Classical Natural Deduction.

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1 Introduction

In the tradition which considers formal logic as an *organon* of thought a central role has been played by the "method of analysis", which amounts to what today, in computer science circles, is called a "bottom-up" or "goal-oriented" procedure[2]. Though the method was largely used in the mathematical practice of the ancient Greeks, its fullest description can be found in Pappus (3rd century A.D.), who writes:

> Now analysis is a method of taking that which is sought as though it were admitted and passing from it through its consequences in order, to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a solution backwards[3].

[1]This paper is based on some ideas I had the privilege to discuss thoroughly with my mentor and friend Marco Mondadori. Although he left us unexpectedly on 4 April 1999, his scientific and philosophical contributions are as alive as ever. I will never be able to express adequately how much I am indebted to him. I am also very much indebted to Dov Gabbay, to whom this paper is dedicated on his 60th birthday, for giving me a job in the first place (back in 1990), for sharing his deep and thrilling views about logic and for teaching me to "think big". Thanks are due also to Krysia Broda with whom I had the opportunity to work on problems related to the contents of this paper. Some of the examples in Sections 2 and 3 are taken from a joint paper with her and Mondadori ([Broda *et al.*,]). We assume the reader is familiar with the natural deduction calculus (in its several variants) as well as with the basic notions of Gentzen's sequent calculi and Smullyan's analytic tableaux. For the reader's convenience the rules of these systems are listed in the Appendix. For the tableau method see [Smullyan, 1968], for natural deduction and its variants the classical references are [Prawitz, 1965] and [Tennant, 1978].

[2]For a general treatment of goal-oriented logical algorithms see [Gabbay and Olivetti, 2000].

[3][Thomas, 1941] pp. 596–599.

This is the so-called *directional* sense of analysis. The idea of an "analytic method", however, is often associated with another sense of "analysis" which is related to the "purity" of the concepts employed to obtain a result and, in the framework of proof-theory, to the *subformula principle* of proofs. In Gentzen's words: "No concepts enter into the proof other than those contained in its final result, and their use was therefore essential to the achievement of that result"[Gentzen, 1935, p. 69] so that "the final result is, as it were, gradually built up from its constituent elements" [Gentzen, 1935, p.88]. Both meanings of analysis have been represented, in the last fifty years, by the notion of a *cut-free* proof in Gentzen-style sequent calculi.

Gentzen introduced the sequent calculi **LK** and **LJ** as well as the natural deduction calculi **NK** and **NJ** in his famous 1935 paper [Gentzen, 1935]. Apparently, his primary interest was in natural deduction, although he considered the sequent calculi as technically more convenient for meta-logical investigation[4]. In the first place he wanted to set up a formal system which "comes as close as possible to actual reasoning"[5]. In this context he introduced the natural deduction calculi where each operator is associated with suitable rules for *introducing* and *eliminating* it. He also suggested that such rules could be seen as *definitions* of the operators themselves. In fact, he argued that the introduction rules alone are sufficient for this purpose and that the elimination rules are "no more, in the final analysis, than consequences of these definitions"[Gentzen, 1935, p. 80]. However, he observed that this "harmony" is exhibited by the intuitionistic calculus but breaks down in the classical case.[6] Indeed, in natural deduction inferences are analysed essentially in a *constructive* way and classical logic is obtained by adding the law of excluded middle in a purely external manner. Gentzen recognized that the special position occupied by this law would have prevented him from proving the *Hauptsatz* (i.e. the subformula principle) in the case of classical logic. So he introduced the sequent calculi as a technical device in order to enunciate and prove the *Hauptsatz* in a convenient form[7] both for intuitionistic and classical logic. These calculi still have a strong deduction-theoretical flavour and Gentzen did not show any sign of considering the relationship between the classical calculus and the semantic notion of entailment which, at the time, was considered as highly suspicious.

Whereas in the natural deduction calculi there are, for each operator, an introduction and an elimination rule, in the sequent calculi there are only

[4] [Gentzen, 1935], p. 69.

[5] [Gentzen, 1935], p. 68.

[6] For a thorough discussion of this subtle meaning-theoretical issue the reader is referred to the writings of Michael Dummett and Dag Prawitz, in particular [Dummett, 1978] and [Prawitz, 1978].

[7] [Gentzen, 1935], p. 69.

introduction rules and the eliminations take the form of introductions in the antecedent. Gentzen seemed to consider the difference between the two formulations as a purely technical aspect.

Gentzen's sequent rules have become a paradigm both in proof-theory and in its applications. This is not without reason. First, like the natural deduction calculi, they provide a precise analysis of the logical operators by specifying how each operator can be introduced in the antecedent or in the succedent of a sequent. Second, their form ensures the validity of the *Hauptsatz*: each proof can be transformed into one which is cut-free, and cut-free proofs enjoy the *subformula property*, that is every sequent in the proof tree contains only subformulas of the formulas in the sequent to be proved.

Therefore, cut-free proofs comply with the *first* meaning of logical analysis, in that they employ no concepts outside those contained in the statement of the theorem. But they comply also with the *second* meaning, in that they can be discovered by means of simple "backwards" procedures like that described by Pappus. Hence, Gentzen-style cut-free methods (which include their semantic-flavoured descendant, known as "the tableau method") were among the first used in automated deduction[8] and are still today very popular in computer science circles and seem to have overturned the once unchallenged supremacy of resolution and its refinements in the area of automated deduction and artificial intelligence.

As a result of Dag Prawitz's later work on normalization ([Prawitz, 1965]) natural deduction also established itself as a respectable "analytic" method in its own right. Not only do normal proofs obey the subformula principle, but they can also be discovered by means of "backwards" procedures similar to those employed for the sequent calculus and there is a close, though far from trivial, relationship between cut-free proofs in the sequent calculus and normal proofs in the natural deduction system. Moreover, natural deduction methods have been devised for a wide variety of logics and constitute the proof-theoretical basis of the taxonomy of logical systems investigated in Dov Gabbay's book on Labelled Deductive Systems.

As Gabbay remarked in that book, logical systems "which may be conceptually far apart (in their philosophical motivation and mathematical definitions), when it comes to automated techniques and proof-theoretical presentation turn out to be brother and sister"[Gabbay, 1996, p. 7]. On the other hand, the same logical system (intended as a set-theoretical definition of a consequence relation) usually admits of a wide variety of proof-theoretical representations which may reveal or hide its similarities with

[8]See, for example, [Beth, 1958], [Wang, 1960], [Kanger, 1963]. See also [Davis, 1983] and [Maslov *et al.*, 1983].

other logical systems. As an example of this approach, Gabbay observes that "it is very easy to move from the truth-table presentation of classical logic to a truth-table system for Lukasiewicz's n-valued logic. It is not so easy to move to an algorithmic system for intuitionistic logic. In comparison, for a Gentzen system presentation, exactly the opposite is true. Intuitionistic and classical logic are neighbours, while Lukasiewicz's logics seem completely different."[Gabbay, 1996, pp. 7–8].

The consideration of these phenomena prompted Gabbay to put forward the view that

> [...] a logical system $\mathbf{L}$ is not just the traditional consequence relation $\vdash$ [...] but a pair $(\vdash, \mathbf{S}_\vdash)$, where $\vdash$ is a mathematically defined consequence relation (i.e. a set of pairs (Δ, Q) such that $\Delta \vdash Q$) [...] and $\mathbf{S}_\vdash$ is an algorithmic system for generating all those pairs. Thus, according to this definition, classical logic $\vdash$ perceived as a set of tautologies together with a Gentzen system $\mathbf{S}_\vdash$ is not the same as classical logic together with the two-valued truth-table decision procedure $\mathbf{T}_\vdash$ for it. In our conceptual framework, $(\vdash, \mathbf{S}_\vdash)$ is *not the same logic* as $(\vdash, \mathbf{T}_\vdash)$[9].

If one takes Gabbay's idea seriously, even civilized areas, such as the elementary proof-theory of classical logic, may still reveal unexplored corners. For instance let us consider classical logic intended as the pair $(\vdash, \mathbf{T}_\vdash)$ where $\vdash$ is the set of all (Δ, Q) such that Δ classically implies Q, and $\mathbf{T}_\vdash$ is the two-valued truth-table decision procedure; we can call this logical system *standard classical logic*. Let us also consider classical logic intended now as the pair $(\vdash, \mathbf{N}_\vdash)$ where $\vdash$ is the same consequence relation as before and $\mathbf{N}_\vdash$ is Gentzen's natural deduction system for it. We shall argue in the sequel (and we think Gabbay would agree) that these two "logical systems" (in his stricter sense) are conceptually, philosophically and algorithmically "far apart". Moreover, natural deduction is not only a specific proof system but a philosophical approach to the meaning of the logical operators. The main problem we address in this paper is the following: can we devise a proof system which can be legitimately called a "natural deduction system" — in that it is constructed in accordance with the basic philosophical principles of Gentzen's and Prawitz' natural deduction — and yet can be perceived as a close "neighbour" of standard classical logic, i.e. the familiar system $(\vdash, \mathbf{T}_\vdash)$?

The interest of this problem is not only philosophical. Natural deduction systems, as remarked above, *do* lend themselves to automated proof search

[9] [Gabbay, 1996], p. 7.

procedure and are at least as "portable" (i.e. easily adaptable to various logics) as the sequent calculus or the tableau method — as shown in Gabbay's book on Labelled Deductive Systems ([Gabbay, 1996]). Moreover, they are built on a solid meaning-theoretical basis and are extensively used in teaching. One of the reasons that their use in computer science and teaching applications is not as widespread as it could be lies perhaps in the fact that their *classical* variants appear to be technically and conceptually contrived, a difficulty which stretches to all the "neighbours" (in Gabbay's sense) of standard classical logic, such as classical modal logics or many-valued logics.

In the sequel we shall argue that *the main drawback of natural deduction*, as a proof-theoretical presentation of classical logic and of its neighbours, does not lie in its being unsuitable to backwards or goal-oriented proof-search, but in the fact that its rules do *not* capture the *classical meaning* of the logical operators. This was already remarked in Prawitz [Prawitz, 1965], where the author pointed out that natural deduction rules are nothing but a reading of Heyting's explanations of the *constructive meaning* of the logical operators ([Heyting, 1956]). In particular, the rules for the *conditional* are based on an interpretation of this operator which has nothing to do with the classical interpretation based on the truth-tables.

This mismatch between the natural deduction rules and the classical meaning of the logical operators is responsible for the fact that rather simple classical tautologies become quite hard to prove within the natural deduction framework, in the sense that their simplest proofs involve "tricks" which are far from being natural.

On the other hand, *the main drawback of the cut-free sequent calculus in all its variants* — still as a proof-theoretical presentation of logics in the neighbourhoods of standard classical logic — is that its rules do *not* capture an essential feature of the classical notions of truth and falsity, namely the *principle of bivalence*, according to which every proposition is determinately true or false, independent of our epistemic means for establishing its truth-value.[10] The lack of a rule expressing bivalence has two main consequences. First, it does not allow the use of "lemmas". As a result, the construction of a cut-free sequent proof (or of a tableau-proof) is a semi-mechanical procedure where the only choice involved concerns the order in which the rules are applied, so restricting significantly the user's (or the computer's) free-

[10]The principle of bivalence is nothing but a semantic reading of the *cut rule* of the sequent calculus. In the Gentzenian tradition, analytic arguments (satisfying the subformula principle) are construed as cut-free proofs. As far as classical logic is concerned, this elimination of cut is, therefore, the *elimination of bivalence*. Indeed, there are 3-valued semantics that characterize the cut-free sequent calculus (and are therefore equivalent to the standard two-valued semantics as far as theoremhood is concerned, but do not validate the cut rule). On these points see [Girard, 1987a] and [D'Agostino, 1990].

dom in devising and representing proofs. A second, related, consequence is that, as emerges from over twenty years of efforts in the area of "automated theorem-proving", the lack of a proper rule expressing the classical principle of bivalence (or "cut", or "transitivity") is responsible for severe inefficiency even if we restrict ourselves to the domain of "analytic" proofs, i.e. proofs satisfying the subformula principle[11].

In the sequel we shall attempt to develop a theory of classical natural deduction which avoids the drawbacks of both standard natural deduction and the cut-free sequent calculus.[12] The underlying philosophical ideas were put forward by Marco Mondadori in the late 1980's, and some of their proof-theoretical consequences were investigated in a series of joint papers by him and the present author.[13] We shall restrict our attention to the propositional operators, since the treatment of quantifiers, though not particularly difficult, requires a good deal of technical details, especially in connection with the introduction rules[14] and will therefore be postponed to a subsequent paper.

Our theory of classical natural deduction makes a neat distinction between *operational rules*, governing the use of logical operators, and *structural rules* dealing with the metaphysical assumptions governing the (classical) notions of truth and falsity, namely the *principle of bivalence* and the *principle of non-contradiction.* The operational rules are, like in standard natural deduction, introduction and elimination rules for the logical operators. Although all the rules have a distinctively "classical" flavour, the elimination rules are shown to be in "harmony" with the introduction rules, in the sense that they can be justified by means of an inversion principle which is very close to Prawitz' one,[15] except that it concerns classical style inference rules, i.e. rules whose premises and conclusions are stated in terms of the truth and falsity of sentences (where truth and falsity are interpreted in a classical, non-epistemic, way), rather than in terms of proofs. These rules can be

[11]On this point and on the advantages on "analytic cut" systems, see [D'Agostino and Mondadori, 1994], [D'Agostino, 1990], [D'Agostino, 1992], [D'Agostino and Mondadori, 1993] and [D'Agostino, 1999].

[12]Another, very different, approach to classical natural deduction which exploits the idea of multiple-conclusion sequents typical of the classical sequent calculus, see [Cellucci, 1988].

[13]See the references in footnote 25.

[14]For a discussion of introduction rules for quantifiers see [Mondadori, 1996].

[15]For a different attempt to develop a concept of "classical harmony" the reader is referred to [Weir, 1986]. We agree with Alan Weir's view that "it would be blind dogmatism to suppose that the classical conception of truth as independent of proof stands in no need of justification. The rebuttal of skeptical attempts to show that the classical notions are incompatible with plausible assumptions in the theory of meaning is a challenge which the classicist ought to try to meet."(p. 461).

seen as assigning a classical meaning to the logical operators, but they are not sufficient to yield the full deductive power of classical logic. The latter can be obtained only with the addition of the metaphysical assumptions expressed by the structural rules. So, the direct inferential role played by the logical operators, via their defined meaning, is clearly separated by the indirect inferential role played by the metaphysical assumptions.

Even with the addition of the structural rules, it is possible to show that the resulting proof system satisfies, like standard natural deduction, the (weak) subformula property. However, since the harmony constraint is satisfied only by the operational rules, a stricter notion of "analytic" proof is obtained with reference to proofs obtained by means of these rules only, without essential applications of the structural rules. We shall argue that the typical idea of an analytic argument as one in which the conclusion is "contained" in the premises applies to proofs which are analytic in this stricter sense. As a sign of the soundness of this approach, it turns out that the recognition of the conclusion's being "contained" in the premises, that is the discovery of a strictly analytic proof of the conclusions from the premises, is a *tractable* problem, as one would expect for any reasonable notion of "containment".

In sections 2 and 3 we shall point out the shortcomings of natural deduction and of the cut-free sequent calculus as formal representations of the notion of classical analytic proof.[16] Next, we shall present our own approach based on classical "intelim" (*int*roduction and *elim*ination) rules, first (section 4) in a format which resembles that of standard natural deduction and then (section 5) in a more convenient format which allows for a more concise representation of proofs. In section 6 we shall address typical normalization issues and show the subformula property for intelim proofs. In section 7, we shall investigate *strictly analytic refutations* — refutations which do not require applications of the principle of bivalence. We shall then propose a graph-based decision procedure for them and show that it runs in quadratic time. Finally, in section 8 we shall point at some future research which may be carried out in connection with this work.

2 Why Gentzen's natural deduction is not a natural formalization of classical logic

The difficulties of natural deduction with classical tautologies depend on the fact that a sentence and its negation are not treated symmetrically, whereas this is exactly what one would expect from the classical meaning of the logical operators.

[16] The example in sections 2 and 3 are taken from [Broda *et al.*,].

Let us consider one of *de Morgan's laws*

$$\neg(P \wedge Q) \to (\neg P \vee \neg Q)$$

and try to prove it in the classical natural deduction system with the *classical reductio ad absurdum* rule. It is not difficult to check that the simplest proof one can obtain is the following:[17]

$$\dfrac{\dfrac{\neg(P \wedge Q)^1 \qquad \dfrac{\dfrac{\dfrac{\neg(\neg P \vee \neg Q)^2 \quad \dfrac{\neg P^3}{\neg P \vee Q}\,3}{\mathbf{F}}\,2,3}{P}\,2 \qquad \dfrac{\dfrac{\neg(\neg P \vee \neg Q)^2 \quad \dfrac{\neg Q^4}{\neg P \vee \neg Q}\,4}{\mathbf{F}}\,2,4}{Q}\,2}{P \wedge Q}\,2}{\dfrac{\mathbf{F}}{\neg P \vee \neg Q}\,1}\,2,1}{\neg(P \wedge Q) \to (\neg P \vee \neg Q)}\,\emptyset$$

where the numerals on the right of the inference lines mean that the conclusion depends on the assumptions marked with the same numerals (see the appendix for a listing of the natural deduction rules).

Notice that we obtain a rather unnatural deduction whatever variant of "natural" deduction we may decide to use, although probably the least bad of all is obtained by means of the variant which *explicitly* incorporates the principle of bivalence in the form of the "classical dilemma" rule:[18]

[17]We adopt a standard propositional language with P, Q, R, etc., possibly with subscripts, as propositional variables, augmented with a constant $\mathbf{F}$ standing for "the absurd".

[18]See [Tennant, 1978].

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(P\wedge Q)^1 \qquad \dfrac{P^2 \quad Q^3}{P\wedge Q}\,2,3}{\mathbf{F}}\,1,2,3}{\neg P}\,1,3}{\neg P\vee\neg Q}\,1,3 \qquad\qquad \dfrac{\neg Q^3}{\neg P\vee\neg Q}\,3}{\neg P\vee\neg Q}\,1}{\neg(P\wedge Q)\rightarrow(\neg P\vee\neg Q)}\,\emptyset
$$

Another example of a tautology whose natural deduction proofs are extremely contrived is Peirce's law: $((P \rightarrow Q) \rightarrow P) \rightarrow P$. The reader can verify that any attempt to prove this classical tautology within the natural deduction framework leads to logical atrocities. Here is a typical proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{(P\rightarrow Q)\rightarrow P^1 \qquad \dfrac{\dfrac{\dfrac{\neg P^2 \quad P^3}{\mathbf{F}}\,2,3}{Q}\,2,3}{P\rightarrow Q}\,2}{P}\,1,2 \qquad\qquad \neg P^2}{\mathbf{F}}\,1,2}{P}\,1}{((P\rightarrow Q)\rightarrow P)\rightarrow P}\,\emptyset
$$

It should be emphasized that the main problem here does not lie in the complexity of proofs (i.e. their length with respect to the length of the tautology to be proved): in fact, if we look at the class of tautologies which generalizes the de Morgan law proved above, namely $T_i = \neg(P_1 \wedge \ldots \wedge P_i) \rightarrow \neg P_1 \vee \ldots \vee \neg P_i$, it is easy to verify that the length of their shortest proofs is linear in the length of the tautology under consideration, both in the system with the classical dilemma and in the system with classical *reductio*. It is rather their *contrived character* which is disturbing, and this depends on the fact that the natural deduction rules do not capture the *classical* meaning of the logical operators. Since these rules are closely related to the *constructive* meaning of the operators, we would not expect such logical atrocities when we use them to prove *intuitionistic* tautologies. (Notice that both the de Morgan law we have considered above and Peirce's law are *not* intuitionistically valid.) Consider, for instance, the other (intuitionistically valid) de Morgan law:

$$\neg P \vee \neg Q \to \neg(P \wedge Q).$$

One can easily obtain a "really natural" intuitionistic proof of this tautology as follows:

$$\dfrac{\dfrac{\neg P \vee \neg Q^{1} \qquad \dfrac{\dfrac{\neg P^{3} \qquad \dfrac{P \wedge Q^{2}}{P}\,2}{\mathbf{F}}\,2,3}{\neg(P \wedge Q)}\,3 \qquad \dfrac{\dfrac{\neg Q^{5} \qquad \dfrac{P \wedge Q^{4}}{Q}\,4}{\mathbf{F}}\,4,5}{\neg(P \wedge Q)}\,5}{\neg(P \wedge Q)}\,1}{\neg P \vee \neg Q \to \neg(P \wedge Q)}\,\emptyset$$

These examples illustrate our claim that *"natural" deduction is really natural only for intuitionistic tautologies* (and not necessarily for all of them).

3 Why cut-free systems are not natural formalizations of classical logic

Contrary to natural deduction, the systems based on the classical (cut-free) sequent calculus treat a sentence and its negation symmetrically. For instance, the analytic tableau method allows for a very simple and natural proof of the non-intuitionistic de Morgan law (see Figure 1, where $\times$ indicates, as usual, a closed branch). We stress once again that it is not the length of proofs which is at issue here. Although the tableau proofs for the class of tautologies which generalizes this de Morgan law are, strictly speaking, shorter than the corresponding natural deduction proofs, the difference in length is negligible, since in both cases one obtains proofs whose length is linear in the length of the given tautology. What is at issue is that the tableau proofs are more natural than those based on the "natural" deduction rules, as a result of the symmetrical treatment of sentences with respect to their negations. The symmetry of the tableau rules, however, is by no means sufficient to guarantee, in general, that such rules always lead to proofs which are natural from the classical point of view. Consider the class of tautologies T_i^* defined as the disjunction of all possible conjunctions which can be obtained from i propositional variables or their negations. For instance $T_2^* = (P_1 \wedge P_2) \vee (P_1 \wedge \neg P_2) \vee (\neg P_1 \wedge P_2) \vee (\neg P_1 \wedge \neg P_2)$. Each T_i^* contains i distinct propositional variables, but 2^i conjunctions. In [D'Agostino, 1992] it is shown that this class of tautologies is sufficient to separate the tableau method (and the cut-free sequent calculus in tree form) from the truth-table method. In fact, it is not difficult to prove that the length of

$$
\begin{array}{c}
\neg(\neg(P \wedge Q) \to (\neg P \vee \neg Q)) \\
| \\
\neg(P \wedge Q) \\
| \\
\neg(\neg P \vee \neg Q) \\
| \\
\neg\neg P \\
| \\
\neg\neg Q \\
| \\
P \\
| \\
Q \\
/ \quad \backslash \\
\neg P \qquad \neg Q \\
\times \qquad \times
\end{array}
$$

Figure 1. A tableau proof of the non-intuitionistic de Morgan Law.

the shortest tableau proofs for the tautologies T_i^* is not bounded above by any polynomial in the length of the tautologies. This implies that analytic tableaux (and the cut-free sequent calculus) cannot p-simulate the truth-table method, since the truth-tables for T_i^* have polynomial size. It also implies that analytic tableaux cannot p-simulate the natural deduction calculus (in any of its variants) since such tautologies do have polynomial size proofs in the latter and such proofs are not at all "difficult" to find.[19]

This embarrassing situation stems from the lack, both in the tableau method and in the cut-free sequent calculus, of a rule expressing bivalence. None of its rules forces a bivalent interpretation of the notion of truth: all the rules are compatible with many-valued interpretations. The fact that all classical tautologies can be proved is the result of the equivalence between such many-valued interpretations and the standard two-valued one. The non-bivalent character of the tableau rules allows for the generation of branches which do not represent *mutually incompatible* valuations and, therefore, give rise to a great deal of undesired redundancy in the enumeration of possible cases. (For a more detailed discussion of

[19]For an excellent survey of the relative complexity of propositional proof systems, the reader is referred to [Urquhart, 1995].

this point the reader is referred to [D'Agostino, 1990, D'Agostino, 1992, D'Agostino and Mondadori, 1994, D'Agostino, 1999].)

4 Classical natural deduction

In order to devise a truly classical variant of natural deduction, we must introduce a set of introduction and elimination rules which correspond closely to the *classical*, rather than to the intuitionistic, reading of the logical operators. From this point of view, as remarked above, the main problematic aspect of natural deduction calculi in the Gentzen-Prawitz tradition is the non-symmetrical treatment of sentences with respect to their negations. The typical involutive character of classical negation is captured in a rather roundabout way, via an *ad hoc* axiom such as the law of excluded middle or a similarly *ad hoc* rule for classical negation. Other problematic aspects are the "discharge rules" for introducing the conditional operator "$\rightarrow$" and for eliminating the disjunction operator "$\vee$", which are both based on the standard *intuitionistic* explanation of these operators. The remaining rules, on the other hand, can apparently be accommodated within the classical truth-functional view.

The problem about negation lies in the fact that, according to the intuitionistic interpretation, $\neg P$ is defined as having the same meaning as $P \rightarrow \mathbf{F}$, where $\mathbf{F}$ is "the absurd". The negation of P can be legitimately asserted only if one is able to provide a refutation of P. On the other hand, classical negation cannot be naturally defined in such a way. Classically speaking, the negation of a proposition P is true if and only if P is false. Thus, the meaning of $\neg P$ is intertwined with the "ontological" notion of falsity, governed by the principle of bivalence (as well as by the principle of non-contradiction), and is not amenable to epistemic reductions.

In intuitionistic natural deduction introduction and elimination rules specify, respectively, the conditions for validly *asserting* a sentence P containing an operator $\sharp$ as its main operator and the consequences of such an assertion. From a classical perspective there should be symmetrical introduction and elimination rules specifying, respectively, the conditions under which P *is true or false* and the consequences of its being true or false. Hence, besides specifying introduction rules determining the conditions under which the truth of P can be inferred, we also need to specify suitable introduction rules determining the conditions under which its *falsity* — i.e. the truth of its negation $\neg P$ — can be inferred. Similarly, we have to specify suitable elimination rules determining the consequences of the falsity, as well as of the truth, of such a sentence P.

Following this approach, the obvious introduction rules for *classical* conjunction and disjunction, which can immediately be read off the classical two-valued truth-tables, are the following:

$$\frac{P \quad Q}{P \wedge Q} \quad \frac{\neg P}{\neg(P \wedge Q)} \quad \frac{\neg Q}{\neg(P \wedge Q)} \quad \frac{P}{P \vee Q} \quad \frac{Q}{P \vee Q} \quad \frac{\neg P \quad \neg Q}{\neg(P \vee Q)}$$

As for negation, since from the classical viewpoint the negation of P means that P is false and, as argued above, classical falsity is a primitive (ontological) notion, there cannot be a rule specifying the conditions under which a proposition $\neg P$ is true apart from the trivial one of the form $\neg P/\neg P$.[20] On the other hand, there is a rule specifying the conditions under which $\neg P$ is *false*, that is $\neg\neg P$ is true:

$$\frac{P}{\neg\neg P}$$

From a natural deduction standpoint such introduction rules should be taken as *definitions* of the logical operators. We claim that they provide a definition of the *classical* logical operators — the operators of what we have called *standard* classical logic — in that they are the only truth-conditional introduction rules which can be read off the two-valued truth tables. Classical elimination rules have therefore to be justified in terms of these definitions. Such rules, whichever they may be, should satisfy Prawitz' *inversion principle*: essentially, by an application of an elimination rule one only recovers a sentence which would have been established if the premiss(es) of the application had been inferred by an application of an introduction rule ([Prawitz, 1965]).

Let us consider the case of disjunction. First we justify the elimination rule for negated disjunctions. Since the conditions for inferring the falsity of a disjunction $P \vee Q$ are that *both* P and Q are false, the elimination rules for negated disjunctions are the following:

$$\frac{\neg(P \vee Q)}{\neg P} \quad \frac{\neg(P \vee Q)}{\neg Q}$$

Moreover, by the same reasoning, the elimination rules for true conjunctions are determined as follows:

[20]If we used *signed formulas* such a rule would no longer be redundant and would have the form: $FP/T\neg P$. The use of signed formulas, though unusual in the context of natural deduction calculi — which is the reason why we have not adopted it in this paper — may have some advantages especially for adapting them to some non-classical logics. On this point see also the final section on "future work".

$$\frac{P \wedge Q}{P} \qquad \frac{P \wedge Q}{Q}$$

Determining the elimination rules for true disjunctions and false conjunctions requires a slightly more complex argument. In order to infer $P \vee Q$ by means of an introduction rule we need either P or Q as premiss. So, what we can infer from $P \vee Q$ is that *either* P is true *or* Q is true, but we have no means for determining which. In some analytic methods developed within the tradition of the cut-free sequent calculus — such as Smullyan's "analytic tableaux" — the "elimination rule" for $\vee$ reflects this non-determinism. For instance, in analytic tableaux, $\vee$-elimination is the following:

$$\begin{array}{ccc} & P \vee Q & \\ & / \quad \backslash & \\ P & & Q \end{array}$$

However, such a non-deterministic elimination rule is not satisfactory from a natural deduction perspective. One could argue that disjunction here is hardly "eliminated" at all. The rule is not an *inference* rule in the standard sense: one cannot describe this rule as establishing that from the truth of $P \vee Q$ we can either infer the truth of P or infer the truth of Q, since this interpretation would be plainly incorrect. The only way of describing this rule as an inference rule would be to say something like: "from $P \vee Q$ we can infer $P \vee Q$" which is a sheer triviality.[21]

The solution of this problem in the framework of standard natural deduction is the well-known "constructive dilemma" elimination rule for disjunction. This rule, like the introduction rule for conditionals and negations, involves the discharge of assumptions. Strictly speaking, such "discharge rules" are not *inference* rules either, since they do not allow to infer a certain sentence as conclusion from the assumption of certain other sentences as premises, but involve further manipulation of the assumption formulas. They are better described as *proof rules* asserting that certain proofs can be combined to yield another proof. In other words, the premises of such proof rules are *sequents*, rather than formulas. Since the truth of a sentence, in intuitionistic terms, is tantamount to its provability, it is quite appropriate to construe logical deduction in terms of such proof rules: understanding the meaning of a sentence means understanding how it can be proved, and since logical deduction is a type of inference which is justified on the sole

[21]For a similar remark, referred to the multiple-conclusion sequent calculus, see [Dummett, 1991], p. 195.

basis of the meaning of the logical operators, it is perfectly natural to see it as a combinatorics of proofs. Hence, the very distinction we have just made between inference rules and proof rules tends to fade, especially considering that in any standard consequence relation (closed under transitivity) every inference rule can be reinterpreted as a proof rule[22] (but not vice-versa).

From the classical viewpoint, however, the distinction makes perfectly good sense. Understanding the meaning of a sentence means understanding how the world should be for the sentence to be true and how it should be for the sentence to be false. Therefore, the premises and the conclusions of logical rules are naturally construed in terms of the truth or falsity of sentences rather than in terms of proofs. As a result, the most direct logical rules are *inference rules* rather than proof rules.

We therefore claim that a *classical inference rule* must be a rule which allows us to infer, as conclusion, the truth or the falsity of a sentence from premises asserting the truth or the falsity of other sentences. *Classical introduction rules* for a logical operator $\sharp$ will be rules expressing the conditions for the truth or falsity of a sentence P containing $\sharp$ as main operator and whose premises assert the truth or the falsity of immediate subformulas of P. *Classical elimination rules* for $\sharp$ will be rules expressing the consequences of the truth or falsity of a sentence P containing $\sharp$ as main operator — possibly with help of auxiliary premises asserting the truth or falsity of some immediate subformula of P — and whose conclusion asserts the truth or falsity of another immediate subformula of P. According to this definition, constructive dilemma is not a classical inference rule.

This does not imply, of course, that constructive dilemma, like other similar discharge rules, is not classically sound. It only implies that if the meaning of the logical operators is to be explained truth-conditionally, in terms of truth and falsity of sentences rather than in terms of proofs (from given assumptions), constructive dilemma is not an argument that stems *directly* from the meaning of disjunction, but one that can be justified only indirectly by (meta-level) reasoning on the concept of classical proof.[23]

However, constructive dilemma is not the *only* elimination rule for true disjunctions that can be justified in terms of the corresponding introduction rules in accordance with the inversion principle. A better solution, in classical terms, arises from the following argument. In order to infer $P \vee Q$ by means of an introduction rule we need either P or Q as premiss. So, if we have valid grounds (in truth-conditional terms) for asserting the truth of

[22]For instance, the validity of the "horizontal" inference rule $P, Q \vdash P \wedge Q$ implies the validity of the "vertical" proof rule which from sequents $\Gamma \vdash P$ and $\Delta \vdash Q$ as premises leads to the sequent $\Gamma, \Delta \vdash P \wedge Q$ as conclusion.

[23]Such an indirect "natural deduction" system for full classical logic is presented in [Smullyan, 1965].

$P \vee Q$, in accordance with the meaning of $\vee$ fixed by the appropriate introduction rules, we must also have valid grounds for asserting the truth of P or for asserting the truth of Q. Suppose now that we have valid grounds for asserting the truth of $\neg P$ (that is the falsity of P). Then, by consistency, the only justification we may have for asserting $P \vee Q$ can only be the truth of Q. This analysis leads to a pair of elimination rules for true disjunctions that correspond to the traditional disjunctive syllogism argument:

$$\frac{P \vee Q \quad \neg P}{Q} \qquad \frac{P \vee Q \quad \neg Q}{P}$$

Similar considerations lead to the dual rules for eliminating false conjunctions:

$$\frac{\neg(P \wedge Q) \quad P}{\neg Q} \qquad \frac{\neg(P \wedge Q) \quad Q}{\neg P}$$

and to the double negation rule for eliminating false negations:

$$\frac{\neg\neg P}{P}$$

Let us now turn our attention to the conditional operator. Since the boolean conditional $P \to Q$ can be defined in terms of the other boolean operators, for instance as $\neg(P \wedge \neg Q)$, the introduction and elimination rules for conditionals are determined as follows:

$$\frac{Q}{P \to Q} \qquad \frac{\neg P}{P \to Q} \qquad \frac{P \quad \neg Q}{\neg(P \to Q)}$$

$$\frac{P \to Q \quad P}{Q} \qquad \frac{P \to Q \quad \neg Q}{\neg P}$$

$$\frac{\neg(P \to Q)}{P} \qquad \frac{\neg(P \to Q)}{\neg Q}$$

(Notice the presence of *modus tollens* as a second elimination rule, besides *modus ponens*, for true conditionals.) Interestingly enough, the elimination rules we have discussed in this section were already discovered by Chrysippus who claimed them to be the fundamental rules of reasoning ("anapodeiktoi"), except that disjunction was interpreted by him in an exclusive sense.

Chrysippus also maintained that his "anapodeiktoi" formed a complete set of inference rules.[24]

We have so specified a set of introduction and elimination rules such that:

1. the introduction rules can be taken as definitions of the classical meaning of the logical operators, specifying — in accordance with the standard two-valued semantics — the conditions for inferring the truth or the falsity of a sentence in terms of the truth or falsity of their immediate subformulas;

2. the elimination rules are justified in terms of this meaning by means of arguments based on a form of the inversion principle;

3. the rules for conjunction and disjunction are dual of each other;

4. sentences and their negations are treated in a symmetric way;

Our approach can be seen as falling half-way between the standard truth-conditional account of classical semantics and a proof-theoretical account, based on the Wittgensteinian meaning-as-use view, which is more typical of intuitionistic logic and its "neighbours".[25] We shall call our inference rules *intelim rules* for the classical operators. When dealing with the two-premiss elimination rules — such as the rules for eliminating true disjunctions, false conjunctions and true conditionals — we call the more complex premiss *major premiss*, while the auxiliary premiss is called *minor premiss*.

However, this is not the end of the story. Despite our claim that the introduction rules can be taken as satisfactory definitions of the classical meaning of the logical operators, and that the elimination rules are in "harmony" with them, as prescribed by the inversion principle, the intelim rules are *not* a complete set of rules for classical propositional logic. The reason, according to our analysis, is that *classical semantics is not completely characterized by the meaning of the logical operators* as it can be defined via the inferential approach. In other words: if we accept (i) the natural deduction stand that the meaning of the logical operators is fixed by specifying the introduction rules for them, (ii) that the introduction rules specified

[24] "The indemonstrables are those of which the Stoics say that they need no proof to be maintained. [...] They envisage many indemonstrable but especially five, from which it seems all others can be deduced". See [Blanché, 1970], pp.115–119 and [Bochensky, 1961], p.126.

[25] Variations on this proof-theoretical theme concerning classical propositional logic, the subformula property and the inversion principle are discussed also in [Mondadori, 1988a], [Mondadori, 1988b], [Mondadori, 1988d], [Mondadori, 1988c], [Mondadori, 1989], [D'Agostino and Mondadori, 1991],[D'Agostino and Mondadori, 1992], [D'Agostino and Mondadori, 1993], [Broda *et al.*,].

above are the "natural" introduction rules for the *classical* operators, and (iii) that the elimination rules specified above are the only *inference* rules (in the strict sense) that can be justified on the basis of the introduction rules by means of the inversion principle, we are bound to conclude that the inferential approach to the meaning of the classical operators *does not characterize classical semantics completely.*

In fact, this conclusion is hardly surprising. The fundamental contrast between classical and intuitionistic logic lies in the underlying notion of truth. While intuitionistic truth is "internal" and can therefore be completely characterized by reference to the meaning of the logical words, as defined by their use in logical inference, classical truth is "external" and requires a reference to a reality which exists independently of human reasoning and perception. So there is no reason to expect the inferential approach to the meaning of the logical operators to provide an exhaustive account of classical deduction. What is missing is the inferential role played by an extra metaphysical assumption — which dates back to Aristotle — namely the assumption of an external reality which makes sentences determinately true or false quite independently of our means of determining their truth value. This assumption is expressed by the classical *principle of bivalence* and we stress that it should not be regarded as an assumption about the meaning of the logical operators — at least if we accept the philosophical stand that this meaning is defined by appropriate introduction rules — but as an assumption about the notions of truth and falsity and their ontological bearing. Bivalence is therefore a metaphysical assumption which plays an *indirect* inferential role in justifying inferences, but cannot be justified by reference to the meaning of the logical operators, even if this meaning is construed in a classical way (i.e. truth-conditionally), unless we give up the basic tenet of natural deduction and of proof-theoretic semantics in general, namely that the introduction rules should be taken as definitions of the corresponding operators and the elimination rules should be in harmony with them. Bivalence, therefore, is not really a logical rule but a *structural* assumption about the relationship between language and world. So, our account neatly separates the inferential role played by the logical operators, whose meaning is defined by their use in inference, and the role played by the bivalence principle.

Bivalence is not the only "metaphysical" assumption of classical logic. Another Aristotelian assumption about the notions of truth and falsity is expressed by the *principle of non-contradiction*, which says that the world is such that a given sentence cannot be true and false at the same time. Observe that non-contradiction is also an intuitionistic principle which stems immediately from the meaning of the negation operator. In intuitionistic terms, however, this principle refers to the internal consistency of our mental representations. In classical terms, the negation of P means that P is false and the falsity of a sentence, like its truth, is a relation between the sentence itself and the postulated external world. Hence, in classical logic the principle of non-contradiction is, like the principle of bivalence, a *structural* assumption about the relationship between language and world.

Let us say that a set Γ of assumptions is *inconsistent* if for some sentence P, both P and $\neg P$ are deducible from Γ. Formally, we can introduce a rule like the following:

$$\frac{P \quad \neg P}{\times}$$

meaning that the tree above $\times$ is a proof that the assumptions on which it depends are mutually inconsistent.

An appropriate rule expressing the classical principle of bivalence is then the following *proof rule*:

$$\frac{\begin{array}{c}[P]\\ \vdots \\ Q/\times\end{array} \qquad \begin{array}{c}[\neg P]\\ \vdots \\ Q/\times\end{array}}{Q/\times}$$

where the notation "$Q/\times$" means that the above tree is either a proof of Q from the undischarged assumptions or a proof that the undischarged assumptions are *inconsistent.* Moreover, the symbol below the line may be equal to "$\times$" only if *both* the symbols immediately above it are equal to "$\times$", that is the whole tree is a a refutation of its undischarged assumption formulas only if both the subtrees above the line are refutations of their undischarged assumption formulas.

The special format of this rule — a proof rule rather than an inference rule — shows that the inferential role it plays is on a different level from that of the other rules. Arguments essentially involving this rule are *indirect* and are not completely justified by the meaning of the logical operators.[26] On the other hand, if the rules of bivalence and non-contradiction are added to the system consisting of the introduction rules, then the elimination rules are not merely "justified" in terms of the introduction rules, but become *derived rules*, as the reader can easily verify.[27]

[26]Unlike bivalence, constructive dilemma is an indirect argument that can be ultimately justified in terms of the meaning of the classical logical operators, though only by means of meta-level reasoning.

[27]Similarly the introduction rules are derived rules in the system consisting of the elimination rules plus the two structural rules. On this point see [D'Agostino, 1990] and [D'Agostino and Mondadori, 1991].

Augmenting the intelim rules with the above rules of *bivalence* and *non-contradiction* we obtain a complete set of rules for classical propositional logic. The completeness proof is entirely routine and will therefore be omitted. The resulting proof system is our candidate for a *classical* theory of natural deduction as opposed to the intuitionistic theory represented by the Gentzen-Prawitz calculi.

5 A more convenient format for classical natural deduction: intelim sequences and trees

The standard tree format of natural deduction proofs, with the conclusion as root and the assumptions as leaves, is certainly perspicuous since it allows us to visualize immediately the inner structure of the proof — the premises and the conclusion of each rule application, the assumptions on which each formula depends, etc.

However, this format involves a good deal of redundancy in the representation of proofs. Whenever a formula, which can be inferred from the assumptions, is used more than once as premiss of further inferences, its proof-tree has to be replicated and this leads to an unnecessary growth of the size of the whole proof. Moreover, the format of the rule of bivalence — the only one involving discharge of assumptions — is not particularly convenient for the transformation of proofs and for the implementation of efficient proof-search algorithms. In this section we propose a different format which, although slightly less perspicuous, appears to be better suited to algorithmic treatment.

In this new format, the tree grows upside-down, the premiss of a rule application do not occur in adjacent branches as immediate predecessors of the conclusion, but on the same branch as the conclusion. So, the application of the intelim rules is *sequential*.

DEFINITION 1. An *intelim sequence* is a sequence of formulas such that each formula is either an assumption or is the conclusion of the application of an intelim rule to preceding formulas. An intelim sequence *based on a set* Γ of assumptions is an intelim sequence such that all its assumption formulas are in Γ.

The *rule of bivalence* (RB for short) splits an intelim sequence into two branches that are developed in parallel:

$$\begin{array}{ccc} & / \quad \backslash & \\ P & & \neg P \end{array}$$

When an intelim tree is expanded in this way we say that RB has been applied to the formula P and will also say that P is *the RB-formula* of this application of RB.

DEFINITION 2. An *intelim tree* is a tree of occurrences of formulas such that each occurrence is either an assumption, or is obtained from the application of an intelim rule to formulas occurring above it in the same branch, or results from an application of RB. An intelim tree *based on a set* Γ of assumptions is a intelim tree such that all its assumption formulas are in Γ.

Observe that each branch of an intelim tree is an intelim sequence.

DEFINITION 3. A branch of an intelim tree is *closed* if it contains both P and $\neg P$ for some sentence P, otherwise it is *open*.

DEFINITION 4. A *classical intelim proof* of P from the assumptions Γ is an intelim tree based on Γ such that P occurs in every *open* branch.

The intelim rules are displayed in Table 1, while the structural rules are displayed in Table 2.

6 Normalization issues

In the sequel we shall use the lower case letters p, q, r, etc., possibly with subscripts, to denote arbitrary *atomic* formulas. By the *complexity* of a formula we shall mean the number of occurrences of logical operators in it (so, for instance, the complexity of atomic formulas is 0 and the complexity of $(p \vee q) \vee (r \wedge s)$ is 3).

DEFINITION 5. A branch is *atomically closed* if it contains both p and $\neg p$ for some atomic formula p.

PROPOSITION 6. *Every closed branch can be expanded into an atomically closed branch by means of applications of intelim rules only.*

Proof. If a branch is closed, it contains both P and $\neg P$ for some sentence P. The proof is an easy induction on the complexity of P and requires the discussion of several cases depending on the logical form of P. We discuss only the case $P = Q \vee R$, the other cases being similar.

If both $Q \vee R$ and $\neg(Q \vee R)$ occur in a branch, then we can apply the elimination rules to $\neg(Q \vee R)$ and append both $\neg Q$ and $\neg R$ to the end of branch. Then the extended branch will contain both $Q \vee R$ and $\neg Q$, so we can apply the appropriate elimination rule to these formulas and append R to the end of the branch. The resulting branch will contain both R and $\neg R$ and the complexity of R is strictly less than the complexity of P. ■

Notice that when a closed branch containing P and $\neg P$ is expanded into an atomically closed one following the procedure described in the proof of the

INTRODUCTION RULES

$$\frac{\begin{array}{c}P\\ Q\end{array}}{P \wedge Q} \qquad \frac{\neg P}{\neg(P \wedge Q)} \qquad \frac{\neg P}{\neg(P \wedge Q)}$$

$$\frac{\begin{array}{c}\neg P\\ \neg Q\end{array}}{\neg(P \vee Q)} \qquad \frac{P}{P \vee Q} \qquad \frac{P}{P \vee Q}$$

$$\frac{\begin{array}{c}P\\ \neg Q\end{array}}{\neg(P \rightarrow Q)} \qquad \frac{\neg P}{P \rightarrow Q} \qquad \frac{Q}{P \rightarrow Q}$$

$$\frac{P}{\neg\neg P}$$

ELIMINATION RULES

$$\frac{\begin{array}{l}P \vee Q\\ \neg P\end{array}}{Q} \qquad \frac{\begin{array}{l}P \vee Q\\ \neg Q\end{array}}{P} \qquad \frac{\neg P \vee Q}{\neg P} \qquad \frac{\neg P \vee Q}{\neg Q}$$

$$\frac{\begin{array}{l}\neg(P \wedge Q)\\ P\end{array}}{\neg Q} \qquad \frac{\begin{array}{l}\neg(P \wedge Q)\\ Q\end{array}}{\neg P} \qquad \frac{P \wedge Q}{P} \qquad \frac{P \wedge Q}{Q}$$

$$\frac{\begin{array}{l}P \rightarrow Q\\ P\end{array}}{Q} \qquad \frac{\begin{array}{l}P \rightarrow Q\\ \neg Q\end{array}}{\neg P} \qquad \frac{\neg(P \rightarrow Q)}{P} \qquad \frac{\neg(P \rightarrow Q)}{\neg Q}$$

$$\frac{\neg\neg P}{P}$$

Table 1. The intelim rules.

$$\frac{\quad}{P \mid \neg P}\ \text{RB} \qquad\qquad \frac{\begin{array}{c}P\\ \neg P\end{array}}{\times}\ \text{RNC}$$

Table 2. The structural rules.

above proposition, its length grows at most linearly in the complexity of P.

PROPOSITION 7. *Every proof* π *of* P *from the set* Γ *of assumptions can be transformed into a proof* π' *of* P *from* Γ *such that RB is applied only to atomic formulas.*

Proof. We use the notation

$$\begin{array}{c} T \\ \mathbf{n} \end{array}$$

to denote either the empty intelim tree or a non-empty intelim tree such that $\mathbf{n}$ is one of its terminal nodes.

The proof is by lexicographic induction on $(\gamma(\pi), \lambda(\pi))$ where $\gamma(\pi)$ is the maximum complexity of an RB-formula in π and $\lambda(\pi)$ is the number of occurrences of RB-formulas of complexity $\gamma(\pi)$.

Let $\gamma(\pi) = k > 0$ and let Q be an RB-formula of complexity k. There are several cases depending on the logical form of Q. We discuss only the case $Q = R \vee S$, the other cases being similar.

If $Q = R \vee S$, then π has the following form:

$$\begin{array}{ccc} & T & \\ & \mathbf{n} & \\ \swarrow & & \searrow \\ R \vee S & & \neg(R \vee S) \\ T_1 & & T_2 \end{array}$$

where T_1 and T_2 are intelim trees such that each of their open branches contains P.

Let π' be the following intelim tree:

$$\begin{array}{ccccc} & & T & & \\ & & \mathbf{n} & & \\ & \swarrow & & \searrow & \\ R & & & & \neg R \\ | & & & \swarrow & & \searrow \\ R \vee S & & S & & & \neg S \\ T_1 & & | & & & | \\ & & R \vee S & & & \neg(R \vee S) \\ & & T_1 & & & T_2 \end{array}$$

Clearly π' is still a classical intelim proof of P from Γ and either $\gamma(\pi') < \gamma(\pi)$ or $\gamma(\pi') = \gamma(\pi)$ and $\lambda(\pi') < \lambda(\pi)$. ∎

DEFINITION 8. An occurrence of a formula P is *idle* in a proof π if it is not used in π as premiss of some application of an intelim rule or of RNC.

DEFINITION 9. Given an intelim tree T, a *path* in T is a finite sequence of nodes such that the first node is the root of T and each of the subsequent nodes is an immediate successor of the previous one. A path is *closed* if it contains both P and $\neg P$ for some formula P.

Notice that, according to the above definition, every branch is a maximal path.

DEFINITION 10. A proof π of P from Γ is *pruned* if it satisfies the following conditions:

1. it contains no idle occurrences of formulas;
2. no branch ϕ of π contains more than one occurrence of the same formula;
3. no branch ϕ of π contains a path ending with P which is properly included in ϕ;
4. no branch ϕ of π contains a closed path which is properly included in ϕ.

We omit the (routine, but long-winded) proof of the following proposition.

PROPOSITION 11. *Every proof π of P from Γ can be transformed into a pruned proof π' of P from Γ.*

DEFINITION 12. An occurrence of a formula P is a *detour* in a proof π if it is at the same time the conclusion of an application of an introduction rule and either the major premiss of an application of an elimination rule or one of the premises of an application of RNC.

Then, we are able to show that:

PROPOSITION 13. *If π is a pruned proof of P from Γ and every closed branch of π is atomically closed, then π contains no detours.*

Proof. Suppose a branch ϕ contains a detour, say an occurrence of $P \vee Q$. Since every closed branch in π is atomically closed, then $P \vee Q$ cannot be a premiss of RNC. So it must be at the same time the conclusion of an introduction and the major premiss of an elimination. Then either (i) ϕ contains two occurrences of P (one as premiss of the introduction of $P \vee Q$ and the other as conclusion of its elimination), or (ii) ϕ contains two occurrences of Q (again one as premiss of its introduction and the other as conclusion of its elimination), or (iii) ϕ contains a closed path which is properly included in it (P as premiss of its introduction and $\neg P$ as minor premiss of its elimination; or Q as premiss of its introduction and $\neg Q$ as minor premiss of its elimination). Then, π is not pruned. ■

DEFINITION 14. A formula Q is a *weak subformula* of a formula P if Q is a subformula of P or the negation of a subformula of P.

DEFINITION 15. Given a proof π of P from Γ, we say that a formula Q is *spurious* in π if Q is not a weak subformula either of P or of some formula in Γ. A proof π of P from Γ has the *weak subformula property* (WSFP for short) if it contains no spurious formulas.

DEFINITION 16. A proof π of P from Γ is *canonical* if (i) every closed branch of π is atomically closed, (ii) RB is applied only to atomic formulas in π , (iii) π is pruned and (iv) π has the WSFP.

LEMMA 17. *Let π be a proof of P from Γ such that (i) π contains no idle occurrences of formulas and (ii) every closed branch of π is atomically closed. If Q is a spurious formula of maximal complexity in π, then Q may only occur in π either as an RB-formula or as a detour.*

Proof. The proof is left to the reader. ■

PROPOSITION 18. *Every proof π of P from Γ can be transformed into a canonical proof π' of P from Γ.*

Proof. Given an arbitrary proof of P from Γ we first expand its closed branches into atomically closed ones (Proposition 6); next we transform it into a proof such that RB is applied only to atomic formulas (Proposition 7); finally, we transform it into a pruned proof (Proposition 11). Thus, we can assume without loss of generality that (i) every closed branch of π is atomically closed, (ii) RB is applied in π only to atomic formulas, and (iii) π is pruned. We then have to show that π can be turned into a proof of P from Γ with the WSFP.

Since π is pruned and every closed branch of π is atomically closed, then π contains no detours (Proposition 13). So, by Lemma 17, the only occurrences of spurious formulas in π may only be (atomic) RB-formulas. We show that all such disturbing occurrences of RB-formulas can be eliminated one by one.

Let $\sigma(\pi)$ be the number of occurrences of spurious RB-formulas in π. Suppose $\sigma(\pi) = k > 0$. Then π has the following form:

$$\begin{array}{ccc} & T & \\ & \mathbf{n} & \\ \swarrow & & \searrow \\ q & & \neg q \\ T_1 & & T_2 \end{array}$$

where (i) q is atomic and spurious, (ii) T_1 and T_2 are intelim trees such that each of their open branches contains P. Since π is pruned, q is used in T_1. Given that q is spurious, it follows that q cannot be used in T_1 as premiss of an introduction rule, because in such a case the conclusion would be a spurious formula, against the conclusion we have just reached that the only spurious formulas in π are atomic RB-formulas. For the same reason, q cannot be used as minor premiss of an elimination rule. Moreover, it is obvious that q cannot be used as major premiss of an elimination rule. Hence, q can be used in a closed branch of T_1 only as premiss of an application of RNC with $\neg q$ as the other premiss. This means that one of the terminal nodes of T_1, say $\mathbf{m}$, contains an occurrence of $\neg q$. Let π' be the following intelim tree:

$$\begin{array}{c} T \\ \mathbf{n} \\ T_1 \\ \mathbf{m} \\ T_2 \ . \end{array}$$

It is easy to check that π' is still a classical intelim proof of P from Γ. Moreover, $\sigma(\pi') < \sigma(\pi)$. ■

The following proposition states the property we anticipated in section 4, namely that if we allow for RB and RNC, the elimination rules become derived rules.

PROPOSITION 19. *If there is a proof of P from Γ than there is a canonical proof π' of P from Γ which contains no application of the elimination rules.*

The above proposition implies that the system consisting of the introduction rules only plus RB and RNC is complete for classical logic and enjoys the WSFP.[28]

The following proposition states a remarkable property of ca-no-ni-cal proofs showing that purely introductive canonical proofs of classical tautologies are nothing but a more concise form of the familiar truth-table procedure.

[28] This is the **KI** system, proposed in [Mondadori, 1988a] and further investigated in [Mondadori, 1996].

PROPOSITION 20. *If P is a tautology, every canonical proof of P from the empty set of assumptions is such that (i) π contains no applications of the elimination rules and (ii) all branches of π are open.*

EXAMPLE 21. The set of formulas

$$\{p \vee q, r \vee s, p \rightarrow t \wedge u, r \rightarrow v \wedge u, \neg u\}$$

classically implies $q \wedge s$. Figure 2(a) shows an intelim sequence for this inference.

EXAMPLE 22. The set

$$\{p \vee q, p \vee \neg q, r \vee s, r \vee \neg s\}$$

classically implies $p \wedge r$. Figure 2(b) shows a classical intelim proof of this inference (for the sake of readability we have omitted the justification of the inference steps).

In most practical applications, canonical proofs may be too restrictive. Indeed, turning a proof into one satisfying the requirement that RB is applied only to atomic formulas may lead to exponential growth in the size of the proof tree, depending on the structure of the conclusion and of the assumption formulas. The weaker requirement that RB-applications are *analytic*, i.e. that their RB formulas are not spurious, may be sufficient in most interesting cases and may lead to essentially shorter proofs.

DEFINITION 23. We say that an application of RB in a proof π of P from Γ is *analytic* if its RB-formula is not spurious in π.

It is not difficult to adapt the proofs of Propositions 7 and 18 to yield a proof of the following:

LEMMA 24. *Every proof π of P from the set Γ of assumptions can be transformed into a proof π' of P from Γ such that all the applications of RB in π' are analytic.*

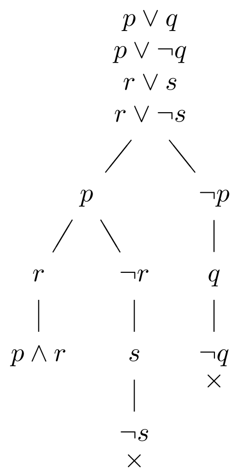
Then, one can easily show that:

PROPOSITION 25. *Let π be a proof of P from Γ such that (i) every closed branch of π is atomically closed, (ii) every application of RB in π is analytic, and (iii) π is pruned. Then π has the WSFP.*

Since the method of proof *ex-absurdo* is the hallmark of classical logic, a systematic use of such proofs, rather than direct ones, is perfectly in tune with the philosophy of *classical* natural deduction. Using intelim trees as a *refutation* method also prompts for comparison with other familiar refutation methods such as Smullyan's analytic tableaux. We have already

1.	$p \vee q$	
2.	$r \vee s$	
3.	$p \to t \wedge u$	
4.	$r \to v \wedge u$	
5.	$\neg u$	
6.	$\neg(t \wedge u)$	I-rule for $\neg\wedge$ to 5
7.	$\neg(v \wedge u)$	I-rule for $\neg\wedge$ to 5
8.	$\neg r$	E-rule for $\to$ to 4 and 7
9.	$\neg p$	E-rule for $\to$ to 3 and 6
10.	q	E-rule for $\vee$ to 1 and 9
11.	s	E-rule for $\vee$ to 2 and 8
12.	$q \wedge s$	I-rule for $\wedge$ to 10 and 11

(a)



(b)

Figure 2. Examples of intelim refutations.

discussed the shortcomings of analytic tableaux in Section 3 and have developed our theory of classical natural deduction so as to avoid them.

It is immediate to check that the completeness of canonical restrictions of the intelim tree method is preserved when the latter is used as a refutation method:

PROPOSITION 26. *If there is a refutation π of Γ (i.e. an intelim tree based on Γ such that all its branches are closed), then there is a canonical refutation π' of Γ.*

So, interpreted as a refutation method, intelim trees can be fruitfully employed as a propositional basis for automated proof techniques. The next section will focus on this aspect and show how this method may be well-suited to the development of efficient and perspicuous refutation algorithms.

Looking at intelim trees from this perspective, we soon discover that we can do without the introduction rules:

PROPOSITION 27. *If there is an intelim proof of P from Γ than there is an intelim refutation π' of $\Gamma, \neg P$ such that (i) all applications of RB in π' are analytic and, (ii) π' contains no application of the introduction rules.*

The above proposition implies that the refutation system consisting of the elimination rules only plus the rule of bivalence (restricted to analytic applications) is a complete system for classical logic. Clearly such a system has the WSFP.[29] Observe that if we have a purely eliminative refutation of $\Gamma, \neg P$ with the WSFP, in order to obtain a purely eliminative proof of P from Γ with the WSFP we only need to apply RB once to the proposition P.

7 Feasible analytic proofs and refutations

Although being incomplete for classical propositional logic, the proof system consisting only of the intelim rules, without RB, is still quite powerful. In accordance with our truth-conditional reformulation of the theory of meaning underlying the natural deduction approach, the consequence relation associated with this system may be construed as the fragment of classical logic which can be justified only by means of the classical meaning of the logical operators as determined by the truth-conditions specified in the introduction rules. The resulting notion of inference, despite being based on "external" concepts of truth and falsity, is not committed to the principle

[29] This is the (analytic restriction) of the system **KE** first proposed in [Mondadori, 1988b] and whose proof-theoretical and computational advantages — especially with respect to standard cut-free refutation systems, such as Smullyan's Tableaux — are discussed extensively in [D'Agostino, 1990], [D'Agostino, 1992] and [D'Agostino and Mondadori, 1994]. See also [D'Agostino, 1999] and [Hähnle, 2001].

of bivalence, namely to what appears to be the most recalcitrant aspect of classical semantics from the point of view of the meaning-as-use approach.

Recall that, for every set of formulas Γ, we call *intelim sequence* based on Γ a (possibly infinite) sequence of formulas each of which is either a formula in Γ or is obtained by preceding formulas by an application of an intelim rule. An intelim sequence is *closed* if it contains both P and $\neg P$ for some formula P.

DEFINITION 28. A proof π of P from Γ is a *strict intelim proof* if it contains no applications of RB, i.e. if π is an intelim sequence. A formula P is *strictly intelim provable* from a set Γ of formulas if there is a strict intelim proof of P from Γ. A set Γ of formulas is *strictly intelim refutable* if there is a closed intelim sequence based on Γ.

Strict intelim proofs[30]and refutations are our candidates for the the set of classical logical arguments whose validity can be determined on the sole basis of the (classical) meaning of the logical operators. They are "analytic" in the particularly strict sense of being justified only by virtue of the way in which the language is immediately used in inference, with no appeal to metaphysical assumptions about the "world" or to forms of indirect reasoning. From this point of view, one could say that the inference process they represent conveys no information at all and therefore complies with the typical requirement of "analytic" arguments, namely that "the conclusion is contained in the premises". Indeed, one cannot get anything out of a set of formulas, by using elimination rules, that is not already in the set, in the sense of being obtainable from what we grasp as the meaning of the formulas involved by means of their definitions in terms of the introduction rules. As we shall show in the remaining of this section, this stricter sense of "analytic inference", unlike the general one,[31] is *tractable*: whether a formula P is strictly intelim provable from a finite set Γ of formulas or whether Γ is inconsistent are both questions that can be decided in polynomial time.

The discussion in the previous section can be easily adapted to show that if there is a strict intelim proof of P from Γ then there is a strict intelim proof π' of P from Γ such that π' has the WSFP. In the remaining of this section we shall concentrate on strict analytic refutations rather than proofs, in order to emphasize their possible use for developments in the area of automated reasoning.

[30]Strict intelim provability (from Γ) is a standard Tarskian consequence relation. It is easy to check that it is closed under identity ($P \vdash P$), monotonicity ($\Gamma \vdash P$ implies that $\Gamma, \Delta \vdash P$) and cut ($\Gamma \vdash P$ and $\Delta, P \vdash Q$ imply $\Gamma, \Delta \vdash Q$). Notice that in this consequence relation there are no tautologies: there is no formula which is provable from the empty set of assumptions.

[31]At least according to the widespread conjecture that $P \neq NP$.

We shall say that an intelim sequence based on Γ is *E-saturated* if it contains all the formulas in Γ and is closed under the application of elimination rules. More precisely,

DEFINITION 29. An intelim sequence ϕ based on Γ is *E-saturated* whenever the following conditions hold true:

1. for every $P \in \Gamma$, P belongs to ϕ;
2. if the premiss (or the premisses) of an elimination rule belong(s) to ϕ, then its conclusion also belongs to ϕ.

Observe that E-saturated intelim sequences based on a finite set Γ of formulas are finite. Moreover, their length is at most linear in the length of Γ (the sum of the lengths of the formulas in Γ). Moreover, E-saturation obviously preserves the WSFP.

We shall also say that an intelim sequence based on Γ is *I-saturated* when it contains all the formulas in Γ and is closed under the application of introduction rules. More precisely:

DEFINITION 30. An intelim sequence ϕ based on Γ is *I-saturated* whenever the following conditions hold true:

1. for every $P \in \Gamma$, P belongs to ϕ;
2. if the premiss(es) of an application of an introduction rule belong to ϕ, then its conclusion also belongs to ϕ.

I-saturated intelim sequence, unlike E-saturated ones, are necessarily infinite, even if based on a finite set Γ, since there is no bound on the application of introduction rules. So, I-saturation clearly does not preserve the WSFP.

We shall say that an intelim sequence based on Γ is *saturated* if it is both E-saturated and I-saturated. Therefore, saturated intelim sequences are necessarily infinite, even if based on a finite set Γ and do not have the WSFP. Clearly, for every Γ, there is exactly one saturated intelim sequence based on Γ and:

PROPOSITION 31. *If the saturated intelim sequence based on* Γ *is open, then every intelim sequence based on* Γ *is also open.*

However, Proposition 26 ensures that if there is a closed intelim sequence for Γ, then there is also a closed intelim sequence for Γ with the WSFP.

We can therefore assume, without loss of completeness, that the application of introduction rules is restricted to the cases in which the conclusion is not a spurious formula.

DEFINITION 32. We say that an intelim sequence based on Γ is *analytically I-saturated* if it is closed under all applications of the introduction rules such that their conclusion is a weak subformula of some formula in Γ.

DEFINITION 33. We say that an intelim sequence based on Γ is *analytically saturated* if it is E-saturated and analytically I-saturated.

Then the propositions in the previous section immediately imply that:

PROPOSITION 34. *If there is a closed intelim sequence based on* Γ*, then the analytically saturated intelim sequence based on* Γ *is also closed.*

It follows that if the analytically saturated intelim sequence based on Γ is open, then there is no closed intelim sequence based on Γ.

DEFINITION 35. We say that P is an immediate subformula of Q if P is a proper subformula of Q (i.e. other than Q itself) and P is not a proper subformula of any proper subformula of Q.

DEFINITION 36 (labelled subformula graph). Let Γ be a set of formulas. The *subformula graph for* Γ, denoted by $S(\Gamma)$ is a directed graph whose vertices are all the subformulas of formulas in Γ and whose arcs are all the pairs

$$\{\langle P, Q\rangle \mid P \text{ is an immediate subformula of } Q\}.$$

A *labelled subformula graph for* Γ is a pair $\langle S(\Gamma), l\rangle$, where $S(\Gamma)$ is the subformula graph for Γ and l is its labelling function, i.e. a partial function mapping vertices into $\{0, 1\}$. A formula is *signed* if $l(P)$ is defined; if $l(P) = 1$ we say that P is *signed as true*, if $l(P) = 0$ we say that P is *signed as false*. For every signed formula P, we define its *unsigned* version P^* as follows: $P^* = P$ if $l(P) = 1$ and $P^* = \neg P$ if $l(P) = 0$.

DEFINITION 37 (intelim graph). An *intelim graph* G *for* Γ is a labelled subformula graph $\langle S(\Gamma), l\rangle$ for Γ satisfying the following conditions:

1. for every $Q \in \Gamma$, $l(Q) = 1$;
2. for every $Q \in S(\Gamma)$ such that $Q \notin \Gamma$, $l(Q)$ is defined *only if* either (i) there is a formula $P \in S(\Gamma)$ such that $l(P)$ is defined and Q^* is the conclusion of a one-premiss intelim rule applied to P^* or (ii) there are formulas $P_1, P_2 \in S(\Gamma)$ such that $l(P_1)$ and $l(P_2)$ are both defined and Q^* is the conclusion of a two-premiss intelim rule applied to P_1^* and P_2^*.

An intelim graph G for Γ is *completed* if it satisfies the two conditions above with the expression "only if" in the second condition being replaced by "if and only if".

EXAMPLE 38. The set of formulas

$$\{p \vee q, r \vee \neg q, p \rightarrow s \wedge t, r \rightarrow u \wedge t, \neg t\}$$

is classically inconsistent and strictly intelim refutable. The initial intelim graph for this set is shown in Figure 3.

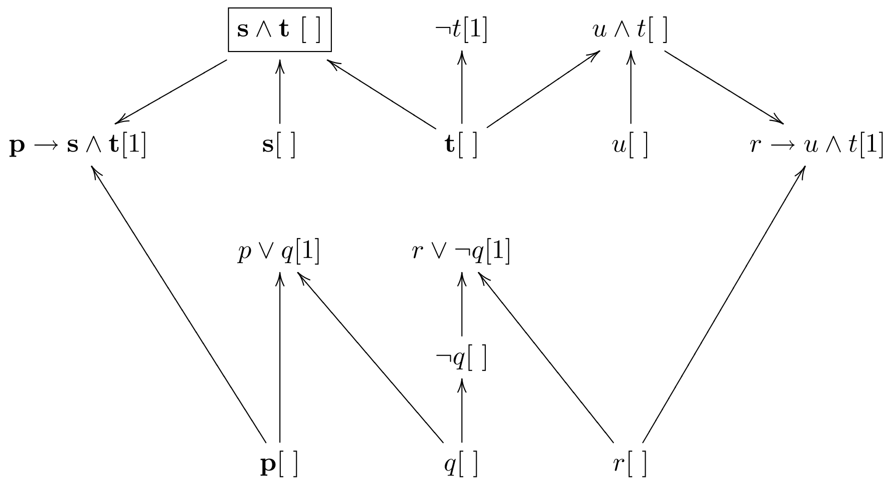


Figure 3. Initial intelim graph for the inconsistent set of formulas in Example 38. The values of the labelling function, when defined, are shown between square brackets. The boldface formulas are the *neighbours* of the boxed formula $s \wedge t$.

An *intelim* graph for Γ corresponds to an open *intelim sequence* based on Γ and a completed intelim graph for Γ to an open intelim sequence based on Γ which is analytically saturated. In both cases, the intelim sequence can be obtained from the intelim graph simply by means of a suitable sequential arrangement of the unsigned versions of the formulas that are signed in G. So, there is a closed intelim sequence based on Γ if and only if there exists no completed intelim graph for Γ.

Given the subformula graph $S(\Gamma)$ of a set of formulas Γ and a formula P in it, the *neighbourhoods of* P, abbreviated as $N(P)$, is defined as the smallest subgraph of G containing, besides P itself, (i) the immediate subformulas of P, (ii) the immediate superformulas of P, and (iii) the immediate subformulas of the immediate superformulas of P. Each neighbourhoods of a formula in an intelim graph is itself an intelim graph (although maybe not *for* the same set of initial formulas) and the overall graph is a completed intelim graph if and only if all the neighbourhoods of the formulas in it are completed intelim graphs. So, an attempt to construct a completed intelim graph for the set of initial formulas Γ consists in trying to "complete", one by one, all the partial intelim graphs corresponding to the neighbourhoods of each signed formula, until we stop either because the completion is impossible or because the graph has become completed.

In order to complete a partial intelim graph we start from a signed formula P, i.e. one for which the labelling function l is defined, and try to complete its neighbourhoods, possibly generating new signed formulas (that is expanding the domain of definition of l). Next we visit another signed formula and repeat the process. If the attempt to complete the neighbourhoods $N(Q)$ of some signed formula Q leads to conflicting assignments of values for the labelling function l, we say that the attempt to build a completed intelim graph has *failed* and stop. Otherwise, we proceed until the neighbourhoods of all signed formulas have been visited (including the ones which have been signed on the way). We call the procedure we have just informally sketched the *completion procedure* for intelim graphs.

Let us denote by $\mathbf{\Gamma}_0$ the class of all finite sets of formulas which are strictly intelim refutable. Then, on the basis of the completion procedure, it is not difficult to show the following:

THEOREM 39. *The complexity of the decision problem for* $\mathbf{\Gamma}_0$ *is* $\mathcal{O}(n^2)$.

Proof.[Sketched] Given any finite set of formulas Γ and an *intelim* graph for Γ, the completion procedure, informally sketched above, always terminates either with a completed graph or with a graph such that for some formula P, its neighbourhoods $N(P)$ cannot be completed. Moreover, the procedure runs in time $\mathcal{O}(n^2)$ where n is the size of the graph.

It is not difficult to see that the procedure always terminates, since the subformula graph is finite and the procedure stops when all the signed formulas of the graph have been visited once. Moreover, if it does not stop before, because the neighbourhoods of some signed formula cannot be completed, the output is completed intelim graph. The simplest way to prove this is by contradiction. Suppose the procedure has terminated (all the signed formulas have been visited) and yet its output graph G is not completed. This means that there must be some formula Q in G such that (i) $l(Q)$ is not defined and (ii) Q is the conclusion of some intelim rule applied to a (pair of) premiss(es), (each of) which is the unsigned version of some signed formula in G. There are, as usual, several cases and, as usual, we shall discuss only of one of them, which we take as being representative of all, leaving the others to the reader.

Suppose both $P \vee Q$ and P are in G. Moreover, $l(P \vee Q) = 1$ and $l(P) = 0$, but $l(Q)$ is undefined. Now since P is signed in G, it must have been already visited once, say at the k-th step of the completion procedure. So, at the end of the k-th step $l(P \vee Q)$ must have been still undefined, otherwise, since both $P \vee Q$ and Q are in the neighbourhoods of P, $l(Q)$ should have been defined and equal to 1 as a result of the completion step. Moreover, since $P \vee Q$ is also signed in G, it must have already been visited once, say at the n-th step of the procedure ($n > k$). So, since P and Q are both in the neighbourhoods of $P \vee Q$, at the end of the n-step, $l(P)$ must have been still undefined, which is impossible since $l(P)$ was assumed to be defined at step k.

The complexity upper bound is a very crude estimate whose justification can be briefly given as follows. Completing the neighbourhoods $N(P)$ of a formula P requires (i) visiting each formula Q in $N(P)$ such that $Q \neq P$ and $l(Q)$ is not defined; (ii) try to assign a value to it according to the *intelim* rules. This, in turn, requires checking at most two "neighbours". More precisely:

1. if Q is an immediate subformula of P, only P and the other immediate subformula of P have to be checked;

2. if Q is an immediate superformula of P, only P and the other immediate subformula of Q have to be checked;

3. if Q is an immediate subformula of an immediate superformula of P, only P itself and the common immediate superformula of Q and P have to be checked.

Hence, completing the neighbourhoods of P requires $\mathcal{O}(n)$ steps, where n is the size of the graph measured by the number of its vertices. Since completing the whole graph requires completing the neighbourhoods of each signed formula in it, the whole completion procedure terminates in $\mathcal{O}(n^2)$ steps. Moreover, the size of the graph is bounded above by the total size of the input Γ (the sum of the number of symbols occurring in each formula). So, the procedure is still $\mathcal{O}(n^2)$ if n is taken to be the input size. Since building the initial subformula graph and initializing the labelling function l also takes $\mathcal{O}(n^2)$ steps, where n is again the input size, it follows that, for any finite set of formulas Γ, whether or not $\Gamma \in \mathbf{\Gamma}_0$ can be decided in $\mathcal{O}(n^2)$ steps. ■

8 Future work

This work could be expanded in several directions, e.g.:

- clarifying the relationship between analytic intelim proofs and normal proofs in Prawitz' style natural deduction;
- comparing intelim trees and graphs, intended as a refutation method, with other known refutation methods (e.g. connection tableaux, see [Hähnle, 2001], Bibel's connection method [Bibel, 1982, Wallen, 1990], and Maslov's inverse method [Maslov *et al.*, 1983]);
- adapting intelim trees and graphs to provide an alternative natural deduction presentation of non-classical systems in the "neighbourhoods" of classical logic, in particular those with an involutive type of negation, such as Girard's linear logic;[32]
- adapting intelim trees and graphs to the neighbours of intuitionistic logic, for instance by making appropriate use of *labelled signed formulas*, as in [D'Agostino and Gabbay, 1994] and [D'Agostino *et al.*, 1999];
- adding suitable rules for quantifiers and modal operators.

Moreover, we are currently investigating "external" semantics (as opposed to the "internal" proof-theoretic semantics discussed in this paper) for the sets of formulas in $\mathbf{\Gamma}_0$, and working out the proof-theoretical and semantical properties of intelim trees with bounded applications of RB (on this point the reader is referred to the forthcoming [D'Agostino and Cilluffo, 2005]).

[32] See [Girard, 1987b] and [Avron, 1988] where a proof-theoretical presentation of (classical) linear logic is given in the framework of Prawitz-style natural deduction.

Appendix

NATURAL DEDUCTION RULES:

$$\frac{P \wedge Q}{P} \qquad \frac{P \wedge Q}{Q} \qquad \frac{P \quad Q}{P \wedge Q}$$

$$\frac{P}{P \vee Q} \qquad \frac{Q}{P \vee Q} \qquad \frac{P \vee Q \qquad \begin{array}{c}\Psi_1, [P] \\ \vdots \\ R\end{array} \qquad \begin{array}{c}\Psi_2, [Q] \\ \vdots \\ R\end{array}}{R}$$

$$\frac{\begin{array}{c}\Psi, [P] \\ \vdots \\ Q\end{array}}{P \rightarrow Q} \qquad \frac{P \rightarrow Q \quad P}{Q}$$

$$\frac{\begin{array}{c}\Psi, [P] \\ \vdots \\ \mathbf{F}\end{array}}{\neg P} \qquad \frac{P \quad \neg P}{\mathbf{F}} \qquad \frac{\mathbf{F}}{Q}$$

The notation $[P]$ means that the conclusion of the rule does not depend on the assumption P. The rules listed above are the intuitionistic rules. Classical logic is obtained by adding one of the following three rules:

$$\frac{}{P \vee \neg P} \qquad \frac{\neg\neg P}{P} \qquad \frac{\begin{array}{c}[\neg P] \\ \vdots \\ \mathbf{F}\end{array}}{P}$$

The last of these rules is the *classical reductio ad absurdum* referred to in the text. Another variant of the classical calculus is described in Tennant 1978 where the author uses the following classical *rule of dilemma*:

$$\frac{\begin{array}{c}[P] \\ \vdots \\ Q\end{array} \qquad \begin{array}{c}[\neg P] \\ \vdots \\ Q\end{array}}{Q}$$

TABLEAU RULES

$$\frac{P \wedge Q}{\begin{array}{l} P \\ Q \end{array}} \qquad \frac{\neg(P \wedge Q)}{\neg P \mid \neg Q} \qquad \frac{P \vee Q}{P \mid Q} \qquad \frac{\neg(P \vee Q)}{\begin{array}{l} \neg P \\ \neg Q \end{array}}$$

$$\frac{P \rightarrow Q}{\neg P \mid Q} \qquad \frac{\neg(P \rightarrow Q)}{\begin{array}{l} P \\ \neg Q \end{array}} \qquad \frac{\neg\neg P}{P}$$

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