Indestructible remarkable cardinals

Victoria Gitman

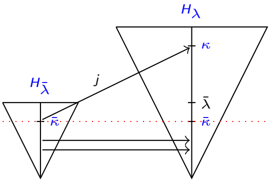
vgitman@nylogic.org
http://boolesrings.org/victoriagitman

5th European Set Theory Conference
August 28, 2015

This is joint work with Yong Cheng (Wuhan University).

The origin of remarkable cardinals

Definition: A cardinal κ is remarkable if in every $\mathrm{Coll}(\omega, {<}\kappa)$-extension $V[G]$, for every regular $\lambda > \kappa$, there an embedding $j : H_{\bar\lambda} \to H_\lambda$, for some V-regular $\bar\lambda < \kappa$, with $\mathrm{cp}(j) = \bar\kappa$ and $j(\bar\kappa) = \kappa$.



Theorem: (Schindler, '00) The following are equiconsistent.

- The theory of $L(\mathbb{R})$ cannot be changed by proper forcing.
- There exists a remarkable cardinal.

A generic version of supercompactness

Theorem: (Magidor, '71) A cardinal κ is supercompact if and only if for every regular $\lambda > \kappa$, there is an embedding $j : H_{\bar{\lambda}} \to H_\lambda$, for some regular $\bar{\lambda} < \kappa$, with $\text{cp}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$.

Absoluteness lemma for countable embeddings: (Folklore) If M is countable and there is an embedding $j : M \to N$, then every transitive model $W \models \mathrm{ZFC}^-$ such that $M, N \in W$ and M is countable in W has an embedding $j^* : M \to N$. If $\text{cp}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$, we can arrange the same to be true for j^*. We can also arrange for j and j^* to agree on finitely many values.

Proof:

- Construct the tree T of finite partial embeddings from M to N.
- T has a branch if and only if there is an embedding from M to N.
- The tree T is in W.
- The tree T is ill-founded in V and hence also in W. $\square$

A generic version of supercompactness (continued)

Observation: A cardinal κ is remarkable if and only if for every regular $\lambda > \kappa$, there is a set-forcing extension $V[H]$ in which there is an embedding $j : H_{\bar{\lambda}} \to H_\lambda$, for some V-regular $\bar{\lambda} < \kappa$, with $\text{cp}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$.

Proof: Suppose $j : H_{\bar{\lambda}} \to H_\lambda$ exists in $V[H]$.

- Let $G \subseteq \mathrm{Coll}(\omega, {<}\kappa)$ be $V[H]$-generic.
- $H_{\bar{\lambda}}$ is countable in $V[G] \subseteq V[H][G]$.
- $j^* : H_{\bar{\lambda}} \to H_\lambda$ exists in $V[G]$ (by the absoluteness lemma).
- Every $\mathrm{Coll}(\omega, {<}\kappa)$-extension has some $j : H_{\bar{\lambda}} \to H_\lambda$ (since $\mathrm{Coll}(\omega, {<}\kappa)$ is weakly homogeneous). $\square$

Other characterizations of remarkable cardinals

Definition: In a $\mathrm{Coll}(\omega, {<}\kappa)$-extension $V[G]$, an embedding $j : H_{\bar{\lambda}} \to H_\lambda$ is $(\bar{\mu}, \bar{\lambda}, \mu, \lambda)$-remarkable if

- $\bar{\lambda}$, λ are V-regular,
- $\mathrm{cp}(j) = \bar{\mu}$ and $j(\bar{\mu}) = \mu$.

Note: A cardinal κ is remarkable if in every $\mathrm{Coll}(\omega, {<}\kappa)$-extension $V[G]$, for every regular $\lambda > \kappa$, there is a $(\bar{\kappa}, \bar{\lambda}, \kappa, \lambda)$-remarkable embedding.

Lemma: If κ is remarkable, then in every $\mathrm{Coll}(\omega, {<}\kappa)$-extension $V[G]$, for every regular $\lambda > \kappa$ and $a \in H_\lambda$, there is a $(\bar{\kappa}, \bar{\lambda}, \kappa, \lambda)$-remarkable $j : H_{\bar{\lambda}} \to H_\lambda$ with $a \in \mathrm{ran}(j)$. In particular, there are such j with $\mathrm{cp}(j)$ arbitrarily high in κ.

Lemma: In a $\mathrm{Coll}(\omega, {<}\kappa)$-extension $V[G]$, if $j : H_{\bar{\lambda}} \to H_\lambda$ is $(\bar{\kappa}, \bar{\lambda}, \kappa, \lambda)$-remarkable, then it lifts to

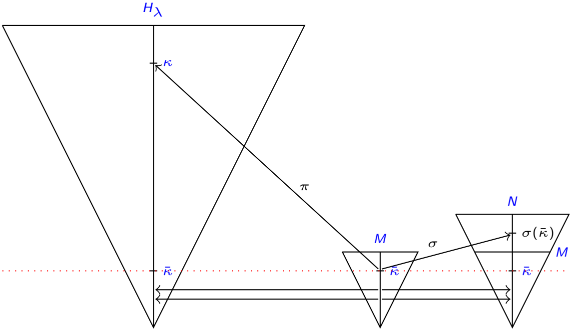
$$j : H_{\bar{\lambda}}[G_{\bar{\kappa}}] \to H_\lambda[G]$$

(where $G_{\bar{\kappa}}$ is the restriction of G to $\mathrm{Coll}(\omega, {<}\bar{\kappa})$).

Other characterizations of remarkable cardinals (continued)

Theorem: (Schindler, '00) A cardinal κ is remarkable if and only if for every regular $\lambda > \kappa$, there are countable transitive models M and N with embeddings

- $\pi : M \to H_\lambda$ with $\pi(\bar{\kappa}) = \kappa$,
- $\sigma : M \to N$ such that
  - $\text{cp}(\sigma) = \bar{\kappa}$,
  - ORD^M is regular in N and $M = H^N_{\mathrm{ORD}^M}$,
  - $\sigma(\bar{\kappa}) > \mathrm{ORD}^M$.



Remarkable cardinals in the hierarchy

Theorem: (Schindler, '00) Strong cardinals are remarkable.

Proof: Suppose κ is strong.

- Fix a regular $\lambda > \kappa$ and $j : V \to M^*$ with $\text{cp}(j) = \kappa$, $j(\kappa) > \lambda$, and $H_\lambda \subseteq M^*$.
- $j : H_\lambda \to H^{M^*}_{j(\lambda)}$ and $H_\lambda \subseteq H^{M^*}_{j(\lambda)}$.
- Take countable $\langle X, Y, h, \in\rangle \prec \langle H^{M^*}_{j(\lambda)}, H_\lambda, j, \in\rangle$.
- Let $\rho : X \to N$ be the collapse map. Then $\rho \restriction Y : Y \to M$ is the collapse of Y.
- Define $\pi : \rho^{-1} : M \to H_\lambda$ and $\sigma = \rho \circ h \circ \rho^{-1} : M \to N$. $\square$

Observation: Measurable cardinals are not necessarily remarkable.

Proof: Remarkable cardinals are totally indescribable and a measurable cardinal is Σ^2_1-describable. $\square$

Theorem: Remarkable cardinals are downward absolute to L.

Proof: In a $\mathrm{Coll}(\omega, {<}\kappa)$-extension $V[G]$, suppose $j : H_{\bar\lambda} \to H_\lambda$ is $(\bar\kappa, \bar\lambda, \kappa, \lambda)$-remarkable.

- $j : L_{\bar\lambda} \to L_\lambda$ in $V[G]$.
- $j^* : L_{\bar\lambda} \to L_\lambda$ exists in $L[G]$ since $L_{\bar\lambda}$ is countable in $L[G]$ (by the absoluteness lemma). $\square$

Remarkable cardinals in the hierarchy (continued)

Definition: A weak κ-model (for a cardinal κ) is a transitive $M \models \mathrm{ZFC}^-$ of size κ and height above κ.

Suppose M is a weak κ-model.

Observation: TFAE.

- There exists $j : M \to N$ with $\mathrm{cp}(j) = \kappa$.
- There exists an M-ultrafilter U with a well-founded ultrapower.
  - U is an M-ultrafilter if $\langle M, \in, U\rangle \models U$ is a normal ultrafilter.
  - $U = \{A \in M \mid \kappa \in j(A)\}$.

Example: A cardinal κ is weakly compact if and only if $2^{<\kappa} = \kappa$ and every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an M-ultrafilter on κ with a well-founded ultrapower.

Definition: An M-ultrafilter U is weakly amenable if for every $X \in M$ with $|X|^M \leq \kappa$, $X \cap U \in M$.

- U is partially internal to M.
- It is needed to iterate the ultrapower construction.

Remarkable cardinals in the hierarchy (continued)

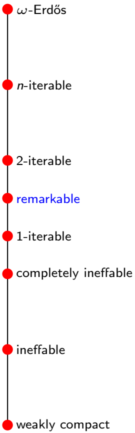
Definition: (G., '07) A cardinal κ is α-iterable ($1 \leq \alpha \leq \omega_1$) if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a weakly amenable M-ultrafilter on κ with α-many well-founded iterated ultrapowers.

Theorem: (G., '07) A 1-iterable cardinal is a stationary limit of completely ineffable cardinals.

Theorem: (G., Welch, '08) An ω-Erdős cardinal implies for every $n \in \omega$, the consistency of a proper class of n-iterable cardinals.

Theorem: (G., Welch, '08)

- A remarkable cardinal is 1-iterable and a stationary limit of 1-iterable cardinals.
- If κ is 2-iterable, then there is a proper class of remarkable cardinals in V_κ.



Laver-like functions

Suppose κ is a large cardinal characterized by the existence of some type of elementary embeddings j.

A Laver-like function $\ell : \kappa \to V_\kappa$ is a guessing function with the property that for every set a, there is some j of the type characterizing the cardinal such that $j(\ell)(\kappa) = a$.

- (supercompact) For every $a \in H_\theta$, there is a θ-supercompactness embedding $j : V \to M$ with $\text{cp}(j) = \kappa$ and $j(\ell)(\kappa) = a$.
- (strong) For every $a \in V_\theta$, there is a θ-strongness embedding $j : V \to M$ with $\text{cp}(j) = \kappa$ and $j(\ell)(\kappa) = a$.
- (extendible) For every $\alpha > \kappa$ and $a \in V_\alpha$, there is $j : V_\alpha \to V_\beta$ with $\text{cp}(j) = \kappa$ and $j(\ell)(\kappa) = a$.
- (strongly unfoldable) For every $a \in V_\theta$, for every $A \subseteq \kappa$, there is a θ-strong unfoldability embedding $j : M \to N$ ($V_\theta \subseteq N$) with $A \in M$ and $j(\ell)(\kappa) = a$.

Additional assumptions:

- $\text{dom}(\ell)$ is contained in the inaccessible cardinals,
- $\ell\,"\,\xi \subseteq V_\xi$.

Laver-like functions (continued)

The existence of Laver-like functions can be forced for most large cardinals, but few have them outright.

Theorem:

- (Laver, '78) Every supercompact cardinal has a Laver function.
- (Gitik, Shelah, '89) Every strong cardinal has a strong Laver function.
- (Corazza, '00) Every extendible cardinal has an extendible Laver function.
- (Džamonja, Hamkins, '06) Not every strongly unfoldable cardinal has a strongly unfoldable Laver function.

Laver-like functions play an important role in indestructibility arguments.

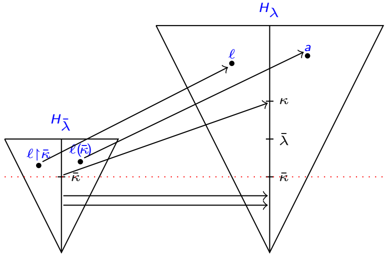
Remarkable Laver functions

Definition: (Cheng, G., '14) A function $\ell : \kappa \to V_\kappa$ is a remarkable Laver function if for every regular $\lambda > \kappa$ and $a \in H_\lambda$, every $\mathrm{Coll}(\omega, {<}\kappa)$-forcing extension $V[G]$ has a $(\bar\kappa, \bar\lambda, \kappa, \lambda)$-remarkable $j : H_{\bar\lambda} \to H_\lambda$ such that

- $\ell \restriction \bar\kappa + 1 \in H_{\bar\lambda}$,
- $\bar\kappa \in \mathrm{dom}(\ell)$,
- $j(\ell(\bar\kappa)) = j(\ell \restriction \bar\kappa + 1)(\kappa) = a$.

Additional assumptions:

- $j(\ell \restriction \bar\kappa) = \ell$,
- $\mathrm{dom}(\ell)$ is contained in the inaccessibles,
- $\ell \text{ "} \xi \subseteq V_\xi$.



Remarkable Laver functions (continued)

Suppose κ is remarkable and $V[G]$ is a $\mathrm{Coll}(\omega, {<}\kappa)$-extension.

Definition: In $V[G]$, suppose $\ell : \xi \to V_\kappa$ ($\xi \leq \kappa$). Say that x is λ-anticipated by ℓ (for λ regular) if

- $x \in H_\lambda$,
- there is a $(\bar{\xi}, \bar{\lambda}, \xi, \lambda)$-remarkable lift $h : H_{\bar{\lambda}}[G_{\bar{\xi}}] \to H_\lambda[G_\xi]$ with $h(\ell \restriction \bar{\xi} + 1)(\xi) = x$.

Lemma: In $V[G]$, if $\ell : \xi \to V_\kappa$ ($\xi{<}\kappa$) and there is λ for which some x is not λ-anticipated by ℓ, then the least such $\lambda < \kappa$.

Definition: Fix a well-ordering W of V_κ of order-type κ. In $V[G]$, define $\ell : \kappa \to V_\kappa$ inductively as follows. Suppose $\ell \restriction \xi$ has been defined.

- Suppose there is λ such that some x is not λ-anticipated by $\ell \restriction \xi$.
  - Let λ' be least such.
  - Let a be W-least set that is not λ'-anticipated by $\ell \restriction \xi$.

  Set $\ell(\xi) = a$.
- Otherwise, $\ell(\xi)$ is undefined.

Remarkable Laver functions exist

Lemma: The definition of ℓ is independent of the $\mathrm{Coll}(\omega, {<}\kappa)$-extension $V[G]$ and $\ell \in V$.

Proof: $\mathrm{Coll}(\omega, {<}\kappa)$ is weakly homogeneous. $\square$

Theorem (Cheng, G., '14) If κ is remarkable, then there is a remarkable Laver function (with all additional properties).

Proof: In a $\mathrm{Coll}(\omega, {<}\kappa)$-extension $V[G]$, suppose λ is least such that some a is not λ-anticipated by ℓ.

- Fix $H_\tau[G]$ that is large enough to see this.
- Fix a $(\bar\kappa, \bar\tau, \kappa, \tau)$-remarkable lift $j : H_{\bar\tau}[G_{\bar\kappa}] \to H_\tau[G]$ and $j(\bar\lambda) = \lambda$.
- $j(\ell \restriction \bar\kappa) = \ell$ (follows from $W \in \mathrm{ran}(j)$).
- By elementarity, $H_{\bar\tau}[G_{\bar\kappa}]$ satisfies that $\bar\lambda$ is least such that some x is not $\bar\lambda$-anticipated by $\ell \restriction \bar\kappa$ and it is correct by absoluteness lemma.
- It follows that $\ell \restriction \bar\kappa + 1 \in H_{\bar\tau}[G_{\bar\kappa}]$ and $\ell(\bar\kappa)$ is not $\bar\lambda$-anticipated by $\ell \restriction \bar\kappa$.
- By elementarity, $j(\ell \restriction \bar\kappa + 1)(\kappa) = y$ is not λ-anticipated by ℓ.
- But y is anticipated by $j' = j \restriction H_{\bar\lambda}[G_{\bar\kappa}]$ and $j' \in H_\tau[G]$. $\rightarrow\leftarrow$ $\square$

Remarkable Laver functions in indestructibility arguments

Lemma: Suppose κ is remarkable and ℓ is a remarkable Laver function. In a $\mathrm{Coll}(\omega, {<}\kappa)$-extension $V[G]$, for every regular $\lambda > \kappa$ and $a \in H_\lambda$, there is a $(\bar\kappa, \bar\lambda, \kappa, \lambda)$-remarkable $j : H_{\bar\lambda} \to H_\lambda$ such that

- $(\bar\kappa, \bar\lambda] \cap \mathrm{dom}(\ell) = \emptyset$,
- $\ell(\bar\kappa) = \langle \bar a, \bar x \rangle$, where $j(\bar a) = a$.

Proof: Let $j(\ell)(\kappa) = \langle a, \lambda + 1 \rangle$ and $j(\bar a) = a$.

- $\ell(\bar\kappa) \notin V_{\bar\lambda}$.
- $\ell\,``\,\xi \subseteq V_\xi$. $\square$

Demonstrating indestructibility of remarkable cardinals

Suppose κ is remarkable and $V[G]$ is an extension by a forcing notion $\mathbb{P}$.

Indestructibility strategy:

Fix $\pi : M \to H_\lambda$ $(\pi(\bar{\kappa}) = \kappa)$ and $\sigma : M \to N$ such that

- $\text{cp}(\sigma) = \bar{\kappa}$,
- ORD^M is regular in N and $M = H^N_{\text{ORD}^M}$,
- $\sigma(\bar{\kappa}) > \text{ORD}^M$.

Lift

- $\pi : M[\bar{G}] \to H_\lambda[G]$,
- $\sigma : M[\bar{G}] \to N[H]$,
- preserve that ORD^M is regular in $N[H]$ and $M[\bar{G}] = H^{N[H]}_{\text{ORD}^M}$.

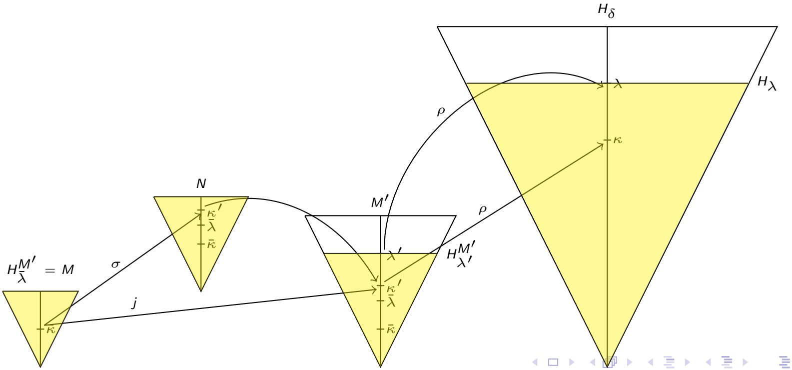
Theorem: (Lifting Criterion) Suppose $j : M \to N$ is an embedding of ZFC^- models having generic extensions $M[G]$ and $N[H]$ by forcing notions $\mathbb{P}$ and $j(\mathbb{P})$ respectively. The embedding j lifts to $j : M[G] \to N[H]$ if and only if $j\,"\,G \subseteq H$.

Choosing a good pair π and σ

Suppose κ is remarkable and $\lambda > \kappa$ is regular.

Fix $\delta > \lambda$ and $\rho : M' \to H_\delta$ with M' countable, $\rho(\kappa') = \kappa$, $\rho(\lambda') = \lambda$, $\rho(\ell') = \ell$.

- Fix an M'-generic $g \subseteq \mathrm{Coll}(\omega, {<}\kappa')^{M'}$.
- In $M'[g]$, choose a $(\bar\kappa, \bar\lambda, \kappa', \lambda')$-remarkable $j : H_{\bar\lambda}^{M'} \to H_{\lambda'}^{M'}$ such that $(\bar\kappa, \bar\lambda) \cap \mathrm{dom}(\ell') = \emptyset$.
- Let $\sigma : H_{\bar\lambda}^{M'} \to N$ with $N = \{\sigma(f)(a) \mid a \in [V_{\kappa'} \cup \{\kappa'\}]^{<\omega}, f \in H_{\bar\lambda}^{M'}\}$.
- Let $M = H_{\bar\lambda}^{M'}$, $\sigma : M \to N$ and $\pi = \rho \circ j : M \to H_\lambda$.



Indestructibility by $\mathrm{Add}(\kappa, \theta)$

Theorem: (Cheng, G., '14) A remarkable cardinal κ can be made indestructible by $\mathrm{Add}(\kappa, \theta)$ for every θ.

Proof:

- $\mathbb{P}_\kappa$ is the κ-length Easton support iteration which forces with $\mathrm{Add}(\xi, \mu)^{V^{\mathbb{P}_\xi}}$ at stage ξ whenever $\ell(\xi) = \langle \mu, x \rangle$ for some x.
- Suppose κ is not remarkable in a $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, \theta)$-extension.
- There is a regular $\lambda > \kappa$ and $q \in \mathbb{P}_\kappa * \mathrm{Add}(\kappa, \theta)$ such that

$$q \Vdash \text{“there are no desired embeddings } \pi \text{ and } \sigma \text{ for } H_\lambda\text{”}.$$

- Fix a good pair $\pi : M \to H_\lambda$ and $\sigma : M \to N$ with
  - $\pi(\bar{q}) = q$, $\pi(\bar{\mathbb{P}}_{\bar{\kappa}}) = \mathbb{P}_\kappa$, $\pi(\bar{\theta}) = \theta$, $\pi(\bar{\ell}) = \ell$
  - $\sigma(\bar{\mathbb{P}}_{\bar{\kappa}}) = \bar{\mathbb{P}}_{\kappa'}$, $\sigma(\bar{\theta}) = \theta'$, $\sigma(\bar{\ell}) = \ell'$,
  - $(\bar{\kappa}, \mathrm{ORD}^M] \cap \mathrm{dom}(\ell') = \emptyset$ and $\bar{\ell}(\bar{\kappa}) = \langle \bar{\theta}, x \rangle$.
- Fix a V-generic $G * g \subseteq \mathbb{P}_\kappa * \mathrm{Add}(\kappa, \theta)$ such that
  - $q \in G * g$,
  - $G * g$ is π " M-generic ($\mathbb{P}_\kappa * \mathrm{Add}(\kappa, \theta)$ is countably closed),
- Let $\bar{G} * \bar{g} = \pi^{-1}(G * g)$.

Indestructibility by $\mathrm{Add}(\kappa, \theta)$ (continued)

Lift π to $M[\bar G][\bar g]$:

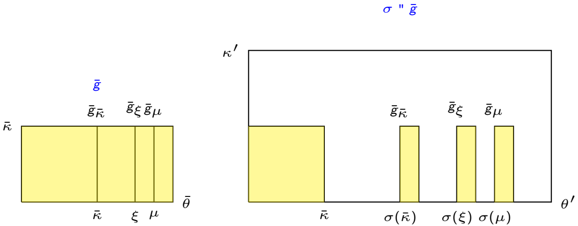
- $\pi\,''\,\bar G * \bar g \subseteq G * g$.
- Lift π to $\pi : M[\bar G][\bar g] \to H_\lambda[G][g]$ by the lifting criterion.

Lift σ to $M[\bar G]$:

- Need an N-generic $G' \subseteq \bar{\mathbb{P}}_{\kappa'}$ with $\sigma\,''\,\bar G = \bar G \subseteq G'$.
- Factor $\bar{\mathbb{P}}_{\kappa'} = \bar{\mathbb{P}}_{\bar\kappa} * \mathrm{Add}(\bar\kappa, \bar\theta) * \dot{\mathbb{P}}_{\text{tail}}$.
- Choose a $N[\bar G][\bar g]$-generic $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ ($N[\bar G][\bar g]$ is countable) and let $G' = \bar G * \bar g * G_{\text{tail}}$.
- Lift σ to $\sigma : M[\bar G] \to N[G']$ by the lifting criterion.

Lift σ to $M[\bar G][\bar g]$:

- Need an $N[G']$-generic $g' \subseteq \mathrm{Add}(\kappa', \theta')^{N[G']}$ with $\sigma\,''\,\bar g \subseteq g'$.



Indestructibility by $\mathrm{Add}(\kappa,\theta)$ (continued)

- Choose any $N[G']$-generic $g' \subseteq \mathrm{Add}(\kappa',\theta')^{N[G']}$ and use the Surgery Method.
- For $p \in \mathrm{Add}(\kappa',\theta')^{N[G']}$, let p^* be the result of altering p to agree with σ " $\bar{g}$.
- Theorem (Woodin):
  - Each $p^* \in \mathrm{Add}(\kappa',\theta')^{N[G']}$.
  - $g^* = \{p^* \mid p \in g'\}$ is $N[G']$-generic for $\mathrm{Add}(\kappa',\theta')^{N[G']}$.
  - Proof needs (1) $\bar{g} \in N[G']$ and (2) $N = \{\sigma(f)(a) \mid a \in [V_{\kappa'} \cup \{\kappa'\}]^{<\omega}, f \in M\}$.
- Lift σ to $\sigma : M[\bar{G}][\bar{g}] \to N[G'][g^*]$ by the lifting criterion.
- There is no forcing in $\bar{\mathbb{P}}_{\kappa'}$ in $(\bar{\kappa}, \mathrm{ORD}^M]$ and $\bar{\mathbb{P}}_{\kappa'}$ is progressively more closed.
  - ORD^M is regular in $N[G'][g^*]$,
  - $M[\bar{G}][\bar{g}] = H^{N[G'][g^*]}_{\mathrm{ORD}^M}$.

Thus, $V[G][g]$ has σ and π as desired, but $q \in G * g$ forces otherwise. $\rightarrow\leftarrow$ $\square$

Indestructibility by $<\kappa$-closed $\leq\kappa$-distributive forcing

Theorem: (Cheng, G., '14) A remarkable cardinal κ can be made indestructible all $<\kappa$-closed $\leq \kappa$-distributive forcing.

Proof:

- $\mathbb{P}_\kappa$ is the κ-length Easton support iteration which forces with $\dot{\mathbb{Q}}_\xi$ at stage ξ whenever $\ell(\xi) = \langle \dot{\mathbb{Q}}_\xi, x\rangle$, where $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_\xi$-name for a $<\xi$-closed $\leq\xi$-distributive poset in $V^{\mathbb{P}_\xi}$ and x is some set.
- Suppose κ is not remarkable in a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$-extension, where $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\kappa$-name for a $<\kappa$-closed $\leq\kappa$-distributive poset.
- There is a regular $\lambda > \kappa$ and $q \in \mathbb{P}_\kappa * \dot{\mathbb{Q}}$ such that

$$q \Vdash \text{"there are no desired embeddings } \pi \text{ and } \sigma \text{ for } H_\lambda\text{"}.$$

- Fix a good pair $\pi : M \to H_\lambda$ and $\sigma : M \to N$ with
  - $\pi(\bar{q}) = q$, $\pi(\bar{\mathbb{P}}_{\bar{\kappa}}) = \mathbb{P}_\kappa$, $\pi(\dot{\mathbb{Q}}_{\bar{\kappa}}) = \dot{\mathbb{Q}}$, $\pi(\bar{\ell}) = \ell$
  - $\sigma(\bar{\mathbb{P}}_{\bar{\kappa}}) = \bar{\mathbb{P}}_{\kappa'}$, $\sigma(\dot{\mathbb{Q}}_{\bar{\kappa}}) = \dot{\mathbb{Q}}_{\kappa'}$, $\sigma(\bar{\ell}) = \ell'$,
  - $(\bar{\kappa}, \mathrm{ORD}^M] \cap \mathrm{dom}(\ell') = \emptyset$ and $\bar{\ell}(\bar{\kappa}) = \langle \dot{\mathbb{Q}}_{\bar{\kappa}}, x\rangle$.
- Fix a V-generic $G * g \subseteq \mathbb{P}_\kappa * \dot{\mathbb{Q}}$ such that
  - $q \in G * g$,
  - $G * g$ is π " M-generic ($\mathbb{P}_\kappa * \dot{\mathbb{Q}}$ is countably closed),
- Let $\bar{G} * \bar{g} = \pi^{-1}(G * g)$.

Indestructibility by $<\kappa$-closed $\leq\kappa$-distributive forcing (continued)

Lift π to $M[\bar{G}][\bar{g}]$:

- $\pi \text{ " } \bar{G} * \bar{g} \subseteq G * g$.
- Lift π to $\pi : M[\bar{G}][\bar{g}] \to H_\lambda[G][g]$ by the lifting criterion.

Lift σ to $M[\bar{G}]$:

- Need an N-generic $G' \subseteq \bar{\mathbb{P}}_{\kappa'}$ with $\sigma \text{ " } \bar{G} = \bar{G} \subseteq G'$.
- Factor $\bar{\mathbb{P}}_{\kappa'} = \bar{\mathbb{P}}_{\bar{\kappa}} * \dot{\mathbb{Q}}_{\bar{\kappa}} * \dot{\mathbb{P}}_{\text{tail}}$.
- Choose a $N[\bar{G}][\bar{g}]$-generic $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ ($N[\bar{G}][\bar{g}]$ is countable) and let $G' = \bar{G} * \bar{g} * G_{\text{tail}}$.
- Lift σ to $\sigma : M[\bar{G}] \to N[G']$ by the lifting criterion.

Indestructibility by $<\kappa$-closed $\leq\kappa$-distributive forcing (continued)

Lift σ to $M[\bar{G}][\bar{g}]$:

- Need an $N[G']$-generic $g' \subseteq (\dot{\mathbb{Q}}_{\kappa'})_{G'} = \mathbb{Q}_{\kappa'}$ with $\sigma\text{ "}\,\bar{g} \subseteq g'$.
- $g' = \langle \sigma\text{ "}\,\bar{g}\rangle$ is the filter generated by $\sigma\text{ "}\,\bar{g}$.
- Clearly g' is $\sigma\text{ "}\,M[\bar{G}]$-generic.
- Indeed, g' is $N[G']$-generic.
  - $N = \{\sigma(f)(a) \mid a \in [V_{\kappa'} \cup \{\kappa'\}]^{<\omega}, f \in M\}$.
  - $\mathbb{Q}_{\kappa'}$ is $\leq\kappa$-distributive in $N[G']$.
- Lift σ to $\sigma : M[\bar{G}][\bar{g}] \to N[G'][g']$ by the lifting criterion.
- There is no forcing in $\bar{\mathbb{P}}_{\kappa'}$ in $(\bar{\kappa}, \mathrm{ORD}^M]$ and $\bar{\mathbb{P}}_{\kappa'}$ is progressively more closed.
  - ORD^M is regular in $N[G'][g']$,
  - $M[\bar{G}][\bar{g}] = H_{\mathrm{ORD}^M}^{N[G'][g']}$.

Thus, $V[G][g]$ has σ and π as desired, but $q \in G * g$ forces otherwise. $\rightarrow\leftarrow$ $\square$

Indestructible remarkable cardinals

Theorem: (Cheng, G., '14) A remarkable κ can be made indestructible by all $<\kappa$-closed $\leq\kappa$-distributive forcing and all two-step iterations $\mathrm{Add}(\kappa,\theta) * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is forced to be $<\kappa$-closed and $\leq \kappa$-distributive.

Proof: $\mathbb{P}_\kappa$ is the κ-length Easton-support iteration which forces with $\dot{\mathbb{Q}}_\xi$ at stage ξ whenever $\ell(\xi) = \langle \dot{\mathbb{Q}}_\xi, x\rangle$ for some set x, where $\dot{\mathbb{Q}}_\xi$ is a $\mathbb{P}_\xi$-name for either

- a $<\xi$-closed $\leq\xi$-distributive forcing, or
- $\mathrm{Add}(\xi,\mu)^{V^{\mathbb{P}_\xi}} * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is forced to be $<\xi$-closed and $\leq\xi$-distributive. $\square$

Applications

Theorem: Any consistent continuum pattern on the regular cardinals can be realized above a remarkable cardinal.

Theorem: It is consistent that κ is remarkable, but not weakly compact in HOD.

Thank you!