5) a) $\sum_{x=0}^{N-1} e^{-2\pi i k x/N} = \sum_{x=0}^{N-1} \left(e^{-2\pi i k/N}\right)^x = \sum_x \alpha^x = \frac{1-\alpha^N}{1-\alpha} =$

$$= \frac{1-e^{-2\pi i k}}{1-e^{-2\pi i k/N}}$$

b) $\frac{1-e^{-2\pi i k}}{1-e^{-2\pi i k/N}} \underset{k\to 0}{\sim} \frac{1-1-2\pi i k}{1-1-\frac{2\pi i k}{N}} = N$ $\quad e^t \approx 1+t+O(t^2)$

Integer k, $k \neq aN$:

$$\frac{1-\left(e^{-2\pi i k}\right)^1}{1-\underbrace{e^{-2\pi i \frac{k}{N}}}_{\text{some complex number}}} = \frac{1-1}{1-e^{-2\pi i \frac{k}{N}}} = 0.$$

c) $f(x) = \sin\left(\frac{2\pi K x}{N}\right) = \frac{1}{2i}\left(e^{\frac{i2\pi K x}{N}} - e^{-\frac{i2\pi K x}{N}}\right)$

K – non-integer

$$F(k) = \frac{1}{2i}\left[\sum_x \left(e^{2\pi i K x/N} - e^{-2\pi i K x/N}\right)e^{-2\pi i k x/N}\right] =$$

$$= \frac{1}{2i}\left[\sum_x e^{-\frac{2\pi i x}{N}(k-K)} + \sum_x e^{-\frac{2\pi i x}{N}(k+K)}\right] =$$

$$= \frac{1}{2i}\left[\frac{1-e^{-2\pi i(k-K)}}{1-e^{-2\pi i(k-K)/N}} + \frac{1-e^{-2\pi i(k+K)}}{1-e^{2\pi i(k+K)/N}}\right]$$

e) Window function $w(x) = 0.5 - 0.5\cos\frac{2\pi x}{N}$

$$W(k) = DFT(w) = \sum_x \frac{1}{2} e^{-\frac{2\pi i kx}{N}} + \sum_x -\frac{1}{2}\cdot\frac{1}{2}\left[e^{\frac{2\pi i x}{N}} + e^{-\frac{2\pi i x}{N}}\right] e^{-2\pi i kx/N} =$$

$$= \frac{1}{2} N \delta_{k,0} = \sum_x -\frac{1}{4} e^{-\frac{2\pi i x}{N}(k-1)} + \sum_x -\frac{1}{4} e^{-\frac{2\pi i x}{N}(k+1)} =$$

$$= \frac{N}{2}\delta_{k,0} - \frac{N}{4}\delta_{k,1} - \frac{N}{4}\underbrace{\delta_{k,-1}}_{=\,\delta_{k,N-1}} \text{ equivalent to}$$

so the DFT of the window function is $\left[\frac{N}{2}, -\frac{N}{4}, 0, \ldots, 0, -\frac{N}{4}\right]$.

$f(x) = \sin\frac{2\pi kx}{N}$, $\quad F(k) = DFT(f)$ - unwindowed Fourier transform

$$DFT(f \cdot w) = \frac{1}{N} F \circledast W$$

Convolution: $(W \circledast F)(n) = \sum_{m=-\infty}^{\infty} \underbrace{W(m)}_{\neq 0 \text{ for } m = -1, 0, 1} F(n-m)$

So for each n $\quad DFT(f \cdot w) = \frac{1}{N}\left[F(n)\cdot\frac{N}{2} - F(n+1)\frac{N}{4} - F(n-1)\frac{N}{4}\right]$

$$= \frac{1}{2}F(n) - \frac{1}{4}F(n+1) - \frac{1}{4}F(n-1)$$

→ get the windowed Fourier transform by combinations of each point in F and its neighbors.