{-\gamma t} \sum_{p=0}^{\lfloor t \rfloor} \left(\frac{e^{\gamma}}{2}\right)^p$$

$$\therefore y(t) = \begin{cases} e^{-\gamma t} \dfrac{1 - \left(\frac{e^{\gamma}}{2}\right)^{\lfloor t \rfloor + 1}}{1 - \left(\frac{e^{\gamma}}{2}\right)} \cdot u(t) & , \gamma \neq \ln(2) \\ (\lfloor t \rfloor + 1) \cdot e^{-\gamma t} \cdot u(t) & , \gamma = \ln(2) \end{cases}$$

(m) $y(t) = \left(2\delta(t) + \delta(t-5)\right) * \sum_{p=0}^{\infty} \left(\frac{1}{2}\right)^p \delta(t-p)$

$= x_1(t) * x_2(t)$





For $\underline{t < 0}$: $y(t) = 0$

$\underline{0 \leq t < 5}$: $y(t) = 2\,\delta(t) * x_2(t-\tau)$

$$= 2x_2(t-\tau)$$

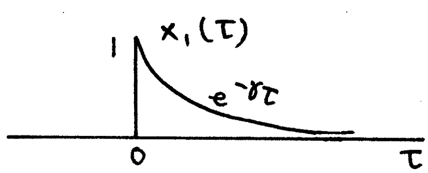
$$= 2\sum_{p=0}^{\infty} \left(\tfrac{1}{2}\right)^p \delta(t-p)$$

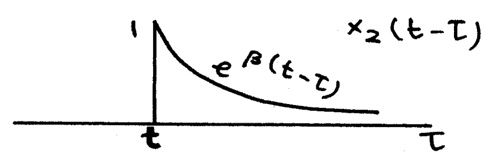
$\underline{t \geq 5}$: $y(t) = \left(2\,\delta(t) + \delta(t-5)\right) * \sum_{p=0}^{\infty} \left(\tfrac{1}{2}\right)^p \delta(t-p)$

$$y(t) = 2\sum_{p=0}^{\infty}\left(\tfrac{1}{2}\right)^p \delta(t-p) + \sum_{p=0}^{\infty}\left(\tfrac{1}{2}\right)^p \delta(t-p-5)$$

$$\therefore y(t) = \begin{cases} 0 & , t < 0 \\ 2\sum_{p=0}^{\infty}\left(\tfrac{1}{2}\right)^p \delta(t-p) & , 0 \leq t < 5 \\ 2\sum_{p=0}^{\infty}\left(\tfrac{1}{2}\right)^p \delta(t-p) + \sum_{p=0}^{\infty}\left(\tfrac{1}{2}\right)^p \delta(t-p-5) & , t \geq 5 \end{cases}$$

(n) $y(t) = e^{-\gamma t}u(t) * e^{\beta t}u(-t) = x_1(t) * x_2(t)$





For $t < 0$: $y(t) = \int_0^{\infty} e^{\beta t} \cdot e^{-(\beta+\gamma)\tau} d\tau$

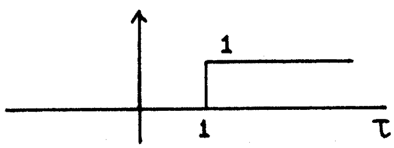
$$= \frac{e^{\beta t}}{\beta + \gamma}$$

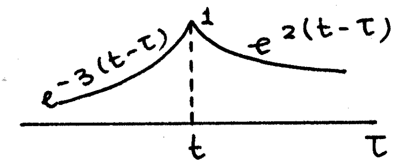
$t \geq 0$: $y(t) = \int_t^{\infty} e^{\beta t} e^{-(\beta+\gamma)\tau} d\tau$

$$= \frac{e^{\beta t}}{\beta+\gamma} e^{-(\beta+\gamma)t} = \frac{e^{-\gamma t}}{\beta + \gamma}$$

$$\therefore y(t) = \begin{cases} e^{\beta t}/\beta+\gamma & , t < 0 \\ e^{-\gamma t}/\beta+\gamma & , t \geq 0 \end{cases}$$

(e) $y(t)\ u(t-1) * h(t)$ $\qquad$ $h(t) = \begin{cases} e^{2t} & , t<0 \\ e^{-3t} & , t \geq 0 \end{cases}$





For $t<1$: $y(t) = \int_1^{\infty} e^{2(t-\tau)} d\tau = e^{2t}\left(\frac{1}{2}e^{-2}\right) = \frac{1}{2}e^{2(t-1)}$

$t \geq 1$: $y(t) = \int_1^{t} e^{-3(t-\tau)} d\tau + \int_t^{\infty} e^{2(t-\tau)} d\tau$

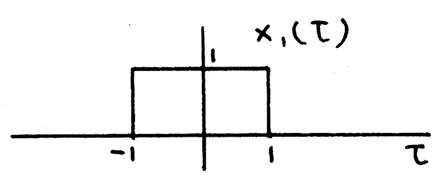
$y(t) = e^{-3t} \cdot \frac{1}{3}\left(e^{3t} - e^{3}\right) + e^{2t} \cdot \frac{1}{2} e^{-2t}$

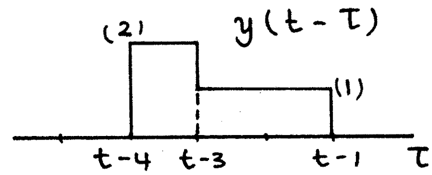
$= \frac{1}{3}\left(1 - e^{3(1-t)}\right) + \frac{1}{2} = \frac{5}{6} - \frac{1}{3}e^{3(1-t)}$

$$\therefore y(t) = \begin{cases} \frac{1}{2}e^{2(t-1)} & , t<1 \\ \frac{5}{6} - \frac{1}{3}e^{3(1-t)} & , t \geq 1 \end{cases}$$

$\boxed{2.6}$

(a) $m(t) = x(t) * y(t)$





For $t-1 < -1$ or $\underline{t<0}$: $m(t) = 0$

$\underline{0 \leq t < 2}$: $m(t) = \int_{-1}^{t-1} (1)(1)\, d\tau = t$

$t \geq 2$ but $t-4 < -1$ or $\underline{2 \leq t < 3}$:

$m(t) = \int_{-1}^{t-3} 2\, d\tau + \int_{t-3}^{1} 1\, d\tau$

$$m(t) = 2(t-2) + 4 - t = t$$

For $t \geqslant 3$ but $t-3 < 1$ or $\underline{3 \leqslant t < 4}$:

$$m(t) = \int_{t-4}^{t-3} 2\, d\tau + \int_{t-3}^{1} 1\, d\tau$$

$$= 2 + 4 - t = 6 - t$$

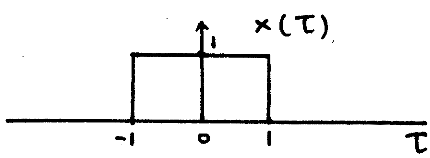
$t \geqslant 4$ but $t - 4 < 1$ or $\underline{4 \leqslant t < 5}$:

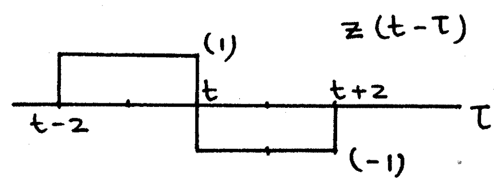
$$m(t) = \int_{t-4}^{1} 2\, d\tau = 2(5-t)$$

$\underline{t \geqslant 5}$: $m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , \quad t < 0 \\ t & , \quad 0 \leqslant t < 3 \\ 6-t & , \quad 3 \leqslant t < 4 \\ 2(5-t) & , \quad 4 \leqslant t < 5 \\ 0 & , \quad t \geqslant 5 \end{cases}$$

(b) $m(t) = x(t) * z(t)$





For $t+2 < -1$ or $\underline{t < -3}$: $m(t) = 0$

$-3 \leqslant t < -1$: $m(t) = \int_{-1}^{t+2} (-1)\, d\tau = -(t+3)$

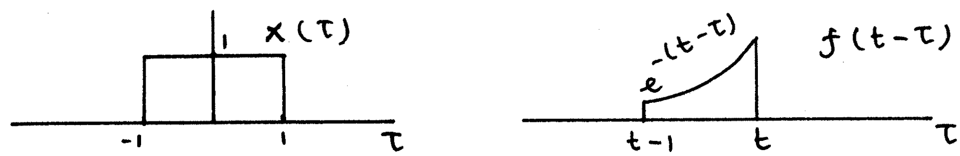
$-1 \leqslant t < 1$: $m(t) = \int_{-1}^{t} (1)\, d\tau + \int_{t}^{1} (-1)\, d\tau = 2t$

$1 \leqslant t < 3$: $m(t) = \int_{t-2}^{1} (1)\, d\tau = 3 - t$

$t \geqslant 3$: $m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , \ t < -3 \\ -t-3 & , \ -3 \le t < -1 \\ 2t & , \ -1 \le t < 1 \\ 3-t & , \ 1 \le t < 3 \\ 0 & , \ t \ge 3 \end{cases}$$

(c) $m(t) = x(t) * f(t)$



For $t < -1 \quad : m(t) = 0$

$-1 \le t < 0 : m(t) = \int_{-1}^{t} e^{-(t-\tau)} d\tau = 1 - e^{-(t+1)}$

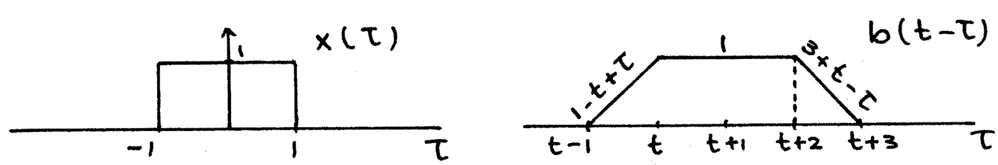
$0 \le t < 1 : m(t) = \int_{t-1}^{t} e^{-(t-\tau)} d\tau = 1 - e^{-1}$

$1 \le t < 2 : m(t) = \int_{t-1}^{1} e^{-(t-\tau)} d\tau = e^{(1-t)} - e^{-1}$

$t \ge 2 \quad : m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , \ t < -1 \\ 1 - e^{-(t+1)} & , \ -1 \le t < 0 \\ 1 - e^{-1} & , \ 0 \le t < 1 \\ e^{(1-t)} - e^{-1} & , \ 1 \le t < 2 \\ 0 & , \ t \ge 2 \end{cases}$$

(d) $m(t) = x(t) * b(t)$



For $t+3<-1$ or $\underline{t<-4}$: $m(t)=0$

$\underline{-4 \leq t < -3}$: $m(t) = \int_{-1}^{t+3} (3+t-\tau)\,d\tau = (t+3)(t-4) - \frac{1}{2}(t+3)^2 + \frac{1}{2}$

$$m(t) = \frac{1}{2}t^2 + 4t + 8$$

$\underline{-3 \leq t < -2}$: $m(t) = \int_{-1}^{t+2} d\tau + \int_{t+2}^{t+3} (3+t-\tau)\,d\tau$

$$= t+3+\frac{1}{2} = t + \frac{7}{2}$$

$\underline{-2 \leq t < -1}$: $m(t) = \int_{-1}^{t+2} d\tau + \int_{t+2}^{1} (3+t-\tau)\,d\tau$

$$= t+3 - \left(\frac{1}{2}t^2 + 2t + \frac{3}{2}\right)$$
$$= -\frac{1}{2}t^2 - t + \frac{3}{2}$$

$\underline{-1 \leq t < 0}$: $m(t) = \int_{-1}^{t} (1-t+\tau)\,d\tau + \int_{t}^{1} d\tau$

$$= -\frac{1}{2}(t^2-1) + 1 - t$$
$$= -\frac{1}{2}t^2 - t + \frac{3}{2}$$

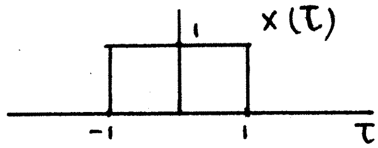
$\underline{0 \leq t < 1}$: $m(t) = \int_{t-1}^{t} (1-t+\tau)\,d\tau + \int_{t}^{1} d\tau = \frac{3}{2} - t$

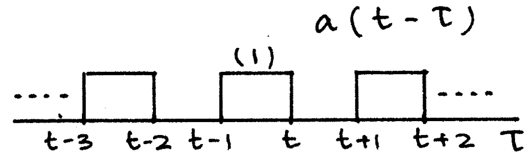
$\underline{1 \leq t < 2}$: $m(t) = \int_{t-1}^{1} (1-t+\tau)\,d\tau = \frac{1}{2}t^2 - 2t + 2$

$t \geq 2$: $m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , t < -4 \\ \frac{1}{2}t^2 + 4t + 8 & , -4 \leq t < -3 \\ t + \frac{7}{2} & , -3 \leq t < -2 \\ -\frac{1}{2}t^2 - t + \frac{3}{2} & , -2 \leq t < 0 \\ \frac{3}{2} - t & , 0 \leq t < 1 \\ \frac{1}{2}t^2 - 2t + 2 & , 1 \leq t < 2 \\ 0 & , t \geq 2 \end{cases}$$

(e) $m(t) = x(t) * a(t)$

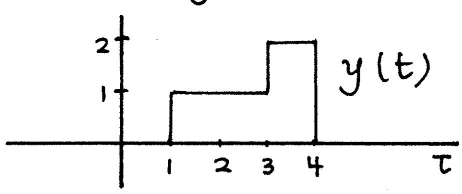


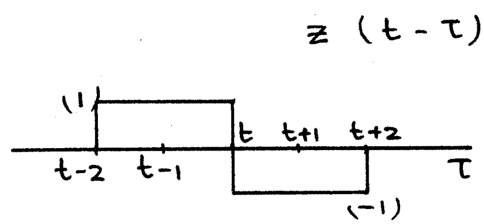


By inspection, since $x(\tau)$ has the width of 2 and the $a(t-\tau)$ has the period of 2 and the duty cicle of $\frac{1}{2}$, we can see that

$$m(t) = 1 \quad \text{for all } t$$

(f) $m(t) = y(t) * z(t)$





For $t+2<1$ or $\underline{t<-1}$: $m(t)=0$

$\underline{-1 \le t < 1}$: $m(t) = \int_{1}^{t+2} (-1)\,d\tau = -(t+1)$

$\underline{2 \le t < 3}$: $m(t) = \int_{1}^{t} (1)\,d\tau + \int_{t}^{3} (-1)\,d\tau + \int_{3}^{4} (-2)\,d\tau$

$$= 2t - 6$$

$\underline{-1 \le t < 2}$: $m(t) = \int_{1}^{t} (1)\,d\tau + \int_{t}^{3} (-1)\,d\tau + \int_{3}^{t+2} (-2)\,d\tau = -2$

$\underline{3 \le t < 4}$: $m(t) = \int_{t-2}^{3} (1)\,d\tau + \int_{3}^{t} (2)\,d\tau + \int_{t}^{4} (-2)\,d\tau$

$$= 3t - 9$$

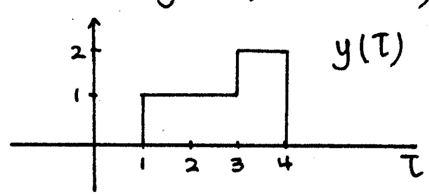
$\underline{4 \le t < 5}$: $m(t) = \int_{t-2}^{3} (1)\,d\tau + \int_{3}^{4} (2)\,d\tau = 7 - t$

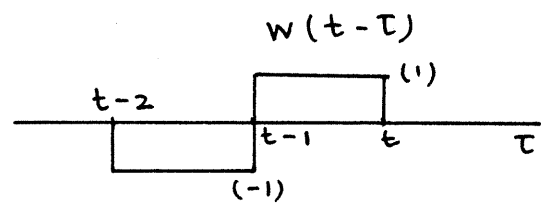
$$5 \leq t < 6 \quad : \quad m(t) = \int_{t-2}^{4} (2)\,d\tau = 2(6-t)$$

$$t \geq 6 \quad : \quad m(t) = 0$$

$$\therefore y(t) = \begin{cases} 0 & , \; t < -1 \\ -(t+1) & , \; -1 \leq t < 1 \\ -2 & , \; 1 \leq t < 2 \\ 2(t-3) & , \; 2 \leq t < 3 \\ 3(t-3) & , \; 3 \leq t < 4 \\ 7-t & , \; 4 \leq t < 5 \\ 2(6-t) & , \; 5 \leq t < 6 \\ 0 & , \; t \geq 6 \end{cases}$$

(g) $m(t) = y(t) * w(t)$





For $t < 1$: $m(t) = 0$

$1 \leq t < 2$: $m(t) = \int_{1}^{t} (1)\,d\tau = t-1$

$2 \leq t < 3$: $m(t) = \int_{1}^{t-1} (-1)\,d\tau + \int_{t-1}^{t} d\tau = 3-t$

$3 \leq t < 4$: $m(t) = \int_{t-2}^{t-1} (-1)\,d\tau + \int_{t-1}^{3} (1)\,d\tau + \int_{3}^{t} (2)\,d\tau$

$$= t-3$$

$4 \leq t < 5$: $m(t) = \int_{t-2}^{3} (-1)\,d\tau + \int_{3}^{t-1} (-2)\,d\tau + \int_{t-1}^{4} (2)\,d\tau$

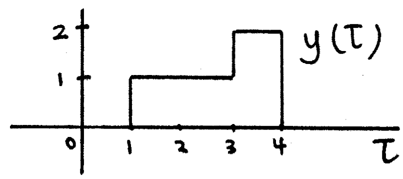
$$= -3t + 13$$

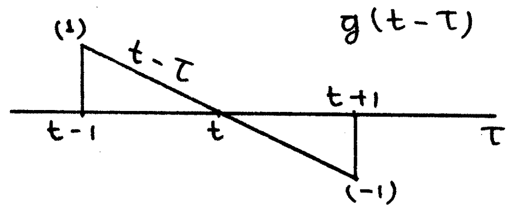
$5 \leq t < 6$: $m(t) = \int_{t-2}^{4} (-2)\,d\tau = 2(t-6)$

$\underline{t \geq 6}$: $m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , \quad t < 1 \\ t-1 & , \quad 1 \leq t < 2 \\ 3-t & , \quad 2 \leq t < 3 \\ t-3 & , \quad 3 \leq t < 4 \\ 13-3t & , \quad 4 \leq t < 5 \\ 2(t-6) & , \quad 5 \leq t < 6 \\ 0 & , \quad t \geq 6 \end{cases}$$

(h) $m(t) = y(t) * g(t)$





For $\underline{t < 0}$: $m(t) = 0$

$\underline{0 \leq t < 2}$: $m(t) = \int_{1}^{t+1} (t-\tau)\, d\tau = \frac{1}{2}t^2 - t$

$\underline{2 \leq t < 3}$: $m(t) = \int_{t-1}^{3} (t-\tau)\, d\tau + \int_{3}^{t+1} 2(t-\tau)\, d\tau$

$m(t) = \frac{1}{2}t^2 - 3t + 4$

$\underline{3 \leq t < 4}$: $m(t) = \int_{t-1}^{3} (t-\tau)\, d\tau + \int_{3}^{4} 2(t-\tau)\, d\tau$

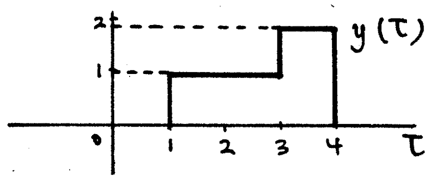
$m(t) = -\frac{1}{2}t^2 + 5t - 11$

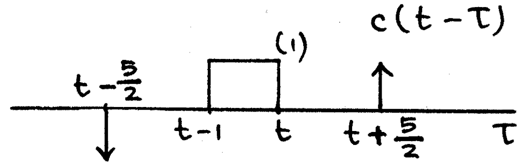
$\underline{4 \leq t < 5}$: $m(t) = \int_{t-1}^{4} 2(t-\tau)\, d\tau = -t^2 + 8t - 15$

$\underline{t \geq 5}$: $m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , \quad t < 0 \\ \frac{1}{2}t^2 - t & , \quad 0 \leq t < 2 \\ \frac{1}{2}t^2 - 3t + 4 & , \quad 2 \leq t < 3 \\ -\frac{1}{2}t^2 + 5t - 11 & , \quad 3 \leq t < 4 \\ -t^2 + 8t - 15 & , \quad 4 \leq t < 5 \\ 0 & , \quad t \geq 5 \end{cases}$$

(i) $m(t) = y(t) * c(t)$





For $t + \frac{5}{2} < 1$ or $\underline{t < -\frac{3}{2}}$: $m(t) = 0$

$\underline{-\frac{3}{2} \leq t < \frac{1}{2}}$: $m(t) = 1$

$\underline{\frac{1}{2} \leq t < 1}$: $m(t) = 2$

$\underline{1 \leq t < \frac{3}{2}}$: $m(t) = \int_1^t d\tau + 2 = t + 1$

$\underline{\frac{3}{2} \leq t < 2}$: $m(t) = \int_1^t d\tau = t - 1$

$\underline{2 \leq t < 3}$: $m(t) = 1\,(1) = 1$

$\underline{3 \leq t < \frac{7}{2}}$: $m(t) = \int_{t-1}^{3} (1)\, d\tau + \int_3^t (2)\, d\tau = t - 2$

$\underline{\frac{7}{2} \leq t < 4}$: $m(t) = t - 2 + (-1) = t - 3$

$\underline{4 \leq t < 5}$: $m(t) = \int_{t-1}^{4} (2)\, d\tau + (-1) = 9 - 2t$

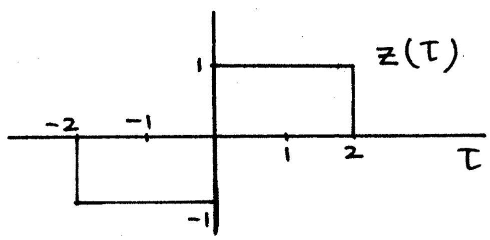
$\underline{5 \leq t < \frac{11}{2}}$: $m(t) = -1$

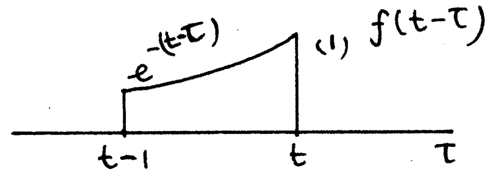
$\underline{\frac{11}{2} \leq t < \frac{13}{2}}$: $m(t) = -2$

$\underline{t \geq \frac{13}{2}}$: $m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , \quad t < -3/2 \\ 1 & , \quad -3/2 \leqslant t < 1/2 \\ 2 & , \quad 1/2 \leqslant t < 1 \\ t+1 & , \quad 1 \leqslant t < 3/2 \\ t-1 & , \quad 3/2 \leqslant t < 2 \\ 1 & , \quad 2 \leqslant t < 3 \\ t-2 & , \quad 3 \leqslant t < 7/2 \\ t-3 & , \quad 7/2 \leqslant t < 4 \\ 9-2t & , \quad 4 \leqslant t < 5 \\ -1 & , \quad 5 \leqslant t < 11/2 \\ -2 & , \quad 11/2 \leqslant t < 13/2 \\ 0 & , \quad t \geqslant 13/2 \end{cases}$$

(j) $m(t) = z(t) * f(t)$





For $\underline{t < -2}$: $m(t) = 0$

$\underline{-2 \leq t < -1}$: $m(t) = \int_{-2}^{t} -e^{-(t-\tau)}\, d\tau = e^{-(t+2)} - 1$

$\underline{-1 \leq t < 0}$: $m(t) = \int_{t-1}^{t} -e^{-(t-\tau)}\, d\tau = e^{-1} - 1$

$\underline{0 \leq t < 1}$: $m(t) = \int_{t-1}^{0} -e^{-(t-\tau)}\, d\tau + \int_{0}^{t} e^{-(t-\tau)}\, d\tau$

$$= 1 + e^{-1} - 2e^{-t}$$

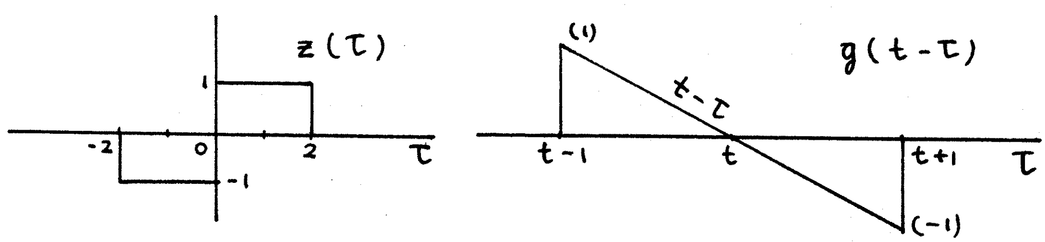
$\underline{1 \leq t < 2}$: $m(t) = \int_{t-1}^{t} e^{-(t-\tau)}\, d\tau = 1 - e^{-1}$

$\underline{2 \leq t < 3}$: $m(t) = \int_{t-1}^{2} e^{-(t-\tau)}\, d\tau = e^{(2-t)} - e^{-1}$

$t \geq 3 \quad : \quad m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , \; t < -2 \\ e^{-(t+2)} - 1 & , \; -2 \leq t < -1 \\ e^{-1} - 1 & , \; -1 \leq t < 0 \\ 1 + e^{-1} - 2e^{-t} & , \; 0 \leq t < 1 \\ 1 - e^{-1} & , \; 1 \leq t < 2 \\ e^{(2-t)} - e^{-1} & , \; 2 \leq t < 3 \\ 0 & , \; t \geq 3 \end{cases}$$

(k) $m(t) = z(t) * g(t)$



For $t+1 < -2$ or $\underline{t < -3}$: $m(t) = 0$

$\underline{-3 \leq t < -1}$: $m(t) = \int_{-2}^{t+1} (\tau - t)\, d\tau = -\frac{1}{2}t^2 - 2t - \frac{3}{2}$

$\underline{-1 \leq t < 1}$: $m(t) = \int_{t-1}^{0} (\tau - t)\, d\tau + \int_{0}^{t+1} (t - \tau)\, d\tau$

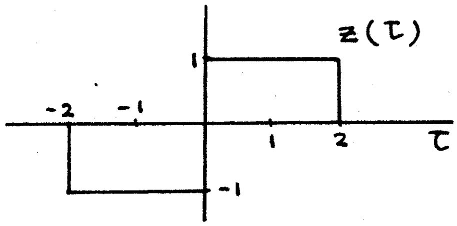
$m(t) = t^2 - 1$

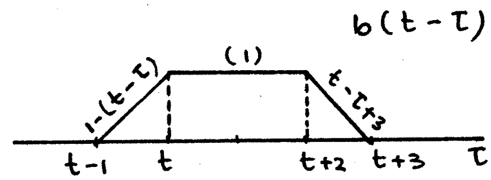
$\underline{1 \leq t < 3}$: $m(t) = \int_{t-1}^{2} (t - \tau)\, d\tau = -\frac{1}{2}t^2 + 2t - \frac{3}{2}$

$\underline{t \geq 3}$: $m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , \; t < -3 \\ -\frac{1}{2}t^2 - 2t - \frac{3}{2} & , \; -3 \leq t < -1 \\ t^2 - 1 & , \; -1 \leq t < 1 \\ -\frac{1}{2}t^2 + 2t - \frac{3}{2} & , \; 1 \leq t < 3 \\ 0 & , \; t \geq 3 \end{cases}$$

(ℓ) $m(t) = z(t) * b(t)$





For $t+3 < -2$ or $\underline{t < -5}$: $m(t) = 0$

$\underline{-5 \leq t < -4}$: $m(t) = \int_{-2}^{t+3} (t+3-\tau)\, d\tau = -\left(\frac{1}{2}t^2 + 5t + \frac{25}{2}\right)$

$\underline{-4 \leq t < -3}$: $m(t) = \int_{-2}^{t+2} (-1)\, d\tau + \int_{t+2}^{t+3} -(t+3-\tau)\, d\tau$

$$m(t) = -\left(t + \frac{9}{2}\right)$$

$\underline{-3 \leq t < -2}$: $m(t) = \int_{-2}^{t+2} (-1)\, d\tau + \int_{t+2}^{0} -(t+3-\tau)\, d\tau + \int_{0}^{t+3} (t+3-\tau)\, d\tau$

$$m(t) = t^2 + 5t + \frac{9}{2}$$

$\underline{-2 \leq t < -1}$: $m(t) = \int_{-2}^{t} (t-1-\tau)\, d\tau + \int_{t}^{0} (-1)\, d\tau + \int_{0}^{t+2} (1)\, d\tau + \int_{t+2}^{t+3} (t+3-\tau)\, d\tau$

$$m(t) = \frac{1}{2}t^2 + 3t + \frac{5}{2}$$

$\underline{-1 \leq t < 0}$: $m(t) = \int_{t-1}^{t} (t-1-\tau)\, d\tau + \int_{t}^{0} (-1)\, d\tau + \int_{0}^{t+2} (1)\, d\tau + \int_{t+2}^{2} (t+3-\tau)\, d\tau$

$$m(t) = -\frac{1}{2}t^2 + t + \frac{3}{2}$$

$\underline{0 \leq t < 1}$: $m(t) = \int_{t-1}^{0} (t-1-\tau)\, d\tau + \int_{0}^{t} (1-t+\tau)\, d\tau$

$$+\int_{t}^{2}(1)\,d\tau$$

$$m(t) = -t^2 + t + \frac{3}{2}$$

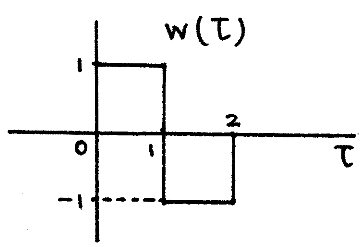
$\underline{1 \le t < 2}$: $m(t) = \int_{t-1}^{t}(1-t+\tau)\,d\tau + \int_{t}^{2}(1)\,d\tau = \frac{5}{2} - t$

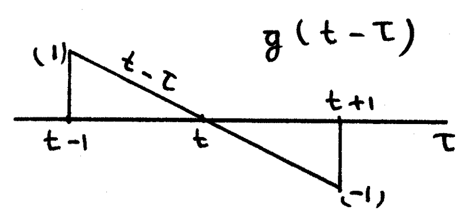
$\underline{2 \le t < 3}$: $m(t) = \int_{t-1}^{2}(1-t+\tau)\,d\tau = \frac{1}{2}t^2 - 3t + \frac{9}{2}$

$\underline{t \ge 3}$: $m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , \quad t < -5 \\ -\left(\frac{1}{2}t^2 + 5t + \frac{25}{2}\right) & , \quad -5 \le t < -4 \\ -\left(t + \frac{9}{2}\right) & , \quad -4 \le t < -3 \\ t^2 + 5t + \frac{9}{2} & , \quad -3 \le t < -2 \\ \frac{1}{2}t^2 + 3t + \frac{5}{2} & , \quad -2 \le t < -1 \\ -\frac{1}{2}t^2 + t + \frac{3}{2} & , \quad -1 \le t < 0 \\ -t^2 + t + \frac{3}{2} & , \quad 0 \le t < 1 \\ \frac{5}{2} - t & , \quad 1 \le t < 2 \\ \frac{1}{2}t^2 - 3t + \frac{9}{2} & , \quad 2 \le t < 3 \\ 0 & , \quad t \ge 3 \end{cases}$$

(m) $m(t) = w(t) * g(t)$





For $t+1 < 0$ or $t < -1$: $m(t) = 0$

$-1 \le t < 0$: $m(t) = \int_{0}^{t+1}(t-\tau)\,d\tau = \frac{1}{2}(t^2-1)$

$0 \leq t < 1$: $m(t) = \int_0^1 (t-\tau)\,d\tau + \int_1^{t+1} (\tau - t)\,d\tau = -\frac{1}{2}t^2 + 2t - \frac{1}{2}$

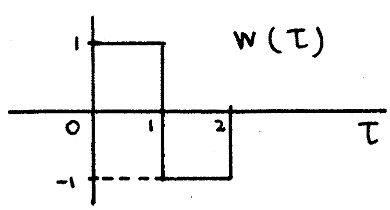
$1 \leq t < 2$: $m(t) = \int_{t-1}^1 (t-\tau)\,d\tau + \int_1^2 (\tau - t)\,d\tau = -\frac{1}{2}t^2 + \frac{3}{2}$

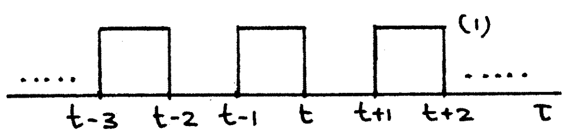
$2 \leq t < 3$: $m(t) = \int_{t-1}^2 (\tau - t)\,d\tau = \frac{1}{2}t^2 - 2t + \frac{3}{2}$

$t \geq 3$: $m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , \ t < -1 \\ \frac{1}{2}(t^2-1) & , \ -1 \leq t < 0 \\ -\frac{1}{2}t^2 + 2t + \frac{1}{2} & , \ 0 \leq t < 1 \\ -\frac{1}{2}t^2 + \frac{3}{2} & , \ 1 \leq t < 2 \\ \frac{1}{2}t^2 - 2t + \frac{3}{2} & , \ 2 \leq t < 3 \\ 0 & , \ t \geq 3 \end{cases}$$

(n) $m(t) = w(t) * a(t)$





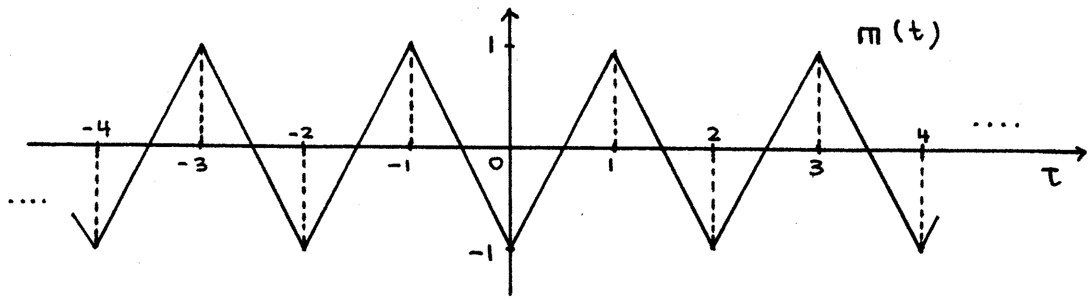
Consider $t-2 \leq \tau \leq \tau$ section only for $a(t-\tau)$, since this section will be replicated with period of 2

$0 \leq t < 1$: $m(t) = t - (1-t) = 2t - 1$

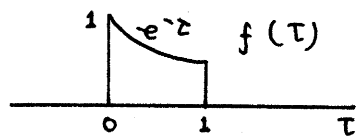
$1 \leq t < 2$: $m(t) = (2-t) - (t-1) = 3 - 2t$

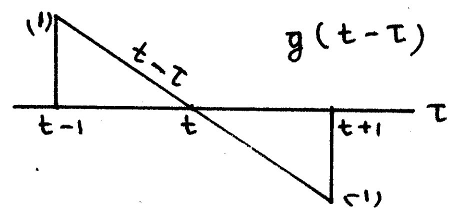
k = integer

$$\therefore m(t) = \begin{cases} 2(t-2k) - 1 & , \ 0 \leq (t-2k) < 1 \\ 3 - 2(t-2k) & , \ 1 \leq (t-2k) < 2 \end{cases}$$



(a) $m(t) = f(t) * g(t)$





For $t+1 < 0$ or $\underline{t < -1}$: $m(t) = 0$

$\underline{-1 \leq t < 0}$: $m(t) = \int_0^{t+1} e^{-\tau}(t-\tau)\, d\tau = t\int_0^{t+1} e^{-\tau} d\tau - \int_0^{t+1} \tau . e^{-\tau} d\tau$

(Integration by parts for the second term)

$$m(t) = 2e^{-(t+1)} + t - 1$$

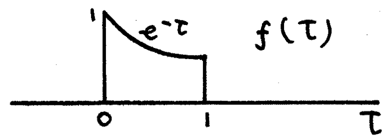
$\underline{0 \leq t < 1}$: $m(t) = \int_0^1 e^{-\tau}(t-\tau)\, d\tau = t(1-e^{-1}) + 2e^{-1} - 1$

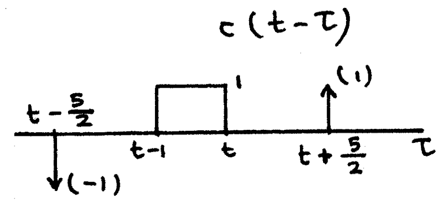
$\underline{1 \leq t < 2}$: $m(t) = \int_{t-1}^1 e^{-\tau}(t-\tau)\, d\tau = e^{-1}(t+2) - 2te^{-(t-1)}$

$\underline{t \geq 2}$: $m(t) = 0$

$$\therefore m(t) = \begin{cases} 0 & , \ t < -1 \\ 2e^{-(t+1)} + t - 1 & , \ -1 \leq t < 0 \\ t(1-e^{-1}) + 2e^{-1} - 1 & , \ 0 \leq t < 1 \\ e^{-1}(t+2) - 2te^{-(t-1)} & , \ 1 \leq t < 2 \\ 0 & , \ t \geq 2 \end{cases}$$

(p) $m(t) = f(t) * c(t)$



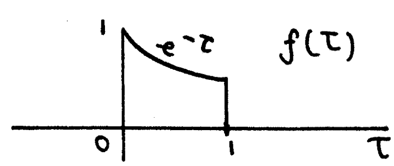


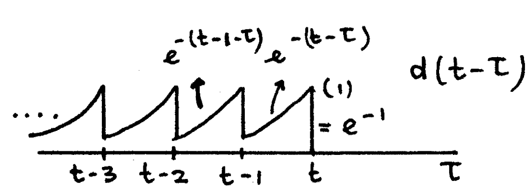
For $t+\frac{5}{2} < 0$ or $t < -\frac{5}{2}$: $m(t) = 0$

$$-\tfrac{5}{2} \leq t < -\tfrac{3}{2} \;:\; m(t) = e^{-(t+\frac{5}{2})}$$
$$-\tfrac{3}{2} \leq t < 0 \;:\; m(t) = 0$$
$$0 \leq t < 1 \;:\; m(t) = \int_0^t e^{-\tau}\,d\tau = 1 - e^{-t}$$
$$1 \leq t < 2 \;:\; m(t) = \int_{t-1}^{1} e^{-\tau}\,d\tau = e^{-(t-1)} - e^{-1}$$
$$2 \leq t < \tfrac{5}{2} \;:\; m(t) = 0$$
$$\tfrac{5}{2} \leq t < \tfrac{7}{2} \;:\; m(t) = -e^{-(t-\frac{5}{2})}$$
$$t \geq \tfrac{7}{2} \;:\; m(t) = 0$$

$$\therefore m(t) = \begin{cases} 0 & , \quad t < -\frac{5}{2} \\ e^{-(t+5/2)} & , \quad -\frac{5}{2} \leq t < -\frac{3}{2} \\ 0 & , \quad -\frac{3}{2} \leq t < 0 \\ 1 - e^{-t} & , \quad 0 \leq t < 1 \\ e^{-(t-1)} - e^{-1} & , \quad 1 \leq t < 2 \\ 0 & , \quad 2 \leq t < \frac{5}{2} \\ -e^{-(t-5/2)} & , \quad \frac{5}{2} \leq t < \frac{7}{2} \\ 0 & , \quad t \geq \frac{7}{2} \end{cases}$$

(q) $m(t) = f(t) * d(t)$





For $t < 0 \quad : \quad m(t) = 0$

$0 \leq t < 1 \quad : \quad m(t) = \int_0^t e^{-(t-\tau)} \cdot e^{-\tau}\, d\tau = t e^{-t}$

$1 \leq t < 2 \quad : \quad m(t) = \int_0^{t-1} e^{-(t-1-\tau)} e^{-\tau}\, d\tau + \int_{t-1}^{1} e^{-(t-\tau)} e^{-\tau}\, d\tau$

$$m(t) = (t-1)\, e^{-(t-1)} + (2-t)\, e^{-t}$$

$2 \leq t < 3 \quad : \quad m(t) = \int_0^{t-2} e^{-(t-2-\tau)} e^{-\tau}\, d\tau + \int_{t-2}^{1} e^{-(t-1-\tau)} e^{-\tau}\, d\tau$

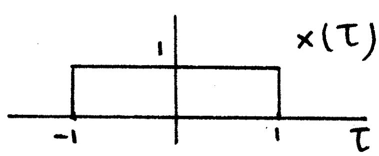
$$m(t) = (t-2)\, e^{-(t-2)} + (3-t)\, e^{-(t-1)}$$

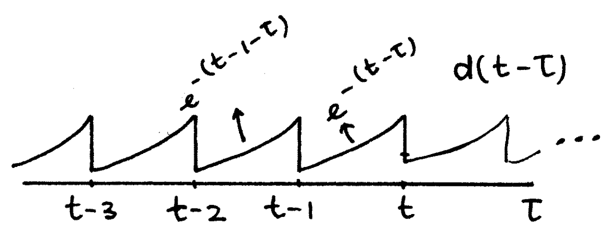
$\vdots$

$1+k \leq t < 2+k \quad : \quad m(t) = (t-(1+k))\, e^{-(t-(1+k))} - (t-(2+k))\, e^{-(t-k)}$

$$\therefore m(t) = \begin{cases} 0 & t < 0 \\ t e^{-t} & 0 \leq t < 1 \\ (t-(1+k))\, e^{-(t-(1+k))} - (t-(2+k))\, e^{-(t-k)} & 1+k \leq t < 2+k \\ & k = 0, 1, 2, \ldots \end{cases}$$

(r) $m(t) = x(t) * d(t)$





$d(t-\tau)$ has period 1

$x(\tau)$ is exactly two periods wide

Hence, $x(\tau)\, d(t-\tau)$ contains exactly two periods of $d(t)$.

$m(t)$ is thus the area under two periods of $d(t)$

$$m(t) = 2\int_0^1 e^{-t}\,dt = 2\left[-e^{-t}\right]_0^1$$

$$= 2 - 2e^{-1}$$

$\boxed{2.7.}$

(a) Distributive : $\underbrace{x[n] * (h[n] + g[n])}_{\text{LHS}} = \underbrace{x[n] * h[n] + x[n] * g[n]}_{\text{RHS}}$

$$\begin{aligned}
\text{LHS} &= x[n] * (h[n] + g[n]) \\
&= \sum_{k=-\infty}^{\infty} x[k]\,(h[n-k] + g[n-k]) && \because \text{definition of convolution sum} \\
&= \sum_{k=-\infty}^{\infty} (x[k].h[n-k] + x[k].g[n-k]) && \because \text{dist. prop. of multiplic.} \\
&= \sum_{k=-\infty}^{\infty} x[k].h[n-k] + \sum_{k=-\infty}^{\infty} x[k].g[n-k] && \because \text{term separation in sum} \\
&= x[n] * h[n] + x[n] * g[n] && \because \text{definition of convolution sum} \\
&= \text{RHS}
\end{aligned}$$

(b) Associative : $\underbrace{x[n] * (h[n] * g[n])}_{\text{LHS}} = \underbrace{(x[n] * h[n]) * g[n]}_{\text{RHS}}$

$$
\begin{aligned}
\text{LHS} &= x[n] * (h[n] * g[n]) \\
&= x[n] * \sum_{k=-\infty}^{\infty} h[k] \, . \, g[n-k] \\
&= \sum_{\ell=-\infty}^{\infty} x[\ell] \, . \left(\sum_{k=-\infty}^{\infty} h[k] \, . \, g[n-k-\ell] \right) \\
&= \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left(x[\ell] \, . \, h[k] \, . \, g[n-k-\ell] \right)
\end{aligned}
$$

Use : $v = k + \ell$ and exchange the order of Σ

$$
\begin{aligned}
&= \sum_{v=-\infty}^{\infty} \left(\sum_{\ell=-\infty}^{\infty} x[\ell] \, . \, h[v-\ell] \right) g[n-v] \\
&= \sum_{v=-\infty}^{\infty} \left(x[v] * h[v] \right) g[n-v] \\
&= (x[n] * h[n]) * g[n] \\
&= \text{RHS}
\end{aligned}
$$

(c) Commutative : $\underbrace{x[n] * h[n]}_{\text{LHS}} = \underbrace{h[n] * x[n]}_{\text{RHS}}$

$$
\text{LHS} = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \, . \, h[n-k]
$$

Use : $k = n - \ell$ substitution

$$
\begin{aligned}
&= \sum_{\ell=-\infty}^{\infty} x[n-\ell] \, . \, h[\ell] = \sum_{\ell=-\infty}^{\infty} h[\ell] \, . \, x[n-\ell] \\
&= h[n] * x[n] \\
&= \text{RHS}
\end{aligned}
$$

$\boxed{2.8}$

(a) $y(t) = x(t) * h(t) = x(t) * \left(\frac{1}{\Delta}\delta(t) - \frac{1}{\Delta}\delta(t-\Delta)\right)$

$y(t) = \frac{1}{\Delta}\left(x(t) - x(t-\Delta)\right)$

(b) when $\Delta \to 0$

$$\lim_{\Delta \to 0} y(t) = \lim_{\Delta \to 0} \frac{x(t) - x(t-\Delta)}{\Delta}$$

is nothing but $\frac{dx}{dt}(t)$ / first derivative of $x(t)$ with respect to time (t)

(c) $g(t) = \lim_{\Delta \to 0} h(t)$

$$h^n(t) = \underbrace{g(t) * g(t) * \ldots\ldots * g(t)}_{n \text{ times}}$$

$$y^n(t) = x(t) * h^n(t)$$

$$= \left(x(t) * g(t)\right) * \underbrace{g(t) * \ldots\ldots * g(t)}_{(n-1) \text{ times}}$$

let $x^{(1)}(t) = x(t) * g(t) = x(t) * \lim_{\Delta \to 0} h(t)$

$$= \lim_{\Delta \to 0} \left(x(t) * h(t)\right) = \frac{dx}{dt}(t) \text{ from (b)}$$

Then $y^n(t) = \left(x^{(1)}(t) * g(t)\right) * \underbrace{g(t) * \ldots\ldots * g(t)}_{(n-2) \text{ times}}$

Analogously, $x^{(1)}(t) * g(t) = \frac{d^2x}{dt^2}(t)$

Doing this repetitively, we will find that

$$y^n(t) = x^{(n-1)}(t) * g(t) = \frac{d^{n-1}x}{dt^{n-1}}(t) * g(t)$$

$$\therefore \; y^n(t) = \frac{d^n x}{dt^n}(t)$$

2.9

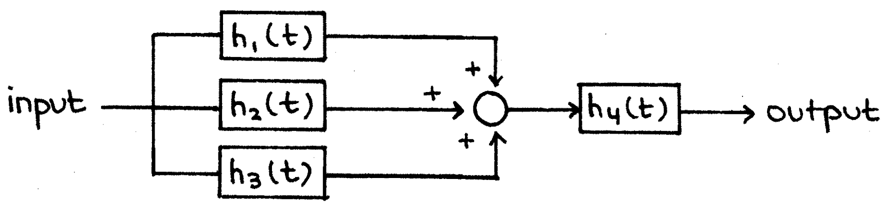
(a) $y(t) = x(t) * \Big(h_1(t) * h_4(t) - (h_2(t) + h_3(t)) \Big) * h_5(t)$

(b) $y[n] = x[n] * h_1[n] * (h_2[n] + h_3[n]) * (h_5[n] - h_4[n])$
$\quad * h_6[n]$

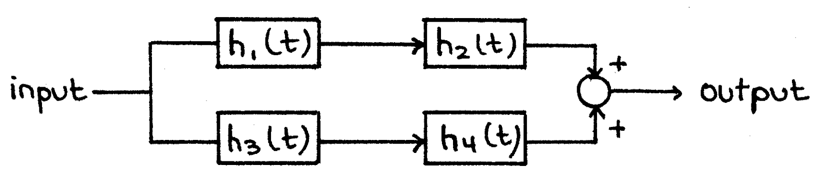
(c) $y[t] = x(t) * \big(\delta(t) - h_1(t) + h_2(t) \big) * h_3(t) * h_4(t)$

2.10

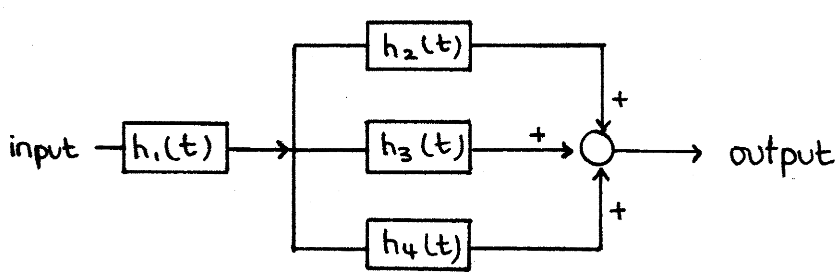
(a) $h(t) = h_1(t) + \{ h_2(t) + h_3(t) \} * h_4(t)$



(b) $h(t) = h_1(t) * h_2(t) + h_3(t) * h_4(t)$



(c) $h(t) = h_1(t) * \{ h_2(t) + h_3(t) + h_4(t) \}$



$\boxed{2.11}$ $h_1[n] = (\frac{1}{2})^n \, (u[n+2] - u[n-3])$
$h_2[n] = \delta[n]$
$h_3[n] = u[n-1]$

(a) $y[n] = x[n] * h[n]$ where $h[n] = h_1[n] * \{h_2[n] + h_3[n]\}$

(b) Let $h_4[n] = h_2[n] + h_3[n] = \delta[n] + u[n-1] = u[n]$
$$h[n] = h_1[n] * h_4[n] = \sum_{k=-\infty}^{\infty} h_1[k]\, h_4[n-k]$$

For $\underline{n < -2}$: $h[n] = 0$
$\underline{-2 \leq n \leq 3}$: $h[n] = \sum_{k=-2}^{n} (\frac{1}{2})^k = 8 - (\frac{1}{2})^n$

$\underline{n \geq 4}$: $h[n] = \sum_{k=-2}^{3} (\frac{1}{2})^k = 8 - (\frac{1}{2})^3 = 7\frac{7}{8}$

$$\therefore h[n] = \begin{cases} 0 & n = -\infty, \dots, -4, -3 \\ 8 - (\frac{1}{2})^n & n = -2, -1, \dots, 3 \\ 7\frac{7}{8} & n = 4, 5, \dots \end{cases}$$

(c) $h_1[n]$
(i) Stable, since $\sum_{k=-\infty}^{\infty} |h_1[n]| = 7\frac{7}{8} < \infty$

(ii) not causal, since $h_1[n] \neq 0$ for $n < 0$
(iii) not memoryless, since $h_1[n] \neq k.\delta[n]$

(d) $h_2[n]$
(i) stable, since $\sum_{k=-\infty}^{\infty} |h_2[n]| = 1 < \infty$

(ii) causal, since $h_2[n] = 0$ for $n < 0$
(iii) memoryless since $h_2[n] = k.\delta[n]$ where $k = 1$

(e) $h_3[n]$
(i) not stable, since $\sum_{k=-\infty}^{\infty} |h_3[n]| \to \infty$

(ii) causal, since $h_3[n] = 0$ for $n < 0$
(iii) not memoryless, since $h_3[n] \neq k\,\delta[n]$

$\boxed{2.12}$ For all problems below, a system $h(t)$ or $h(n)$ is:
(i) memoryless, iff $h(t) = k\,\delta(t)$ or $h[n] = k\,\delta[n]$
(ii) causal, iff $h(t) = 0$ for $t < 0$ or $h[n] = 0$ for $n < 0$
(iii) stable, iff $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$ or $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

(a) $h(t) = e^{-2|t|}$
(i) not memoryless
(ii) not causal
(iii) stable, since $\int_{-\infty}^{\infty} |h(t)|\,dt = 2\int_{0}^{\infty} e^{-2t}\,dt = 2 < \infty$

(b) $h(t) = e^{2t}\,u(t-1)$
(i) not memoryless
(ii) causal
(iii) not stable, since $\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{1}^{\infty} e^{2t}\,dt \quad \infty$

(c) $h(t) = u(t+1) - 2u(t-1)$
(i) not memoryless
(ii) not causal
(iii) not stable

(d) $h(t) = 3\,\delta(t)$
(i) memoryless
(ii) causal
(iii) stable

(e) $h(t) = \cos(\pi t)\,u(t)$
(i) not memoryless
(ii) causal
(iii) not stable, since $\int_{0}^{\infty} |\cos(\pi t)|\,dt \longrightarrow \infty$

(f) $h[n] = 2^n u[-n]$

(i) not memoryless

(ii) not causal

(iii) stable, since $\sum_{k=-\infty}^{0} 2^k = 2 < \infty$

(g) $h[n] = e^{2n} u[n-1]$

(i) not memoryless

(ii) causal

(iii) not stable, since $\sum_{k=1}^{\infty} (e^2)^k \to \infty$

(h) $h[n] = \cos(\frac{\pi}{8} n) \{u[n] - u[n-10]\}$

(i) not memoryless

(ii) causal

(iii) stable, since $\sum_{k=0}^{k=9} \cos(\frac{\pi}{8} k) < \infty$

(i) $h[n] = 2u[n] - 2u[n-1] = 2\delta[n]$

(i) memoryless

(ii) causal

(iii) stable

(j) $h[n] = \sin(\frac{\pi}{2} n)$

(i) not memoryless

(ii) not causal

(iii) not stable

(k) $h[n] = \delta[n] + \sin(\pi n) = \delta[n]$ since $\sin(\pi n) = 0$ for all n

(i) memoryless

(ii) causal

(iii) stable

$\boxed{2.13}$ $y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$

$$|y[0]| = \left|\sum_{k=-\infty}^{\infty} h[k]\, x[-k]\right|$$

let $x[-k] = \begin{cases} 1 & h[k] \geq 0 \\ -1 & h[-k] < 0 \end{cases}$ 　Take sign of $h[n]$ and reflect it. Clearly $|x[k]| = 1 < \infty$

Then $h[k]\,x[-k] = |h[k]|$, so

$$|y[0]| = \left|\sum_{k=-\infty}^{\infty} |h[k]|\right|$$

$$= \sum_{k=-\infty}^{\infty} |h[k]|$$

Hence there exists an input for which $|y[n]| = \sum_{k=-\infty}^{\infty} |h[k]|$ and

$\therefore \sum_{k=-\infty}^{\infty} |h[k]| < \infty$ is a necessary condition for stability

$\boxed{2.14}$ $s[n] = \sum_{k=-\infty}^{n} h[k]$

(a) $h[n] = \left(\frac{1}{2}\right)^n u[n]$

For $\underline{n<0}$: $s[n] = 0$

$\underline{n \geq 0}$: $s[n] = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^n$

$$= 2\left(1 - \left(\tfrac{1}{2}\right)^{n+1}\right) = 2 - \left(\tfrac{1}{2}\right)^n$$

$\therefore s[n] = \left(2 - \left(\frac{1}{2}\right)^n\right) u[n]$

(b) $h[n] = \delta[n] - \delta[n-1]$

For $\underline{n<0}$: $s[n] = 0$

$\underline{n=0}$: $s[n] = 1$

$\underline{n \geq 1}$: $s[n] = 0$

$\therefore s[n] = \underline{\underline{\delta[n]}}$

(c) $h[n] = (-1)^n \{u[n+2] - u[n-3]\}$

For $\underline{n<-2}$: $s[n] = 0$

$\underline{-2 \leq n \leq 2}$: $s[n] = \begin{cases} 1 & n = -2, 0, 2 \\ 0 & n = -1, 1 \end{cases}$

$\underline{n \geq 3}$: $s[n]$

$\therefore s[n] = \delta[n+2] + \delta[n] + \delta[n-2]$

(d) $h[n] = u[n]$

For $\underline{n<0}$: $s[n] = 0$

$\underline{n \geq 0}$: $s[n] = n+1$

$\therefore s[n] = \underline{\underline{(n+1)\,u[n]}}$

(e) $h(t) = e^{-|t|}$

For $\underline{t<0}$: $s(t) = \int_{-\infty}^{t} e^{\tau} d\tau = e^{t}$

$\underline{t \geq 0}$: $s(t) = \int_{-\infty}^{0} e^{\tau} d\tau + \int_{0}^{t} e^{-\tau} d\tau = 2 - e^{-t}$

$\therefore s(t) = \begin{cases} e^{t} & , t<0 \\ 2-e^{-t} & , t \geq 0 \end{cases}$

(f) $h(t) = \delta(t) - \delta(t-1)$

For $\underline{t < 0}$: $s(t) = 0$
$\underline{0 \leq t < 1}$: $s(t) = 1$
$\underline{t \geq 1}$: $s(t) = 0$

$\therefore s(t) = \underline{u(t) - u(t-1)}$

(g) $h(t) = u(t+1) - u(t-1)$

For $\underline{t < -1}$: $s(t) = 0$
$\underline{-1 \leq t < 1}$: $s(t) = t+1$
$\underline{t \geq 1}$: $s(t) = 2$

$$\therefore s(t) = \begin{cases} 0 & , \quad t < -1 \\ t+1 & , \quad -1 \leq t < 1 \\ 2 & , \quad t \geq 1 \end{cases}$$

(h) $h(t) = t\,u(t)$

For $\underline{t < 0}$: $s(t) = 0$
$\underline{t \geq 0}$: $s(t) = \int_0^t \tau \, d\tau = \frac{1}{2} t^2$

$\therefore s(t) = \underline{\underline{\frac{1}{2} t^2 u(t)}}$

$\boxed{2.15}$ The frequency response is determined as follow :

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \qquad \text{for continuous time}$$

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n] . e^{-j\Omega n} \qquad \text{for discrete time}$$

(a) $h[n] = (\frac{1}{2})^n u[n]$

$$H(e^{j\Omega}) = \sum_{n=0}^{\infty} (\tfrac{1}{2})^n e^{-j\Omega n} = \sum_{n=0}^{\infty} (\tfrac{1}{2} e^{-j\Omega})^n = \frac{1}{1-\frac{1}{2} e^{-j\Omega}}$$

(b) $h[n] = \delta[n] - \delta[n-1]$

$$H(e^{j\Omega}) = (1) - e^{-j\Omega} = 1 - e^{-j\Omega}$$

(c) $h[n] = (-1)^n \{u[n+2] - u[n-3]\}$

$$H(e^{j\Omega}) = e^{j2\Omega} - e^{j\Omega} + 1 - e^{-j\Omega} + e^{-j2\Omega}$$
$$H(e^{j\Omega}) = 1 + 2\cos(2\Omega) - 2\cos(\Omega)$$

(d) $h[n] = (.9)^n e^{j\frac{\pi}{2}n} u[n]$

$$H(e^{j\Omega}) \sum_{n=0}^{\infty} (0.9\, e^{-j(\Omega - \pi/2)})^n = \frac{1}{1+0.9\, e^{-j(\Omega-\frac{\pi}{2})}}$$

(e) $h(t) = e^{-|t|}$

$$H(j\omega) = \int_{-\infty}^{0} e^{t} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt$$

$$H(j\omega) = \frac{1}{1-j\omega} + \frac{1}{1+j\omega} = \frac{2}{1+\omega^2}$$

(f) $h(t) = -\delta(t+1) + \delta(t) - \delta(t-1)$

$$H(j\omega) = -e^{j\omega} + 1 - e^{-j\omega} = 1 - 2\cos(\omega)$$

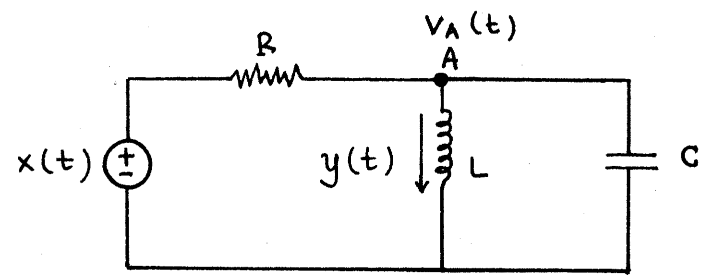
(g) $h(t) = \cos(\pi t)\{u(t+3) - u(t-3)\}$
$$H(j\omega) = \int_{-3}^{3} \cos(\pi t) e^{-j\omega t}\, dt = \frac{2\omega}{\pi^2 - \omega^2} \sin(3\omega)$$

(h) $h(t) = e^{2t}\, u(-t)$

$$H(j\omega) = \int_{-\infty}^{0} e^{2t} \cdot e^{-j\omega t}\, dt = \frac{1}{2 - j\omega}$$

2.16

(a)



Writing a node equation for node A

$$y(t) + \frac{V_A(t) - x(t)}{R} + C\,\frac{dV_A(t)}{dt} = 0 \qquad \ldots .(1)$$

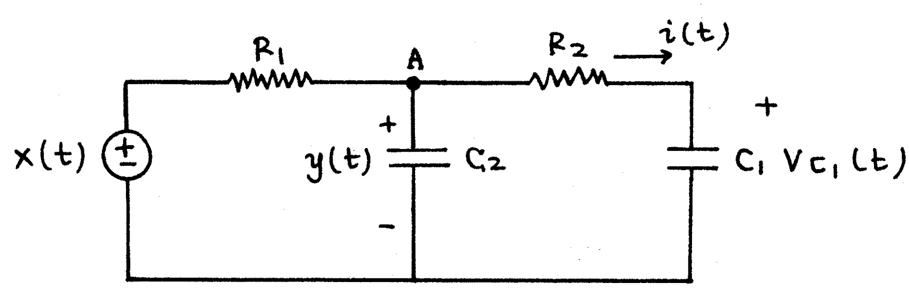
For the inductor L :

$$V_A(t) = L\,\frac{dy(t)}{dt} \qquad \ldots .(2)$$

Combining (1) and (2) ; we have :

$$\frac{d^2}{dt^2} y(t) + \frac{1}{RC}\frac{d}{dt} y(t) + \frac{1}{LC} y(t) = \frac{1}{RLC} x(t)$$

(b)



(b) Node equation for node A :

$$C_2 \cdot \frac{d}{dt} y(t) + \frac{y(t) - x(t)}{R_1} + i(t) = 0 \quad \ldots (1)$$

For capacitor $C_1 \Rightarrow \frac{d}{dt} V_{C_1}(t) = \frac{i(t)}{C_1}$

and $y(t) = i(t) \cdot R_2 + V_{C_1}(t)$

$$\Leftrightarrow \frac{d}{dt} y(t) = R_2 \frac{d}{dt} i(t) + \frac{i(t)}{C_1} \quad \ldots (2)$$

Combining (1) & (2) :

$$\frac{d^2}{dt^2} y(t) + \left(\frac{1}{C_2R_2} + \frac{1}{C_2R_1} + \frac{1}{C_1R_2}\right)\frac{d}{dt} y(t) + \frac{1}{C_1C_2R_1R_2} y(t)$$

$$= \frac{1}{C_1C_2R_1R_2} x(t) + \frac{1}{C_2R_1} \frac{d}{dt} x(t)$$

$\boxed{2.17}$

(a) $5 \frac{d}{dt} y(t) + 10 y(t) = 0$ with $y(t) = c e^{rt}$

characteristic equation : $5r + 10 = 0 \rightarrow r = -2$

$y(t) = c \cdot e^{-2t}$

I.C : $y(0) = 3 \longrightarrow c = 3$

$\therefore y(t) = 3e^{-2t}$

(b) $\frac{d^2}{dt^2} y(t) + 5 \frac{d}{dt} y(t) + 6 y(t) = 0$

characteristic equation : $r^2 + 5r + 6 = 0 \rightarrow r_1 = -2$
$r_2 = -3$

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

I.C (i) $y(0) = 2 \longrightarrow 2 = c_1 + c_2$ ....(1)

(ii) $\frac{d}{dt} y(0) = 1 \longrightarrow 1 = -2c_1 - 3c_2$ ....(2)

Solving (1)&(2) : $c_1 = 7$, $c_2 = -5$

$$\therefore y(t) = 7e^{-2t} - 5e^{-3t}$$

(c) $\frac{d^2}{dt^2} y(t) + 3\frac{d}{dt} y(t) + 2y(t) = 0$

Characteristic equation : $r^2 + 3r + 2 = 0 \longrightarrow r_1 = -1$, $r_2 = -2$

$$y(t) = c_1 e^{-t} + c_2 e^{-2t}$$

I.C (i) $y(0) = 0 \longrightarrow 0 = c_1 + c_2$ ....(1)

(ii) $\frac{d}{dt} y(0) = 1 \longrightarrow 1 = -c_1 - 2c_2$ ....(2)

Solving (1) and (2) : $c_1 = 1$, $c_2 = -1$

$$\therefore y(t) = e^{-t} - e^{-2t}$$

(d) $\frac{d^2}{dt^2} y(t) + 2\frac{d}{dt} y(t) + y(t) = 0$

Characteristic equation : $r^2 + 2r + 1 = 0 \longrightarrow r_1 = r_2 = -1$

$$y(t) = c_1 t e^{-t} + c_2 e^{-t}$$

I.C (i) $y(0) = 1 \longrightarrow 1 = c_2$

(ii) $\frac{d}{dt} y(0) = 1 \longrightarrow 1 = c_1 - c_2$

Solving : $c_1 = 2$, $c_2 = 1$

$\therefore \; y(t) = 2t\,e^{-t} + e^{-t}$

(e) $\frac{d^2}{dt^2} y(t) + 4y(t) = 0$

Characteristic equation : $r^2 + 4 = 0 \longrightarrow r = \pm j2$

$y(t) = c_1 \cos(2t) + c_2 \sin(2t)$

I.C : (i) $y(0) = 1 \longrightarrow -1 = c_1$
(ii) $\frac{d}{dt} y(0) = 1 \longrightarrow 1 = 2c_2$

Solving : $c_1 = -1$, $c_2 = 1/2$

$\therefore \; y(t) = -\cos(2t) + \frac{1}{2}\sin(2t)$

(f) $\frac{d^2}{dt^2} y(t) + 2\frac{d}{dt} y(t) + 2y(t) = 0$

Characteristic equation : $r^2 + 2r + 2 = 0 \longrightarrow r = -1 \pm j1$

$y(t) = e^{-t}\{c_1 \cos(t) + c_2 \sin(t)\}$

I.C : (i) $y(0) = 1 \longrightarrow 1 = c_1$
(ii) $\frac{d}{dt} y(0) = 1 \longrightarrow 0 = -c_1 + c_2$

Solving : $c_1 = 1$, $c_2 = 1$

$\therefore \; y(t) = e^{-t}\{\cos(t) + \sin(t)\}$

$\boxed{2.18}$

(a) $y[n] - \alpha y[n-1] = 0$

Characteristic equation : $1 - \alpha r^{-1} = 0 \rightarrow r = \alpha$

$\therefore y[n] = c(\alpha)^n$

I.C. : $y[-1] = 3 \rightarrow y[0] = \alpha(3) = 3\alpha \rightarrow 3\alpha = c(\alpha)^0 \rightarrow c = 3\alpha$

$\therefore y[n] = 3\alpha(\alpha)^n = 3(\alpha)^{n+1}$

(b) $y[n] - \frac{9}{10} y[n-2] = 0$

Characteristic equation : $1 - \frac{9}{10} r^{-2} = 0 \rightarrow r = \pm \frac{3}{4}$

$$y[n] = c_1\left(\tfrac{3}{4}\right)^n + c_2\left(-\tfrac{3}{4}\right)^n$$

I.C : $y[0] = \frac{9}{16} y[-2] = -\frac{9}{16}$; $y[-1] = \frac{9}{16} y[-1] = \frac{9}{16}$

$$\left.\begin{aligned} -\frac{9}{16} &= c_1 + c_2 \\ \frac{9}{16} &= \left(\tfrac{3}{4}\right)c_1 - \left(\tfrac{3}{4}\right)c_2 \end{aligned}\right\} \text{ solve } \quad \begin{aligned} c_1 &= 3/32 \\ c_2 &= -21/32 \end{aligned}$$

$$\therefore y[n] = \frac{3}{32}\left(\frac{3}{4}\right)^n - \frac{21}{32}\left(-\frac{3}{4}\right)^n$$

(c) $y[n] - \frac{1}{4} y[n-1] - \frac{1}{8} y[n-2] = 0$

Characteristic equation : $1 - \frac{1}{4} r^{-1} - \frac{1}{8} r^{-2} = 0 \rightarrow r_{1,2} = \frac{1}{2}, -\frac{1}{4}$

$$y[n] = c_1\left(\tfrac{1}{2}\right)^n + c_2\left(-\tfrac{1}{4}\right)^n$$

I.C . $y[0] = \frac{1}{4} y[1] + \frac{1}{8} y[-2] = \frac{1}{8}$

$y[1] = \frac{1}{4} y[0] + \frac{1}{8} y[-1] = \frac{1}{32}$

$$\left.\begin{aligned}\frac{1}{8} &= c_1 + c_2 \\ \frac{1}{32} &= c_1\left(\frac{1}{2}\right) - c_2\left(\frac{1}{4}\right)\end{aligned}\right\} \quad \begin{aligned} c_1 &= 1/12 \\ c_2 &= 1/24\end{aligned}$$

$$\therefore\ y[n] = \frac{1}{12}\left(\frac{1}{2}\right)^n + \frac{1}{24}\left(-\frac{1}{4}\right)^n$$

(d) $y[n] + \frac{9}{16}\, y[n-2] = 0$

Characteristic equation : $1 + \frac{9}{16} r^{-2} = 0 \longrightarrow r = \pm j\frac{3}{4}$

$$y[n] = c_1\left(j\frac{3}{4}\right)^n + c_2\left(-j\frac{3}{4}\right)^n$$

I.C : $y[0] = -\frac{9}{16}\, y[-2] = \frac{9}{16}$; $y[1] = -\frac{9}{16}\, y[-1] = -\frac{9}{16}$

$$\left.\begin{aligned}\frac{9}{16} &= c_1 + c_2 \\ -\frac{9}{16} &= c_1\left(j\frac{3}{4}\right) + c_2\left(-j\frac{3}{4}\right)\end{aligned}\right\} \quad \begin{aligned} c_1 &= \frac{9}{32} + j\frac{3}{8} \\ c_2 &= \frac{9}{32} - j\frac{3}{8}\end{aligned}$$

$$\therefore\ y[n] = \left(\frac{9}{32} + j\frac{3}{8}\right)\left(j\frac{3}{4}\right)^n + \left(\frac{9}{32} - j\frac{3}{8}\right)\left(-j\frac{3}{4}\right)^n$$

(e) $y[n] + y[n-1] + \frac{1}{2}\, y[n-2] = 0$

Characteristic equation : $1 + r^{-1} + \frac{1}{2} r^{-2} = 0 \longrightarrow r_{1,2} = \frac{-1 \pm j}{2}$

$$y[n] = c_1\left(\frac{-1+j}{2}\right)^n + c_2\left(\frac{-1-j}{2}\right)^n$$

I.C : $y[0] = -y[-1] - \frac{1}{2}\, y[-2] = 1 - \frac{1}{2} = \frac{1}{2}$; $y[1] = -\frac{1}{2} + \frac{1}{2} = 0$

$$\left.\begin{aligned}\frac{1}{2} &= c_1 + c_2 \\ 0 &= c_1\left(\frac{-1+j}{2}\right) + c_2\left(\frac{-1-j}{2}\right)\end{aligned}\right\} \quad \begin{aligned} c_1 &= \frac{1}{4}(1-j) \\ c_2 &= \frac{1}{4}(1+j)\end{aligned}$$

$$\therefore\ y[n] = \frac{1}{4}(1-j)\left(\frac{-1+j}{2}\right)^n + \frac{1}{4}(1+j)\left(\frac{-1-j}{2}\right)^n$$

$\boxed{2.19}$

(a) $5\frac{d}{dt}y(t) + 10\,y(t) = 2x(t)$

Natural, characteristic equation : $5r + 10 = 0 \rightarrow r = -2$

$y^n(t) = c\,e^{-2t}$

(i) $x(t) = 2\,u(t)$

Particular : $y^P(t) = K \longrightarrow 10K = 4 \rightarrow K = \frac{2}{5}$

Forced : $y^f(t) = \frac{2}{5} + c\,e^{-2t}$

use $y(0) = 0 \longrightarrow c = -\frac{2}{5}$

$\therefore\; y^f(t) = \frac{2}{5}\,(1 - e^{-2t})$

(ii) $x(t) = e^{-t}u(t)$

Particular : $y^P(t) = K\,e^{-t} \rightarrow -5K + 10K = 2$

$K = \frac{2}{5}$

Forced : $y^f(t) = \frac{2}{5}\,e^{-t} + c\,e^{-2t}$

use $y(0) = 0 \rightarrow 0 = \frac{2}{5} + c \rightarrow c = -\frac{2}{5}$

$\therefore\; y^f(t) = \frac{2}{5}\,(e^{-t} - e^{-2t})$

(iii) $x(t) = \cos(3t)\,u(t)$

Particular : $y^P(t) = K_1\cos(3t) + K_2\sin(3t)$

substituting will give : $K_1 = \frac{1}{10}$; $K_2 = \frac{1}{15}$

Forced : $y^f(t) = \frac{1}{10}\cos(3t) + \frac{1}{15}\sin(3t) + c\,e^{-2t}$

use $y(0) = 0 \rightarrow 0 = \frac{1}{10} + c \rightarrow c = -\frac{1}{10}$

2.19 con't

$$\therefore y^f(t) = \frac{1}{10}(\cos(3t)-1) + \frac{1}{15}\sin(3t)$$

(b) $\frac{d^2}{dt^2} y(t) + 5\frac{d}{dt} y(t) + 6y(t) = 2x(t) + \frac{d}{dt}x(t)$

Natural : Characteristic equation : $r^2 + 5r + 6 < 0$

$r = -2, -3$

$$y^n(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

(i) $x(t) = -2u(t)$

Particular : $y^P(t) = K \rightarrow 6K = -4 \rightarrow K = -\frac{2}{3}$

Forced : $y^f(t) = -\frac{2}{3} + c_1 e^{-2t} + c_2 e^{-3t}$

I.C : $\left.\begin{array}{l} y(0) = 0 \rightarrow \frac{7}{3} = c_1 + c_2 \\ y'(0) = 0 \rightarrow 0 = -2c_1 - 3c_2 \end{array}\right\}$ solve : $c_1 = 2$, $c_2 = -\frac{1}{3}$

$$\therefore y^f(t) = -\frac{1}{3} + 2e^{-2t} - \frac{1}{3}e^{-3t}$$

(ii) $x(t) = 2e^{-t}u(t)$ RHS $= 4e^{-t} - 2e^{-t} = 2e^{-t}$

Particular : $y^P(t) = K.e^{-t}$

substitution gives :

$K - 5K + 6K = 2 \rightarrow K = 1$

Forced : $y^f(t) = e^{-t} + c_1 e^{-2t} + c_2 e^{-3t}$

I.C : $\left.\begin{array}{l} y(0) = 0 \rightarrow 0 = 1 + c_1 + c_2 \\ y'(0) = 0 \rightarrow 0 = -1 - 2c_1 - 3c_2 \end{array}\right\}$ $c_1 = -2$, $c_2 = 1$

$$\therefore y^f(t) = e^{-t} - 2e^{-2t} + e^{-3t}$$

(iii) $x(t) = (\sin(3t))u(t)$
RHS $= 2\sin(3t) + 3\cos(3t)$

<u>Particular</u> : $y^P(t) = K_1\cos(3t) + K_2\sin(3t)$
substitution gives : $K_1 = -\frac{1}{6}$
$K_2 = \frac{1}{6}$

<u>Forced</u> : $y^f(t) = c_1e^{-2t} + c_2e^{-3t} - \frac{1}{6}\cos(3t) + \frac{1}{6}\sin(3t)$

I.C : $y(0) = 0 \rightarrow 0 = c_1 + c_2 - \frac{1}{6}$
$y'(0) = 0 \rightarrow 0 = -2c_1 - 3c_2 + \frac{1}{2}$ $\Big\}$ $c_1 = 0$, $c_2 = \frac{1}{6}$

$\therefore y^f(t) = \frac{1}{6}\left(e^{-3t} - \cos(3t) + \sin(3t)\right)$

(iv) $x(t) = 5e^{-2t}u(t)$
RHS : $10e^{-2t} - 10e^{-2t} = 0$

<u>Particular</u> : $y^P(t) = 0$

<u>Forced</u> : $y^f(t) = c_1e^{-2t} + c_2e^{-3t}$

I.C : $y(0) = 0 \longrightarrow 0 = c_1 + c_2$
$y'(0) = 0 \longrightarrow 0 = -2c_1 - 3c_2$ $\Big\}$ $c_1 = 0$, $c_2 = 0$

$\therefore y^f(t) = 0$

(c) $$\frac{d^2y(t)}{dt^2} + 3\frac{d}{dt}y(t) + 2y(t) = \underbrace{x(t) + \frac{d}{dt}x(t)}_{\text{RHS}}$$

<u>Natural</u> : Characteristic equation : $r^2 + 3r + 2 = 0$;
$r = -2, -1$

2.19 con't

$$y^n(t) = c_1 e^{-2t} + c_2 e^{-t}$$

(i) $x(t) = 5\,u(t)$ RHS $= 5$

Particular : $y^p(t) = K$
substitution gives $K = \frac{5}{2}$

Forced : $y^f(t) = \frac{5}{2} + c_1 e^{-2t} + c_2 e^{-t}$

I.C : $\left.\begin{array}{l} y(0) = 0 \rightarrow 0 = \frac{5}{2} + c_1 + c_2 \\ y'(0) = 0 \rightarrow 0 = -2c_1 - c_2 \end{array}\right\}$ $\begin{array}{l} c_1 = 2.5 \\ c_2 = -5 \end{array}$

$\therefore y^f(t) = 2.5\left(1 + e^{-2t} - 2e^{-t}\right)$

(ii) $x(t) = e^{2t}\,u(t)$ RHS $= 3e^{2t}$

Particular : $y^p(t) = K e^{2t}$
substitution gives $K = \frac{1}{4}$

Forced : $y^f(t) = \frac{1}{4} e^{2t} + c_1 e^{-2t} + c_2 e^{-t}$

I.C : $\left.\begin{array}{l} y(0) = 0 \rightarrow 0 = \frac{1}{4} + c_1 + c_2 \\ y'(0) = 0 \rightarrow 0 = \frac{1}{2} - 2c_1 - c_2 \end{array}\right\}$ $\begin{array}{l} c_1 = 3/4 \\ c_2 = -1 \end{array}$

$\therefore y^f(t) = \frac{1}{4} e^{2t} + \frac{3}{4} e^{-2t} - e^{-t}$

(iii) $x(t) = \cos(t) + \sin(t)$ RHS $= 2\cos(t)$

Particular : $y^p(t) = K_1 \cos(t) + K_2 \sin(t)$
substitution gives $K_1 = 1$
$K_2 = \frac{1}{3}$

<u>Forced</u> : $y^f(t) = \cos(t) + \frac{1}{3}\sin(t) + c_1 e^{-2t} + c_2 e^{-t}$

I.C : $y(0) = 0 \rightarrow 0 = 1 + c_1 + c_2$, $y'(0) = 0 \rightarrow 0 = \frac{1}{3} - 2c_1 - c_2$ $\Big\}$ $c_1 = \frac{4}{3}$, $c_2 = -\frac{7}{3}$

$\therefore y^f(t) = \cos(t) + \frac{1}{3}\sin(t) + \frac{4}{3}e^{-2t} - \frac{7}{3}e^{-t}$

(iv) $x(t) = e^{-t}u(t)$ RHS $= 0$

<u>Particular</u> : $y^p(t) = 0$ since RHS $= 0$
<u>Forced</u> : $y^f(t) = 0$

(d) $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = \frac{d}{dt}x(t)$

<u>Natural</u> : Characteristic equation : $r^2 + 2r + 1 = 0$
; $r_1 = r_2 = -1$

$y^n(t) = (c_1 t + c_2)e^{-t}$

(i) $x(t) = e^{-3t}u(t)$ RHS $= -3e^{-3t}$

<u>Particular</u> : $y^p(t) = Ke^{-3t}$
substitution gives : $K = -\frac{3}{4}$

<u>Forced</u> : $y^f(t) = -\frac{3}{4}e^{-3t} + (c_1 t + c_2)e^{-t}$

I.C : $y(0) = 0 \longrightarrow 0 = -\frac{3}{4} + c_2$, $y'(0) = 0 \longrightarrow 0 = \frac{9}{4} + c_1 - c_2$ $\Big]$ $c_1 = -\frac{3}{2}$, $c_2 = \frac{3}{4}$

$\therefore y^f(t) = -\frac{3}{4}e^{-3t} + \left(\frac{3}{4} - \frac{3}{2}t\right)e^{-t}$

(ii) $x(t) = 2e^{-t}u(t)$ RHS $= -2e^{-t}$
<u>Particular</u> : $y^p(t) = Kt^2e^{-t}$

since $r=-1$ is a double-root of the characteristic equation

substitution gives $K=-1$

Forced : $y^f(t) = -t^2 e^{-t} + (c_1 t + c_2)e^{-t}$

$$\left.\begin{array}{l} \text{I.C} : y(0) = 0 \longrightarrow 0 = c_2 \\ \quad\;\; y'(0) = 0 \longrightarrow 0 = c_1 - c_2 \end{array}\right\} c_1 = c_2 = 0$$

$\therefore\ y^f(t) = -t^2 e^{-t}$

(iii) $x(t)\ 2\sin(t)$ RHS $= 2\cos(t)$

Particular : $y^p(t) = K_1 \cos(t) + K_2 \sin(t)$

substitution gives : $K_1 = 0$

$K_2 = 1$

Forced : $y^f(t) = \sin(t) + (c_1 t + c_2)e^{-t}$

$$\left.\begin{array}{l} \text{I.C} : y(0) = 0 \longrightarrow 0 = c_2 \\ \quad\;\; y'(0) = 0 \longrightarrow 0 = 1 - c_2 + c_1 \end{array}\right\} \begin{array}{l} c_1 = -1 \\ c_2 = 0 \end{array}$$

$\therefore\ y^f(t) = \sin(t) - t e^{-t}$

$\boxed{2.20}$

(a) $y[n] - \frac{2}{5} y[n-1] = 2x[n]$

Natural : $x[n] = 0$ Charateristic equation :

$1 - \frac{2}{5} r^{-1} = 0\ ,\ r = \frac{2}{5}$

$y^n[n] = c\left(\frac{2}{5}\right)^n$

2.20 con't

(i) $x[n] = 2u[n]$ RHS $= 4.u[n]$

<u>Particular</u> : $y^p[n] = K u[n]$
using $n \geq 1$, $\frac{3}{5} K = 4 \rightarrow K = \frac{20}{3}$

<u>Forced</u> : $y^f[n] = \frac{20}{3} + c\left(\frac{2}{5}\right)^n$

I.C : $y[0] = \frac{2}{5} y[-1] + 2x[0]$

$y[0] = 4$

$4 = \frac{20}{3} + c \rightarrow c = -\frac{8}{3}$

$\therefore y^f[n] = \frac{20}{3} - \frac{8}{3}\left(\frac{2}{5}\right)^n, \quad n \geq 0$

(ii) $x[n] = -\left(\frac{1}{12}\right)^n u[n]$ RHS $= -2\left(\frac{1}{2}\right)^n u[n]$

<u>Particular</u> : $y^p[n] = K\left(\frac{1}{2}\right)^n$
$K\left(\frac{1}{2}\right)^n - \frac{2}{5} K \left(\frac{1}{2}\right)^{n-1} = -2\left(\frac{1}{2}\right)^n$
$K = -10$

<u>Forced</u> : $y^f[n] = c\left(\frac{2}{5}\right)^n - 10\left(\frac{1}{2}\right)^n$

I.C : $y[-1] = 0 \rightarrow y[0] = -2 = c - 10$
$c = 8$

$\therefore y^f[n] = 8\left(\frac{2}{5}\right)^n - 10\left(\frac{1}{2}\right)^n, \quad n \geq 0$

(iii) $x[n] = \cos\left(\frac{\pi}{5} n\right)$ RHS : $2\cos\left(\frac{\pi}{5} n\right)$

<u>Particular</u> : $y^p[n] = K_1 \cos\left(\frac{\pi}{5} n\right) + K_2 \sin\left(\frac{\pi}{5} n\right)$
substitution gives : $K_1 = 2.638$; $K_2 = 0.917$

2.20 con't

Forced : $y^f[n] = c\left(\frac{2}{5}\right)^n + 2.638\cos\left(\frac{\pi}{5}n\right) + 0.917\sin\left(\frac{\pi}{5}n\right)$

I.C : $y[0] = 2$

$2 = c + 2.638$

$c = -0,638$

$\therefore y^f[n] = -0,638\left(\frac{2}{5}\right)^n + 2.638\cos\left(\frac{\pi}{5}n\right)$

$+ 0.917\sin\left(\frac{\pi}{5}n\right), \quad n \geq 0$

(b) $y[n] - \frac{9}{16}y[n-2] = x[n-1]$

Natural : Characteristic equation : $1 - \frac{9}{16}r^{-2} = 0$;

$r_{1,2} = \pm\frac{3}{4}$

$y^n[n] = c_1\left(\frac{3}{4}\right)^n + c_2\left(-\frac{3}{4}\right)^n$

(i) $x[n] = u[n] \longrightarrow$ RHS $= u[n-1]$

Particular : $y^p[n] = K\,u[n]$

for $n \geq 2$, we can calculate K as follow :

$\left(1 - \frac{9}{16}\right)K = 1 \rightarrow K = \frac{16}{7}$

Forced : $y^f[n] = \frac{16}{7} + c_1\left(\frac{3}{4}\right)^n + c_2\left(-\frac{3}{4}\right)^n$

$y[-1] = y[-2] = 0$; $y[0] = 0 + 0 = 0$

$y[1] = 0 + 1 = 1$

$$\left.\begin{aligned} 0 &= \frac{16}{7} + c_1 + c_2 \\ 1 &= \frac{16}{7} + c_1\left(\frac{3}{4}\right) - c_2\left(\frac{3}{4}\right) \end{aligned}\right\} \begin{aligned} c_1 &= -2 \\ c_2 &= -0.2857 \end{aligned}$$

$\therefore y^f[n] = \frac{16}{7} - 2(0.75)^n - 0.2857(-0.75)^n, \quad n \geq 0$

2.20 con't

(ii) $x[n] = -\left(\frac{1}{2}\right)^n u[n] \longrightarrow \text{RHS} = -\left(\frac{1}{2}\right)^{n-1} u[n-1]$

<u>Particular</u> : $y^P[n] = K\left(\frac{1}{2}\right)^n u[n]$
for $n \geq 2$, $K\left(\frac{1}{2}\right)^n - \frac{9}{16} K\left(\frac{1}{2}\right)^{n-2} = -\left(\frac{1}{2}\right)^{n-1}$

$$\left(1 - 4 \times \frac{9}{16}\right) K = -2$$
$$K = \frac{8}{5}$$

<u>Forced</u> : $y^f[n] = \frac{8}{5}\left(\frac{1}{2}\right)^n + c_1\left(\frac{3}{4}\right)^n + c_2\left(-\frac{3}{4}\right)^n$

I.C : $\left.\begin{array}{l} y[0] = 0 \\ y[1] = -1 \end{array}\right\} \begin{array}{l} 0 = \frac{8}{5} + c_1 + c_2 \\ -1 = \frac{8}{5}\left(\frac{1}{2}\right) + c_1\left(\frac{3}{4}\right) - c_2\left(\frac{3}{4}\right) \end{array}$

solve : $c_1 = -2$; $c_2 = \frac{2}{5}$

$\therefore y^f[n] = \frac{4}{5}\left(\frac{1}{2}\right)^n - 2\left(\frac{3}{4}\right)^n + \frac{2}{5}\left(-\frac{3}{4}\right)^n$, $n \geq 0$

(iii) $x[n] = \left(\frac{3}{4}\right)^n u[n] \longrightarrow \text{RHS} = \left(\frac{3}{4}\right)^{n-1} u[n-1]$

<u>Particular</u> : $y^P[n] = K n \left(\frac{3}{4}\right)^n u[n]$
since $\left(\frac{3}{4}\right)^n$ is included in natural respons
for $n \geq 2$, substitution gives $K = \frac{2}{3}$

<u>Forced</u> : $y^f[n] = \frac{2}{3} n\left(\frac{3}{4}\right)^n + c_1\left(\frac{3}{4}\right)^n + c_2\left(-\frac{3}{4}\right)^n$

I.C : $y[0] = 0$, $y[1] = 1$

$$\left.\begin{array}{l} 0 = c_1 + c_2 \\ 1 = \frac{1}{2} + c_1\left(\frac{3}{4}\right) - c_2\left(\frac{3}{4}\right) \end{array}\right\} \begin{array}{l} c_1 = \frac{1}{3} \\ c_2 = -\frac{1}{3} \end{array}$$

$\therefore y^f[n] = \frac{2}{3} n\left(\frac{3}{4}\right)^n + \frac{1}{3}\left(\frac{3}{4}\right)^n - \frac{1}{3}\left(-\frac{3}{4}\right)^n$, $n \geq 0$

(c) $y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] + x[n-1]$

Natural : Characteristic equation : $1 - \frac{1}{4}r^{-1} - \frac{1}{8}r^{-2} = 0$

$; \; r = -\frac{1}{4}, \frac{1}{2}$

$$y^n[n] = c_1\left(-\tfrac{1}{4}\right)^n + c_2\left(\tfrac{1}{2}\right)^n$$

(i) $x[n] = -2u[n] \longrightarrow \text{RHS} = -2u[n] - 2u[n-1]$

Particular : $y^P[n] = K\,u[n]$
for $n \geq 2 : \left(1 - \frac{1}{4} - \frac{1}{8}\right)K = 2$;
$K = \frac{16}{5}$

Forced : $y^f[n] = \frac{16}{5} + c_1\left(-\frac{1}{4}\right)^n + c_2\left(\frac{1}{2}\right)^n$

I.C : $y[0] = 0 + 0 + 1 = 1$
$y[1] = 0 + \frac{1}{4}(1) + 2 = \frac{9}{4}$

$$\left.\begin{aligned} 1 &= \tfrac{16}{5} + c_1 + c_2 \\ \tfrac{9}{4} &= \tfrac{16}{5} + c_1\left(-\tfrac{1}{4}\right) + c_2\left(\tfrac{1}{2}\right) \end{aligned}\right\} \quad \begin{aligned} c_1 &= -\tfrac{1}{5} \\ c_2 &= -2 \end{aligned}$$

$$\therefore y^f[n] = \frac{16}{5} - \frac{1}{5}\left(-\frac{1}{4}\right)^n - 2\left(\frac{1}{2}\right)^n, \quad n \geq 0$$

(ii) $x[n] = \left(\frac{1}{8}\right)^n u[n] \longrightarrow \text{RHS} = \left(\frac{1}{8}\right)^n u[n] + \left(\frac{1}{8}\right)^{n-1} u[n-1]$

Particular : $y^P[n] = K\left(\frac{1}{8}\right)^n u[n]$
for $n \geq 2$; substitution gives $K = -1$

Forced : $y^f[n] = -\left(\frac{1}{8}\right)^n + c_1\left(-\frac{1}{4}\right)^n + c_2\left(\frac{1}{2}\right)^n$

I.C : $y[0] = 0 + 0 + 1 = 1$

$$y[1] = 0 + \frac{1}{4} + 1 + \frac{1}{8} = \frac{11}{8}$$

$$\left.\begin{aligned} 1 &= -1 + c_1 + c_2 \\ \frac{11}{8} &= -\frac{1}{8} + c_1\left(-\frac{1}{4}\right) + c_2\left(\frac{1}{2}\right) \end{aligned}\right\} \quad \begin{aligned} c_1 &= -\frac{2}{3} \\ c_2 &= \frac{8}{3} \end{aligned}$$

$$\therefore y^f[n] = -\left(\frac{1}{8}\right)^n - \frac{2}{3}\left(-\frac{1}{4}\right)^n + \frac{8}{3}\left(\frac{1}{2}\right)^n, \; n \geq 0$$

(iii) $x[n]\, e^{j\frac{\pi}{4}n}\, u[n] \longrightarrow \text{RHS} = e^{j\frac{\pi}{4}n} u[n] + e^{j\frac{\pi}{4}(n-1)} u[n-1]$

Particular : $y^p[n] = K\, e^{j\frac{\pi}{4}n}\, u[n]$

for $n \geq 2$, substitution gives

$K = 0.959 - j\,2.070$

Forced : $y^f[n] = K e^{j\frac{\pi}{4}n} + c_1\left(-\frac{1}{4}\right)^n + c_2\left(\frac{1}{2}\right)^n$

I.C :

$y[0] = 0 + 0 + 1 = 1$

$y[1] = 0 + \frac{1}{4} + 1 + e^{j\frac{\pi}{4}} = \frac{5}{4} + e^{j\frac{\pi}{4}}$

$$\left.\begin{aligned} 1 &= K + c_1 + c_2 \\ \frac{5}{4} + e^{j\frac{\pi}{4}} &= K e^{j\frac{\pi}{4}} + c_1\left(-\frac{1}{4}\right) + c_2 \frac{1}{2} \end{aligned}\right\}$$

solve : $c_1 = 0.273 - j0.610$

$c_2 = -0.233 + j\,2.679$

$$\begin{aligned} \therefore y^f[n] = {} & (0.959 - j\,2.070)\, e^{j\frac{\pi}{4}n} \\ & + (0.273 - j0.610)\left(-\frac{1}{4}\right)^n \\ & + (-0.233 + j2.679)\left(\frac{1}{2}\right)^n, \; n \geq 0 \end{aligned}$$

(iv) $x[n] = \left(\frac{1}{2}\right)^n u[n] \longrightarrow \text{RHS} = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-1} u[n-1]$

<u>Particular</u> : $y^p[n] = K n(\frac{1}{2})^n u[n]$
for $n \geq 2$, substitution gives $K = 1$

<u>Forced</u> : $y^f[n] = n(\frac{1}{2})^n + c_1(-\frac{1}{4})^n + c_2(\frac{1}{2})^n$

I.C : $y[0] = 1$
$y[1] = 0 + \frac{1}{4} + 1 + \frac{1}{2} = \frac{7}{4}$

$$\left.\begin{aligned} 1 &= c_1 + c_2 \\ \frac{7}{4} &= \frac{1}{2} - \frac{1}{4}c_1 + \frac{1}{2}c_2 \end{aligned}\right\} \begin{aligned} c_1 &= -1 \\ c_2 &= 2 \end{aligned}$$

$$\therefore y^f[n] = n(\tfrac{1}{2})^n - (-\tfrac{1}{4})^n + 2(\tfrac{1}{2})^n, \ n \geq 0$$

(d) $y[n] + y[n-1] + \frac{1}{2}y[n-2] = x[n] + 2x[n-1]$

Natural : Characteristic equation :
$1 + r^{-1} + \frac{1}{2}r^{-2} = 0 \rightarrow r_{1,2} = \frac{-1 \pm j}{2}$

$$y^n[n] = c_1\left(\frac{-1+j}{2}\right)^n + c_2\left(\frac{-1-j}{2}\right)^n$$

(i) $x[n] = u[n] \longrightarrow RHS = u[n] + 2u[n-1]$

<u>Particular</u> : $y^p[n] = K u[n]$
$n \geq 2$, substitution gives $K = \frac{6}{5}$

<u>Forced</u> : $y^f[n] = \frac{6}{5} + c_1\left(\frac{-1+j}{2}\right)^n + c_2\left(\frac{-1-j}{2}\right)^n$

I.C : $y[0] = 1$
$y[1] = -1 + 3 = 2$

$$\left.\begin{aligned} 1 &= \frac{6}{5} + c_1 + c_2 \\ 2 &= \frac{6}{5} + c_1\left(-\frac{1+j}{2}\right) + c_2\left(-\frac{1-j}{2}\right) \end{aligned}\right\}$$

Solve : $c_1 = -\frac{1}{10} - j\frac{7}{10}$; $c_2 = -\frac{1}{10} + j\frac{7}{10}$

$$\therefore y^f[n] = \frac{6}{5} + \left(-\frac{1}{10} - j\frac{7}{10}\right)\left(\frac{-1+j}{2}\right)^n + \left(-\frac{1}{10} + j\frac{7}{10}\right)\left(\frac{-1-j}{2}\right)^n$$

$, n \geqslant 0$

(ii) $x[n] = \left(-\frac{1}{2}\right)^n u[n] \longrightarrow RHS = \left(-\frac{1}{2}\right)^n u[n] + 2\left(-\frac{1}{2}\right)^{n-1} u[n-1]$

Particular : $y^p[n] = K\left(-\frac{1}{2}\right)^n$

substitution gives $K = -3$

Forced : $y^f[n] = c_1\left(\frac{-1+j}{2}\right)^n + c_2\left(\frac{-1-j}{2}\right)^n - 3\left(-\frac{1}{2}\right)^n$

I.C : $y[0] = 1$

$y[1] = -1 + 1 + 2\left(-\frac{1}{2}\right) = -1$

$$\left.\begin{aligned} 1 &= c_1 + c_2 - 3 \\ -1 &= c_1\left(\frac{-1+j}{2}\right) + c_2\left(\frac{-1-j}{2}\right) + \frac{3}{2} \end{aligned}\right\} \begin{aligned} c_1 &= 2 + j\,0.5 \\ c_2 &= 2 - j\,0.5 \end{aligned}$$

$$\therefore y^f[n] = (2 + j0.5)\left(\frac{1+j}{2}\right)^n + (2 - j0.5)\left(\frac{-1-j}{2}\right)^n - 3\left(-\frac{1}{2}\right)^n$$

$, n \geqslant 0$

$\boxed{2.21}$

(a) $\frac{d}{dt} y(t) + 10\, y(t) = 2\, x(t)$, $y(0) = 1$, $x(t) = u(t)$

$t \geqslant 0$: natural : Characteristic equation :

$r + 10 = 0 \longrightarrow r = -10$

$y^n(t) = c.\, e^{-10t}$

particular : $RHS = 2\, u(t)$

$y^p(t) = K\, u(t)$

by substitution we have $K = \frac{1}{5}$

$$y(t) = \frac{1}{5} + c.e^{-10t}$$

I.C: $y(0) = 1$: $1 = \frac{1}{5} + c \longrightarrow c = \frac{4}{5}$

$$\therefore y(t) = \frac{1}{5}(1 + 4e^{-10t}) \quad , t \geqslant 0$$

(b) $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 4y = \dfrac{dx}{dt}$ $\qquad y(0) = 0$, $\dot{y}(0) = 1$,

$x(t) = e^{-2t}u(t)$

$t \geqslant 0$: natural , Characteristic Equation :

$r^2 + 5r + 4 = 0 \longrightarrow r = -4, -1$

$y^n(t) = C_1 e^{-4t} + C_2 e^{-t}$

particular : RHS $= -2e^{-2t}$

$y^P(t) = Ke^{-2t}$

by substitution we have : $K = 1$

$$y(t) = e^{-2t} + C_1 e^{-4t} + C_2 e^{-t}$$

IC : $\left.\begin{array}{l} y(0) = 0 \longrightarrow 0 = 1 + C_1 + C_2 \\ \dot{y}(0) = 1 \longrightarrow 1 = -2 - 4C_1 - C_2 \end{array}\right\} \begin{array}{l} C_1 = -1/3 \\ C_2 = -2/3 \end{array}$

$$\therefore y(t) = e^{-2t} - \frac{1}{3}(e^{-4t} + 2e^{-t}) \quad , t \geqslant 0$$

(c) $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 2x$

$y(0) = -1$, $\dot{y}(0) = 1$, $x(t) = \cos(t)\, u(t)$

$t \geqslant 0$ natural : Characteristic equation :

$$r^2 + 3r + 2 = 0 \longrightarrow r = -2, -1$$
$$y^n(t) = c_1 e^{-2t} + c_2 e^{-t}$$

<u>Particular</u> : RHS $= 2\cos(t)\,u(t)$

$$y^p(t) = K_1 \cos(t) + K_2 \sin(t)$$
by substitution we have : $K_1 = \frac{1}{5}$, $K_2 = \frac{3}{5}$

$$\Rightarrow \quad y(t) = \frac{1}{5}\cos(t) + \frac{3}{5}\sin(t) + c_1 e^{-2t} + c_2 e^{-t}$$

$$\text{IC} : \begin{array}{ll} y(0) = -1 : & -1 = \frac{1}{5} + c_1 + c_2 \\ \dot{y}(0) = 1 : & 1 = \frac{3}{5} - 2c_1 - c_2 \end{array} \Big\} \quad \begin{array}{l} c_1 = \frac{4}{5} \\ c_2 = -\frac{1}{5} \end{array}$$

$$\therefore\ y(t) = \frac{1}{5}\left(\cos(t) + 3\sin(t) + 4e^{-2t} - e^{-t}\right), \quad t \geq 0$$

(d) $$\frac{d^2y}{dt^2} + y = 3\frac{dx}{dt}$$

$y(0) = -1$, $\dot{y}(0) = 1$, $x(t) = 2e^{-t}u(t)$

$t \geq 0$, <u>natural</u> : Characteristic equation : $r^2 + 1 = 0$
$$r = \pm j$$
$$y^n(t) = c_1 \cos(t) + c_2 \sin(t)$$

<u>particular</u> : RHS $= -6e^{-t}\,u(t)$
$$y^p(t) = K e^{-t}$$
by substitution we have : $K = -3$

$$\Rightarrow y(t) = -3e^{-t} + c_1\cos(t) + c_2 \sin(t)$$

$$\text{IC} : \begin{array}{ll} y(0) = -1 : & -1 = -3 + c_1 \\ \dot{y}(0) = 1 : & 1 = 3 + c_2 \end{array} \Big\} \quad \begin{array}{l} c_1 = 2 \\ c_2 = -2 \end{array}$$

$$\therefore y(t) = -3e^{-t} + 2(\cos(t) - \sin(t)), \quad t \geq 0$$

$\boxed{2.22}$

(a) $y[n] - \frac{1}{2} y[n-1] = 2x[n]$, $y[-1] = 3$, $x[n] = 2(-\frac{1}{2})^n u[n]$

Natural : Characteristic equation : $1 - \frac{1}{2} r^{-1} = 0$

$$r = \frac{1}{2}$$

$$y^n[n] = c\left(\frac{1}{2}\right)^n$$

Particular : RHS $= 4\left(-\frac{1}{2}\right)^n u[n]$

$n \geq 1$: $y^P[n] = K\left(-\frac{1}{2}\right)^n u[n]$

substitution gives : $K = 2$

$$\Longrightarrow y[n] = 2\left(-\frac{1}{2}\right)^n + c\left(\frac{1}{2}\right)^n$$

IC : $y[0] = \frac{1}{2} y[-1] + 2x[0] = \frac{1}{2}(3) + 2(2) = \frac{11}{2}$

$$\frac{11}{2} = 2 + c$$

$$c = \frac{7}{2}$$

$$\therefore y[n] = 2\left(-\frac{1}{2}\right)^n + \frac{7}{2}\left(\frac{1}{2}\right)^n, \quad n \geq 0$$

(b) $y[n] - \frac{1}{9} y[n-2] = x[n-1]$, $y[-1] = 1$, $y[-2] = 0$,

$x[n] = u[n]$

Natural : $1 - \frac{1}{9} r^{-2} = 0 \rightarrow r = \pm \frac{1}{3}$

$$y^n[n] = c_1\left(\frac{1}{3}\right)^2 + c_2\left(-\frac{1}{3}\right)^n$$

Particular : RHS $= u[n-1]$

$n \geq 2$: $y^P[n] = K u[n]$

by substitution we have : $K = \frac{9}{8}$

$$\Rightarrow y[n] = \frac{9}{8} + c_1\left(\frac{1}{3}\right)^n + c_2\left(-\frac{1}{3}\right)^n$$

IC : $y[0] = \frac{1}{9}\, y[-2] + x[-1] = 0$

$y[-1] = \frac{1}{9}\, y[-1] + x[0] = \frac{1}{9} + 1 = \frac{10}{9}$

$$\left.\begin{aligned} 0 &= \frac{9}{8} + c_1 + c_2 \\ \frac{10}{9} &= \frac{9}{8} + c_1\left(\frac{1}{3}\right) - c_2\left(\frac{1}{3}\right) \end{aligned}\right\} \quad \begin{aligned} c_1 &= -\frac{7}{12} \\ c_2 &= -\frac{13}{24} \end{aligned}$$

$$\therefore y[n] = \frac{9}{8} - \frac{7}{12}\left(\frac{1}{3}\right)^n - \frac{13}{24}\left(-\frac{1}{3}\right)^n, \quad n \geq 0$$

(c) $y[n] - \frac{1}{4}\, y[n-1] - \frac{1}{8}\, y[n-2] = x[n] + x[n-1]$

$y[-1] = 2,\ y[-2] = -1,\ x[n] = 2^n u[n]$

<u>Natural</u> : $1 - \frac{1}{4} r^{-1} - \frac{1}{8} r^{-2} = 0 \longrightarrow r = \frac{1}{2}, -\frac{1}{4}$

$y^n[n] = c_1\left(\frac{1}{2}\right)^n + c_2\left(-\frac{1}{4}\right)^n$

<u>Particular</u> : RHS $= 2^n u[n] + 2^{n-1} u[n-1]$
$n \geq 2$: $y^P[n] = K\, 2^n u[n]$
substitution gives : $K = \frac{16}{9}$

$$\Rightarrow y[n] = \frac{16}{9} \cdot 2^n + c_1\left(\frac{1}{2}\right)^n + c_2\left(-\frac{1}{4}\right)^n$$

IC : $y[0] = \frac{1}{8}\, y[-2] + \frac{1}{4}\, y[-1] + x[0] = -\frac{1}{8} + \frac{1}{2} + 1 = \frac{11}{8}$

$y[1] = \frac{1}{8}\, y[-1] + \frac{1}{4}\, y[0] + x[1] + x[0]$

$= \frac{1}{4} + \frac{11}{32} + 2 + 1 = \frac{115}{32}$

$$\left.\begin{aligned} \frac{11}{8} &= \frac{16}{9} + c_1 + c_2 \\ \frac{115}{32} &= \frac{32}{9} + c_1\left(\frac{1}{2}\right) - c_2\left(\frac{1}{4}\right) \end{aligned}\right\} \quad \begin{aligned} c_1 &= -\frac{1}{12} \\ c_2 &= -\frac{23}{72} \end{aligned}$$

$$\therefore\ y[n] = \frac{16}{9}\cdot 2^n - \frac{1}{12}\left(\frac{1}{2}\right)^n - \frac{23}{72}\left(-\frac{1}{4}\right)^n \quad , \ n \geqslant 0$$

(d) $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$

$$y[-1] = 1 \ , \ y[-2] = -1 \ , \ x[n] = 2u[n]$$

<u>Natural</u> : $1 - \frac{3}{4}r^{-1} + \frac{1}{8}r^{-2} = 0 \rightarrow r = \frac{1}{2}, \frac{1}{4}$

$$\therefore\ y^n(n) = c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{4}\right)^2$$

<u>Particular</u> : RHS $= 4u[n]$
$n \geqslant 2$: $y^p[n] = K u[n]$
substitution gives : $\left(1 - \frac{3}{4} + \frac{1}{8}\right)K = 4$
$K = \frac{32}{3}$

$$\Longrightarrow y[n] = \frac{32}{3} + c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{4}\right)^n$$

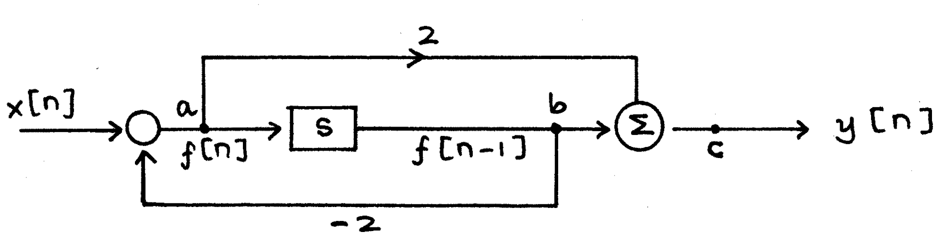
IC : $y[0] = \frac{3}{4}(1) - \frac{1}{8}(-1) + 4 = 4\frac{7}{8}$

$$y[1] = \frac{3}{4}\left(4\frac{5}{8}\right] - \frac{1}{8}(1) + 4 = \frac{241}{32}$$

$$\left.\begin{aligned} 4\frac{7}{8} &= \frac{32}{3} + c_1 + c_2 \\ \frac{241}{32} &= \frac{32}{3} + c_1\left(\frac{1}{2}\right) + c_2\left(\frac{1}{4}\right) \end{aligned}\right\} \quad \begin{aligned} c_1 &= -6\frac{3}{4} \\ c_2 &= \frac{23}{24} \end{aligned}$$

$$\therefore\ y[n] = \frac{32}{3} - \frac{27}{4}\left(\frac{1}{2}\right)^n + \frac{23}{24}\cdot\left(\frac{1}{4}\right)^n \quad , \ n \geqslant 0$$

$\boxed{2.23}$
(a)



Node a : $f[n] = -2f[n-1] + x[n]$ .... (1)

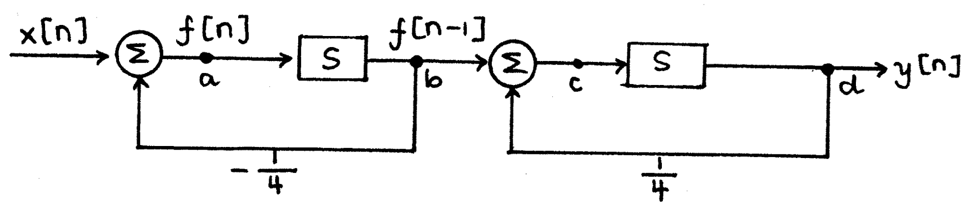
Node c : $y[n] = 2f[n] + f[n-1]$ .... (2)

This is a "direct form II" implementation which is equivalent to (interchange the cascading order)

$$y[n] = -2y[n-1] + 2x[n] + x[n-1]$$

$$\therefore y[n] + 2y[n-1] = 2x[n] + x[n-1]$$

(b)



Node a : $f[n] = x[n] - \frac{1}{4} f[n-1]$ ....(1)

Node c : $y[n] = f[n-2] + y[n-1]$ ....(2)

To have a causal system, we choose to modify (1) and (2) :

(1) $f[n-1] = x[n-1] - \frac{1}{4} f[n-2]$

$f[n-2] = x[n-2] - \frac{1}{4} f[n-3]$ ....(3)

(2) $f[n-2] = y[n] - y[n-1]$

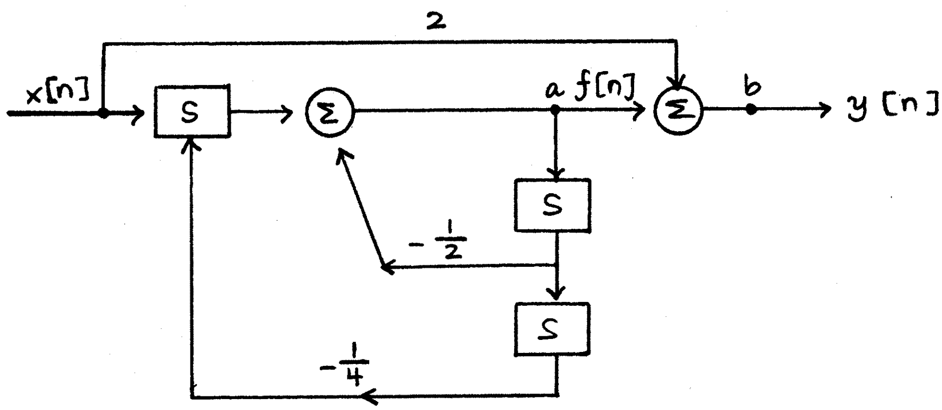
$f[n-3] = y[n-1] - y[n-2]$ ....(4)

(3) and (4) :

$$\left(y[n] - y[n-1]\right) = x[n-2] - \frac{1}{4}\left(y[n-1] - y[n-2]\right)$$

$$\therefore y[n] - \frac{3}{4} y[n-2] - \frac{1}{4} y[n-2] = x[n-2]$$

(c)



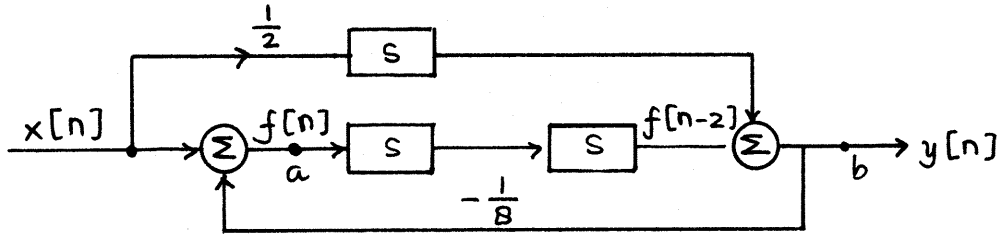
$$\underline{\text{Node } a} : f[n] = x[n-1] - \tfrac{1}{2} f[n-1] - \tfrac{1}{4} f[n-2] \quad \ldots\ldots(1)$$
$$\underline{\text{Node } b} : y[n] = f[n] + 2x[n] \quad \ldots\ldots(2)$$

(2)
$$\begin{aligned} f[n] &= y[n] - 2x[n] \\ f[n-1] &= y[n-1] - 2x[n-1] \\ f[n-2] &= y[n-2] - 2x[n-2] \end{aligned}$$

$$\begin{aligned} y[n] - 2x[n] &= x[n-1] - \tfrac{1}{2}\left(y[n-1] - 2x[n-1]\right) \\ &\quad - \tfrac{1}{4}\left(y[n-2] - 2x[n-2]\right) \end{aligned}$$

$$\therefore\; y[n] + \tfrac{1}{2} y[n-1] + \tfrac{1}{4} y[n-2] = 2x[n] + 2x[n-1] + \tfrac{1}{2} x[n-2]$$

(d)



$$\text{Node } a : f[n] = x[n] - \tfrac{1}{8} y[n] \quad \ldots\ldots(1)$$
$$\text{Node } b : y[n] = \tfrac{1}{2} x[n-1] + f[n-2] \quad \ldots\ldots(2)$$

(1) $f[n-2] = x[n-2] - \frac{1}{8} y[n-2]$

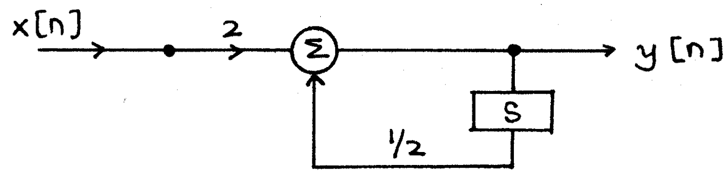
$$\left(y[n]-\frac{1}{2}x[n-1]\right) = x[n-2]-\frac{1}{8}y[n-2]$$

$$\therefore\ y[n]+\frac{1}{8}y[n-2] = \frac{1}{2}x[n-1]+x[n-2]$$

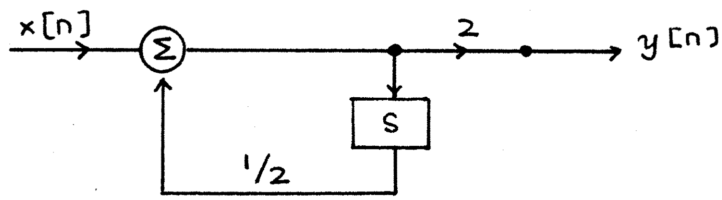
$\boxed{2.24}$

(a) $y[n]-\frac{1}{2}y[n-1] = 2x[n]$

Direct from I : $y[n] = 2x[n]+\frac{1}{2}y[n-1]$

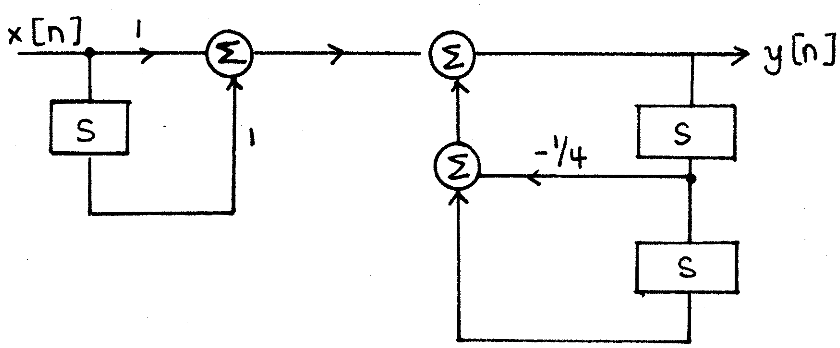


Direct from II :

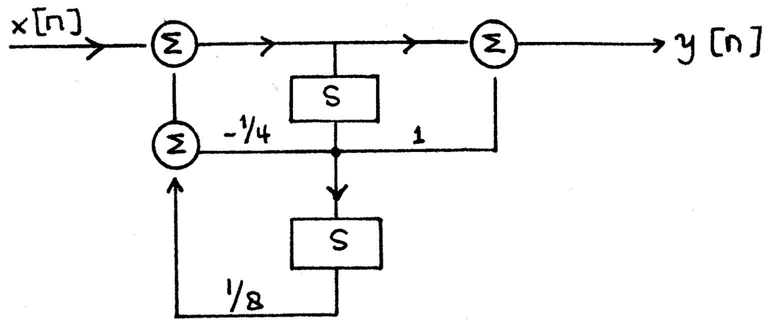


(b) $y[n]+\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2] = x[n]+x[n-1]$

Direct from I : $y[n] = x[n]+x[n-1]-\frac{1}{4}y[n-1]+\frac{1}{8}y[n-2]$

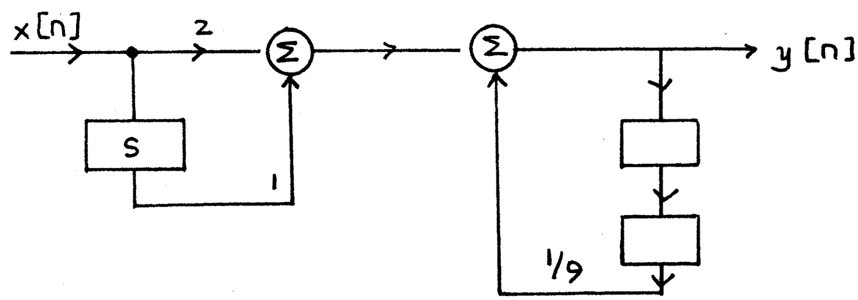


Direct from II :

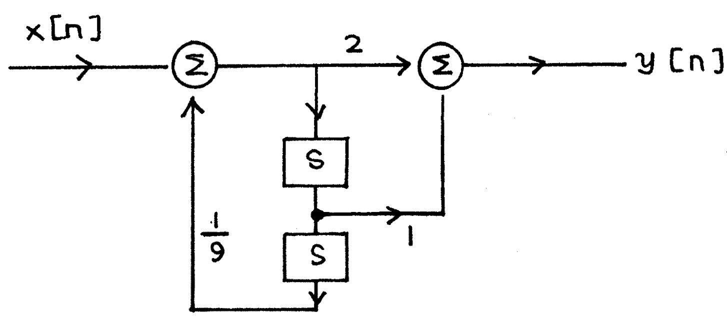


(c) $y[n] - \frac{1}{9}\, y[n-2] = 2x[n] + x[n-1]$

Direct Form 1 : $y[n] = 2x[n] + x[n-1] + \frac{1}{9}\, y[n-2]$

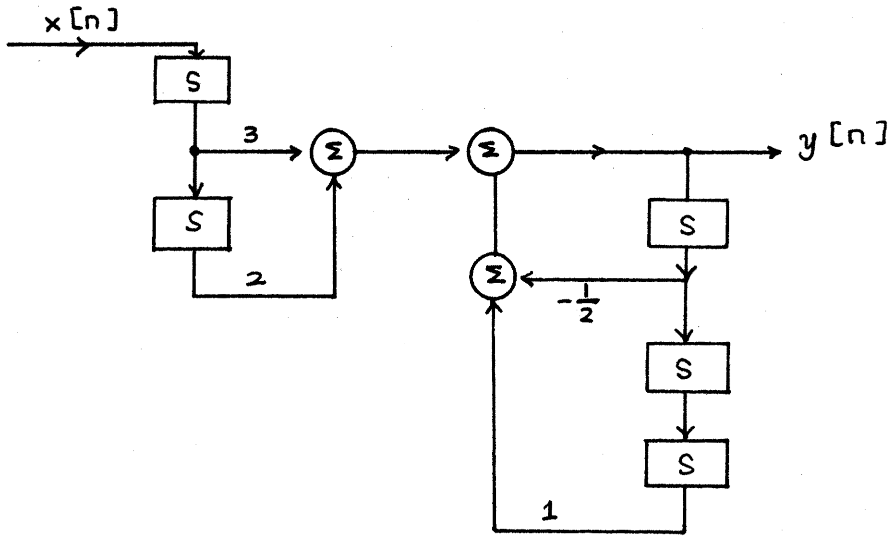


Direct Form II :

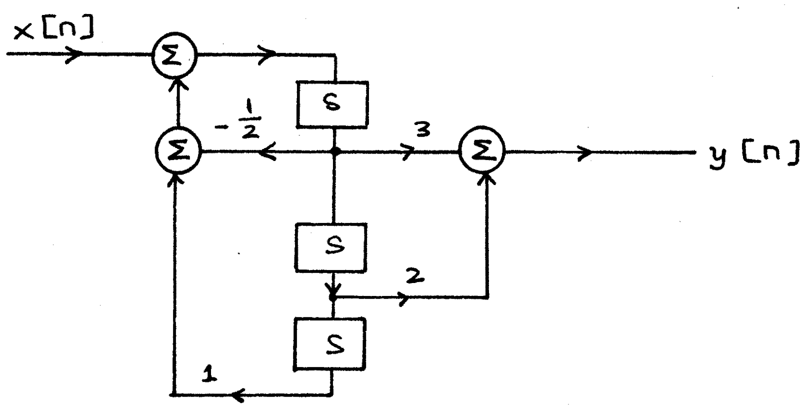


(d) $y[n] + \frac{1}{2}\, y[n-1] - y[n-3] = 3x[n-1] + 2x[n-2]$

Direct Form I : $y[n] = 3x[n-1] + 2x[n-2] - \frac{1}{2} y[n-1]$
$+ y[n-3]$

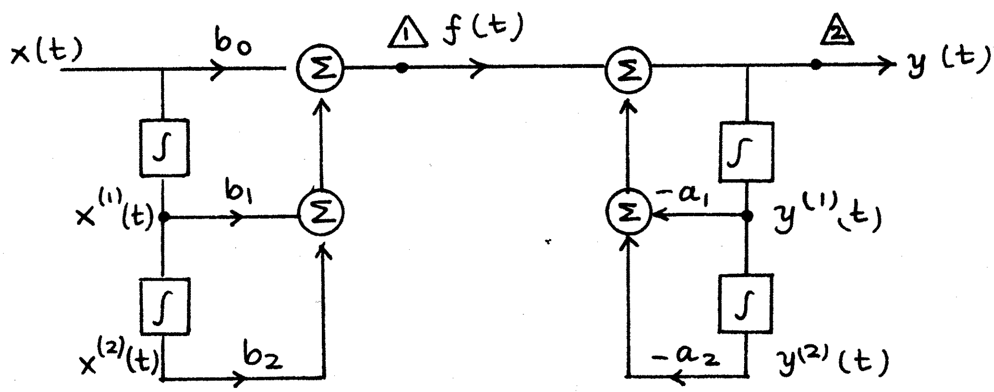


Direct Form II :



2.25 It is clear that 'direct form II' depicted in fig. 2.27 (b) is equivalent to 'direct form I' in fig. 2.27 (a) from topological observation (see Sec. 2.5)

Now, it is left for us to show 'direct form I' in Fig. 2.27 (a) implement Eq (2.42)



Node 1 : $f(t) = b_0 x(t) + b_1 x^{(1)}(t) + b_2 x^{(2)}(t)$

Node 2 : $y(t) = f(t) - a_1 y^{(1)}(t) - a_2 y^{(2)}(t)$

So, $y(t) = b_0 x(t) + b_1 x^{(1)}(t) + b_2 x^{(2)}(t) - a_1 y^{(1)}(t) - a_2 y^{(2)}(t)$

$$y(t) = -a_1 y^{(1)}(t) - a_2 y^{(2)}(t) + b_0 x(t) + b_1 x^{(1)}(t) + b_2 x^{(2)}(t) \quad \text{.....(2.42)}$$

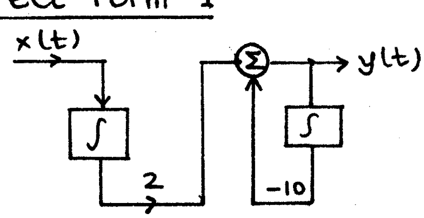
$\boxed{2.26}$

(a) $\frac{d}{dt} y(t) + 10 y(t) = 2x(t)$

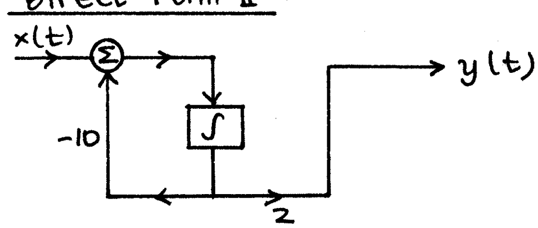
$\Leftrightarrow y(t) + 10 y^{(1)}(t) = 2x^{(1)}(t)$

$y(t) = 2x^{(1)}(t) - 10 y^{(1)}(t)$

Direct Form I :



Direct Form II :

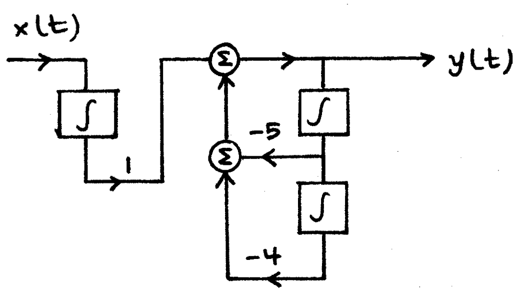


(b) $\frac{d^2}{dt^2} y(t) + 5\frac{d}{dt} y(t) + 4y(t) = \frac{d}{dt} x(t)$

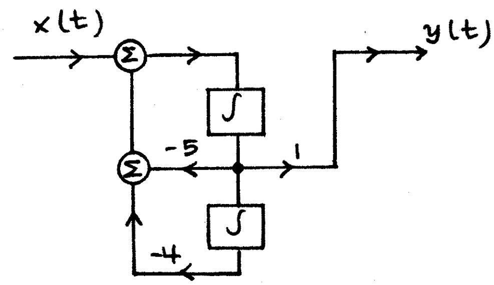
$\Longleftrightarrow \; y(t) + 5\, y^{(1)}(t) + 4\, y^{(2)}(t) = x^{(1)}(t)$

$y(t) = x^{(1)}(t) - 5y^{(1)}(t) - 4\, y^{(2)}\, t$

Direct Form I :



Direct Form II :

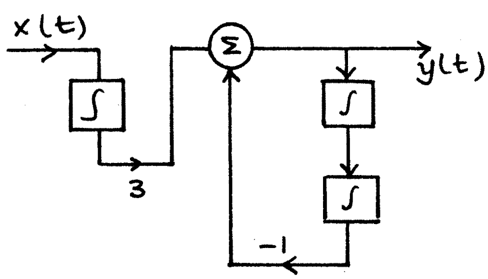


(c) $\frac{d^2}{dt^2} y(t) + y(t) = 3\frac{d}{dt} x(t)$

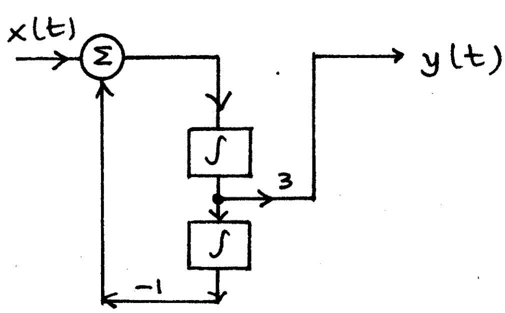
$\Longleftrightarrow \; y(t) + y^{(2)}(t) = 3x^{(1)}(t)$

$y(t) = 3x^{(1)}(t) - y^{(2)}(t)$

Direct Form I :



Direct Form II :

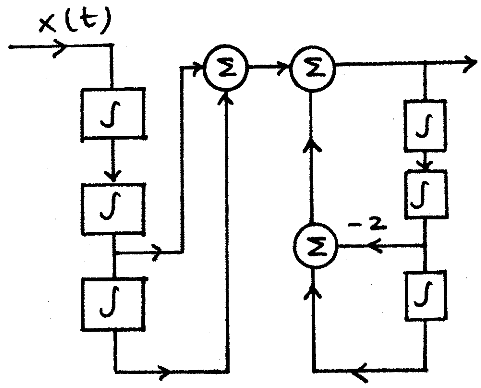


(d) $\frac{d^3}{dt^3} y(t) + 2\frac{d}{dt} y(t) + 3y(t) = x(t) + 3\frac{d}{dt} x(t)$

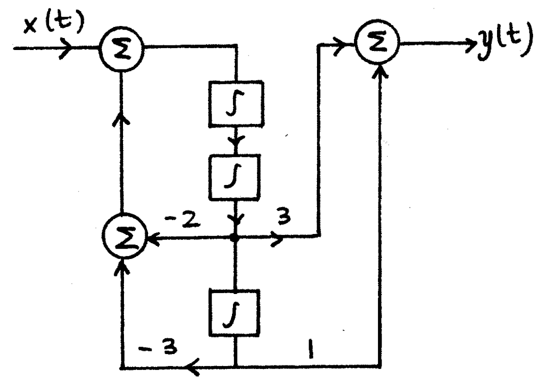
$\Longleftrightarrow \; y(t) + 2y^{(2)}(t) + 3y^{(3)}(t) = x^{(3)}(t) + 3x^{(2)}(t)$

$y(t) = x^{3}(t) + 3x^{(2)}(t) - 2y^{(2)}(t) - 3y^{(3)}(t)$

Direct Form I :



Direct Form II :



$\boxed{2.27}$

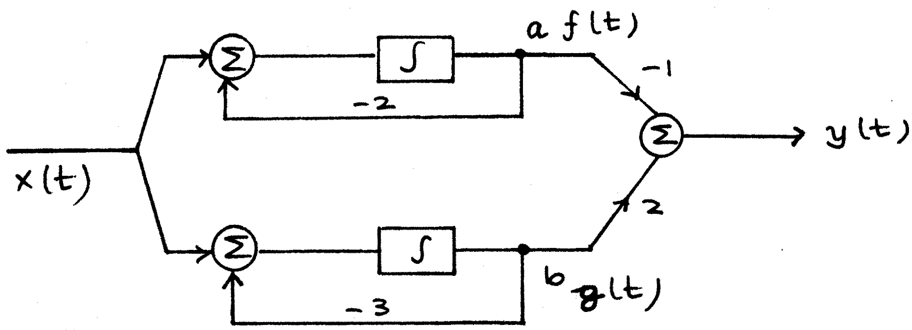
(a) $y(t) = x^{(1)}(t) + 2y^{(1)}(t)$

$$\Leftrightarrow \frac{d}{dt} y(t) - 2y(t) = x(t)$$

(b) $y(t) = x^{(1)}(t) + 2y^{(1)}(t) - y^{(2)}(t)$

$$\Longleftrightarrow \frac{d^2}{dt^2} y(t) - 2\frac{d}{dt} y(t) + y(t) = \frac{d}{dt} x(t)$$

(c)



Here, we have a system of differential equations as follow :

(1) $y(t) = -f(t) + 2g(t)$

(2) $f(t) = x^{(1)}(t) - 2f^{(-1)}(t) \Leftrightarrow \frac{d}{dt} f(t) + 2f(t) = x(t)$

(3) $g(t) = x^{(1)}(t) - 3g^{(1)}(t) \Leftrightarrow \frac{d}{dt} g(t) + 3g(t) = x(t)$

If, we want to go further :

(2) $\frac{df(t)}{dt} + 2f(t) = x(t)$ multiply both sides with e^{2t}

$$\frac{d}{dt}\left(e^{2t} f(t)\right) = x(t)\, e^{2t}$$

$$\therefore f(t) = e^{-2t} \int_a^t x(\tau) e^{2\tau} d\tau$$, 'a' is some constant

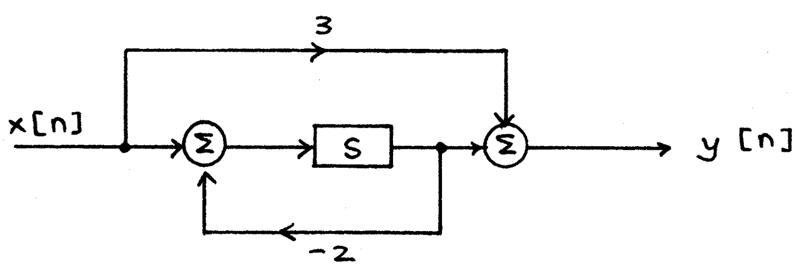
(3) Similarly :

$$g(t) = e^{-3t} \int_b^t x(\tau) e^{3\tau} d\tau$$

$$\therefore y(t) = -e^{-2t}\left(\int_a^t x(\tau) e^{2\tau} d\tau\right) + 2e^{-3t}\left(\int_b^t x(\tau) e^{3\tau} d\tau\right)$$

2.28

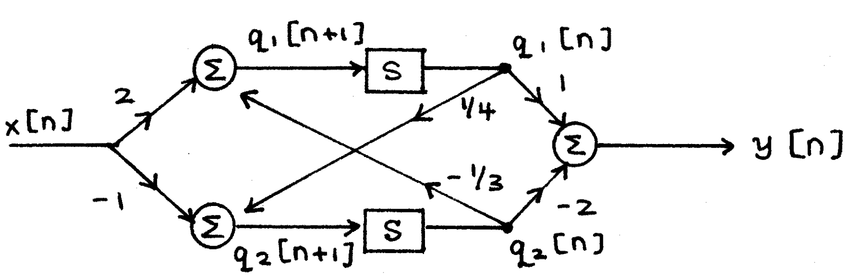
(a)



$$q[n+1] = -2\, q[n] + x[n]$$
$$y[n] = 3\, x[n] + q[n]$$

$$\therefore \bar{A} = [-2],\ \bar{b} = [1],\ \bar{c}[1],\ D[3]$$

(b)

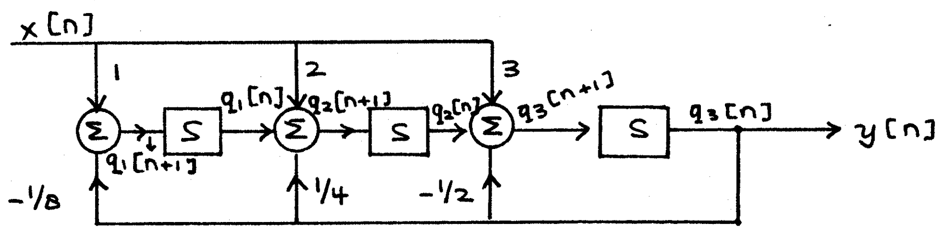


$$q_1[n+1] = 2x[n] - \tfrac{1}{3} q_2[n]$$
$$q_2[n+1] = -x[n] + \tfrac{1}{4} q_1[n]$$
$$y[n] = q_1[n] - 2q_2[n]$$

$$\therefore \quad \bar{A} = \begin{bmatrix} 0 & -1/3 \\ 1/4 & 0 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\bar{c} = \begin{bmatrix} 1 & -2 \end{bmatrix}, \quad \bar{D} = [\,0\,]$$

(c)



$$q_1[n+1] = x[n] - \tfrac{1}{8} \cdot q_3[n]$$

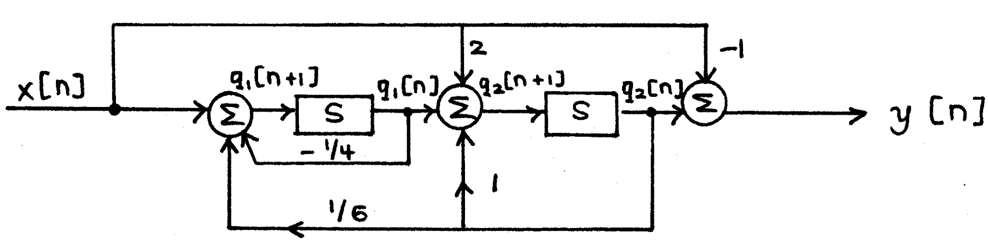
$$q_2[n+1] = q_1[n] + 2x[n] + \tfrac{1}{4} q_3[n]$$
$$q_3[n+1] = q_2[n] + 3x[n] - \tfrac{1}{2} q_3[n]$$

$$y[n] = q_3[n]$$

$$\therefore \quad \bar{A} = \begin{bmatrix} 0 & 0 & -1/8 \\ 1 & 0 & 1/4 \\ 0 & 1 & -1/2 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\bar{c} = [\,0 \quad 0 \quad 1\,], \quad \bar{D} = [\,0\,]$$

(d)



$$q_1[n+1] = x[n] - \frac{1}{4} q_1[n] + \frac{1}{6} q_2[n]$$

$$q_2[n+1] = 2x[n] + q_1[n] + q_2[n]$$

$$y[n] = -x[n] + q_2[n]$$

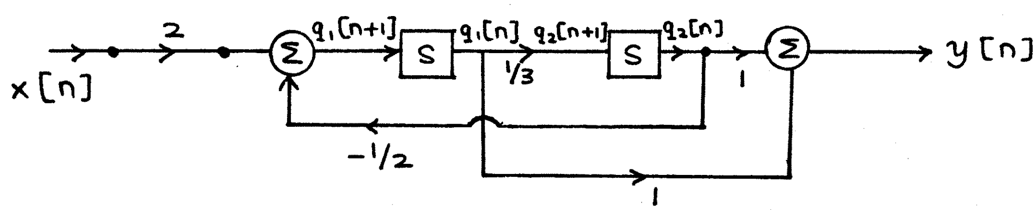
$$\therefore \quad \bar{A} = \begin{bmatrix} -1/4 & 1/6 \\ 1 & 1 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\bar{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \bar{D} = [1]$$

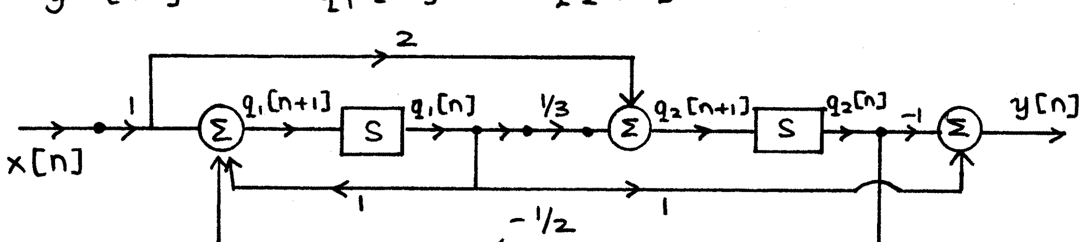
$\boxed{2.29}$

To synthesize the block diagram for the system, draw a unit delay for each state variable, then work on the interconnections.

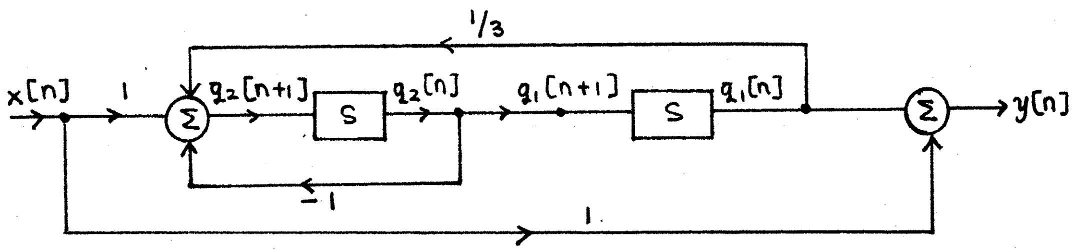
(a)
$$\begin{aligned} q_1[n+1] &= \qquad\qquad - \tfrac{1}{2} q_2[n] + 2x[n] \\ q_2[n+1] &= \tfrac{1}{3} q_1[n] \\ y[n] &= q_1[n] - q_2[n] \end{aligned}$$



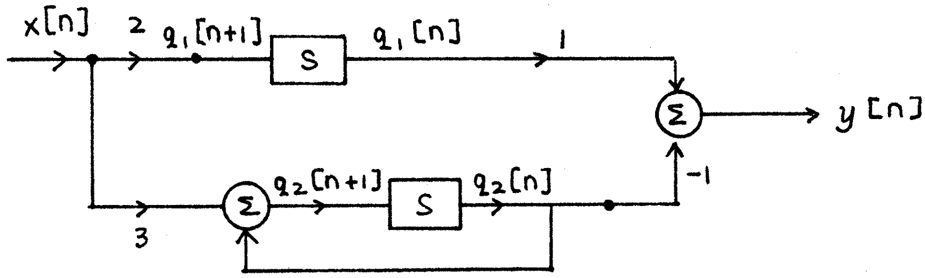
(b)
$$\begin{aligned} q_1[n+1] &= q_1[n] - \tfrac{1}{2} q_2[n] + x[n] \\ q_2[n+1] &= \tfrac{1}{3} q_1[n] \qquad\qquad + 2x[n] \\ y[n] &= q_1[n] - q_2[n] \end{aligned}$$



(c) $q_1[n+1] = -\frac{1}{2} q_2[n]$
$q_2[n+1] = \frac{1}{3} q_1[n] - q_2[n] + x[n]$
$y[n] = q_1[n] + x[n]$

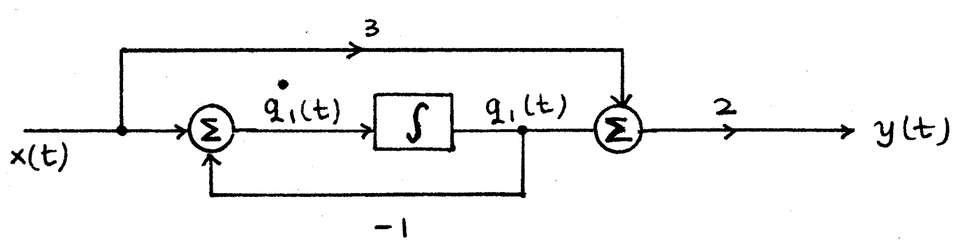


(d) $q_1[n+1] = 2x[n]$
$q_2[n+1] = q_2[n] + 3x[n]$
$y[n] = q_1[n] - q_2[n]$



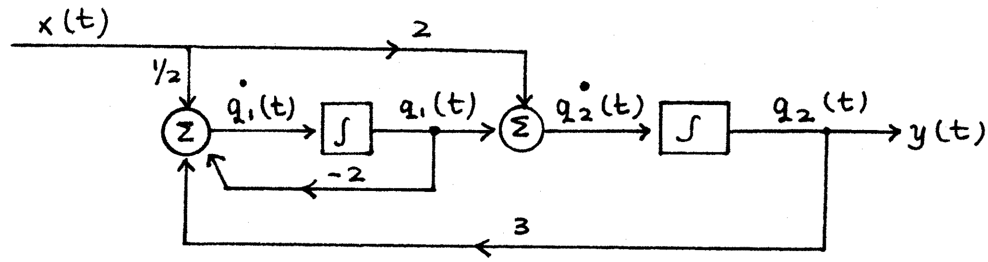
$\boxed{2.30}$

(a) $\dot{q}_1(t) = x(t) - q_1(t)$
$y(t) = 6x(t) + 2q_1(t)$



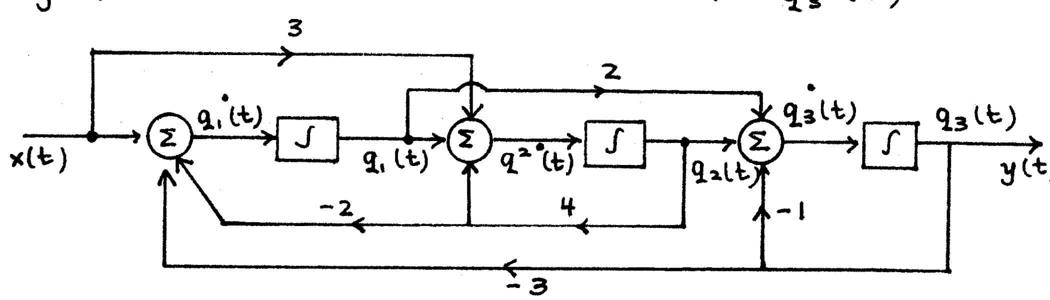
$\therefore \bar{A} = [-1], \bar{b} = [1], \bar{c} = [2], \bar{D} = [6]$

(b) $\dot{q}_1(t) = \frac{1}{2}\, x(t) - 2\, q_1(t) + 3\, q_2(t)$
$\dot{q}_2(t) = 2\, x(t) + q_1(t)$
$y(t) = q_2(t)$



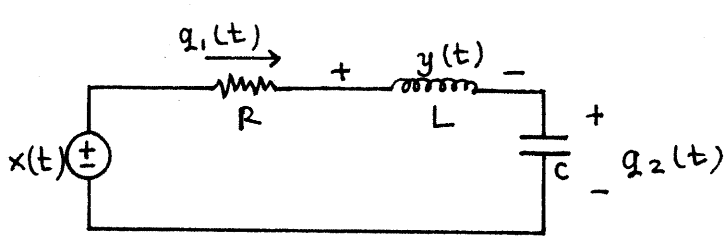
$$\therefore\ \bar{A} = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix},\quad \bar{b} = \begin{bmatrix} 1/2 \\ 2 \end{bmatrix},\quad \bar{c} = \begin{bmatrix} 0 & 1 \end{bmatrix},\quad \bar{d} = \begin{bmatrix} 0 \end{bmatrix}$$

(c) $\dot{q}_1(t) = x(t) - 8\, q_2(t) - 3\, q_3(t)$
$\dot{q}_2(t) = 3\, x(t) + q_1(t) + 4\, q_2(t)$
$\dot{q}_3(t) = 2\, q_1(t) + q_2(t) - q_3(t)$
$y(t) = + q_3(t)$



$$\therefore\ \bar{A} = \begin{bmatrix} 0 & -8 & -3 \\ 1 & 4 & 0 \\ 2 & 1 & -1 \end{bmatrix},\quad \bar{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix},\quad \bar{c} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},\quad \bar{D} = \begin{bmatrix} 0 \end{bmatrix}$$

(d)



$$x(t) = q_1(t).R + L\frac{dq_1(t)}{dt} + q_2(t)$$

$$\frac{dq_1(t)}{dt} = -\frac{R}{L}q_1(t) - \frac{1}{L}q_2(t) + \frac{1}{L}x(t) \qquad \ldots.(1)$$

$$q_1(t) = C\frac{dq_2(t)}{dt}$$

$$\frac{dq_2(t)}{dt} = \frac{1}{C}q_1(t) \qquad \ldots.(2)$$

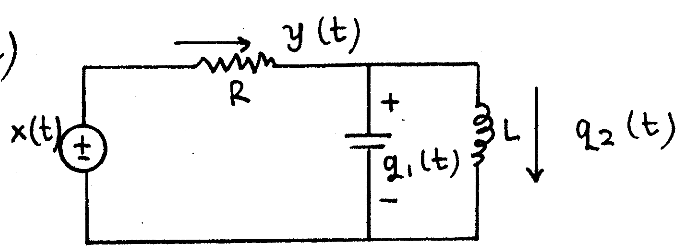
$$x(t) = q_1(t).R + y(t) + q_2(t)$$

$$y(t) = -R.q_1(t) - q_2(t) + x(t) \qquad \ldots.(3)$$

Combining (1), (2), (3) :

$$\bar{A} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 1/L \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} -R & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \end{bmatrix}$$

(e)



$$x(t) = y(t).R + q_1(t)$$

$$y(t) = -\frac{1}{R}q_1(t) + \frac{1}{R}x(t) \ldots..(3)$$

$$y(t) = C\frac{dq_1(t)}{dt} + q_2(t)$$

$$-\frac{1}{R}q_1(t) + \frac{1}{R}x(t) = C\frac{dq_1(t)}{dt} + q_2(t)$$

$$\frac{dq_1(t)}{dt} = -\frac{1}{RC}q_1(t) - \frac{1}{C}q_2(t) + \frac{1}{RC}x(t) \quad \ldots.(1)$$

$$q_1(t) = L\,\frac{dq_2(t)}{dt}$$

$$\frac{dq_2(t)}{dt} = \frac{1}{L}\,q_1(t) \qquad \ldots(2)$$

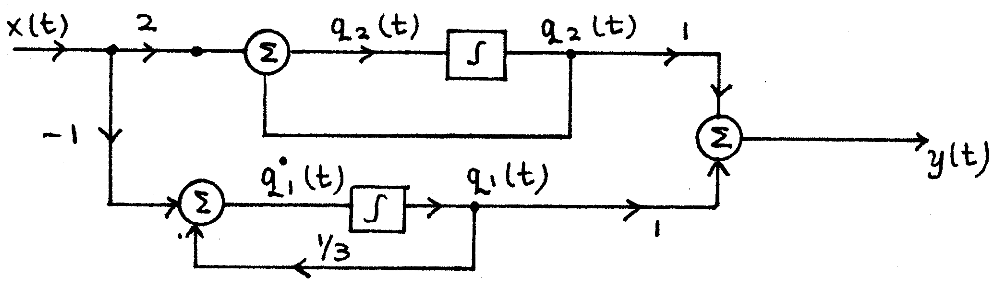
Combining (1), (2) and (3):

$$\therefore\ \bar{A} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix},\quad \bar{b} = \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix},\quad \bar{c} = \begin{bmatrix} -\frac{1}{R} & 0 \end{bmatrix},\quad \bar{D} = \begin{bmatrix} \frac{1}{R} \end{bmatrix}$$

2.31

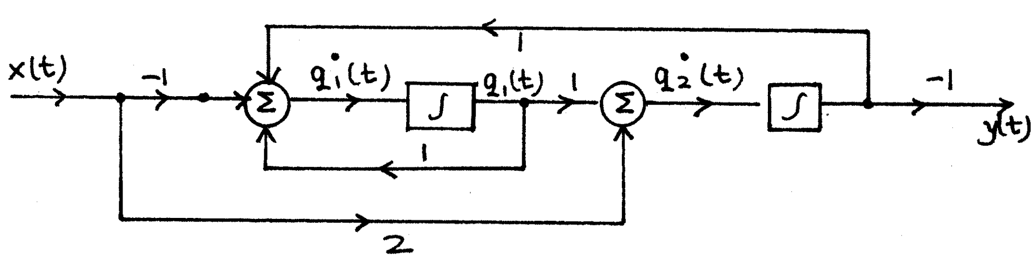
(a)

$$\dot{q}_1(t) = \frac{1}{3}q_1(t) - x(t)$$
$$\dot{q}_2(t) = -\frac{1}{2}q_2(t) + 2x(t)$$
$$y(t) = q_1(t) + q_2(t)$$

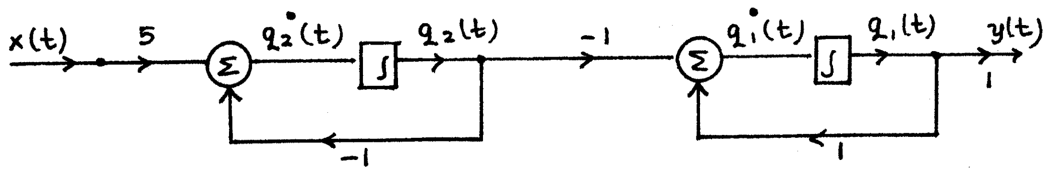


(b)

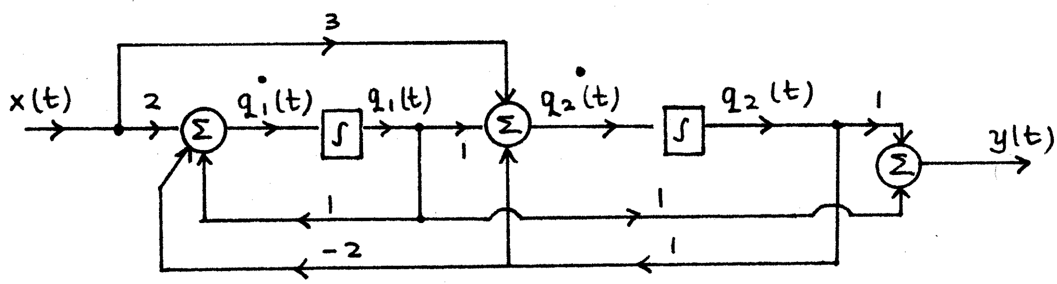
$$\dot{q}_1(t) = q_1(t) + q_2(t) - x(t)$$
$$\dot{q}_2(t) = q_1(t) + 2x(t)$$
$$y(t) = -q_2(t)$$



(c) $$\begin{aligned} \dot{q}_1(t) &= q_1(t) - q_2(t) \\ \dot{q}_2(t) &= \qquad\qquad - q_2(t) + 5x(t) \\ y(t) &= q_1(t) \end{aligned}$$



(d) $$\begin{aligned} \dot{q}_1(t) &= q_1(t) - 2q_2(t) + 2x(t) \\ \dot{q}_2(t) &= q_1(t) + q_2(t) + 3x(t) \\ y(t) &= q_1(t) + q_2(t) \end{aligned}$$



$\boxed{2.32}$ $\bar{A} = \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\bar{c} = [1 \ \ -1]$, $\bar{} = [0]$

(a) $q_1'[n] = 2q_1[n]$, $q_2'[n] = 3q_2[n]$

$$\bar{q}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \bar{q} \quad \Longrightarrow \quad T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$T^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\bar{A}' = T\bar{A}T^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 1/3 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\bar{A}' = \begin{bmatrix} 1 & -1/3 \\ 1/2 & 0 \end{bmatrix}$$

$$\bar{b}' = T\bar{b} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$\bar{c}' = cT^{-1} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/3 \end{bmatrix}$$

$$D' = D = [0]$$

(b) $q_1'[n] = 3q_2[n]$, $q_2'[n] = 2q_1[n]$

$$\bar{q}' = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \bar{q} \implies T = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$$

$$T^{-1} = -\frac{1}{6} \begin{bmatrix} 0 & -3 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1/3 & 0 \end{bmatrix}$$

$$\bar{A}' = T\bar{A}T^{-1} = \begin{bmatrix} 0 & 1/2 \\ -1/3 & 1 \end{bmatrix}$$

$$\bar{b}' = T\bar{b} = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$\bar{c}' = cT^{-1} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ 1/3 & 0 \end{bmatrix} = \begin{bmatrix} -1/3 & 1/2 \end{bmatrix}$$

$$D' = [0]$$

(c) $q_1'[n] = q_1[n] + q_2[n]\ ,\ q_2'[n] = q_1[n] - q_2[n]$

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow T' = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\bar{A}' = T\bar{A}T^{-1} = \begin{bmatrix} 5/12 & 11/12 \\ 1/12 & 7/12 \end{bmatrix}$$

$$\bar{b} = Tb = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\bar{c}' = \bar{c}T^{-1} = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$D' = [0]$$

$\boxed{2.33}$

(a)
$$\begin{aligned} \dot{q}_1(t) &= b_1 x(t) + \alpha_1 q_1(t) \\ \dot{q}_2(t) &= b_2 x(t) + \alpha_2 q_2(t) \\ y(t) &= c\, q_1(t) + c_2 q_2(t) \end{aligned}$$

<u>so</u> : $\bar{A} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$, $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $\bar{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$, $\bar{D} = [0]$

(b) For : $q_1'(t) = q_1(t) - q_2(t)$ and $q_2'(t) = 2q_1(t)$

$$T = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \Rightarrow T^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ -1 & 1/2 \end{bmatrix}$$

$$\bar{A}' = T\bar{A}T^{-1} = \begin{bmatrix} \alpha_2 & \frac{1}{2}(\alpha_1 - \alpha_2) \\ 0 & \alpha_1 \end{bmatrix}$$

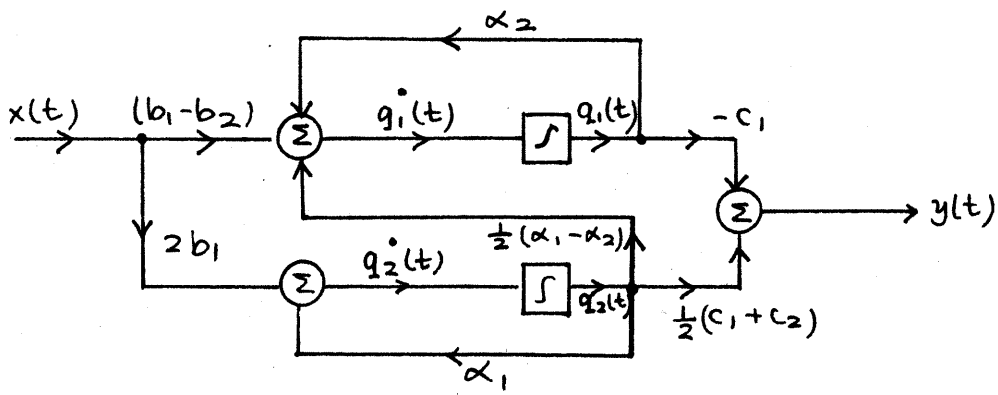
$$\bar{b}' = T\bar{b} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 - b_2 \\ 2b_1 \end{bmatrix}$$

$$\bar{c}' = cT^{-1} = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ -1 & 1/2 \end{bmatrix} = \begin{bmatrix} -c_1 & \frac{1}{2}(c_1+c_2) \end{bmatrix}$$

$$D' = [0]$$

(c) For a system in (b):

$$\begin{aligned} \dot{q}_1(t) &= d_2\, q_1(t) + \tfrac{1}{2}(\alpha_1 - \alpha_2) q_2(t) + (b_1 - b_2) x(t) \\ \dot{q}_2(t) &= \alpha_1\, q_2(t) + 2b_1 x(t) \\ y(t) &= -c_1 q_1(t) + \tfrac{1}{2}(c_1 c_2) q_2(t) \end{aligned}$$



(d) New states : $q_1'(t) = \frac{1}{b_1} q_1(t)$, $q_2'(t) = b_2 q_1(t) - b_1 q_2(t)$

$$T = \begin{bmatrix} \frac{1}{b_1} & 0 \\ b_2 & -b_1 \end{bmatrix} \longrightarrow T^{-1} = \begin{bmatrix} b_1 & 0 \\ b_2 & -\frac{1}{b_1} \end{bmatrix}$$

$$\bar{A}' = T\bar{A}T^{-1} = \begin{bmatrix} \frac{1}{b'} & 0 \\ b_2 & -b_1 \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ b_2 & -\frac{1}{b_1} \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 \\ b_1 b_2(\alpha_1 - \alpha_2) & \alpha_2 \end{bmatrix}$$

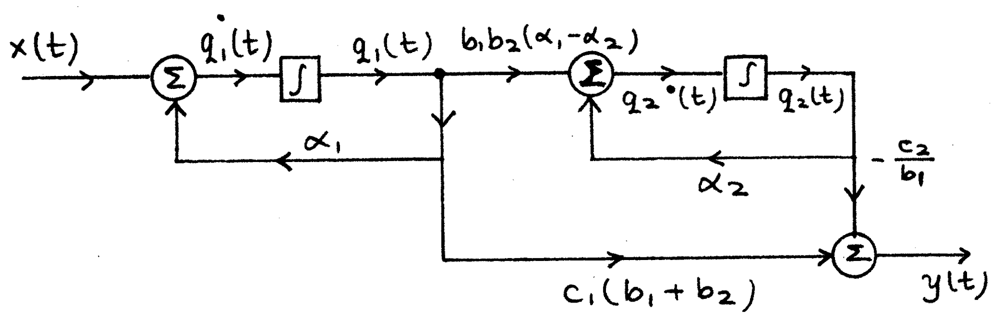
$$\bar{b}' = T\bar{b} = \begin{bmatrix} 1/b_1 & 0 \\ b_2 & -b_1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{c}' = cT^{-1} = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ b_2 & -1/b_1 \end{bmatrix} = \begin{bmatrix} c_1(b_1+b_2) & -\frac{c_2}{b_1} \end{bmatrix}$$

$$\bar{D}' = [0]$$

(e) The corresponding differential equations are :

$$\begin{aligned} \dot{q}_1(t) &= \alpha_1 q_1(t) + x(t) \\ \dot{q}_2(t) &= b_1 b_2(\alpha_1 - \alpha_2) q_1(t) + \alpha_2 q_2(t) \\ y(t) &= c_1(b_1+b_2)\, q_1(t) - \frac{c_2}{b_1} q_2(t) \end{aligned}$$



$\boxed{2.34}$

(a) $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(k.\Delta).g_\Delta(t-k.\Delta)\Delta$$

As Δ decreases and approaches zero, the approximation quality improves

(b) $g_\Delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h_\Delta(t)$

$\tilde{x}(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = \ldots ?$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(k.\Delta)\, g_\Delta(t-k.\Delta)\Delta$$

Let the system be represented by an operator S such that $H\{g_\Delta(t)\} = h_\Delta(t)$

$$H\{\tilde{x}(t)\} = H\left\{\sum_{k=-\infty}^{\infty} x(k\Delta)\, g_\Delta(t-k\Delta)\Delta\right\}$$

$$= \sum_{k=-\infty}^{\infty} \left(H\{x(k.\Delta) . g_\Delta(t-k\Delta)\Delta\right) \because \text{linearity of } S$$

$$= \sum_{k=-\infty}^{\infty} \left(x(k.\Delta).H\{g_\Delta(t-k\Delta)\}\Delta\right) \because \text{linearity of } S$$

$$= \sum_{k=-\infty}^{\infty} x(k\Delta).h_\Delta(t-k\Delta).\Delta \because \text{time - invariance of } S$$

$$\therefore H\{\tilde{x}(t)\} = \sum_{k=-\infty}^{\infty} x(k\Delta).h_\Delta(t-k.\Delta)\Delta$$

(c) when $\Delta \to 0$

$$h(t) = \lim_{\Delta\to 0} h_\Delta(t)$$

$$\lim_{\Delta\to 0} H\{x^2(t)\} = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k.\Delta).h_\Delta(t-k\Delta).\Delta$$

As $\Delta \to 0$, the limit is a Riemann sum, which represents an integral

$$I \cong \int_{-\infty}^{\infty} x(\tau).h_\Delta(t-\tau)\,d\tau$$

Using the fact that $h(t) = \lim_{\Delta\to 0} h_\Delta(t)$

$$I \cong \int_{-\infty}^{\infty} x(\tau).h(t-\tau)\,d\tau$$

$$= x(t) * h(t)$$

$\boxed{2.35}$ $y_u(t) = \{1 - e^{-t/RC}\}\, u(t)$

(a) $x(t) = g_\Delta t = \frac{1}{\Delta}\left(u(t+\frac{\Delta}{2}) - u(t-\frac{\Delta}{2})\right)$

(b) The response of the system to $x(t)$ is $y(t)$:

$$y(t) = H\{x(t)\} = H\left\{\frac{1}{\Delta}\left(u(t+\tfrac{\Delta}{2}) - u(t-\tfrac{\Delta}{2})\right)\right\}$$

$$= \frac{1}{\Delta}\left(H\{u(t+\tfrac{\Delta}{2})\} - H\{u(t-\tfrac{\Delta}{2})\}\right) \therefore \text{ linearity}$$

$$\therefore y(t) = \frac{1}{\Delta}\left(y_u(t+\tfrac{\Delta}{2}) - y_u(t-\tfrac{\Delta}{2})\right) \because \text{ time - invariance}$$

(c) $$h(t) = \lim_{\Delta \to 0} \frac{y_u(t+\frac{\Delta}{2}) - y_u(t-\frac{\Delta}{2})}{\Delta}$$

$$= \frac{d}{dt} y_u(t)$$

$$\underline{\underline{h(t) = \frac{1}{RC} e^{-t/RC} . u(t)}}$$

$\boxed{2.36}$

(a) $$r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau) . y(\tau - t)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, y(-(t-\tau))\, d\tau \qquad \ldots (1)$$

First, assume that $r_{xy}(t)$ can be expressed in terms of a convolution integral, i.e :

$$r_{xy}(t) = \int_{-\infty}^{\infty} f_1(\tau) . f_2(t-\tau)\, d\tau = f_1(t) * f_2(t) \ldots (2)$$

By (1), we can see that (1) and (2) can be equivalent if choosing : $f_1(v_1) \equiv x(v_1)$ and $f_2(v_2) \equiv y(-v_2)$ where v_1, v_2 are arguments, in this case are $v_1 = \tau$

and $v_2 = t - \tau$

Then : $r_{xy}(t) = f_1(t) * f_2(t) = x(t) * y(-t)$

$$\therefore r_{xy}(t) = x(t) * y(-t)$$

(b) Procedure :

1. Graph both $x(\tau)$ and $y(\tau - t)$ as a function of the independent variable τ. To obtain $y(\tau - t)$, shift $y(\tau)$ by t to the right

2. Begin with the shift t at $-\infty$

3. Write mathematical representation for $x(\tau) \cdot y(\tau - t)$

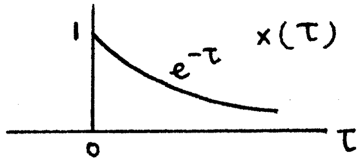
4. Increase the shift t until the mathematical representation for $x(\tau)\, y(\tau - t)$ changes. The value t at which the change occurs defines the end of the current set and the beginning of a new set

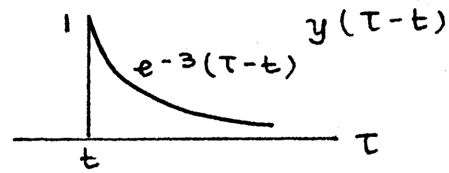
5. Let t be in the new set. Repeat (3) and (4) 'til all set of shifts t and the corresponding representations for $x(\tau) \cdot y(\tau - t)$ are identified. This usually implies increasing t until $t = \infty$

6. For each set of shifts t, integrate $x(\tau) \cdot y(\tau - t)$ from $\tau = -\infty$ to $\tau = \infty$ to obtain $r_{xy}(t)$ on each set

(c) Cross correlation :

(i) $x(t) = e^{-t}\,u(t)\,.\;y(t) = e^{-3t}\,u(t)$



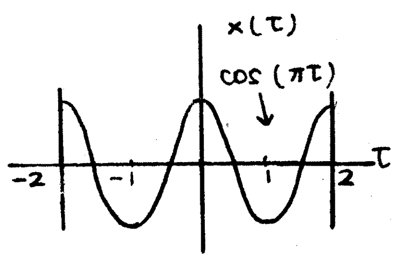


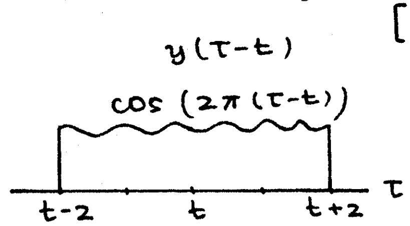
$\underline{t<0}$: $r_{xy}(t) = \int_0^{\infty} e^{3t}\,e^{-4\tau}\,d\tau = \frac{1}{4}e^{3t}$

$\underline{t\geq 0}$: $r_{xy}(t) = \int_0^{\infty} e^{3t}\,e^{-4\tau}\,d\tau = \frac{1}{4}e^{-t}$

$$\therefore r_{xy}(t) = \begin{cases} 1/4\,.\,e^{3t} & ,\ t<0 \\ 1/4\,.\,e^{-t} & ,\ t\geq 0 \end{cases}$$

(ii) $x(t) = \cos(\pi t)[u(t+2) - u(t-2)]$, $y(t) = \cos(2\pi t)$ $[u(t+2) - u(t-2)]$





$\underline{t<-4}$: $r_{xy}(t) = 0$

$\underline{-4\leq t<0}$: $r_{xy}(t) = \int_{-2}^{t+2} \cos(\pi\tau)\,.\,\cos(2\pi\tau - 2\pi t)\,d\tau$

$$= \frac{1}{3\pi}\left(2\sin(2\pi t) - \sin(\pi t)\right)$$

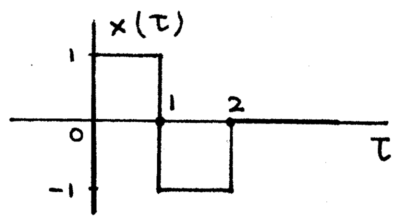
$\underline{0\leq t<4}$: $r_{xy}(t) = \int_{t-2}^{2} \cos(\pi\tau)\,.\,\cos(2\pi\tau - 2\pi t)\,d\tau$

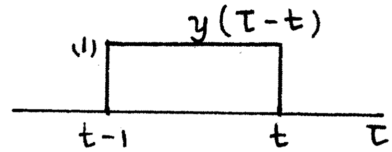
$$= \frac{1}{3\pi}\left(-2\sin(2\pi t) + \sin(\pi t)\right)$$

$\underline{t\geq 4}$: $r_{xy}(t) = 0$

$$\therefore r_{xy}(t) = \begin{cases} 0 & , \quad t < -4 \\ \frac{1}{3\pi}\left(2\sin(2\pi t) - \sin(\pi t)\right) & , \quad -4 \leq t < 0 \\ \frac{1}{3\pi}\left(-2\sin(2\pi t) + \sin(\pi t)\right) & , \quad 0 \leq t < 4 \\ 0 & , \quad t \geq 4 \end{cases}$$

(iii) $x(t) = u(t) - 2u(t-1) + u(t-2)$, $y(t) = u(t+1) - u(t)$





$t < 0$: $r_{xy}(t) = 0$

$0 \leq t < 1$: $r_{xy}(t) = t$

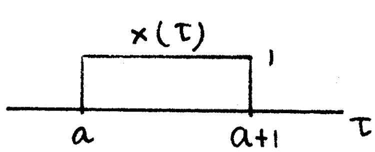
$1 \leq t < 2$: $r_{xy}(t) = (2-t) - (t-1) = 3-2t$

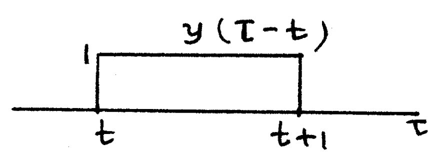
$2 \leq t < 3$: $r_{xy}(t) = -(3-t) = t-3$

$t \geq 3$: $r_{xy}(t) = 0$

$$\therefore r_{xy}(t) = \begin{cases} 0 & , \quad t < 0 \\ t & , \quad 0 \leq t < 1 \\ 3-2t & , \quad 1 \leq t < 2 \\ t-3 & , \quad 2 \leq t < 3 \\ 0 & , \quad t \geq 3 \end{cases}$$

(iv) $x(t) = u(t-a) - u(t-(a+1))$, $y(t) = u(t) - u(t-1)$





$t < a-1$: $r_{xy}(t) = 0$

$a-1 \leq t < a$: $r_{xy}(t) = t+1-a$

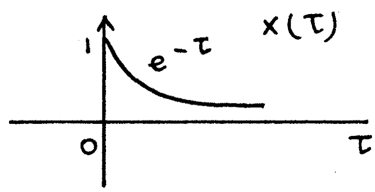
$a \leq t < a+1$: $r_{xy}(t) = a+1-t$

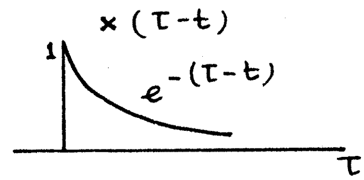
$t \geq a+1$: $r_{xy}(t) = 0$

$$\therefore r_{xy}(t) = \begin{cases} 0 & , \; t < a-1 \\ t+1-a & , \; a-1 \leqslant t < a \\ a+1-t & , \; a \leqslant t < a+1 \\ 0 & , \; t \geqslant a+1 \end{cases}$$

(d) Auto correlation

(i) $x(t) = e^{-t}\,u(t)$



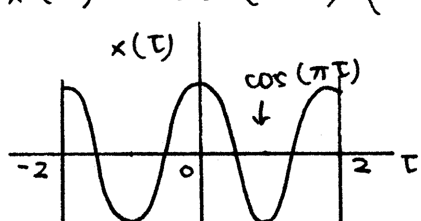


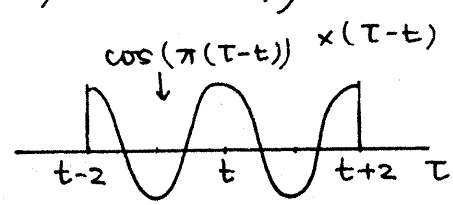
$t < 0 \quad : r_x(t) = \int_0^{\infty} e^{t} . e^{-2\tau} d\tau = \frac{1}{2} e^{t}$

$t \geqslant 0 \quad : r_x(t) = \int_0^{\infty} e^{t} . e^{-2\tau} d\tau = \frac{1}{2} e^{-t}$

$$\therefore r_{xx}(t) = \begin{cases} \frac{1}{2} e^{t} & , \; t < 0 \\ \frac{1}{2} e^{-t} & , \; t \geqslant 0 \end{cases}$$

(ii) $x(t) = \cos(\pi t)\left(u(t+2) - u(t-2) \right)$





$t < -4 \quad : r_{xx}(t) = 0$

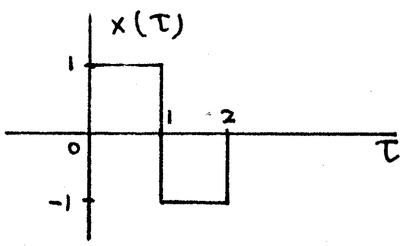
$-4 \leqslant t < 0 \quad : r_{xx}(t) = \int_{-2}^{t+2} \cos^2(\pi t)\, dt = \frac{1}{2}(t+4) + \frac{1}{4\pi}\sin(2\pi t)$

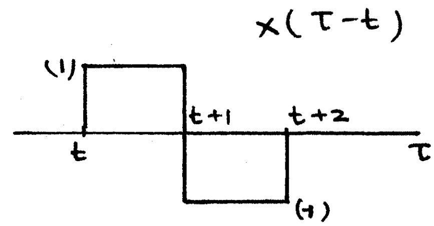
$0 \leqslant t < 4 \quad : r_{xx}(t) = \int_{t-2}^{2} \cos^2(\pi t)\, dt = \frac{1}{2}(4-t) - \frac{1}{4\pi}\sin(2\pi t)$

$t \geqslant 4 \quad : r_{xx}(t) = 0$

$$\therefore r_{xx}(t) = \begin{cases} 0 & , \quad t < -4 \\ \frac{1}{2}(t+4) + \frac{1}{4}\sin(2\pi t) & , \; -4 \leq t < 0 \\ \frac{1}{2}(t+4) - \frac{1}{4}\sin(2\pi t) & , \quad 0 \leq t < 4 \\ 0 & , \quad t \geq 4 \end{cases}$$

(iii) $x(t) = u(t) - 2u(t-1) + u(t-2)$





$t < -2$: $r_{xx}(t) = 0$

$-2 \leq t < -1$: $r_{xx}(t) = -(t+2)$

$-1 \leq t < 0$: $r_{xx}(t) = (t+1) + t + (t+1) = 3t+2$

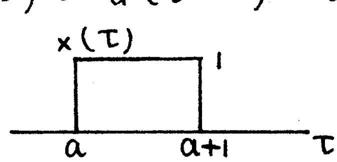
$0 \leq t < 1$: $r_{xx}(t) = (1-t) - (t) + (1-t) = 2-3t$

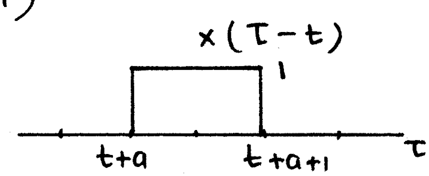
$1 \leq t < 2$: $r_{xx}(t) = t-2$

$t \geq 2$: $r_{xx}(t) = 0$

$$\therefore r_{xx}(t) = \begin{cases} 0 & , \quad t < -2 \\ -(t+2) & , \; -2 \leq t < -1 \\ 3t+2 & , \; -1 \leq t < 0 \\ 2-3t & , \quad 0 \leq t < 1 \\ t-2 & , \quad 1 \leq t < 2 \\ 0 & , \quad t \geq 2 \end{cases}$$

(iv) $x(t) = u(t-a) - u(t-a-1)$





$t < -1$: $r_{xx}(t) = 0$

$-1 \leq t < 0 \quad : r_{xx}(t) = t+1$

$0 \leq t < 1 \quad : r_{xx}(t) = 1-t$

$t \geq 1 \quad : r_{xx}(t) = 0$

$$\therefore r_{xx}(t) = \begin{cases} 0 & , \quad t < -1 \\ t+1 & , \quad -1 \leq t < 0 \\ 1-t & , \quad 0 \leq t < 1 \\ 0 & , \quad t \geq 1 \end{cases}$$

(e) $r_{xy}(t) = r_{yx}(-t)$

$$r_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)\, y(\tau - t)\, d\tau \leftarrow u = \tau - t,\ du = d\tau$$

$$r_{xy}(t) = \int_{-\infty}^{\infty} y(u) \,.\, x(u+t)\, du$$

'u' is a dummy variable, can be replaced by 'τ'

$$r_{xy}(t) = \int_{-\infty}^{\infty} y(\tau) \,.\, x(\tau + t)\, d\tau$$

$$r_{xy}(t) = r_{yx}(-t)$$

(f) $r_{xx}(t) = r_{xx}(-t)$

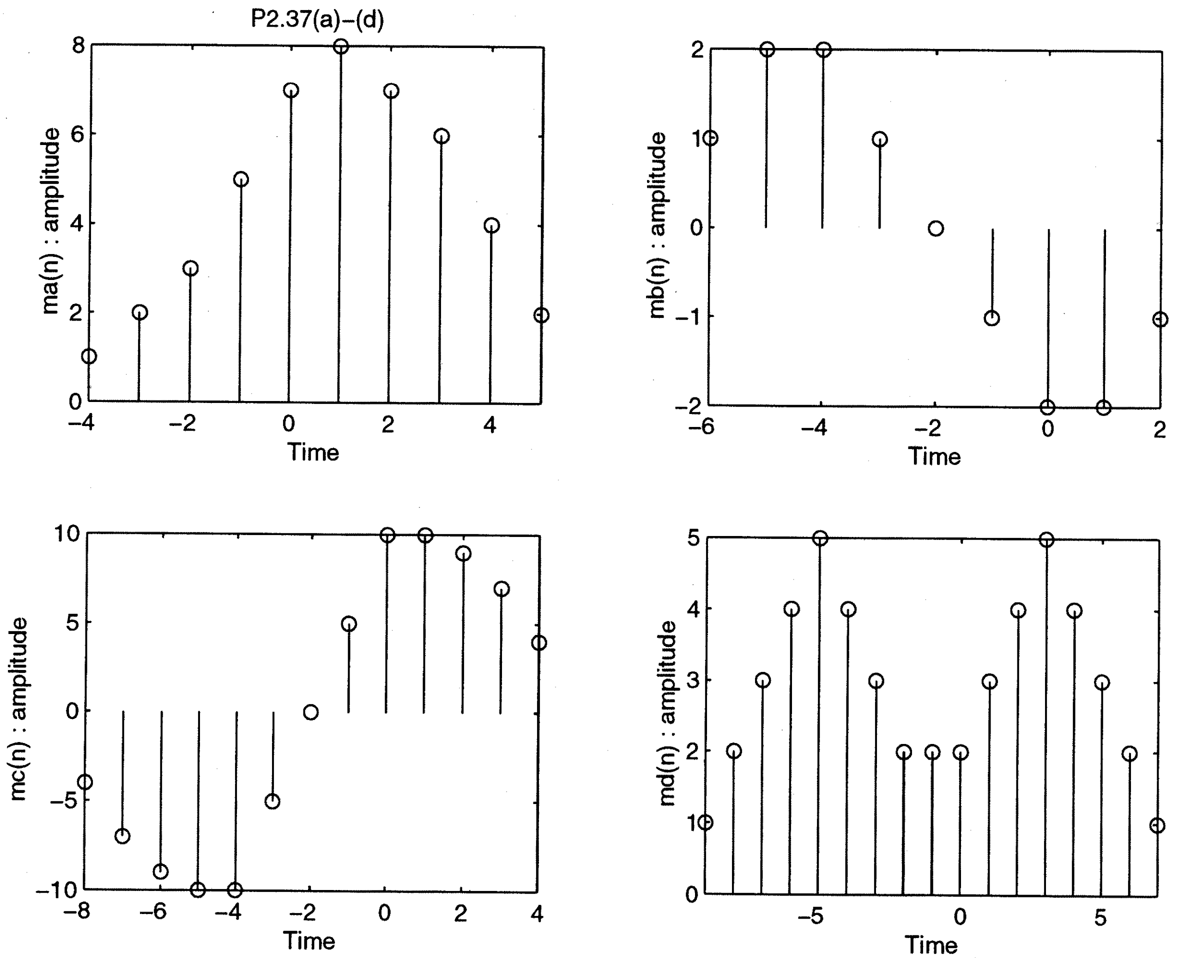
$$r_{xx}(t) = \int_{-\infty}^{\infty} x(\tau) \,.\, x(\tau - t)\, d\tau \leftarrow u = \tau - t,\ du = d\tau$$

$$r_{xx}(t) = \int_{-\infty}^{\infty} x(u) \,.\, x(u+t)\, du$$

$$r_{xx}(t) = r_{xx}(-t)$$

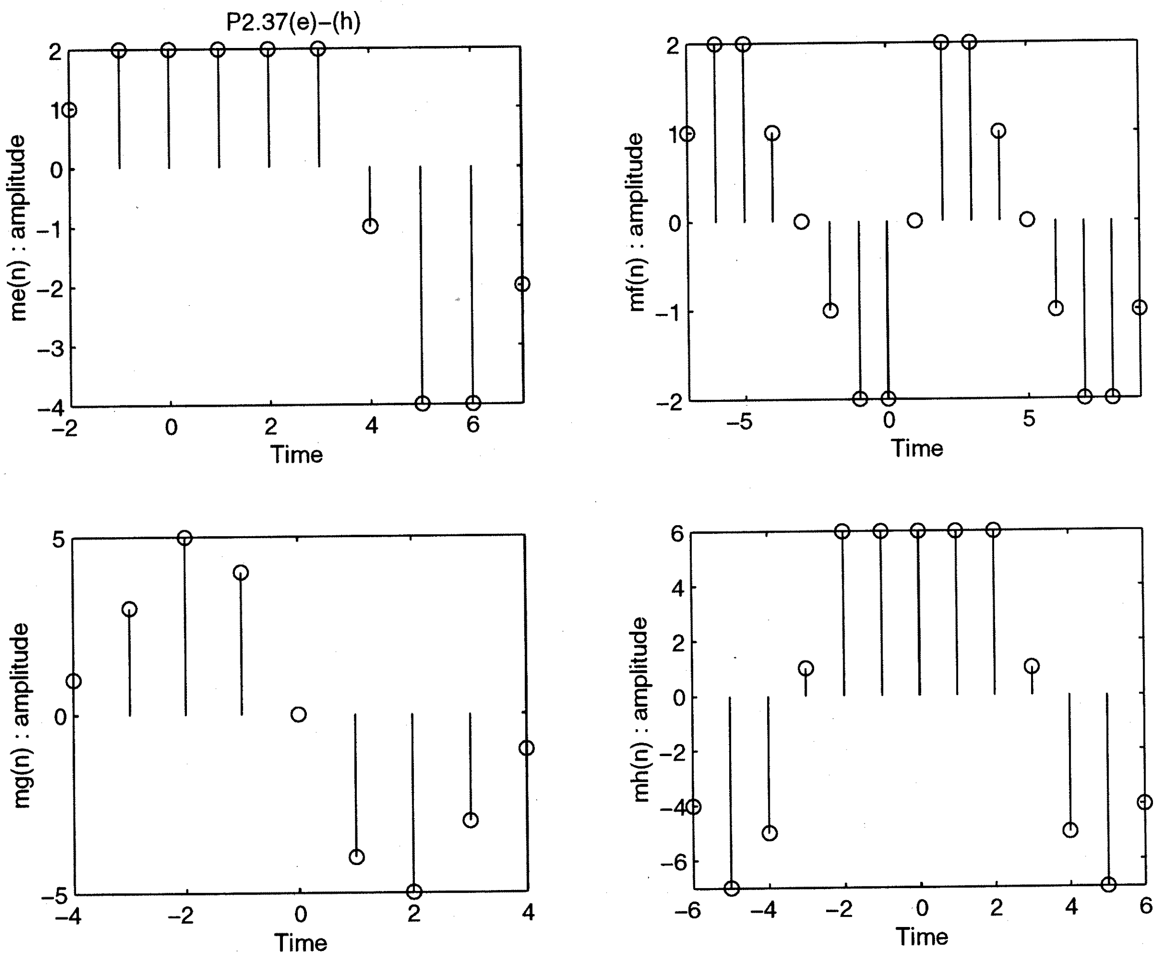
P 2.37

- Plot 1 of 3 -



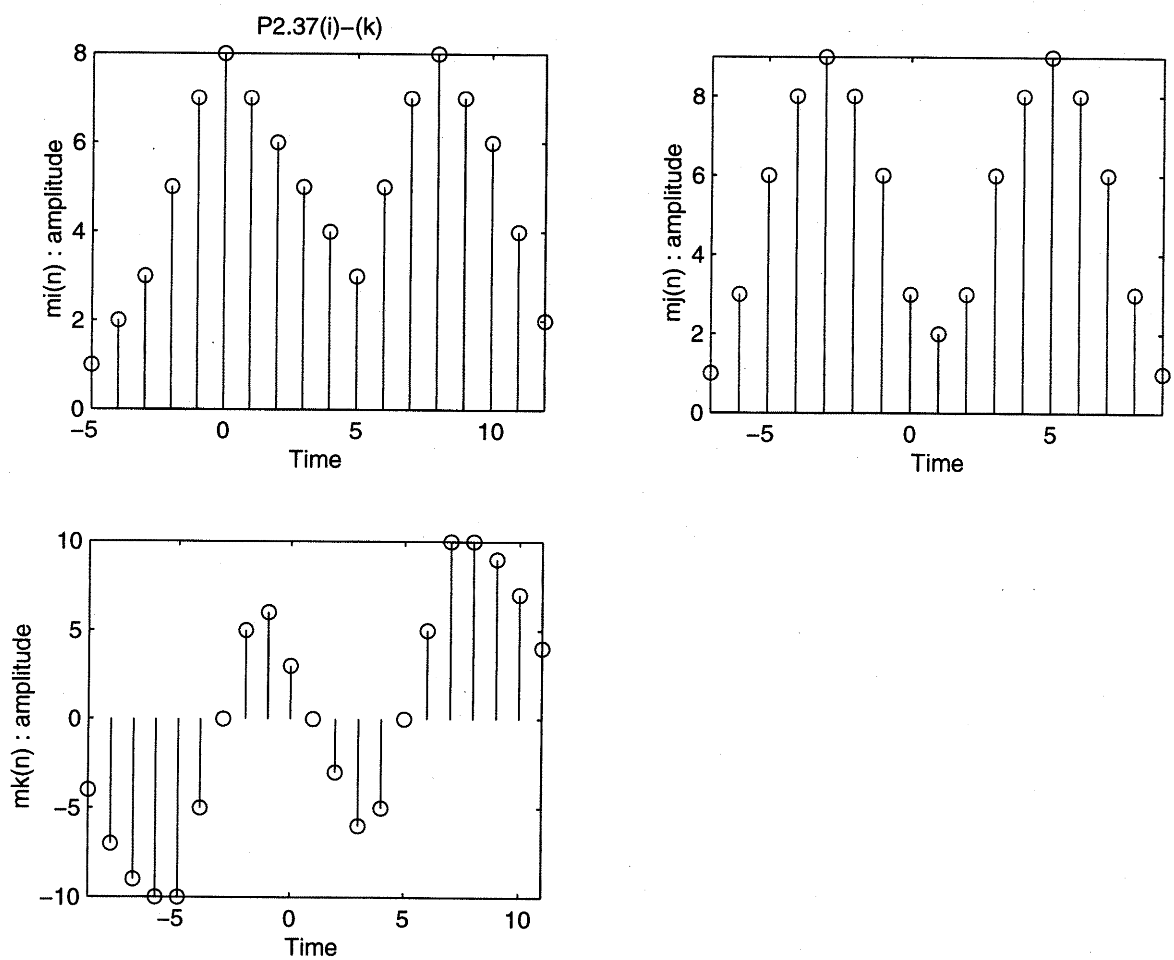
P 2.37

- Plot 2 of 3 -



P. 2.37

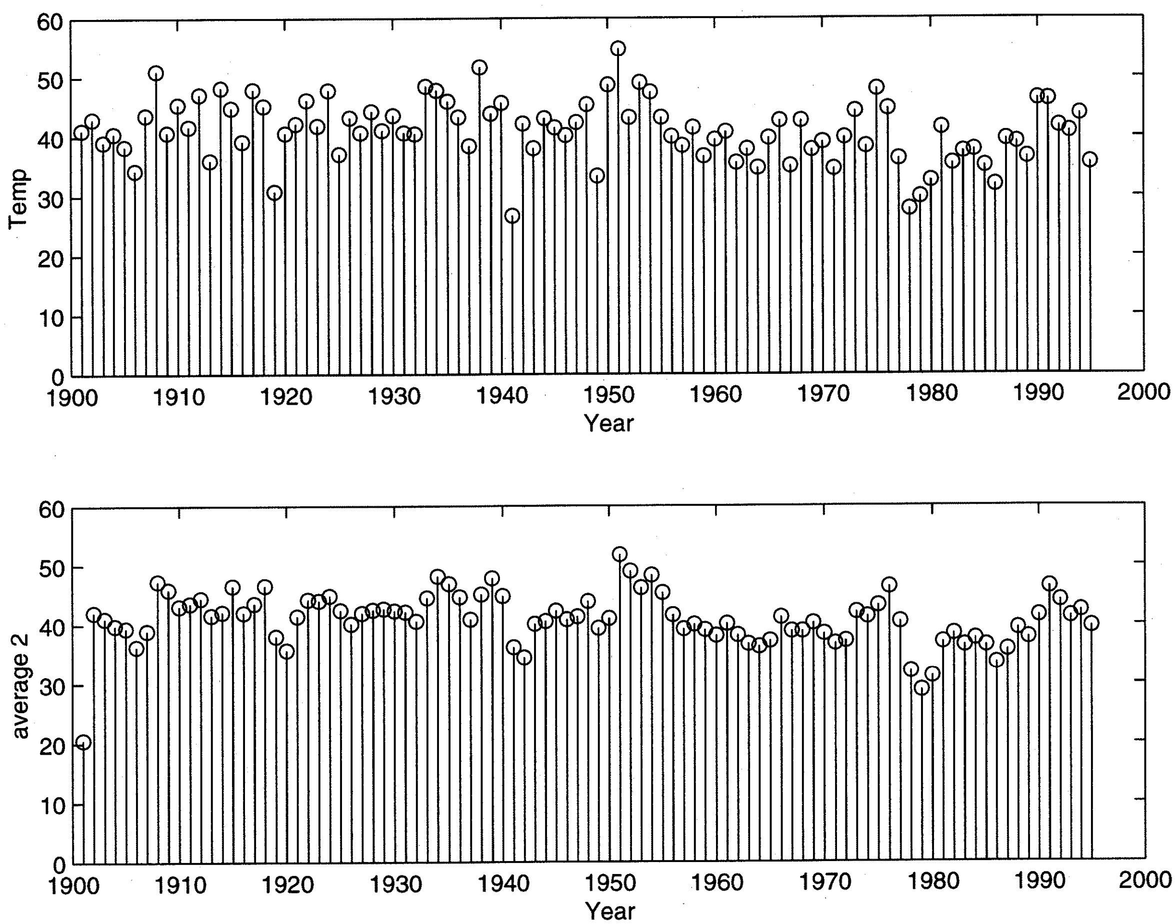
- Plot 3 of 3 -



P 2.38

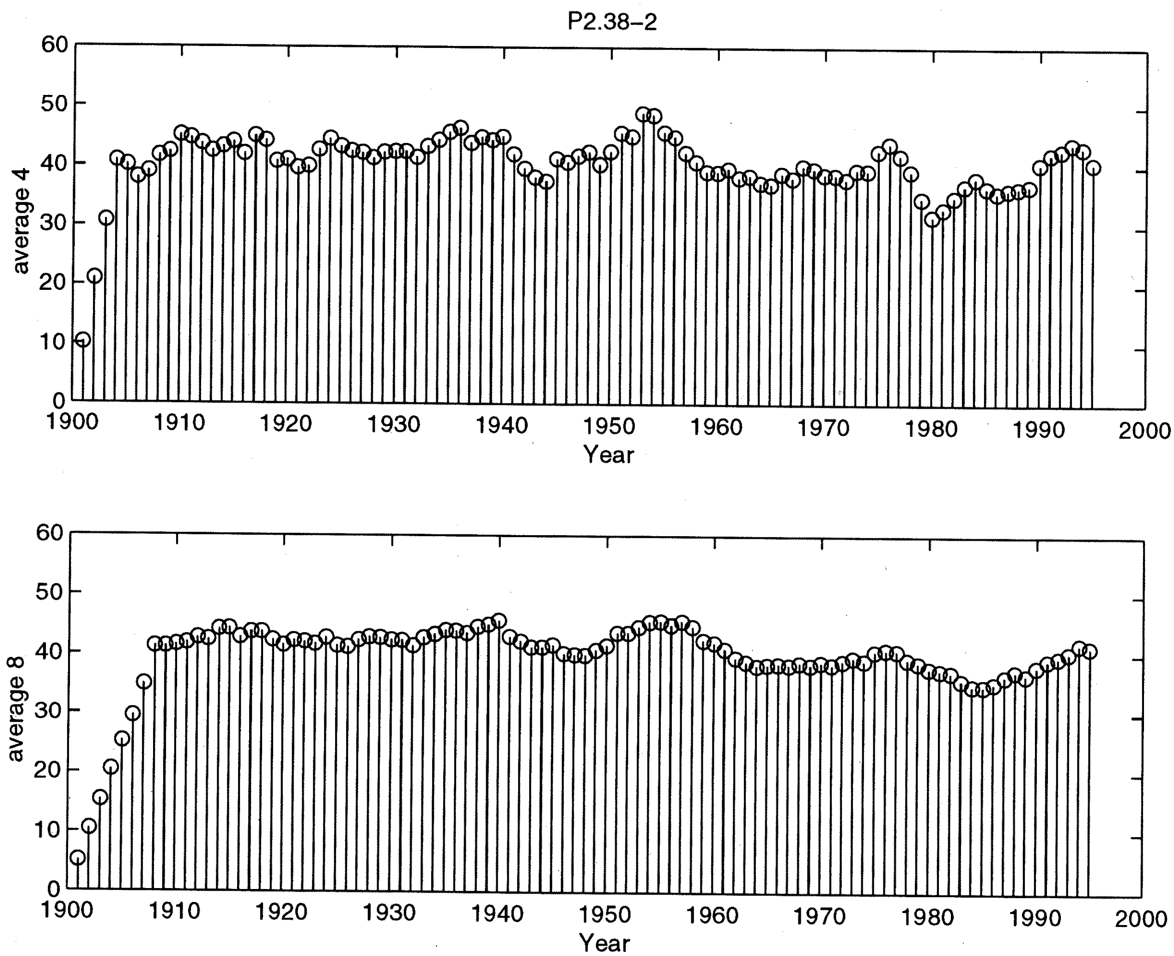
- Plot 1 of 2 -

P2.38–1



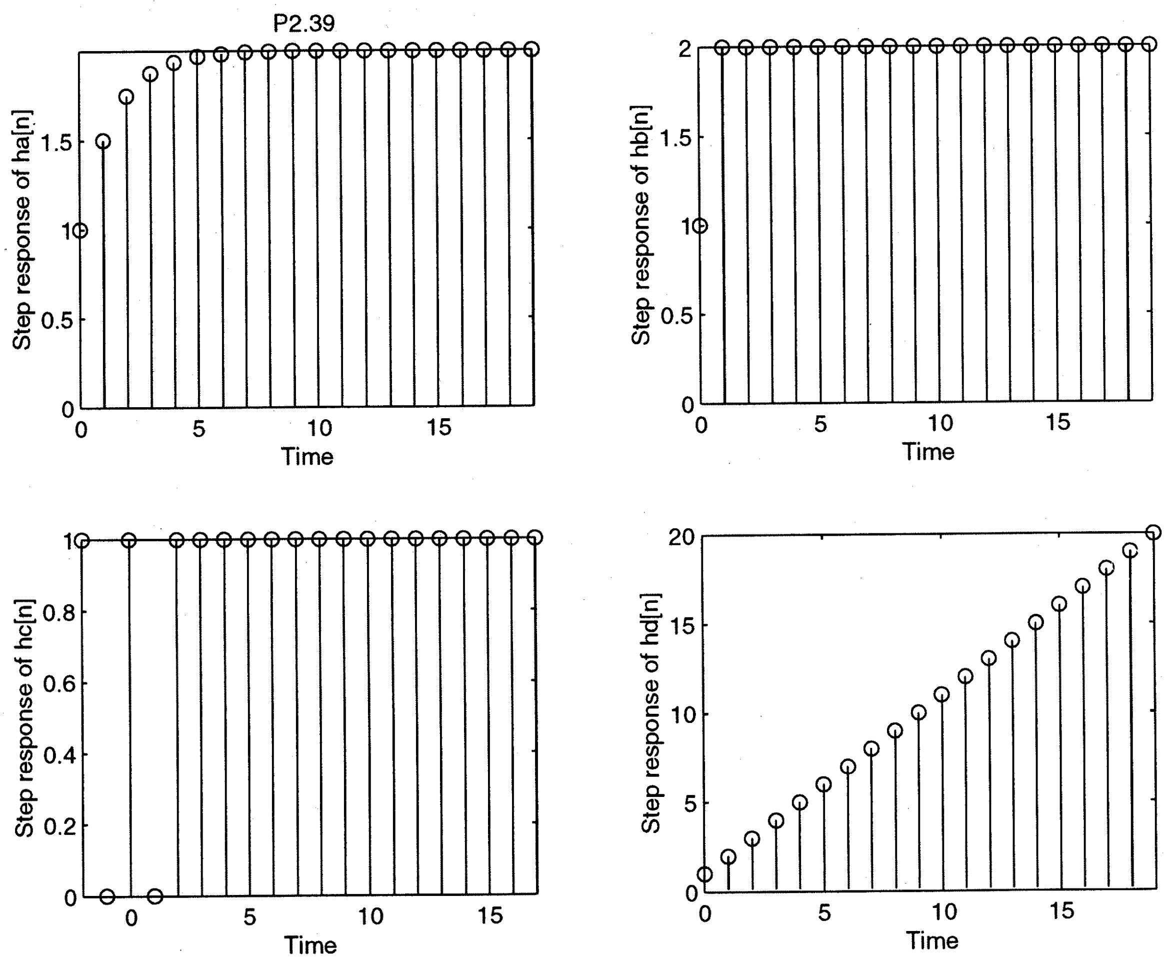
P. 2.38

- Plot 2 of 2 -



P 2.39

— Plot 1 of 1 —



2.40

(b) $$|H(e^{j\Omega_0})|^2 = \frac{|\mathrm{Re}(\text{output})|^2 + |\mathrm{Im}(\text{output})|^2}{|\mathrm{Re}(\text{input})|^2 + |\mathrm{Im}(\text{input})|^2}\Bigg|_{\text{at a particular } n}$$

$$\measuredangle H(e^{j\Omega_0}) = \underbrace{\measuredangle\,\text{output} - \measuredangle\,\text{input}}_{\text{time shift between input and output} \times \Omega_0}\Big|_{\text{at a particular } n}$$

	$\Omega_0 = \pi/3$		$\Omega_0 = 2\pi/3$	
	$\lvert H(\,)\rvert$	$\measuredangle H(.)$ rad	$\lvert H(.)\rvert$	$\measuredangle H(..)$ rad
h_a	0.866	-0.5236	0.500	-1.0472
h_b	0.4330	-1.5708	0.2500	0
h_c	0.2165	-0.5236	0.1250	-1.0472

(c) $\underline{h}_a = [\underset{\uparrow}{1} \;\; 1]/2$, $\underline{h}_b = [\underset{\uparrow}{1} \;1\;1\;1]/4$, $\underline{h}_c = [\underset{\uparrow}{1}\;1\;1\;1\;\;1\;1\;1\;1]/8$

$$H_a(e^{j\Omega}) = \frac{1}{2}(1 + e^{-j\Omega}) = e^{-j\frac{\Omega}{2}} \cos\frac{\Omega}{2} \rightarrow |H_a(e^{j\Omega})| = \left|\cos\frac{\Omega}{2}\right|$$

$$H_b(e^{j\Omega}) = \frac{1}{4}(1 + e^{-j\Omega} + e^{-j2\Omega} + e^{-j3\Omega})$$

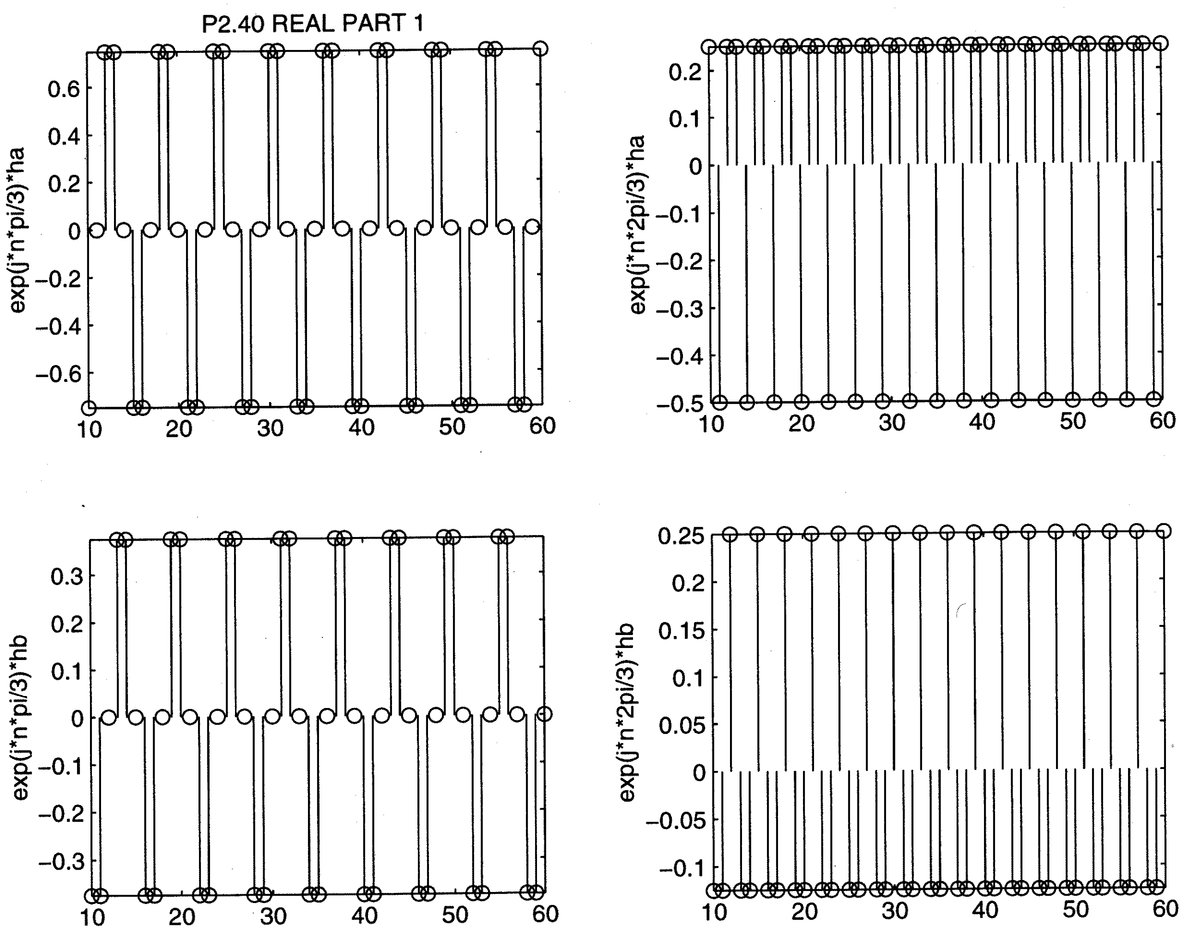
$$= \cos\frac{\Omega}{2} \cdot \cos\Omega \cdot e^{-j\frac{3}{2}\Omega} \rightarrow |H_b(e^{j\Omega})| = \left|\cos\frac{\Omega}{2} \cdot \cos\Omega\right|$$

$$H_c(e^{j\Omega}) = \frac{1}{8}\sum_{n=0}^{7} e^{-j\Omega n} = \frac{1}{8}\,\frac{1 - e^{-j8\Omega}}{1 - e^{-j\Omega}} = \frac{1}{8} e^{-j\frac{7}{2}\Omega}\,\frac{\sin 4\Omega}{\sin\frac{\Omega}{2}}$$

$$\rightarrow |H_c(e^{j\Omega})| = \frac{1}{8}\left|\frac{\sin 4\Omega}{\sin\frac{\Omega}{2}}\right| = \left|\cos\frac{\Omega}{2} \cdot \cos\Omega \cdot \cos 2\Omega\right|$$

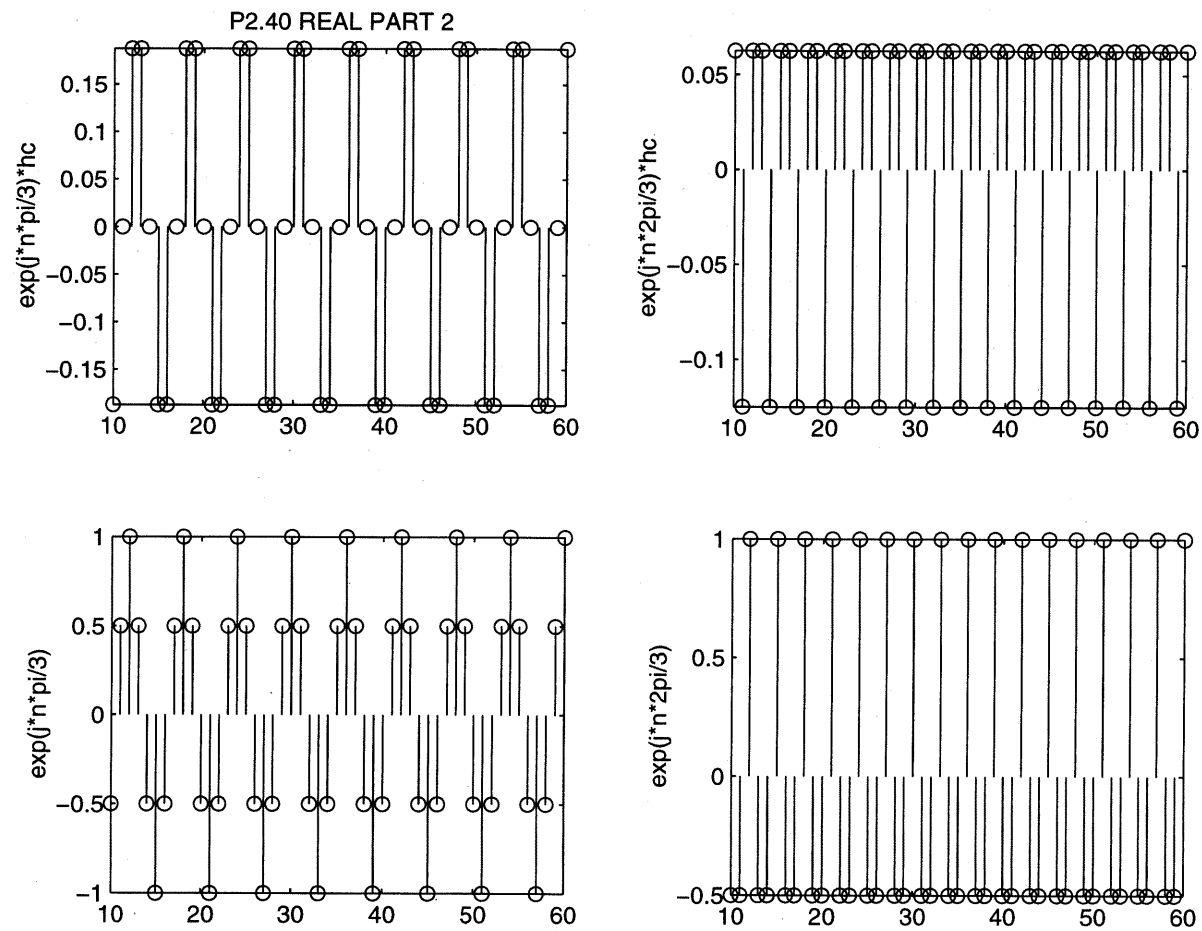
P 2.40

- Plot 1 of 5 -



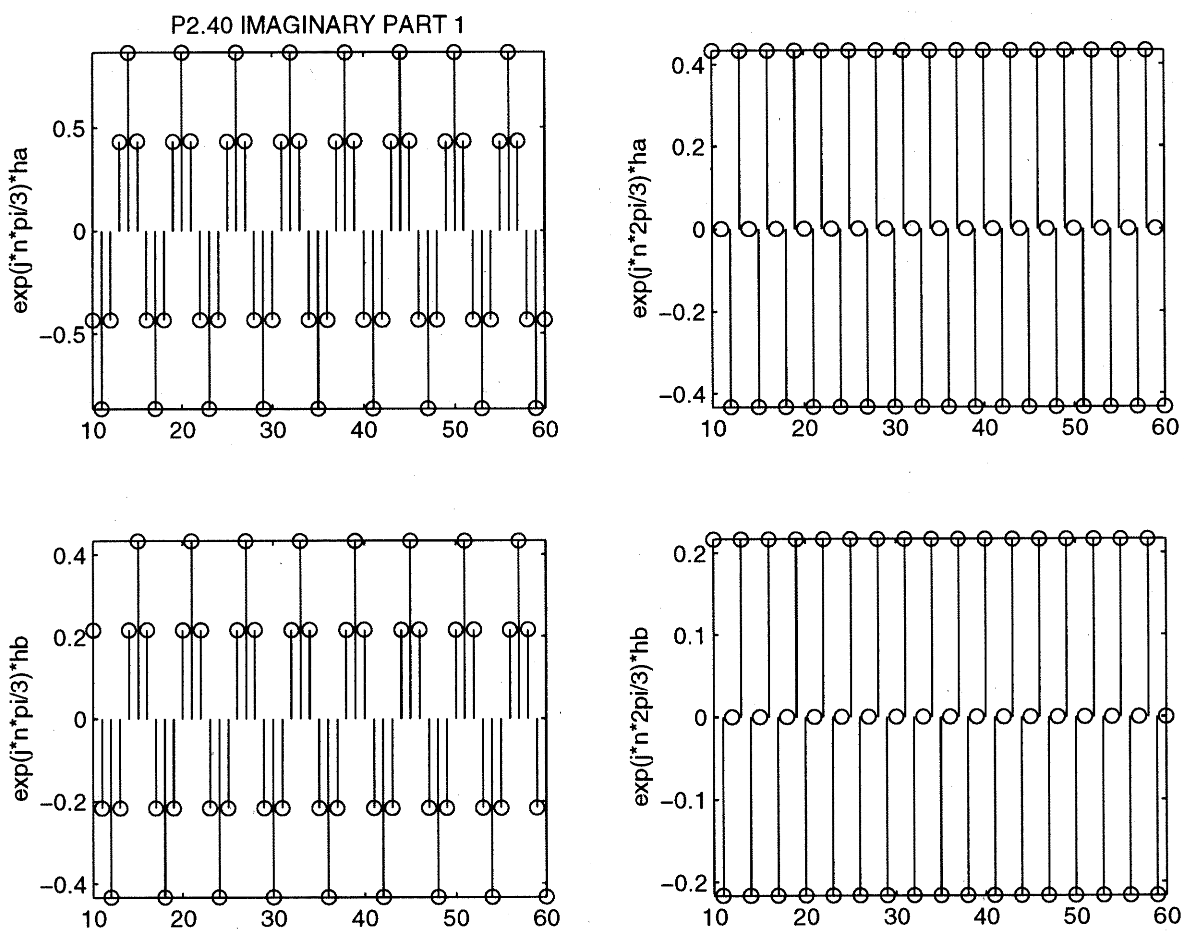
P 2.40

– Plot 2 of 5 –



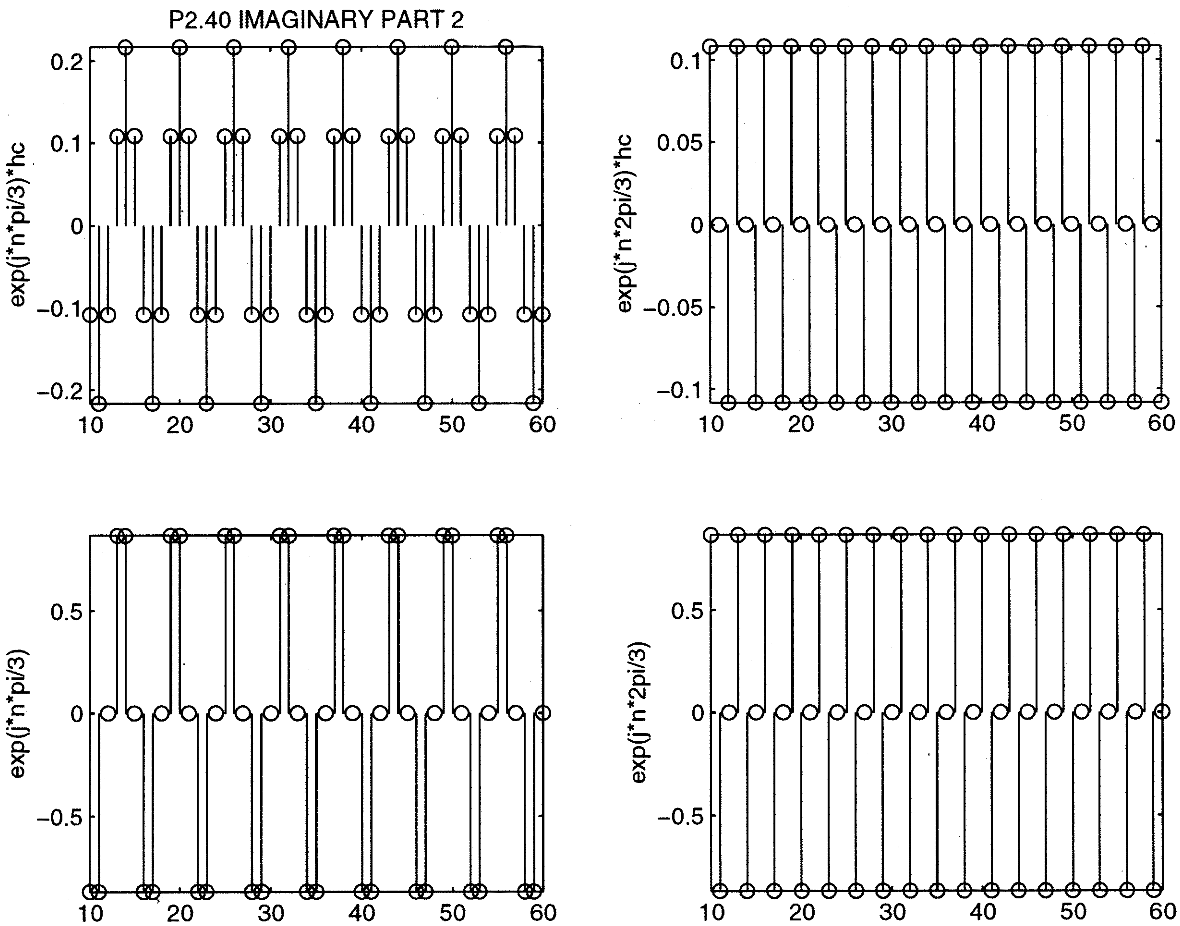
P 2.40

— Plot 3 of 5 —



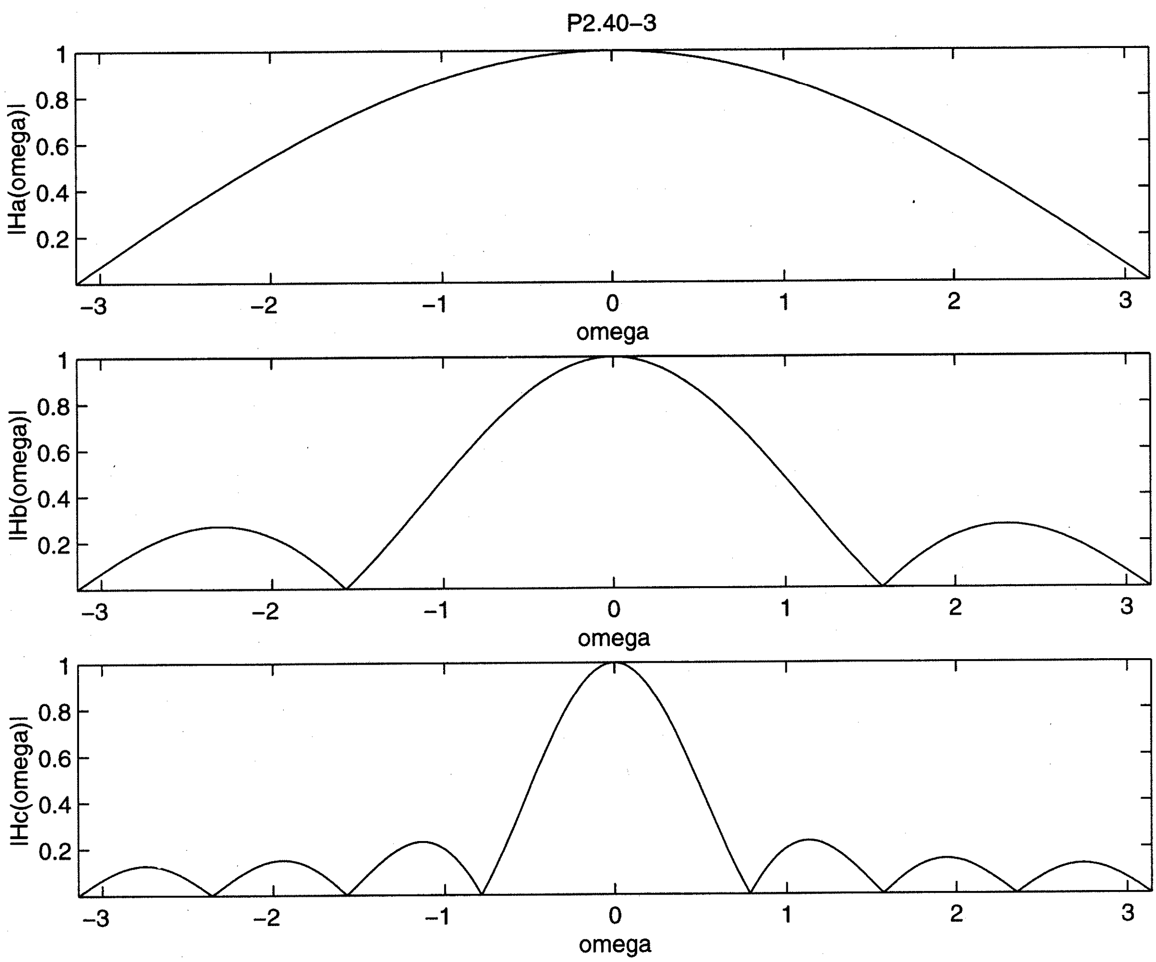
P 2.40

– Plot 4 of 5

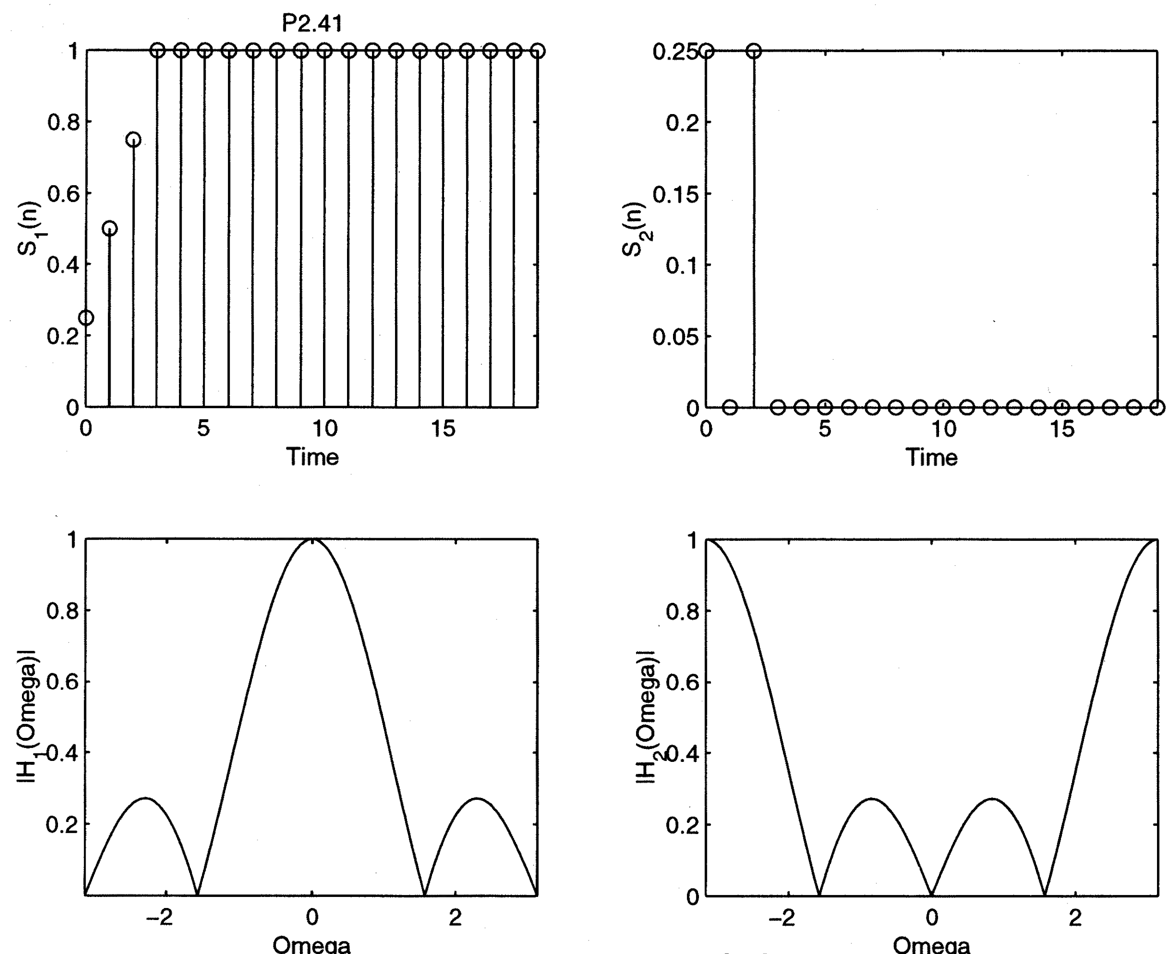


P 2.40

– Plot 5 of 5 –



P 2.41
– Plot 1 of 1 –



$$H_1(e^{j\Omega}) = \frac{1}{4} + \frac{1}{4}e^{-j\Omega} + \frac{1}{4}e^{-j2\Omega} + \frac{1}{4}e^{-j3\Omega}$$

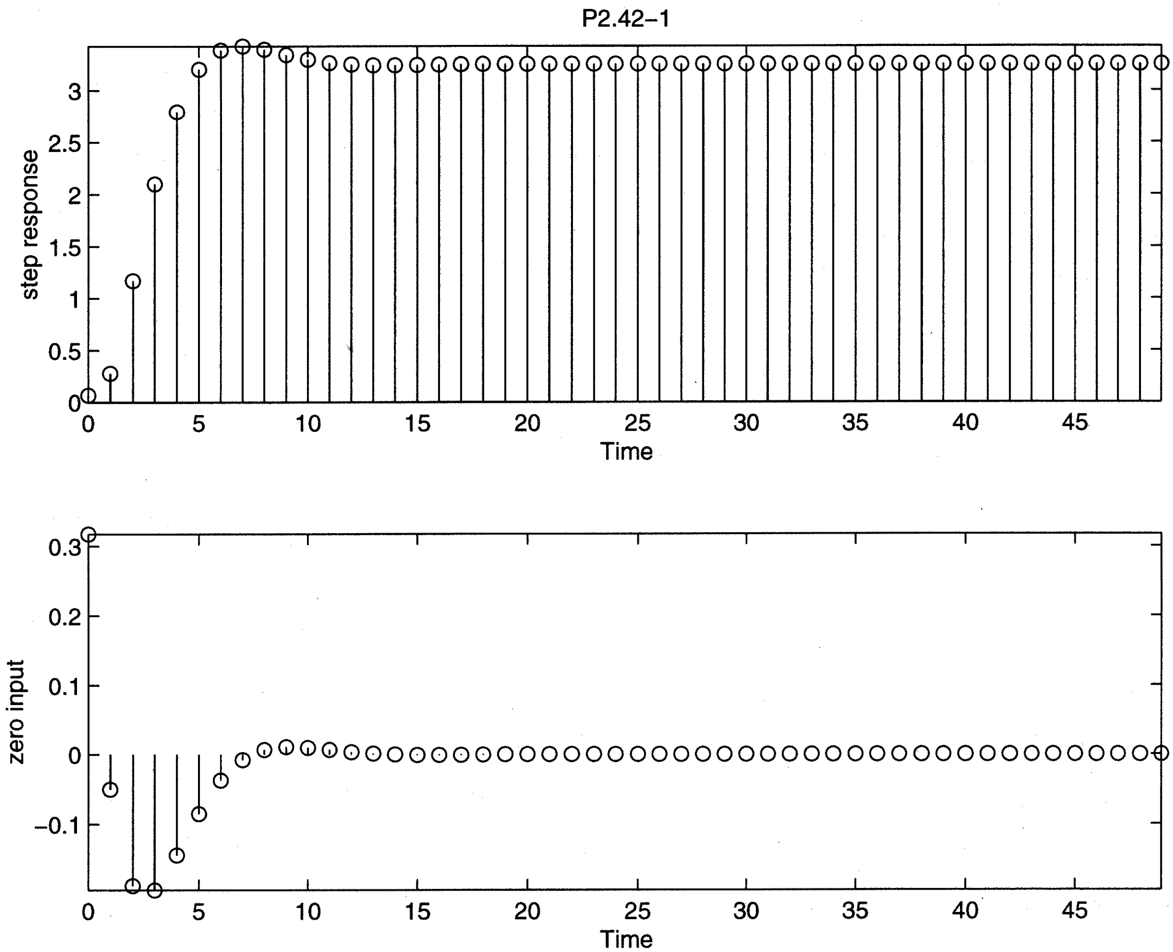
$$= \frac{e^{-j\frac{3\Omega}{2}}}{2}\left(\frac{e^{j\frac{3\Omega}{2}} + e^{j\Omega/2} + e^{-j\Omega/2} + e^{-j\frac{3\Omega}{2}}}{2}\right)$$

$$= e^{-j3\Omega}\left(\frac{1}{2}\cos\frac{3\Omega}{2} + \frac{1}{2}\cos\Omega/2\right)$$

$$H_2(e^{j\Omega}) = \frac{1}{4} - \frac{1}{4}e^{-j\Omega} + \frac{1}{4}e^{-j2\Omega} - \frac{1}{4}e^{-j3\Omega}$$

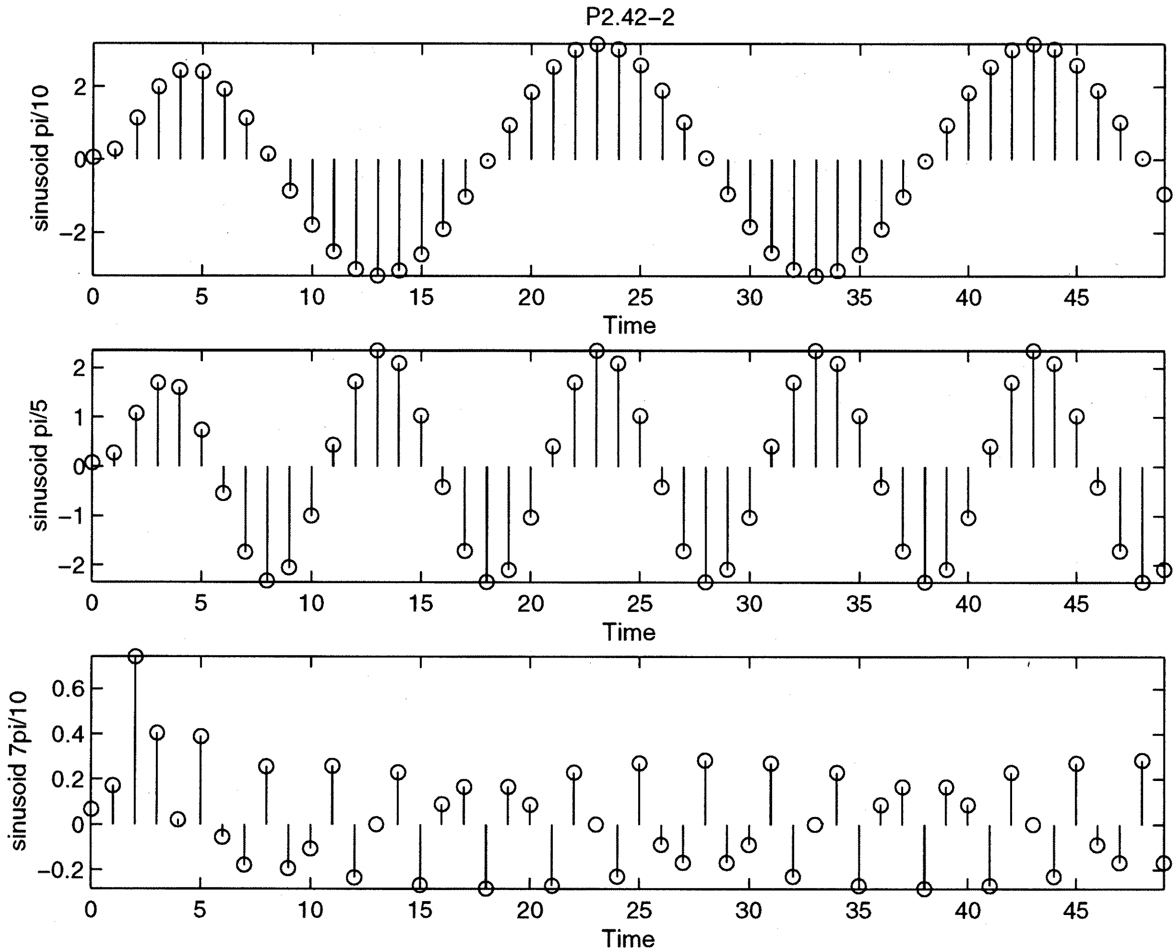
$$= \frac{je^{-j\frac{3\Omega}{2}}}{2}\left(\sin\frac{3\Omega}{2} + \sin\Omega/2\right)$$

P 2.42
– Plot 1 of 3 –



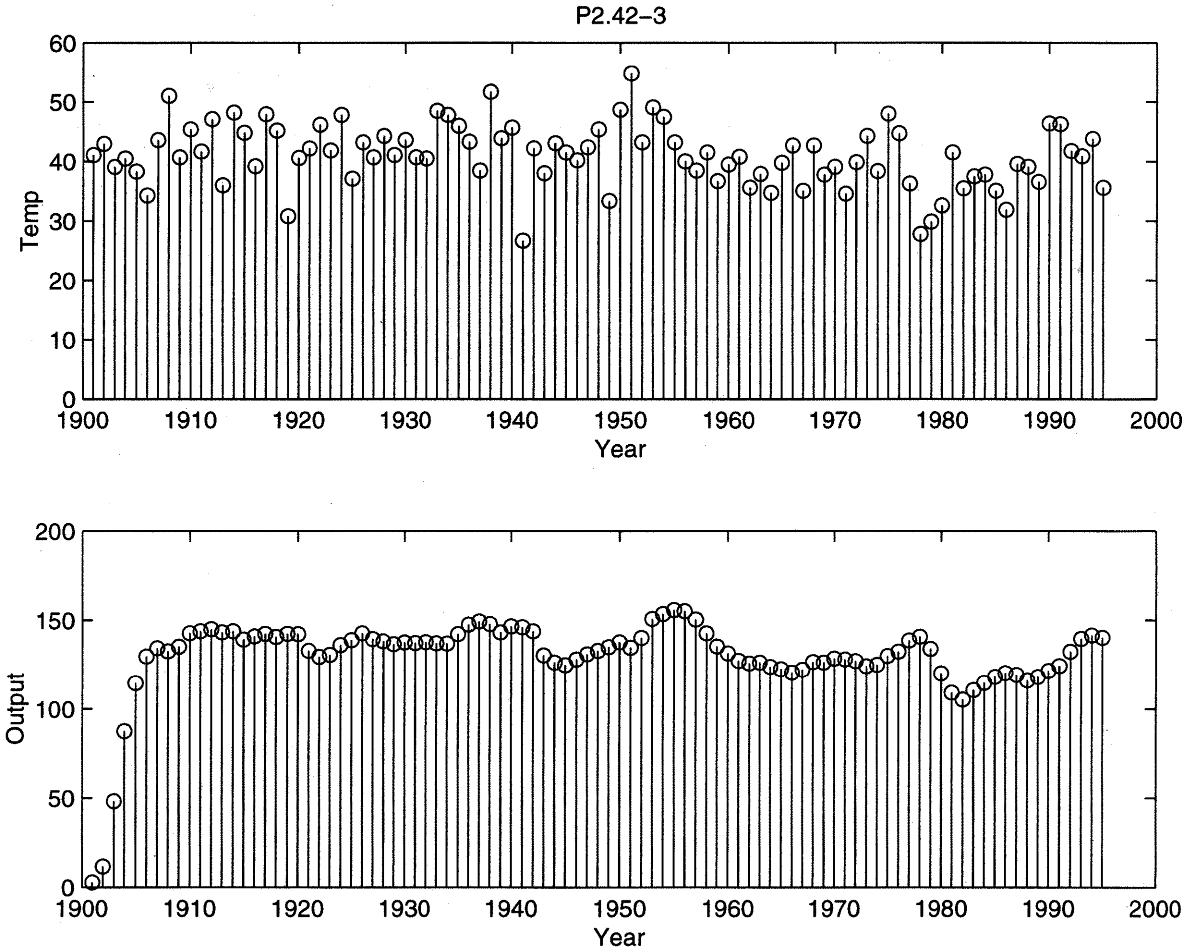
P 2.42

– Plot 2 of 3 –



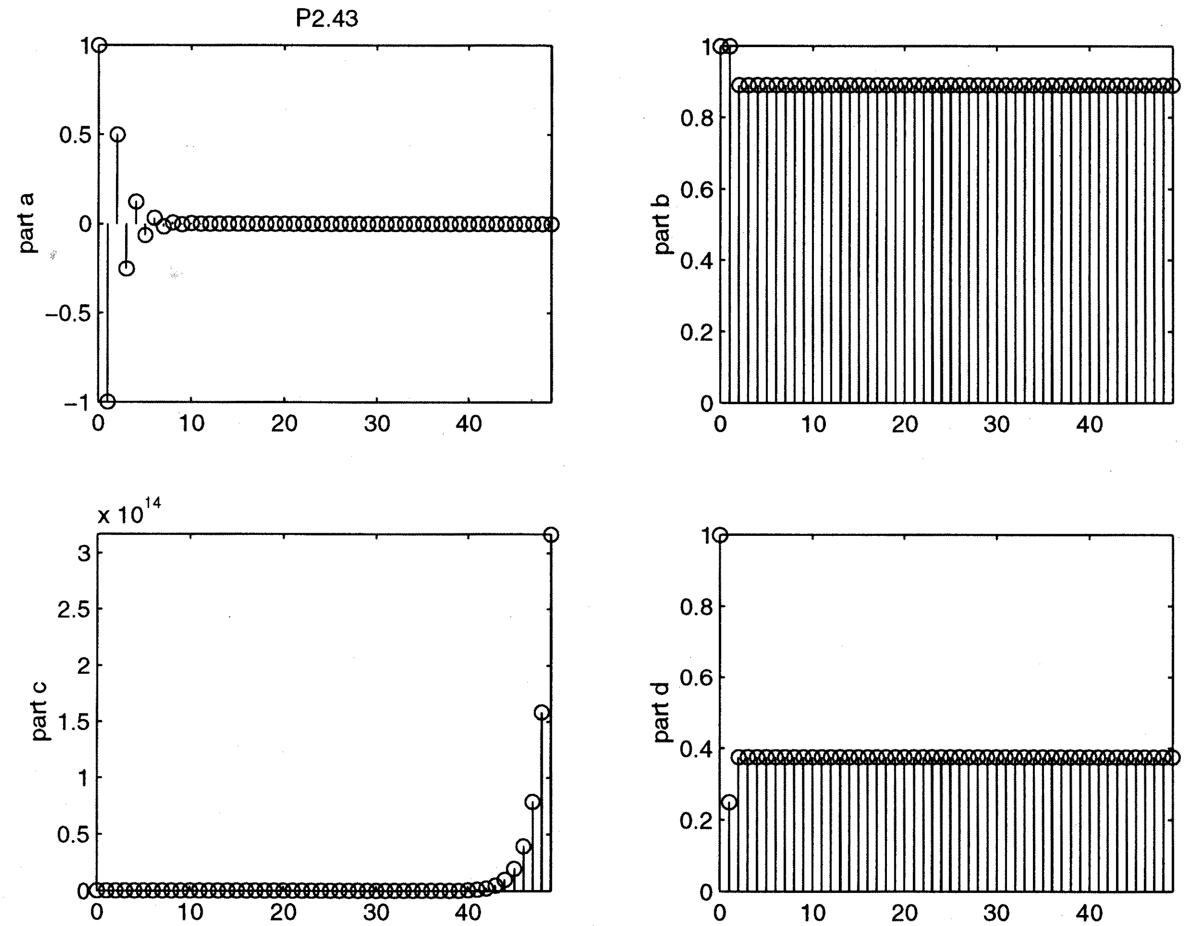
P 2.42

— Plot 3 of 3 —



P 2.43

— Plot 1 of 1 —



$\boxed{2.44}$

(a) To find n_0 :

$$|c_1 \alpha_1^{n_0}| \leqslant |c_1 / 1000| \longrightarrow |\alpha_1|^{n_0} \leqslant 10^{-3}$$

$\longrightarrow n_0 \geqslant 15.6$, choose $\boxed{n_0 = 16}$

Note that $|\alpha_2| = |\alpha_1|$

(b) Note that $H(e^{j\Omega}) = c_0$ derived in part (a)

and $|y[n_0]| \leqslant |c_1 \alpha_1^{n_0}| + |c_2 \alpha_2^{n_0}| + |c_0 e^{j\Omega n}|$

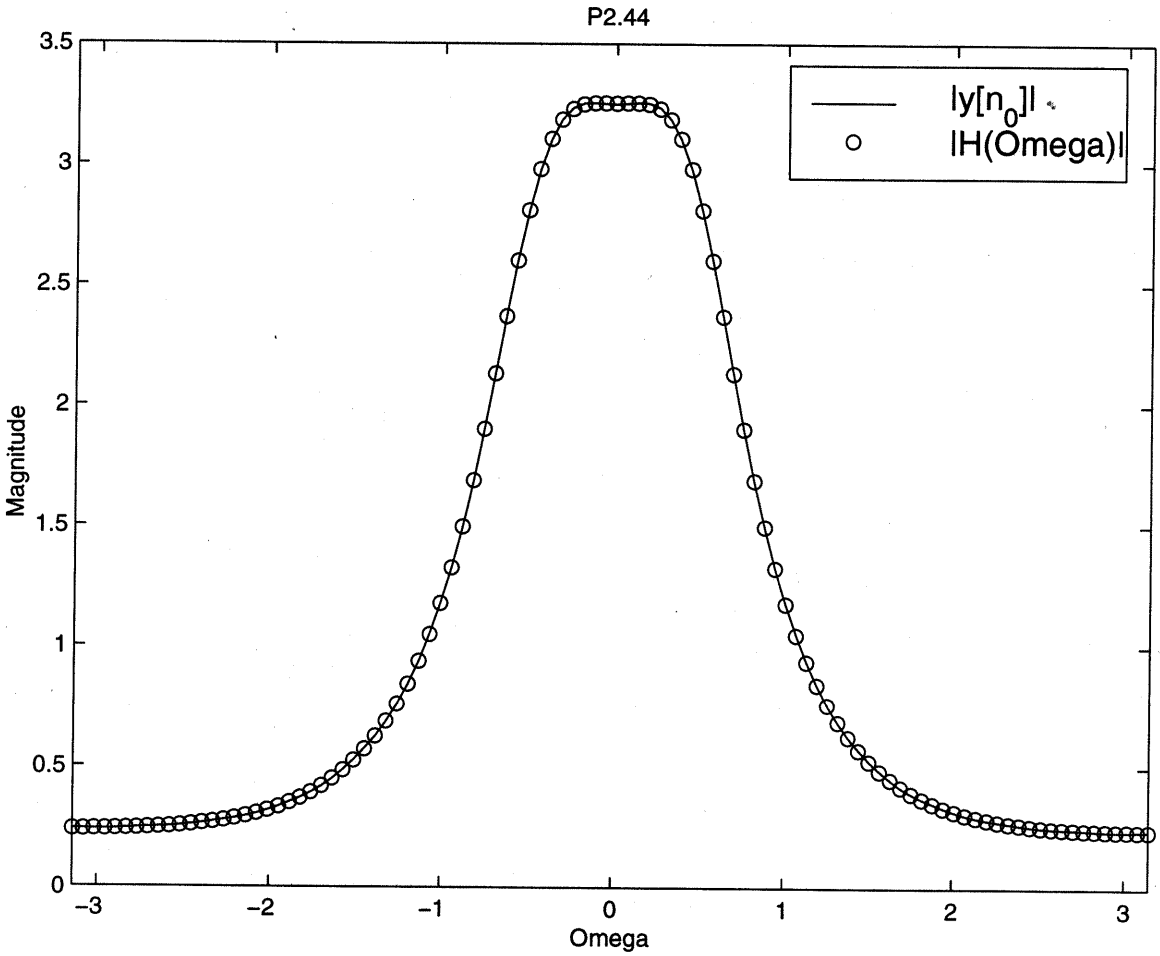
$$\leqslant \frac{|c_1| + |c_2|}{1000} + |H(e^{j\Omega})|$$

Hence, $|y[n_0]| - |H(e^{j\Omega})| \leqslant \dfrac{|c_1| + |c_2|}{1000}$

If $\dfrac{|c_1| + |c_2|}{1000}$ is small , $|y[n_0]| \simeq |H(e^{j\Omega})|$

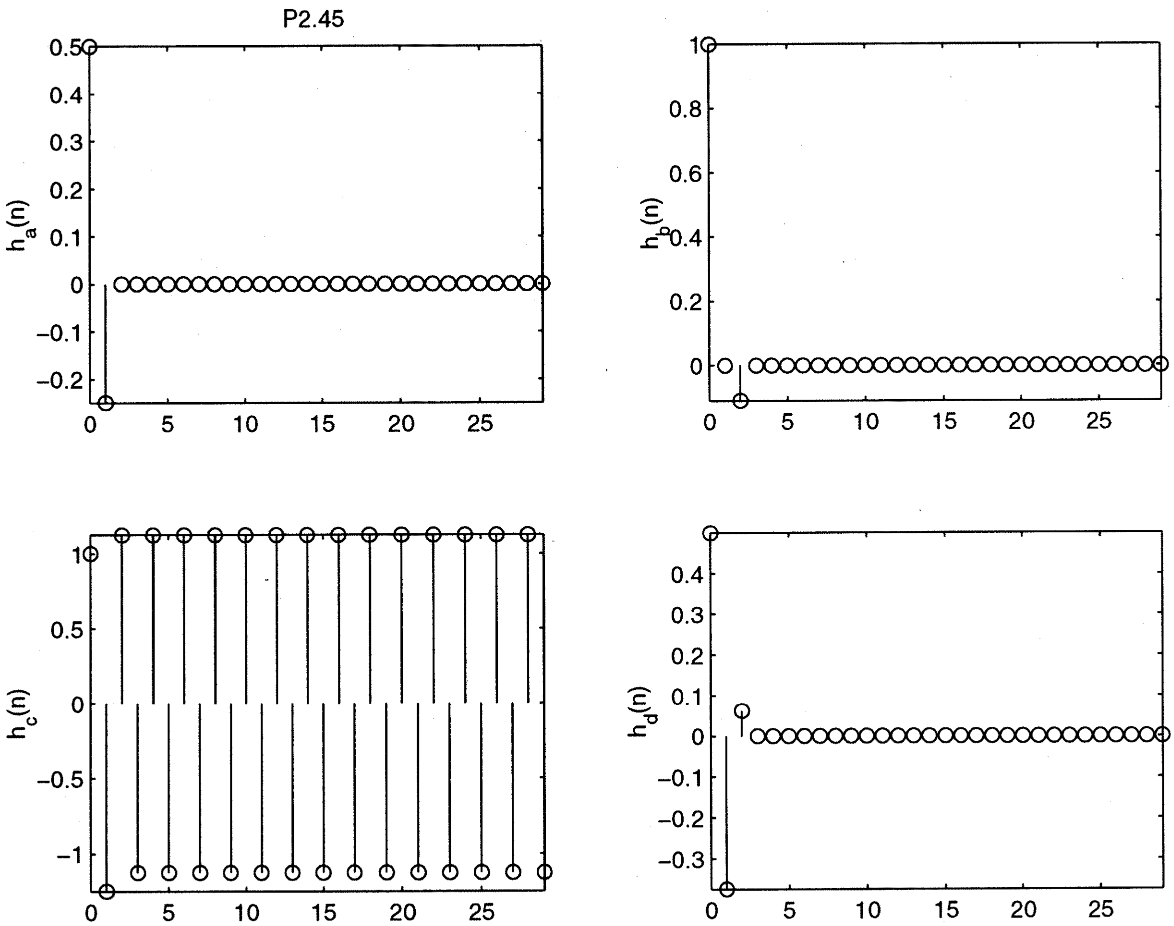
P 2.44

- Plot 1 of 1 -



P 2.45

– Plot 1 of 1 –



2.46

```
P2.46 :
=======

Part (a) :
==========
a =
                          x1           x2
           x1        1.00000     -0.33333
           x2        0.50000            0

b =
                          u1
           x1        2.00000
           x2        6.00000

c =
                          x1           x2
           y1        0.50000     -0.33333

d =
                          u1
           y1              0

Sampling time: unspecified
Discrete-time system.

Part (b) :
==========
a =
                          x1           x2
           x1              0      0.50000
           x2       -0.33333      1.00000

b =
                          u1
           x1        6.00000
           x2        2.00000

c =
                          x1           x2
           y1       -0.33333      0.50000

d =
                          u1
           y1              0

Sampling time: unspecified
Discrete-time system.

Part (c) :
==========
a =
                          x1           x2
           x1        0.41667      0.91667
           x2        0.08333      0.58333

b =
                          u1
           x1        3.00000
           x2       -1.00000


c =
                          x1           x2
           y1              0      1.00000

d =
                          u1
           y1              0

Sampling time: unspecified
Discrete-time system.
```

P 2.47

– Plot 1 of 1 –