Assignment 7

1) $$\frac{f(t+dt,x)-f(t-dt,x)}{2dt} = -v\frac{f(t,x+dx)-f(t,x-dx)}{2dx}$$

$$f(x,t) = \xi^t e^{ikx}$$

$$\rightarrow \frac{\xi^{t+dt}-\xi^{t-dt}}{2dt} = \frac{-v}{2dx}\,\xi^t\left(e^{ikdx}-e^{-ikdx}\right)$$

$$\xi^{dt}-\xi^{-dt} = -\underbrace{\frac{vdt}{dx}}_{a}\,2i\sin(kdx) \quad \Big|\cdot\xi^{dt}$$

$$\xi^{2dt} + 2ia\sin kdx\,\xi^{dt} - 1 = 0 \;; \qquad \xi^{dt} \hat{=} A$$

$$A^2 + 2ia\sin kdx\,A - 1 = 0$$

$$A = -ia\sin kdx \pm \sqrt{1-a^2\sin^2 kdx}$$

If CFL condition is satisfied, $a \leq 1 \Rightarrow \sqrt{(\ldots)}$ is real

$$A^2 = (\mathrm{Re}A)^2 + (\mathrm{Im}A)^2 = 1-a^2\sin^2 kdx + a^2\sin^2 kdx = \underline{\underline{1}}$$

$\xi^{2dt} = 1$ and $dt \neq 0$, so $\xi = 1$

$f(x,t) = F(k)e^{ikx}$, $|F(K)|^2 =$ const in time

$\Rightarrow$ energy of each Fourier mode is conserved.

2) a. $V(r) = A + B\ln r$, $r^2 = i^2+j^2$

$\rho[0,0] = 1$, $V[0,0] = 1$.

Origin: $$\underset{1}{\rho[0,0]} = \underset{1}{V[0,0]} - \frac{1}{4}\Big(\underbrace{V[0,1]+V[0,-1]+V[1,0]+V[-1,0]}_{=4A+4B\ln 1 = 0}\Big)$$

$$\Rightarrow A = 0$$

$[1,0]$: $$V[1,0] = \frac{1}{4}\left(V[0,0]+V[2,0]+V[1,1]+V[1,-1]\right)$$

$$B\ln 1 = \frac{1}{4}\left(1 + B\ln 2 + 2B\ln\sqrt{2}\right)$$

$$0 = \underline{1+2B\ln 2} \Rightarrow B \simeq -0.721, \quad V = B\ln r$$

$V[1,0] = 0$, $V[2,0] = B\ln 2 = -0.5$; $V[5,0] \simeq -1{,}16$.