The Virtual Large Cardinal Hierarchy

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Abstract. We continue the study of the virtual large cardinal hierarchy, initiated in [GS18], by analysing virtual versions of superstrong, Woodin, Vopěnka, and Berkeley cardinals. Gitman and Schindler showed that virtualizations of strong and supercompact cardinals yield the same large cardinal notion [GS18]. We show the same result for a (weak) virtualization of Woodin and a virtualization of Vopěnka cardinals. We show that a virtually Berkeley cardinal implies that the virtual Vopěnka Principle holds. We also show that if the virtual Vopěnka principle holds and On is not Mahlo, then there is a virtually Berkely cardinal, and if there is a model of ZFC with a virtually Berkeley cardinal, then there is a model of ZFC in which the virtual Vopěnka principle holds and On is not Mahlo.

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1 Introduction

The study of generic large cardinals, being cardinals that are critical points of elementary embeddings existing in generic extensions, goes back to the 1970's. At that time, the primary interest was the existence of *precipitous* and *saturated* ideals on small cardinals like ω_1 and ω_2. Research in this area later moved to the study of more general generic embeddings, both defined on V, but also on rank-initial segments of V — these were investigated by e.g. [DL89] and [FG10].

The move to *virtual* large cardinals happened when [Sch00] introduced the *remarkable cardinals*, which it turned out later were precisely a virtualization of supercompactness. Various other virtual large cardinals were first investigated in [GS18]. The key difference between virtual large cardinals and generic versions of large cardinals studied earlier is that in the virtual case we require the embedding to be between sets with the target model being a subset of the ground model. These assumptions imply that virtual large cardinals are actual large cardinals: they are at least ineffable, but small enough to exist in L. These large cardinals are special because they allow us to work with embeddings as in the higher reaches of the large cardinal hierarchy while being consistent with $V = L$, which enables equiconsistencies at these "lower levels".

To take a few examples, [Sch00] has shown that the existence of a remarkable cardinal is equiconsistent with the statement that the theory of $L(\mathbb{R})$ cannot be changed by proper forcing, which was improved to semi-proper forcing in [Sch04]. [Wil19] has shown that the existence of a virtually Vopěnka cardinal is equiconsistent with the hypothesis

$$\mathsf{ZF} + \ulcorner\boldsymbol{\Sigma}^1_2 \text{ is the class of all } \omega_1\text{-Suslin sets}\urcorner + \Theta = \omega_2,$$

and [SW18] has shown that the existence of a virtually Shelah cardinal is equiconsistent with the hypothesis

$$\mathsf{ZF} + \ulcorner\text{every universally Baire set of reals has the perfect set property}\urcorner.$$

Kunen's Inconsistency fails for virtual large cardinals in the sense that a forcing extension can have elementary embeddings $j : V_\alpha^V \to V_\alpha^V$ with α much larger than the supremum of the critical sequence. In the theory of large cardinals, Kunen's Inconsistency is for instance used to prove that requiring that $j(\kappa) > \lambda$ in the definition of κ being λ-strong is superfluous. It turns out that the use of Kunen's Inconsistency in that argument is actually essential because versions of virtual strongness with and without that condition are not equivalent; see e.g. Corollary 3.11. The same holds for virtual versions of other large cardinals where this condition is used in the embeddings characterization. Each of these virtual large cardinals therefore has two non-equivalent versions, with and without the condition.

In this paper, we continue the study of virtual versions of various large cardinals.

In Section 3 we establish some new relationships between virtual large cardinals that were previously studied. We prove the Gitman-Schindler result, alluded to in [GS18], that virtualizations of strong and supercompact cardinals are equivalent to remarkability.

We show how the existence of virtual supercompact cardinals without the $j(\kappa) > \lambda$ condition is related to the existence of virtually rank-into-rank cardinals.

In Section 4 we study virtual versions of Woodin and Vopěnka cardinals. We show that inaccessible cardinals are virtually Vopěnka if and only if they are *faintly pre-Woodin*, a weakening of the virtual notion of Woodinness — see Definition 4.2.

In Section 5 we study a virtual version of *Berkeley* cardinals, a large cardinal known to be inconsistent with ZFC. We show that the existence of a virtually Berkeley cardinal is equiconsistent to the virtual Vopěnka principle holding while On is not Mahlo, which is in turn equivalent to On being virtually pre-Woodin but not virtually Woodin. The equiconsistency with a virtually Berkeley cardinal improves on a result of [GH19].

2 Preliminaries

We will denote the class of ordinals by On. For sets X and Y we denote by ${}^XY$ the set of all functions from X to Y. For an infinite cardinal κ, we let H_κ be the set of sets X such that the cardinality of the transitive closure of X has size less than κ. The symbol $\lightning$ will denote a contradiction and $\mathscr{P}(X)$ will denote the power set of X. We will say that an elementary embedding $j : \mathcal{M} \to \mathcal{N}$ is **generic** if it exists in a forcing extension of V. We will use H_λ and V_λ to denote these sets as defined in the ground model V, while the H_λ or V_λ of any other universe will have a superscript indicating which universe it comes from.

A key folklore lemma which we will frequently need when dealing with generic elementary embeddings is the following.

Lemma 2.1 (Countable Embedding Absoluteness). *Let $\mathcal{M}$ and $\mathcal{N}$ be transitive sets and assume that $\mathcal{M}$ is countable. Let $\pi\colon \mathcal{M} \to \mathcal{N}$ be an elementary embedding, $\mathcal{P}$ a transitive class with $\mathcal{M}, \mathcal{N} \in \mathcal{P}$ and $\mathcal{P} \models \mathsf{ZF}^- + \mathsf{DC} + \ulcorner\mathcal{M}$ is countable$\urcorner$.*[1] *Then $\mathcal{P}$ has an elementary embedding $\pi^*\colon \mathcal{M} \to \mathcal{N}$ which agrees with π on any desired finite set and on the critical point if it exists.*

The following proposition is an almost immediate corollary of Countable Embedding Absoluteness.

Proposition 2.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be transitive models and assume that there is a generic elementary embedding $\pi : \mathcal{M} \to \mathcal{N}$. Then $V^{\mathrm{Col}(\omega,\mathcal{M})}$ has an elementary embedding $\pi^* : \mathcal{M} \to \mathcal{N}$ which agrees with π on any desired finite set and has the same critical point if it exists.*

[1] The theory ZF^- consists of the axioms of ZF without the powerset axiom and with the collections scheme instead of the replacement scheme.

For proofs, see [GS18] (Section 3).

3 Virtually supercompact cardinals

In this section, we establish some relationships between virtual large cardinals related to virtually supercompact cardinals. We start with definitions of the relevant virtual large cardinal notions.

Definition 3.1. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is

- **faintly θ-measurable** if, in a forcing extension, there is a transitive set $\mathcal{N}$ and an elementary embedding $\pi\colon H_\theta \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$,
- **faintly θ-strong** if it is faintly θ-measurable, $H_\theta \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$,
- **faintly θ-supercompact** if it is faintly θ-measurable, ${}^{<\theta}\mathcal{N} \cap V \subseteq \mathcal{N}$ and $\pi(\kappa) > \theta$.

We further replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we attach a "**pre**" if we leave out the assumption $\pi(\kappa) > \theta$, and when we do not mention θ we mean that it holds for all regular $\theta > \kappa$. For instance, a faintly pre-strong cardinal is a cardinal κ such that for all regular $\theta > \kappa$, κ is faintly θ-measurable with $H_\theta \subseteq \mathcal{N}$. $\dashv$

Observe that whenever we have a virtual large cardinal that has its defining property for all regular θ, we can assume that the target of the embedding is an element of the ground model V and not just a subset of V. Suppose, for instance, that κ is virtually measurable and fix a regular $\theta > \kappa$ and set $\lambda := (2^{<\theta})^+$. Take a generic elementary embedding $\pi : H_\lambda \to \mathcal{M}_\lambda$ witnessing that κ is virtually λ-measurable. The restriction $\pi \restriction H_\theta\colon H_\theta \to \pi(H_\theta)$ witnesses that κ is virtually θ-measurable and the target model $\mathcal{M}_\theta := \pi(H_\theta)$ is in V because $\mathcal{M}_\lambda \subseteq V$ by assumption. Thus, the weaker assumption that the target model $\mathcal{M}_\theta \subseteq V$ only affects level-by-level virtual large cardinals. Indeed, as we will see in later sections, even further weakening the assumption $\mathcal{N} \subseteq V$ to $H_\theta = H_\theta^{\mathcal{N}}$ in the definition of virtually strong (or supercompact) cardinals yields the same notion (again, we do not know whether this holds level-by-level).

Small cardinals such as ω_1 can be generically measurable and hence faintly measurable. However, virtual large cardinals are large cardinals in the usual sense, as the following shows.

Recall from [GW11] that a cardinal κ is **1-iterable** if for every $A \subseteq \kappa$ there is a transitive $\mathcal{M} \models \mathsf{ZFC}^-$ with $\kappa, A \in \mathcal{M}$ and a weakly amenable $\mathcal{M}$-ultrafilter μ on κ with a well-founded ultrapower. Recall that μ is an **$\mathcal{M}$-ultrafilter** on κ if the structure $\langle M, \in, \mu\rangle$ satisfies that μ is a normal ultrafilter on κ, and such a μ is **weakly amenable** if $\mu \cap X \in \mathcal{M}$ for every $X \in \mathcal{M}$ of $\mathcal{M}$-cardinality $\leq \kappa$. It is not difficult to see that an

$\mathcal{M}$-ultrafilter μ on κ with a well-founded ultrapower is weakly amenable if and only if the ultrapower embedding $j : \mathcal{M} \to \mathcal{N}$ is **κ-powerset preserving**, meaning that $\mathcal{M}$ and $\mathcal{N}$ have the same subsets of κ. 1-iterable cardinals are weakly ineffable limits of ineffable cardinals, and hence, in particular, weakly compact [GW11].

Proposition 3.2. *For any regular uncountable cardinal θ, every virtually θ-measurable cardinal is a 1-iterable limit of 1-iterable cardinals.*

The proof is essentially the same as the proof of Theorem 4.8 in [GS18].

Schindler observed in [Sch14] that remarkable cardinals can be viewed as a version of virtual supercompact cardinals via Magidor's characterization of supercompactness. Later Gitman and Schindler observed in [GS18] that remarkables are precisely the virtually supercompacts, and indeed surprisingly, they are also precisely the virtually strongs. So, in particular, virtually strong and virtually supercompact cardinals are equivalent. We give the proof of the equivalences, which was omitted in [GS18], here.

Definition 3.3. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **virtually θ-supercompact ala Magidor** if there are $\bar{\kappa} < \bar{\theta} < \kappa$ and a generic elementary embedding $\pi\colon H_{\bar{\theta}} \to H_\theta$ such that $\operatorname{crit}\pi = \bar{\kappa}$ and $\pi(\bar{\kappa}) = \kappa$. $\dashv$

Theorem 3.4 (G.-Schindler). *For an uncountable cardinal κ, the following are equivalent.*

(i) *κ is virtually strong.*
(ii) *κ is virtually supercompact.*
(iii) *κ is virtually supercompact ala Magidor.*

Proof. $(ii) \Rightarrow (i)$ is simply by definition.

$(i) \Rightarrow (iii)$: Fix a regular uncountable $\theta > \kappa$. By (i) there exists a generic elementary embedding $\pi\colon H_{(2^{<\theta})^+} \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$, $H_{(2^{<\theta})^+} \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq V$. We can restrict the embedding π to $\pi : H_\theta \to H^{\mathcal{M}}_{\pi(\theta)}$. Since $H_\theta, H^{\mathcal{M}}_{\pi(\theta)} \in \mathcal{M}$, Countable Embedding Absoluteness 2.1 implies that $\mathcal{M}$ has a generic elementary embedding $\pi^* : H_\theta \to H^{\mathcal{M}}_{\pi(\theta)}$ with $\operatorname{crit}\pi^* = \kappa$ and $\pi^*(\kappa) = \pi(\kappa) > \theta$. Since $H_\theta = H^{\mathcal{M}}_\theta$ as $\mathcal{M} \subseteq V$ and $H_\theta \subseteq \mathcal{M}$, elementarity of π now implies that $H_{(2^{<\theta})^+}$ has ordinals $\bar{\kappa} < \bar{\theta} < \kappa$ and a generic elementary embedding $\sigma\colon H_{\bar{\theta}} \to H_\theta$ with $\operatorname{crit}\sigma = \bar{\kappa}$ and $\sigma(\bar{\kappa}) = \kappa$. This shows (iii).

$(iii) \Rightarrow (ii)$: Fix a regular uncountable $\theta > \kappa$ and let $\delta = (2^{<\theta})^+$. By (iii) there exist ordinals $\bar{\kappa} < \bar{\delta} < \kappa$ and a generic elementary embedding $\pi\colon H_{\bar{\delta}} \to H_\delta$ with $\operatorname{crit}\pi = \bar{\kappa}$ and $\pi(\bar{\kappa}) = \kappa$. Let $\pi(\bar{\theta}) = 2^{<\theta}$ (we can assume that θ is in the range of π by taking it to be largest so that $(2^{<\theta})^+ = \delta$). We will argue that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H_{\bar\delta}$, so that by elementarity κ will be virtually θ-supercompact in H_δ, and hence also in V. Consider the restriction $\sigma := \pi\colon H_{\bar\theta} \to H_\theta$. Note that H_θ is closed under $<\bar\theta$-sequences (and more) in V. We can assume without loss that σ lives in a $\mathrm{Col}(\omega, H_{\bar\theta})$-extension. Let $\dot\sigma$ be a $\mathrm{Col}(\omega, H_{\bar\theta})$-name for σ. Now define

$$X := \bar\theta{+}1 \cup \{x \in H_\theta \mid \exists y \in H_{\bar\theta}\, \exists p \in \mathrm{Col}(\omega, H_{\bar\theta})\, p \Vdash \dot\sigma(\check y) = \check x\} \in V.$$

Note that $|X| = |H_{\bar\theta}| = 2^{<\bar\theta}$ and that $\operatorname{ran}\sigma \subseteq X$. Now let $\overline{X} \prec H_\theta$ be such that $X \subseteq \overline{X}$ and $\overline{X}$ is closed under $<\bar\theta$-sequences. Note that we can find such an $\overline{X}$ of size $(2^{<\bar\theta})^{<\bar\theta} = 2^{<\bar\theta}$. Let $\mathcal{M}$ be the transitive collapse of $\overline{X}$, so that $\mathcal{M}$ is still closed under $<\bar\theta$-sequences and we still have that $|\mathcal{M}| = 2^{<\bar\theta} < \bar\delta$, making $\mathcal{M} \in H_{\bar\delta}$.

Countable Embedding Absoluteness 2.1 then implies that $H_{\bar\delta}$ has a generic elementary embedding $\sigma^*\colon H_{\bar\theta} \to \mathcal{M}$ with $\operatorname{crit}\sigma^* = \bar\kappa$, showing that $\bar\kappa$ is virtually $\bar\theta$-supercompact in $H_{\bar\delta}$. ■

Remark 3.5. The above proof shows that if κ is virtually $(2^{<\theta})^+$-strong, then it is virtually θ-supercompact, and if it is virtually $(2^{<\theta})^+$-supercompact ala Magidor, then it is virtually θ-supercompact. It is open whether they are equivalent level-by-level (see Question 6.1).

As a corollary of the proof, we obtain the following weaker characterization of virtually strong cardinals.

Proposition 3.6. *A cardinal κ is virtually strong if and only if for every $\theta > \kappa$ in a forcing extension, there is a transitive set $\mathcal{N}$ and an elementary embedding $\pi : H_\theta \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$ and $H_\theta = H_\theta^{\mathcal{N}}$.*

Next, we define a virtualisation of the α-superstrong cardinals.

Definition 3.7. Let θ be a regular uncountable cardinal and α be an ordinal. Then a cardinal $\kappa < \theta$ is **faintly (θ, α)-superstrong** if it is faintly θ-measurable, $H_\theta \subseteq \mathcal{N}$ and $\pi^\alpha(\kappa) \leq \theta$.[2] We replace "faintly" by **virtually** when $\mathcal{N} \subseteq V$, we say that κ is **faintly α-superstrong** if it is faintly (θ, α)-superstrong for *some* θ, and lastly κ is simply **faintly superstrong** if it is faintly 1-superstrong. ⊣

[2]Here we set $\pi^\alpha(\kappa) := \sup_{\xi<\alpha} \pi^\xi(\kappa)$ when α is a limit ordinal.

Proposition 3.8 (N.). *If κ is faintly superstrong, then H_κ has a proper class of virtually strong cardinals.*

Proof. Fix a regular $\theta > \kappa$ and a generic elementary embedding $\pi\colon H_\theta \to \mathcal{N}$ with $\operatorname{crit}\pi = \kappa$, $H_\theta \subseteq \mathcal{N}$ and $\pi(\kappa) \leq \theta$. Then $\pi(\kappa)$ is a V-cardinal. Let's argue that $H_{\pi(\kappa)}$ thinks that κ is virtually strong. Fixing $\kappa < \delta < \pi(\kappa)$, we have that $\pi : H_\delta \to H^{\mathcal{N}}_{\pi(\delta)}$ with $H_\delta \subseteq H_\theta \subseteq H^{\mathcal{N}}_{\pi(\delta)}$. We can build a transitive $\mathcal{M} \prec H^{\mathcal{N}}_{\pi(\delta)}$ such that $\pi'' H_\delta, H_\delta \subseteq \mathcal{M}$ and $\mathcal{M} \in H_{\pi(\kappa)}$. This gives a generic elementary embedding $\pi^* : H_\delta \to \mathcal{M}$ witnessing that κ is virtually δ-strong in $H_{\pi(\kappa)}$. Now since $H_\kappa \prec H_{\pi(\kappa)}$, we have that H_κ thinks that there is a proper class of virtually strong cardinals. ■

The following theorem shows that the existence of "Kunen inconsistencies" is precisely what is stopping pre-strongness from being equivalent to strongness.

Theorem 3.9 (N.). *Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is virtually θ-pre-strong if and only if one of the following holds.*

(i) κ is virtually θ-strong, or

(ii) κ is virtually (θ, ω)-superstrong.

Proof. ($\Leftarrow$) is trivial, so we show ($\Rightarrow$). Let κ be virtually θ-pre-strong. Assume (i) fails, meaning that there is a generic elementary embedding $\pi\colon H_\theta \to \mathcal{N}$ for some transitive $\mathcal{N} \subseteq V$ with $H_\theta \subseteq \mathcal{N}$, $\operatorname{crit}\pi = \kappa$ and $\pi(\kappa) \leq \theta$.

First, assume that there is some $n < \omega$ such that $\pi^n(\kappa) = \theta$. The proof of Proposition 3.8 shows that κ is virtually strong in $H_{\pi(\kappa)}$. It follows that $\pi(\kappa)$ is virtually strong in $H_{\pi^2(\kappa)}$ by elementarity, and by applying elementarity repeatedly, we get that $\pi^n(\kappa) = \theta$ is virtually strong in $\mathcal{N}$. Note that the condition $\pi^n(\kappa) = \theta$ implies that θ is inaccessible in $\mathcal{N}$, and hence also in H_θ. Thus, H_θ satisfies that there is no largest cardinal, and so by elementarity $\mathcal{N}$ does not have a largest cardinal also. Let $\delta = (\theta^+)^{\mathcal{N}}$. In particular, θ is virtually δ-strong in $\mathcal{N}$, and so $\mathcal{N}$ has a generic elementary embedding $\sigma : H^{\mathcal{N}}_\delta \to \mathcal{M}$ with $\operatorname{crit}\sigma = \theta$ and $H_\theta \subseteq H^{\mathcal{N}}_\delta \subseteq \mathcal{M}$. Thus, $H_\theta \prec H^{\mathcal{M}}_{\sigma(\theta)}$, from which it follows that κ is virtually strong in $H^{\mathcal{M}}_{\sigma(\theta)}$, and, in particular, virtually θ-strong. But $H^{\mathcal{M}}_{\sigma(\theta)}$ must be correct about this since $H^{\mathcal{M}}_\theta = H^{\mathcal{N}}_\theta = H_\theta$. But then κ is actually virtually θ-strong, contradicting our assumption that (i) fails.

Next, assume that there is a least $n < \omega$ such that $\pi^{n+1}(\kappa) > \theta$. Since $\pi(\kappa) \leq \theta$, we have as before that κ is virtually strong in $H_{\pi(\kappa)}$. Since, by elementarity, $H_{\pi^i(\kappa)} \prec H_{\pi^{i+1}(\kappa)}$, we have that κ is virtually strong in $H_{\pi^n(\kappa)}$. Repeatedly applying elementarity to the statement that κ is virtually strong in $H_{\pi(\kappa)}$, we also get that $\pi^n(\kappa)$ is virtually strong in $H^{\mathcal{N}}_{\pi^{n+1}(\kappa)}$. This means that there is some generic elementary embedding $\sigma\colon H_\theta \to \mathcal{M}$ with $H_\theta \subseteq \mathcal{M}$, $\mathcal{M} \subseteq H^{\mathcal{N}}_{\pi^{n+1}(\kappa)}$, $\operatorname{crit}\sigma = \pi^n(\kappa)$ and $\sigma(\pi^n(\kappa)) > \theta$. Thus, by elementarity, we get $H_{\pi^n(\kappa)} \prec H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))}$. Since, as we already argued, κ is virtually strong in $H_{\pi^n(\kappa)}$ this means that κ is also virtually strong in $H^{\mathcal{M}}_{\sigma(\pi^n(\kappa))}$, and as $H^{\mathcal{M}}_\theta = H^{\mathcal{N}}_\theta = H_\theta$, this means that κ is actually virtually θ-strong, contradicting our assumption that (i) fails.

Finally, assume $\pi^n(\kappa) < \theta$ for all $n < \omega$ and let $\lambda = \sup_{n<\omega} \pi^n(\kappa)$. Since $\lambda \leq \theta$, we have that κ is virtually (θ, ω)-superstrong by definition. ■

We then get the following consistency result.

Corollary 3.10 (N.). *For any uncountable regular cardinal θ, the existence of a virtually θ-strong cardinal is equiconsistent with the existence of a faintly θ-measurable cardinal.*

Proof. The above Proposition 3.8 and Theorem 3.9 show that virtually θ-pre-strongs are equiconsistent with virtually θ-strongs. Let us now argue that if κ is faintly θ-measurable in L, then κ is virtually θ-pre-strong in L. Suppose that $\pi : L_\theta \to \mathcal{M}$ is a generic elementary embedding with $\mathcal{M}$ transitive and $\operatorname{crit}\pi = \kappa$. By elementarity, $\mathcal{M}$ satisfies $V = L$, and hence by absoluteness of the construction of L and transitivity of $\mathcal{M}$, $\mathcal{M} = L_\beta$ for some cardinal $\beta \geq \theta$. But then trivially we have $L_\theta \subseteq \mathcal{M}$.

■

Recall that a cardinal κ is **virtually rank-into-rank** if there exists a cardinal $\theta > \kappa$ and a generic elementary embedding $\pi\colon H_\theta \to H_\theta$ with $\operatorname{crit}\pi = \kappa$. We then have the following corollary.

Corollary 3.11 (N.). *The following are equivalent.*

(i) For every regular uncountable cardinal θ, every virtually θ-pre-strong cardinal is virtually θ-strong.

(ii) There are no virtually rank-into-rank cardinals.

Proof. $(\Leftarrow)$: Note first that being virtually ω-superstrong is equivalent to being virtually rank-into-rank. Indeed, every virtually rank-into-rank cardinal is virtually ω-superstrong by definition, and if κ is virtually ω-superstrong, as witnessed by a generic elementary embedding $\pi : H_\theta \to \mathcal{M}$ with $\pi^\omega(\kappa) \leq \theta$, and we let $\lambda = \pi^\omega(\kappa)$, then restriction $\pi\colon H_\lambda \to H_\lambda$ witnesses that κ is virtually rank-into-rank. The above Theorem 3.9 then implies $(\Leftarrow)$.

($\Rightarrow$): Here we have to show that if there exists a virtually rank-into-rank cardinal, then there exists a $\theta > \kappa$ and a virtually θ-pre-strong cardinal which is not virtually θ-strong. Let $\langle\kappa, \theta\rangle$ be the lexicographically least pair such that κ is virtually rank-into-rank as witnessed by a generic embedding $\pi : H_\theta \to H_\theta$, which trivially makes κ virtually θ-pre-strong. If κ was also virtually θ-strong, then we would have a generic elementary embedding $\pi^* : H_\theta \to \mathcal{M}$ with $\operatorname{crit}\pi^* = \kappa$, $\pi^*(\kappa) > \theta$, and $\mathcal{M} \subseteq V$. By Countable Embedding Absoluteness 2.1, $\mathcal{M}$ sees that κ virtually rank-into-rank, but then, using elementarity, this reflects down below κ, showing that the pair $\langle\kappa, \theta\rangle$ could not have been least. ■

As a final result of this section, we separate virtual and faint large cardinals. The separation is trivial in general, as successor cardinals can be faintly measurable and are never virtually measurable, but the separation still holds true if we rule out this successor case.

For a slightly more fine-grained distinction let us prepend a **power-** adjective whenever the generic embedding (with critical point κ) is κ-powerset preserving. Note that the proof of Lemma 3.2 shows that faintly power-measurables are also 1-iterable. Our separation result is then the following.

Theorem 3.12 (G.). *For $\Phi \in \{measurable, pre\text{-}strong, strong\}$, if κ is virtually Φ, then there exist forcing extensions $V[G]$ and $V[G^*]$ such that*

(i) in $V[G]$, κ is inaccessible and faintly Φ, but not faintly power-Φ, and

(ii) in $V[G^*]$, κ is faintly power-Φ, but not virtually κ^{++}-pre-strong.

Proof. We start with (i). Let us assume that κ is virtually measurable. This implies, in particular, that for every regular $\theta > \kappa$, we have generic elementary embeddings $\pi : H_\theta \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$ such that $\mathcal{M} \in V$. Thus, by Proposition 2.2, we can assume that each generic embedding π exists in a $\operatorname{Col}(\omega, H_\theta)$-extension.

Let $\mathbb{P}_\kappa$ be the Easton support iteration that adds a Cohen subset to every regular $\alpha < \kappa$, and let $G \subseteq \mathbb{P}_\kappa$ be V-generic. Standard computations show that $\mathbb{P}_\kappa$ preserves all inaccessible cardinals.

Fix a regular $\theta \gg \kappa$ and let $h \subseteq \operatorname{Col}(\omega, H_\theta)$ be $V[G]$-generic. In $V[h]$, we must have an elementary embedding $\pi : H_\theta \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$ and $\mathcal{M} \in V$, and we can assume without loss that $\mathcal{M}$ is countable. Obviously, $\pi \in V[G][h]$. Working in $V[G][h]$, we will now lift π to an elementary embedding on $H_\theta[G] = H_\theta^{V[G]}$. To ensure that such a lift exists, it suffices to find in $V[G][h]$ an $\mathcal{M}$-generic filter for $\pi(\mathbb{P}_\kappa)$ containing $\pi''G$.[3] Observe first that $\pi''G = G$ since the critical point of π is κ and we

[3] This standard lemma is referred to in the literature as the **lifting criterion**.

can assume that $\mathbb{P}_\kappa \subseteq V_\kappa$. Next, observe that $\pi(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\text{tail}}$ is the forcing beyond κ. Since $\mathcal{M}[G]$ is countable, we can build an $\mathcal{M}[G]$-generic filter G_{tail} for $\mathbb{P}_{\text{tail}}$ in $V[G][h]$. Thus, $G * G_{\text{tail}}$ is $\mathcal{M}$-generic for $\pi(\mathbb{P}_\kappa)$, and so we can lift π to $\pi : H_\theta[G] \to \mathcal{M}[G][G_{\text{tail}}]$. Since θ was chosen arbitrarily, we have just shown that κ is faintly measurable in $V[G]$.

Now suppose that κ is faintly power-measurable in $V[G]$. Fix a regular $\theta < \bar{\theta}$ and a generic elementary embedding $\sigma : H_{\bar{\theta}}[G] \to \mathcal{N}$ with $\operatorname{crit}\sigma = \kappa$ and $\mathscr{P}(\kappa)^{V[G]} = \mathscr{P}(\kappa)^{\mathcal{N}}$. By elementarity, $H^{\mathcal{N}}_{\sigma(\theta)} = \sigma(H_\theta)[\sigma(G)]$ is a forcing extension of $K = \sigma(H_\theta)$ by $\sigma(G) = G * \bar{G}_{\text{tail}} \subseteq \sigma(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \bar{\mathbb{P}}_{\text{tail}}$. Thus, we have the restrictions $\sigma : H_\theta \to K$ and $\sigma : H_\theta[G] \to K[G][\bar{G}_{\text{tail}}]$. Let us argue that $\mathscr{P}^{V[G]}(\kappa) \subseteq \mathscr{P}^{K[G]}(\kappa)$, and hence we have equality. Suppose $A \subseteq \kappa$ in $V[G]$ and let $\dot{A}$ be a nice $\mathbb{P}_\kappa$-name for A, which can be coded by a subset of κ. Since $\operatorname{crit}\sigma = \kappa$, we have that $\dot{A} \in K$, and hence $A = \dot{A}_G \in K[G]$. But now it follows that the $K[G]$-generic for $\operatorname{Add}(\kappa, 1)$, the forcing at stage κ in $\sigma(\mathbb{P}_\kappa)$, cannot be in $V[G]$. Thus, we have reached a contradiction, showing that κ cannot be faintly power-measurable in $V[G]$.

If $\Phi =$ measurable, then we are done at this point. For $\Phi =$ pre-strong we simply note that $G \in \mathcal{M}[G * G_{\text{tail}}]$ so that $H_\theta^{V[G]} \subseteq \mathcal{N}[G * G_{\text{tail}}]$ as well, and since we lifted π, we still have $\pi(\kappa) > \theta$ in the $\Phi =$ strong case.

For (ii), we change $\mathbb{P}_\kappa$ to only add Cohen subsets to *successor* cardinals $\lambda < \kappa$ and call the resulting forcing $\mathbb{P}^*_\kappa$. Let $G^* \subseteq \mathbb{P}^*_\kappa$ be V-generic. We verify that κ is faintly-Φ as above by lifting an embedding $\pi : H_\theta \to \mathcal{M}$, with $\mathcal{M} \subseteq V$, to $\pi : H_\theta[G^*] \to \mathcal{M}[G^*][G_{\text{tail}}]$ in a collapse extension $V[G^*][h]$. The lifted embedding is κ-powerset preserving because a subset of κ from $\mathcal{M}[G^*][G_{\text{tail}}]$ has to already be in $\mathcal{M}[G^*]$ as $\mathbb{P}_{\text{tail}}$ is $\leq\kappa$-closed, and $\mathcal{M}[G^*] \subseteq V[G^*]$. So it remains to show that κ is not virtually κ^{++}-pre-strong. Suppose it is and fix a generic embedding $\sigma : H_{\kappa^{++}}[G^*] \to K[G^*][G_{\text{tail}}]$ with $\operatorname{crit}\sigma = \kappa$ and $H_{\kappa^{++}}[G^*] \subseteq K[G^*][G_{\text{tail}}]$. It follows that the generic subset of κ^+ added at stage κ^+ by the tail forcing must be $V[G^*]$-generic, which is contradictory.

■

Starting from much stronger hypothesis, it can be shown that a power-measurable cardinal need not even be virtually κ^+-measurable. Now, first, we observe that virtually θ-measurable cardinals κ are Π^2_1-indescribable for all $\theta > \kappa$. The proof is identical to the standard Hanf-Scott proof that measurable cardinals are Π^2_1-indescribable; see e.g. Proposition 6.5 in [Kan08]. It should be noted that we crucially need the "virtual" property for the proof to go through. Using this indescribability fact, the proof of the following theorem is precisely the same as Hamkins' Proposition 8.2 in [HS18].

Theorem 3.13 (Virtualised Hamkins). *Assuming κ is a κ^{++}-tall cardinal,*[4] *there is a forcing extension in which κ is not virtually κ^+-measurable, but becomes measurable in a further $Add(\kappa^+, 1)$-generic extension.*

This then immediately gives the separation result.

Corollary 3.14. *Assuming κ is a κ^{++}-tall cardinal, it is consistent that κ is faintly power-measurable, but not virtually κ^+-measurable.*

In contrast to the above separation result, we will show in Theorem 4.6 and Theorem 4.10 that the faint-virtual distinction vanishes when we are dealing with virtual pre-Woodin or Woodin cardinals.

4 Virtual Woodin and Vopěnka cardinals

In this section we will analyse the virtualisations of Woodin and Vopěnka cardinals, which can be seen as "boldface" variants of strongs and supercompacts.

Definition 4.1. Let θ be a regular uncountable cardinal. Then a cardinal $\kappa < \theta$ is **faintly (θ, A)-strong** for a set $A \subseteq H_\theta$ if there exists a generic elementary embedding $\pi\colon (H_\theta, \in, A) \to (\mathcal{M}, \in, B)$ such that $\operatorname{crit} \pi = \kappa$, $\pi(\kappa) > \theta$, $H_\theta \subseteq \mathcal{M}$, and $B \cap H_\theta = A$. We say that κ is **faintly (θ, A)-supercompact** if we further have that ${}^{<\theta}\mathcal{M} \cap V \subseteq \mathcal{M}$. We say that κ is **faintly (θ, A)-extendible** if it further holds that $\mathcal{M} = H_\mu$ for some V-cardinal μ. We will leave out θ if the property holds of all regular $\theta > \kappa$. ⊣

Definition 4.2. A cardinal δ is **faintly Woodin** if, given any $A \subseteq H_\delta$, there exists a faintly $({<}\delta, A)$-strong cardinal $\kappa < \delta$. ⊣

As with previous definitions, for both of the above two definitions we substitute "faintly" for **virtually** when $\mathcal{M} \subseteq V$, and substitute "strong", "supercompact", "extendible", and "Woodin" for **pre-strong**, **pre-supercompact**, **pre-extendible** and **pre-Woodin** when we do not require that $\pi(\kappa) > \theta$.

We note in the following proposition that, in analogy with Woodin cardinals, virtually Woodin cardinals are Mahlo. This property fails for virtually pre-Woodin cardinals since [Wil19], together with Theorem 4.10 below, shows that they can be singular.

Proposition 4.3 (Virtualised folklore). *Virtually Woodin cardinals are Mahlo.*

[4] Recall that κ is **κ^{++}-tall** if there is an elementary embedding $j\colon V \to M$ with $\operatorname{crit} j = \kappa$, ${}^\kappa M \subseteq M$ and $j(\kappa) > \kappa^{++}$.

Proof. Let δ be virtually Woodin. Note that δ is a limit of weakly compact cardinals by Proposition 3.2, making δ a strong limit. As for regularity, assume that we have a cofinal increasing function $f\colon \alpha \to \delta$ with $f(0) > \alpha$ and $\alpha < \delta$, and note that f cannot have any closure points. Fix a virtually $({<}\delta, f)$-strong cardinal $\kappa < \delta$. We claim that κ is a closure point for f, which will yield the desired contradiction.

Let $\gamma < \kappa$ and choose a regular $\theta \in (f(\gamma), \delta)$ above κ. We then have a generic elementary embedding $\pi\colon (H_\theta, \in, f \cap H_\theta) \to (\mathcal{N}, \in, f^+)$ with $H_\theta \subseteq \mathcal{N}$, $\mathcal{N} \subseteq V$, $\operatorname{crit}\pi = \kappa$, $\pi(\kappa) > \theta$, and f^+ a function such that $f^+ \cap H_\theta = f \cap H_\theta$. But then $f^+(\gamma) = f(\gamma) < \pi(\kappa)$ by our choice of θ, so elementarity implies that $f(\gamma) < \kappa$, making κ a closure point for f, a contradiction. Thus, δ is inaccessible.

Next, let us show that κ is Mahlo. Let $C \subseteq \delta$ be a club and let $\kappa < \delta$ be a virtually $({<}\delta, C)$-strong cardinal. Let $\theta \in (\min C, \delta)$ be above κ and let $\pi\colon (H_\theta, \in, C \cap H_\theta) \to (\mathcal{N}, \in, C^+)$ be the associated generic elementary embedding having $C^+ \cap \theta = C$. Then for every $\gamma < \kappa$ there exists an element of C^+ below $\pi(\kappa)$, namely $\min C$, so by elementarity κ is a limit of elements of C, making it an element of C. As κ is regular, this shows that δ is Mahlo. ■

The well-known equivalence of the "function definition" and "A-strong" definition of Woodin cardinals holds for virtually Woodin cardinals, and the analogue of the equivalence between virtually strongs and virtually supercompacts allows us to strengthen this:

Theorem 4.4 (D.-G.-N.). *For an uncountable cardinal δ, the following are equivalent.*

- *(i)* *δ is virtually Woodin.*
- *(ii)* *For every $A \subseteq H_\delta$ there exists a virtually $({<}\delta, A)$-supercompact $\kappa < \delta$.*
- *(iii)* *For every $A \subseteq H_\delta$ there exists a virtually $({<}\delta, A)$-extendible $\kappa < \delta$.*
- *(iv)* *For every function $f\colon \delta \to \delta$ there are regular cardinals $\kappa < \theta < \delta$, such that κ is a closure point of f, and a generic elementary embedding $\pi\colon H_\theta \to \mathcal{M}$ such that $\operatorname{crit}\pi = \kappa$, $H_\theta \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\pi(f \restriction \kappa)(\kappa) < \theta$.*
- *(v)* *For every function $f\colon \delta \to \delta$ there are regular cardinals $\kappa < \theta < \delta$, such that κ is a closure point of f, and a generic elementary embedding $\pi\colon H_\theta \to \mathcal{M}$ such that $\operatorname{crit}\pi = \kappa$, ${}^{<\theta}\mathcal{M} \cap V \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$ and $\pi(f \restriction \kappa)(\kappa) < \theta$.*
- *(vi)* *For every function $f\colon \delta \to \delta$ there are regular cardinals $\bar\theta < \kappa < \theta < \delta$, such that κ is a closure point of f, and a generic elementary embedding $\pi\colon H_{\bar\theta} \to H_\theta$ with $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) < \bar\theta$ and $f \restriction \kappa \in \operatorname{ran}\pi$.*

Proof. Firstly note that $(iii) \Rightarrow (ii) \Rightarrow (i)$ and $(v) \Rightarrow (iv)$ are simply by definition.

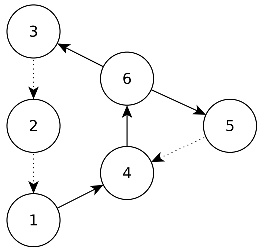

Figure 1: Proof strategy of Theorem 4.4, dotted lines are trivial implications.

$\boxed{(i) \Rightarrow (iv)}$ Assume δ is virtually Woodin, and fix a function $f\colon \delta \to \delta$. Let $\kappa < \delta$ be virtually $({<}\delta, f)$-strong and let $\theta < \delta$ be a regular cardinal such that $\sup_{\alpha \leq \kappa} f(\alpha) < \theta$. Then there is a generic elementary embedding $\pi\colon (H_\theta, \in, f \cap H_\theta) \to (\mathcal{M}, \in, f^+)$ such that $H_\theta \subseteq \mathcal{M}$, $f \cap H_\theta = f^+ \cap H_\theta$, $\mathcal{M} \subseteq V$, and $\pi(\kappa) > \theta$. Note that, by our choice of θ, $f \restriction \kappa \in H_\theta$ and $\pi(f \restriction \kappa)(\kappa) = f^+(\kappa) = f(\kappa) < \theta$.

So it suffices to show that κ is a closure point for f. Let $\alpha < \kappa$. Then

$$f(\alpha) = f^+(\alpha) = \pi(f \restriction \kappa)(\alpha) = \pi(f \restriction \kappa)(\pi(\alpha)) = \pi(f(\alpha)),$$

so π fixes $f(\alpha)$ for every $\alpha < \kappa$. Now, if κ was not a closure point of f then, letting $\alpha < \kappa$ be the least such that $f(\alpha) \geq \kappa$, we have

$$\theta > f(\alpha) = \pi(f(\alpha)) > \theta,$$

a contradiction. Note that we used that $\pi(\kappa) > \theta$ here, so this argument would not work if we had only assumed δ to be virtually pre-Woodin.

$\boxed{(iv) \Rightarrow (vi)}$ Assume (iv) holds, let $f\colon \delta \to \delta$ be given and define $g\colon \delta \to \delta$ as $g(\alpha) := (2^{<\gamma_\alpha})^+$, where γ_α is the least regular cardinal above $|f(\alpha)|$. By (iv) there is a $\kappa < \delta$ which is a closure point of g (and so also a closure point of f), and there is a regular $\lambda \in (\kappa, \delta)$ for which there is a generic elementary embedding $\pi\colon H_\lambda \to \mathcal{M}$ with $\operatorname{crit}\pi = \kappa$, $H_\lambda \subseteq \mathcal{M}$, $\mathcal{M} \subseteq V$, and $\pi(g \restriction \kappa)(\kappa) < \lambda$.

Let θ be the least regular cardinal above $|\pi(f \restriction \kappa)(\kappa)|$, and note that $H_\theta \in H_\lambda$ by our definition of g. Thus, both H_θ and $H^{\mathcal{M}}_{\pi(\theta)}$ are elements of $\mathcal{M}$. An application of Countable Embedding Absoluteness 2.1 then yields that $\mathcal{M}$ has a generic elementary embedding $\pi^*\colon H^{\mathcal{M}}_\theta \to H^{\mathcal{M}}_{\pi(\theta)}$ such that $\operatorname{crit}\pi^* = \kappa$, $\pi^*(\kappa) = \pi(\kappa)$, $\pi(f \restriction \kappa) \in \operatorname{ran}\pi^*$, and $\pi(f \restriction \kappa)(\kappa) < \theta$. By elementarity of π, H_λ has an ordinal $\bar\theta < \kappa$ and a generic elementary embedding $\sigma\colon H_{\bar\theta} \to H_\theta$ with $\sigma(\operatorname{crit}\sigma) = \kappa$, $f \restriction \kappa \in \operatorname{ran}\sigma$ and $f(\operatorname{crit}\sigma) < \bar\theta$, which is what we wanted to show.

$\boxed{(vi) \Rightarrow (v)}$ Assume (vi) holds and let $f\colon \delta \to \delta$ be given. Define $g\colon \delta \to \delta$ as $g(\alpha) := \langle (2^{<\gamma_\alpha})^+, f(\alpha)\rangle$, where γ_α is the least regular cardinal above $|f(\alpha)|$. In particular, g codes f. By (vi) there exist regular $\bar\kappa < \bar\lambda < \kappa < \lambda$ such that κ is a closure point of g (so also a closure point of f) and there exists a generic elementary embedding $\pi\colon H_{\bar\lambda} \to H_\lambda$ with $\operatorname{crit}\pi = \bar\kappa$, $\pi(\bar\kappa) = \kappa$, $g(\bar\kappa) < \bar\lambda$, and $g \restriction \kappa \in \operatorname{ran}\pi$. Since f is definable from g and $g \restriction \kappa \in \operatorname{ran}\pi$, it follows that $f \restriction \kappa \in \operatorname{ran}\pi$. So let $\pi(\bar f) = f \restriction \kappa$ with $\bar f\colon \bar\kappa \to \bar\kappa$. Now observe that $\bar f = f \restriction \bar\kappa$ since for $\alpha < \bar\kappa$, we have $f(\alpha) = \pi(\bar f)(\alpha) = \pi(\bar f(\alpha)) = \bar f(\alpha)$.

Let $\bar\theta$ be the least regular cardinal above $|f(\bar\kappa)|$. By the definition of g, we have $H_{\bar\theta} \in H_{\bar\lambda}$. Now, following the $(iii) \Rightarrow (ii)$ direction in the proof of Theorem 3.4 we get that $H_{\bar\lambda}$ has a generic elementary embedding $\sigma\colon H_{\bar\theta} \to \mathcal{M}$ with $\mathcal{M}$ closed under $<\bar\theta$-sequences from V, $\operatorname{crit}\sigma = \bar\kappa$, $\sigma(\bar\kappa) > \bar\theta$, and $\sigma(\bar f)(\bar\kappa) < \bar\theta$. Let $\pi(\bar\theta) = \theta$ and $\pi(\mathcal{M}) = \mathcal{N}$. Now by elementarity of π, we get that there is a generic elementary embedding $\sigma^* : H_\theta \to \mathcal{N}$ with $\operatorname{crit}\sigma^* = \kappa$, $\sigma^*(\kappa) > \theta$, and $\sigma^*(\pi(\bar f))(\kappa) = \sigma^*(f \restriction \kappa)(\kappa) < \theta$.

$\boxed{(vi) \Rightarrow (iii)}$ Let C be the club of all cardinals α such that

$$(H_\alpha, \in, A \cap H_\alpha) \prec (H_\delta, \in, A).$$

Let $f\colon \delta \to \delta$ be given as $f(\alpha) := \langle \gamma_0^\alpha, \gamma_1^\alpha\rangle$, where γ_0^α is the first limit point of C above α and the γ_1^α are chosen such that $\{\gamma_1^\alpha \mid \alpha < \beta\}$ encodes $A \cap H_\beta$ for inaccessible cardinals $\beta < \delta$.

Let $\kappa < \delta$ be a closure point of f such that there are regular cardinals $\bar\theta < \kappa < \theta$ and a generic elementary embedding $\pi\colon H_{\bar\theta} \to H_\theta$ such that $\pi(\operatorname{crit}\pi) = \kappa$, $f(\operatorname{crit}\pi) < \bar\theta$, and $f \restriction \kappa \in \operatorname{ran}\pi$. Let $\bar\kappa = \operatorname{crit}\pi$. We claim that $\bar\kappa$ is virtually $(<\delta, A)$-extendible. Since $\kappa \in C$ because it is a closure point of f, it suffices by the definition of C to show that

$$(H_\kappa, \in, A \cap H_\kappa) \models \ulcorner \bar\kappa \text{ is virtually } (A \cap H_\kappa)\text{-extendible}\urcorner. \tag{1}$$

Let β be the least element of C above $\bar\kappa$ but below $\bar\theta$, and note that β exists as $f(\bar\kappa) < \bar\theta$, and the definition of f says that the first coordinate of $f(\bar\kappa)$ is a limit point of C above $\bar\kappa$. It then holds that

$$(H_{\bar\kappa}, \in, A \cap H_{\bar\kappa}) \prec (H_\beta, \in, A \cap H_\beta)$$

as both $\bar\kappa$ and β are elements of C. Since f encodes A in the manner previously described and $\pi(f \restriction \bar\kappa) = f \restriction \kappa$, we get that $\pi(A \cap H_{\bar\kappa}) = A \cap H_\kappa$, and thus

$$(H_\kappa, \in, A \cap H_\kappa) \prec (H_{\pi(\beta)}, \in, A^*) \tag{2}$$

for $A^* := \pi(A \cap H_\beta)$. Now, as $(H_\gamma, \in, A \cap H_\gamma)$ and $(H_{\pi(\gamma)}, \in, A^* \cap H_{\pi(\gamma)})$ are elements of $H_{\pi(\beta)}$ for every $\gamma < \kappa$, Countable Embedding Absoluteness 2.1 implies that $H_{\pi(\beta)}$ sees that $\bar{\kappa}$ is virtually $(<\kappa, A^*)$-extendible, which by (2) then implies (1), which is what we wanted to show. ■

As a corollary of the proof, we now have an analogue of Proposition 3.6 for virtually Woodin cardinals.

Proposition 4.5. *A cardinal δ is virtually Woodin if and only if, for every $A \subseteq H_\delta$, there is a cardinal κ satisfying the weakening of virtual $(<\delta, A)$-strongness where $H_\theta = H_\theta^{\mathcal{N}}$ holds in place of $\mathcal{N} \subseteq V$, with $\mathcal{N}$ being the target of the generic embedding.*

Theorem 4.6. *Faintly Woodin cardinals are virtually Woodin.*

Proof. Using Proposition 4.5, it suffices to observe that if

$$\pi : (H_\theta, \in, A) \to (\mathcal{M}, \in, B)$$

is a faintly (θ, A)-strong embedding such that A codes the sequence of H_λ for $\lambda < \theta$, then $H_\theta = H_\theta^{\mathcal{M}}$. ■

Next, we start our analysis of virtually Vopěnka cardinals. We will work with the class-sized version here, but all the following results work equally well for inaccessible virtually Vopěnka cardinals[5]. Our formal setting for working with classes throughout this article will be the second-order Godël-Bernays set theory GBC whose axioms consist of ZFC together with class axioms consisting of extensionality for classes, class replacement asserting that every class function when restricted to a set is a set, global choice asserting that there is a class well-order of sets, and a weak comprehension scheme asserting that every first-order formula defines a class.

Definition 4.7. The **virtual Vopěnka Principle** states that for every class $\mathcal{C}$ consisting of structures in a common language, there are distinct $\mathcal{M}, \mathcal{N} \in \mathcal{C}$ for which there is a generic elementary embedding $\pi\colon \mathcal{M} \to \mathcal{N}$. ⊣

Analogously to the Vopěnka Principle, the virtual Vopěnka Principle can be restated in terms of the existence of elementary embeddings between elements of *natural sequences*.

[5]Note however that we have to require inaccessibility here: see [Wil19] for an analysis of the singular virtually Vopěnka cardinals.

Definition 4.8. Say that a class function $f\colon \mathsf{On} \to \mathsf{On}$ is an **indexing function** if it satisfies that $f(\alpha) > \alpha$ and $f(\alpha) \leq f(\beta)$ for all $\alpha < \beta$. ⊣

Definition 4.9. Say that an On-sequence $\vec{\mathcal{M}} = \langle \mathcal{M}_\alpha \mid \alpha < \mathsf{On}\rangle$ is **natural** if there exists an indexing function $f^{\vec{\mathcal{M}}}\colon \mathsf{On} \to \mathsf{On}$ and unary relations $R^{\vec{\mathcal{M}}}_\alpha \subseteq V_{f^{\vec{\mathcal{M}}}(\alpha)}$ such that $\mathcal{M}_\alpha = (V_{f^{\vec{\mathcal{M}}}(\alpha)}, \in, \{\alpha\}, R^{\vec{\mathcal{M}}}_\alpha)$ for every α. ⊣

The following theorem shows that inaccessible cardinals are virtually Vopěnka if and only if they are virtually pre-Woodin if and only if they are just faintly pre-Woodin. In particular, we get that virtually pre-Woodin and faintly pre-Woodin cardinals are equivalent, and we already showed in Theorem 4.6 that the equivalence holds for virtually Woodin and faintly Woodin cardinals.

Theorem 4.10 (D.-G.-N.). *The following are equivalent.*

(i) *The virtual Vopěnka Principle holds.*

(ii) *For any natural* On*-sequence* $\vec{\mathcal{M}}$ *there exists a generic elementary embedding* $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ *for some* $\alpha < \beta$.

(iii) On *is virtually pre-Woodin.*

(iv) On *is faintly pre-Woodin.*

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are trivial.

$(iv) \Rightarrow (i)$: Assume that On is faintly pre-Woodin and fix some class $\mathcal{C}$ of structures in a common language. Let κ be $(<\mathsf{On}, \mathcal{C})$-pre-strong. Fix some regular $\theta > \kappa$ such that H_θ has a structure M from $\mathcal{C}$ of the κ-th rank among elements of $\mathcal{C}$, and fix a generic elementary embedding

$$\pi\colon (H_\theta, \in, \mathcal{C} \cap H_\theta) \to (\mathcal{N}, \in, \mathcal{C}^*)$$

with $\operatorname{crit}\pi = \kappa$, $H_\theta \subseteq \mathcal{N}$, and $\mathcal{C} \cap H_\theta = \mathcal{C}^* \cap H_\theta$. Now we have that $\pi : M \to \pi(M)$ is an elementary embedding (note that the language is finite and therefore can be assumed to be fixed by π), and $\pi(M) \neq M$ because $\pi(M)$ is a structure in $\mathcal{C}^*$ of rank $\pi(\kappa)$ among elements of $\mathcal{C}^*$. The structure $(\mathcal{N}, \in, \mathcal{C}^*)$ believes that both M and $\pi(M)$ are elements of $\mathcal{C}^*$, and, by Countable Embedding Absoluteness 2.1, $\mathcal{N}$ has a generic elementary embedding between M and $\pi(M)$. Therefore, $(\mathcal{N}, \in, \mathcal{C}^*)$ satisfies that there is a generic elementary embedding between two distinct elements of $\mathcal{C}^*$, and hence $(H_\theta, \in, \mathcal{C} \cap H_\theta)$ satisfies that there is a generic elementary embedding between two distinct elements of $\mathcal{C}$, and it must be correct about this.

$(ii) \Rightarrow (iii)$: Assume (ii) holds and assume that On is not virtually pre-Woodin, which means that there exists some class A for which there are no virtually A-pre-strong cardinals. Define a function $f\colon \mathsf{On}\to\mathsf{On}$ by setting $f(\alpha)$ to be the least regular $\eta>\alpha$ such that α is not virtually (η, A)-pre-strong. Also define $g\colon \mathsf{On}\to\mathsf{On}$ by setting $g(\alpha)$ to be the least strong limit cardinal above α which is a closure point of f. Note that g is an indexing function, so we can let $\vec{\mathcal{M}}$ be the natural sequence induced by g and $R_\alpha := A\cap H_{g(\alpha)}$. (ii) supplies us with $\alpha<\beta$ and a generic elementary embedding

$$\pi\colon (H_{g(\alpha)}, \in, A\cap H_{g(\alpha)})\to(H_{g(\beta)}, \in, A\cap H_{g(\beta)}).$$

Let us argue that π is not the identity map, so that it must have a critical point. Since $g(\alpha)$ is a closure point of f, the structure $(H_{g(\alpha)}, \in, A\cap H_{g(\alpha)})$ can define f correctly. So it must satisfy that f does not have a strong limit closure point above α, but the structure $(H_{g(\beta)}, \in, A\cap H_{g(\beta)})$ does have such a closure point, namely $g(\alpha)$. Thus, π must have a critical point. Now since $g(\alpha)$ is a closure point of f, it holds that $f(\operatorname{crit}\pi)<g(\alpha)$, so fixing a regular $\theta\in(f(\operatorname{crit}\pi), g(\alpha))$ we get that $\operatorname{crit}\pi$ is virtually (θ, A)-pre-strong, contradicting the definition of f. Hence On is virtually pre-Woodin. ■

5 Virtual Berkeley cardinals

Berkeley cardinals were introduced by W. Hugh Woodin at the University of California, Berkeley around 1992, as a large cardinal candidate that would potentially be inconsistent with ZF. They trivially imply Kunen's Inconsistency and are therefore at least inconsistent with ZFC, but they have not to date been shown to be inconsistent with ZF. In the virtual setting the virtually Berkeley cardinals, like all the other virtual large cardinals, are small large cardinals that are downwards absolute to L.

The theorems of this section show that virtually Berkeley cardinals are precisely the large cardinals which separate virtually pre-Woodin cardinals from virtually Woodin cardinals, as well as separating inaccessible virtually Vopěnka cardinals from Mahlo virtually Vopěnka cardinals.

Definition 5.1. A cardinal δ is **virtually proto-Berkeley** if for every transitive set $\mathcal{M}$ such that $\delta\subseteq\mathcal{M}$ there exists a generic elementary embedding $\pi\colon\mathcal{M}\to\mathcal{M}$ with $\operatorname{crit}\pi<\delta$.

If $\operatorname{crit}\pi$ can be chosen arbitrarily large below δ, then δ is **virtually Berkeley**, and if $\operatorname{crit}\pi$ can be chosen as an element of any club $C\subseteq\delta$ we say δ is **virtually club Berkeley**. ⊣

Proposition 5.2. *Virtually (proto)-Berkeley cardinals and virtually club Berkeley cardinals are downward absolute to L. If $0^{\#}$ exists, then every Silver indiscernible is virtually club Berkeley.*

Proof. Downward absoluteness to L follows by Countable Embedding Absoluteness 2.1. Suppose $0^{\#}$ exists and δ is a limit Silver indiscernible. Fix a transitive set $\mathcal{M} \in L$ such that $\delta \subseteq \mathcal{M}$ and a club $C \subseteq \delta$ in L. Let $\lambda \gg \operatorname{rank}(\mathcal{M})$ be a regular cardinal in V, so that every element of L_λ is definable from indiscernibles below λ. In V, we can define a shift of indicernibles embedding $\pi : L_\lambda \to L_\lambda$ with $\operatorname{crit}\pi \in C$ fixing indiscernibles involved in the definition of $\mathcal{M}$ (these are easy to avoid since there are finitely many). It follows that $\pi(\mathcal{M}) = \mathcal{M}$. By Countable Embedding Absoluteness 2.1, L must have such a generic elementary embedding $\sigma : L_\lambda \to L_\lambda$ and it restricts to $\sigma : \mathcal{M} \to \mathcal{M}$. Thus, δ is virtually club Berkeley, but then so is every Silver indiscernible. ■

Virtually (proto-)Berkeley cardinals turn out to be equivalent to their "boldface" versions. The proof is a straightforward virtualisation of Lemma 2.1.12 and Corollary 2.1.13 in [Cut17].

Proposition 5.3 (Virtualised Cutolo). *If δ is virtually proto-Berkeley, then for every transitive set $\mathcal{M}$ such that $\delta \subseteq \mathcal{M}$ and every subset $A \subseteq \mathcal{M}$ there exists a generic elementary embedding $\pi\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with $\operatorname{crit}\pi < \delta$. If δ is virtually Berkeley then we can furthermore ensure that $\operatorname{crit}\pi$ is arbitrarily large below δ.*

Proof. Let $\mathcal{M}$ be transitive with $\delta \subseteq \mathcal{M}$ and $A \subseteq \mathcal{M}$. Let

$$\mathcal{N} := \mathcal{M} \cup \{A, \{\{A, x\} \mid x \in \mathcal{M}\}\} \cup \{\{A, x\} \mid x \in \mathcal{M}\}$$

and note that $\mathcal{N}$ is transitive. Further, both A and $\mathcal{M}$ are definable in $\mathcal{N}$ without parameters: the set A is defined as the unique set such that there is a set B (namely $\{\{A, x\} \mid x \in \mathcal{M}\}$) all of whose elements are pairs of the form $\{A, x\}$, and every set x in $\mathcal{N}$ which does not contain A and which is equal to neither A nor B, must satisfy that $\{A, x\} \in B$. M is defined in N from A as the class containing exactly all elements x such that A is not an element of the transitive closure of x.

But this means that a generic elementary embedding $\pi\colon\ \mathcal{N} \to \mathcal{N}$ fixes both $\mathcal{M}$ and A, giving us a generic elementary $\sigma\colon (\mathcal{M}, \in, A) \to (\mathcal{M}, \in, A)$ with $\operatorname{crit}\sigma = \operatorname{crit}\pi$, yielding the desired conclusion. ■

The following is a straight-forward virtualisation of the usual definition of the Vopěnka filter (see e.g. [Kan08]).

Definition 5.4. The **virtual Vopěnka filter** F on On consists of those classes X for which there is an associated natural On-sequence $\vec{\mathcal{M}}^X$ such that $\operatorname{crit}\pi \in X$ for any $\alpha < \beta$ and any generic elementary $\pi\colon\ \mathcal{M}^X_\alpha \to \mathcal{M}^X_\beta$. ⊣

Theorem 4.10 shows that the virtual Vopěnka filter is proper if and only if the virtual Vopěnka principle holds. The proof of Proposition 24.14 in [Kan08] also shows that if the virtual Vopěnka principle holds, then the virtual Vopěnka filter is normal. However, as we will see shortly, unlike the Vopěnka filter, the virtual Vopěnka filter might be proper, but not uniform. This means that there could be an ordinal δ such that every natural sequence of structures has an elementary embedding between two of its structures with critical point below δ. Standard proofs of the uniformity of the Vopěnka filter fail in the virtual context once again because of the absence of Kunen's Inconsistency. It appears that Kunen's Inconsistency is essential to the proof that every class of structures can be coded by a natural On-sequence, so that in particular we would be able to code structures in a language with arbitrarily many constants. Indeed, we will show that if the virtual Vopěnka filter is uniform, then there are no Berkeley cardinals, and if there is model of ZFC with a Berkeley cardinal, then there is a model of ZFC in which the virtual Vopěnka filter is proper, but not uniform. It will follows that the virtual Vopěnka Principle holds and On is Mahlo precisely when On is virtually Woodin and precisely when the virtual Vopěnka filter is uniform.

Lemma 5.5 (N.). *Assume that the virtual Vopěnka principle holds and that there are no virtually Berkeley cardinals. Then the virtually Vopěnka filter F on On contains every class club C.*

Proof. The crucial extra property we get by assuming that there are no virtually Berkeley cardinals is that F becomes uniform, i.e., contains every tail $(\delta, \mathsf{On}) \subseteq \mathsf{On}$. Indeed, assume that δ is the least cardinal such that $(\delta, \mathsf{On}) \notin F$ (note that $(\gamma, \mathsf{On}) \in F$ up to at least the first inaccessible cardinal because a critical point of an elementary embedding $\pi : V_\alpha \to V_\beta$ must be inaccessible). Let $\mathcal{M}$ be a transitive set with $\delta \subseteq \mathcal{M}$ and $\gamma < \delta$ a cardinal. As $(\gamma, \mathsf{On}) \in F$ by minimality of δ, we may fix a natural sequence $\vec{\mathcal{N}}$ witnessing this. Let $\vec{\mathcal{M}}$ be the natural sequence induced by the indexing function $f\colon \mathsf{On} \to \mathsf{On}$ given by

$$f(\alpha) := \max(\operatorname{rank}(\mathcal{N}_\alpha) + 5, \operatorname{rank}(\mathcal{M}) + 5)$$

and unary relations $R_\alpha := \{\langle\mathcal{M}, \vec{\mathcal{N}}_\alpha\rangle\}$, where $\vec{N}_\alpha := \langle N_\beta \mid \beta \leq \alpha\rangle$. Suppose $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is a generic elementary embedding with $\operatorname{crit}\pi \leq \delta$, which exists as $(\delta, \mathsf{On}) \notin F$. Since π respects the relations R_α, we must have $\pi \restriction \mathcal{M}\colon \mathcal{M} \to \mathcal{M}$ and $\pi(\vec{\mathcal{N}}_\alpha) = \vec{\mathcal{N}}_\beta$. We also get that $\operatorname{crit}\pi > \gamma$, as

$$\pi \restriction \mathcal{N}_{\operatorname{crit}\pi}\colon \mathcal{N}_{\operatorname{crit}\pi} \to \mathcal{N}_{\pi(\operatorname{crit}\pi)}$$

is an embedding between two structures in $\vec{\mathcal{N}}$ and $(\gamma, \mathsf{On}) \in F$. This means that δ is virtually Berkeley, a contradiction. Thus $\operatorname{crit}\pi > \delta$, implying that $(\delta, \mathsf{On}) \in F$ $\lightning$.

From here the proof of Lemma 8.11 in [Jec06] shows what we wanted. ■

Theorem 5.6 (N.). *If there are no virtually Berkeley cardinals, then* On *is virtually pre-Woodin if and only if* On *is virtually Woodin.*

Proof. Assume On is virtually pre-Woodin, so the virtual Vopěnka principle holds by Theorem 4.10. Let F be the virtual Vopěnka filter, which must be proper. The assumption that there are no virtually Berkeley cardinals implies that for any class A we not only get a virtually A-prestrong cardinal, but we get stationarily many such. Indeed, assume this fails — we will follow the proof of Theorem 4.10.

Failure means that there is some class A and some class club C such that there are no virtually A-pre-strong cardinals in C. Since there are no virtually Berkeley cardinals, Lemma 5.5 implies that $C \in F$, so there exists some natural sequence $\vec{\mathcal{N}}$ such that whenever $\pi\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$ is an elementary embedding between two distinct structures of $\vec{\mathcal{N}}$ it holds that $\operatorname{crit}\pi \in C$. Define $f\colon \mathsf{On} \to \mathsf{On}$ as follows: if $\alpha \in C$, then let $f(\alpha)$ be the least cardinal $\eta > \alpha$ such that α is not virtually (η, A)-pre-strong, and otherwise let $f(\alpha) = \alpha$. Also, define $g\colon \mathsf{On} \to \mathsf{On}$ to have $g(\alpha)$ be the least strong limit cardinal in C above α and $\operatorname{rank}(\mathcal{N}_\alpha)$ which is a closure point of f.

Now let $\vec{\mathcal{M}}$ be the natural sequence induced by g and relations R_α coding $A \cap H_{g(\alpha)}$ and the singleton $\{\mathcal{N}_\alpha\}$. Apply the virtual Vopěnka principle to get $\alpha < \beta$ and a generic elementary embedding $\pi\colon \mathcal{M}_\alpha \to \mathcal{M}_\beta$, which restricts to

$$\pi\colon (H_{g(\alpha)}, \in, A \cap H_{g(\alpha)}) \to (H_{g(\beta)}, \in, A \cap H_{g(\beta)}),$$

making $\operatorname{crit}\pi$ virtually $(g(\alpha), A)$-pre-strong, which means that $\operatorname{crit}\pi \notin C$. But as we also get the embedding $\pi \restriction \mathcal{N}_\alpha\colon \mathcal{N}_\alpha \to \mathcal{N}_\beta$, we have that $\operatorname{crit}\pi \in C$ by definition of $\vec{\mathcal{N}}$, $\lightning$.

Now fix any class A and some large $n < \omega$ and define the class

$$C := \{\kappa \in \text{Card} \mid (H_\kappa, \in, A \cap H_\kappa) \prec_{\Sigma_n} (V, \in, A)\}.$$

This is a club and we can therefore find a virtually A-pre-strong cardinal $\kappa \in C$. Assume that κ is not virtually A-strong and let θ be least such that it is not virtually (θ, A)-strong. Fix a generic elementary embedding

$$\pi\colon (H_\theta, \in, A \cap H_\theta) \to (M, \in, B)$$

with $\operatorname{crit}\pi = \kappa$, $H_\theta \subseteq M$, $M \subseteq V$, $A \cap H_\theta = B \cap H_\theta$ and $\pi(\kappa) \leq \theta$.

Now we have that $\pi(\kappa)$ is inaccessible and the structure

$$(H_{\pi(\kappa)}, \in, A \cap H_{\pi(\kappa)}) = (H^M_{\pi(\kappa)}, \in, B \cap H^M_{\pi(\kappa)})$$

believes that κ is virtually $(A \cap H_{\pi(\kappa)})$-strong as in the proof of Theorem 3.9, meaning that $(H_\kappa, \in, A \cap H_\kappa)$ believes that there is a proper class of virtually $(A \cap H_\kappa)$-strong cardinals. But $\kappa \in C$, which means that

$$(V, \in, A) \models \ulcorner\text{There exists a proper class of virtually } A\text{-strong cardinals}\urcorner,$$

implying that On is virtually Woodin. ■

We get the following immediate corollaries from Theorem 4.10.

Corollary 5.7. *If the virtual Vopěnka Principle holds and On is not Mahlo, then there is a virtually Berkeley cardinal.*

Corollary 5.8. *The existence of a virtually pre-Woodin cardinal is equiconsistent with the existence of a virtually Woodin cardinal.*

Theorem 5.9 (N.). *If there is a virtually Berkeley cardinal, then the virtual Vopěnka Principle holds.*

Proof. Suppose that δ is a virtually Berkeley cardinal. To show that the virtual Vopěnka principle holds, we show that On is virtually pre-Woodin, which is equivalent by Theorem 4.10. Fixing a class A, we have to show that there exists a virtually A-pre-strong cardinal. Using that δ is virtually Berkeley, for every cardinal $\theta \geq \delta$ there exists a generic elementary embedding

$$\pi_\theta\colon (H_\theta, \in, A \cap H_\theta) \to (H_\theta, \in, A \cap H_\theta)$$

with $\operatorname{crit}\pi_\theta < \delta$. By the pigeonhole principle we thus get some $\kappa < \delta$ which is the critical point of proper class many π_θ, showing that κ is virtually A-pre-strong, making On virtually pre-Woodin.

■

Corollary 5.10. *If there is a model of ZFC with a virtually Berkeley cardinal, then there is a model of ZFC in which the virtual Vopěnka principle holds and On is not Mahlo.*

Proof. Using, Theorem 5.9, it suffices to show that there is a model of ZFC with a virtually Berkeley cardinal in which On is not Mahlo. Take any model with a virtually Berkeley cardinal, call it δ. If On is not Mahlo there, then we are done. Otherwise, On is Mahlo, so we can let κ be the least inaccessible cardinal above δ. Note that δ is virtually Berkeley in $H_\kappa \models \mathsf{ZFC}$ and On is not Mahlo in H_κ. Thus, H_κ is a model of ZFC with a virtually Berkeley cardinal in which On is not Mahlo.

■

By Corollaries 5.7 and 5.10 we then get the following corollary.

Corollary 5.11. *The following are equiconsistent:*

(i) *There is a virtually Berkeley cardinal.*

(ii) *The virtual Vopěnka principle holds and On is not Mahlo.*

Question 1.7 in [Wil18] asks whether the existence of a non-Σ_2-reflecting *weakly remarkable* cardinal always implies the existence of an ω-Erdős cardinal. Here a weakly remarkable cardinal is a rewording of a virtually pre-strong cardinal, and Lemmata 2.5 and 2.8 in the same paper also show that:

Theorem 5.12 (Wilson). *An ω-Erdős is equivalent to a virtually club Berkeley. The least such is also the least virtually Berkeley cardinal.*[6]

Furthermore, Wilson also showed that a non-Σ_2-reflecting virtually pre-strong cardinal is equivalent to a virtually pre-strong cardinal which is not virtually strong. We can therefore reformulate Wilson's question to the following equivalent question.

[6]Note that this also shows that virtually club Berkeley cardinals and virtually Berkeley cardinals are equiconsistent, which is an open question in the non-virtual context.

Question 5.13 (Wilson). If there exists a virtually pre-strong cardinal which is not virtually strong, is there then a virtually Berkeley cardinal?

Wilson also showed, in [Wil18], that his question has a positive answer in L, which in particular shows that they are equiconsistent. Applying our Theorem 3.9 we can ask the following related question, where a positive answer to that question would imply a positive answer to Wilson's question.

Question 5.14. If there exists a cardinal κ which is virtually (θ, ω)-superstrong for arbitrarily large cardinals $\theta > \kappa$, is there then a virtually Berkeley cardinal?

Our results above at least give a partially positive result:

Corollary 5.15. *If for every class A, there exists a virtually A-pre-strong cardinal and there are no virtually A-strong cardinals, then there exists a virtually Berkeley cardinal.*

Proof. The assumption implies by definition that On is virtually pre-Woodin but not virtually Woodin, so Theorem 5.9 supplies us with the desired result. ■

The assumption that there is a virtually A-pre-strong cardinal for every class A in the above corollary may seem a bit strong, but Theorem 5.6 shows that this is necessary, which might lead one to think that the question could have a negative answer.

6 Questions

Question 6.1. Are virtually θ-strong cardinals, virtually θ-supercompacts and virtually θ-supercompacts ala Magidor all equivalent?

Question 6.2 (Wilson). If there exists a virtually ω-superstrong cardinal, is there then also a virtually Berkeley cardinal?

A Chart of virtual large cardinals

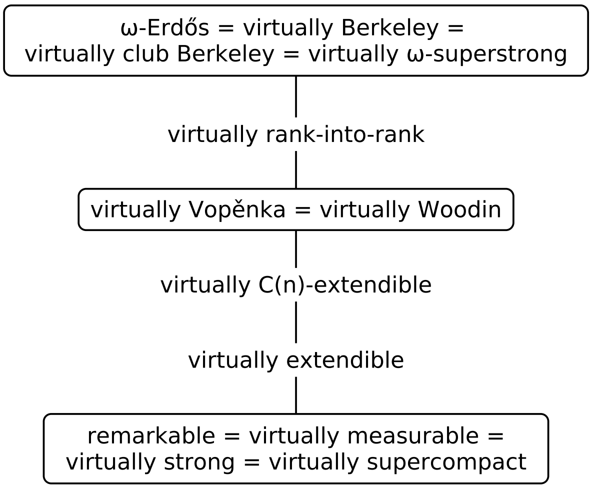

Figure 2: Relative consistency implications between some virtual large cardinals.

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