BABY MEASURABLE CARDINALS

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1. Introduction

Measurable cardinals and most stronger large cardinals are defined by the existence of elementary embeddings $j : V \to M$ from the universe V into an inner model M with that cardinal as the critical point. Stronger large cardinal axioms impose additional assumptions on the target model M that allow it to capture more and more sets from V. Weakly compact and many other smaller large cardinals are defined via combinatorial properties, often involving existence of large homogeneous sets for colorings. But almost all of them also have elementary embedding characterizations that, like the larger large cardinals, follow prescribed patterns as consistency strength grows. These smaller large cardinals κ are characterized by elementary embeddings of models $\langle M, \in\rangle \models \mathrm{ZFC}^-$ of size κ, most often transitive, with critical point κ, and usually, but not always, with well-founded targets. ZFC^- is the theory ZFC with the powerset axiom removed, the collection scheme in place of the replacement scheme, and the version of the axiom of choice which states that every set can be well-ordered.[1] In many cases, the existence of these embeddings can be equivalently expressed in terms of the existence of certain ultrafilters on the subsets of κ contained in M.

Suppose $\langle M, \in\rangle \models \mathrm{ZFC}^-$ is a transitive model of size κ with $\kappa \in M$ (we will give precise definitions and drop the transitivity requirement in the next section). We call such structures *weak κ-models*. We will call a filter U, on the subsets of κ of a weak κ-model M, an *M-ultrafilter* if the structure $\langle M, \in, U\rangle$, together with a predicate for U, satisfies that U is a uniform, normal ultrafilter on κ. What this says is that U is an ultrafilter on the subsets of κ that live in M and if a κ-length sequence of elements of U is an element of M, then its diagonal intersection is in U. The Łoś-Theorem holds for ultrapowers by an M-ultrafilter, but the ultrapower need not be well-founded. Note that the filter U may not be countably complete for sequences outside of M. The M-ultrafilter U, is in most interesting cases, external to M and even though $\langle M, \in\rangle \models \mathrm{ZFC}^-$, separation and replacement can fail completely in the structure $\langle M, \in, U\rangle$ once we let M know about U. In the same way that making the inner model M closer to V in the characterizations of larger large cardinals increases strength, for the smaller large cardinals, we increase strength by making M be more compatible with U, which amounts to M being more correct about the properties of U or to having more of the ZFC^- axioms in the structure $\langle M, \in, U\rangle$.

Let's look at some examples. A cardinal κ is weakly compact if and only if it is inaccessible and every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is a (externally) countably complete M-ultrafilter U (in particular the ultrapower

[1]Without the powerset axiom, collection and replacement schemes are not equivalent and neither are the various forms of the axiom of choice. See [Zar82] and [GHJ16].

is well-founded). A cardinal κ is completely ineffable if and only if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is an M-ultrafilter U such that $\langle M, \in, U\rangle$ satisfies Σ_0-separation (but with a potentially ill-founded ultrapower) [HL21]. A cardinal κ is 1-iterable if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is an M-ultrafilter U such that $\langle M, \in, U\rangle$ satisfies Σ_0-separation and the ultrapower is well-founded [GW11]. A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is an M-ultrafilter U such that $\langle M, \in, U\rangle$ satisfies Σ_0-separation and which is (externally) countably complete [Mit79].

In this article, motivated by the work of Bovykin and McKenzie [BM], we consider a hierarchy of large cardinal notions characterized by the existence, for weak κ-models M, of M-ultrafilters U such that the structure $\langle M, \in, U\rangle$ satisfies fragments up to full ZFC^-. Following Bovykin and McKenzie, we call such cardinals *baby measurable*. We show that increasing levels of replacement and separation for the structures $\langle M, \in, U\rangle$ increases large cardinal strength even if the ultrapowers are ill-founded. For example, adding even Σ_0-replacement to the characterization of completely ineffable cardinals pushes consistency strength well beyond a Ramsey cardinal. We apply the concepts derived from baby measurable cardinals to Holy's and the second author's game Ramsey cardinals [HS18] to obtain strong small large cardinal notions below a measurable cardinal. We show that these new large cardinals are robust under forcing. Finally, we show that these notions are naturally connected to models of Kelley-Morse set theory in which Ord is measurable and help us to understand the structure of these second-order models.

2. Preliminaries

Given a set X, when we say that X is a model of set theory, we will tacitly assume that the membership relation is the actual membership $\in$ restricted to X, so that X is really the model $\langle X, \in\rangle$.

Definition 2.1. Suppose that κ is an inaccessible cardinal.

- A *weak κ-model* is a transitive set $M \models \mathrm{ZFC}^-$ of size κ with $\kappa, V_\kappa \in M$.
- A *κ-model* is a weak κ-model such that $M^{<\kappa} \subseteq M$.
- An *imperfect weak κ-model* is a (not necessarily transitive) set $M \models \mathrm{ZFC}^-$ of size κ such that $M \prec_{\Sigma_0} V$ and $V_\kappa \cup \{V_\kappa\} \cup \{\kappa\} \subseteq M$.
- An *imperfect κ-model* is an imperfect weak κ-model such that $M^{<\kappa} \subseteq M$.

In each case, we will say that M is *simple* if κ is the largest cardinal in M.[2]

Our canonical examples of imperfect models will be elementary substructures of size κ of some large H_θ for a regular θ.[3] For simple models, the notions of imperfect weak κ-models and weak κ-models coincide.

Proposition 2.2. *If M is a simple imperfect weak κ-model, then M is a weak κ-model.*

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[2]Note that these definitions differ slightly from those appearing in earlier literature. For instance, in the definition of weak κ-model, it is not usually assumed that κ is inaccessible and $V_\kappa \in M$.

[3]H_θ for a cardinal θ is the collection of all sets whose transitive closure has size less than θ and given that θ is regular, we have $H_\theta \models \mathrm{ZFC}^-$.

Proof. We just need to show that M is transitive. Fix $a \in M$. Since M is simple, it thinks that there is a bijection $f : \kappa \to a$. By Σ_0-elementarity, f is really a bijection between κ and a. Thus, since $\kappa \subseteq M$, $a \subseteq M$. □

Given an imperfect weak κ-model M, we will let $P^M(\kappa)$ denote the collection of all the subsets of κ that are elements of M. Note that in most cases, $P^M(\kappa)$ will be a class, but not an element of M.

Proposition 2.3. *Suppose that M is an imperfect weak κ-model. If $\bar{M} \in M$ and M thinks that $\bar{M}$ is an imperfect κ-model, then $\bar{M}$ is an imperfect κ-model.*

Proof. The model $\bar{M}$ is an imperfect weak κ-model model by the Σ_0-elementarity of M, so it remains to check closure. Let $f : \bar{M} \to V_\kappa$ be a bijection in M, which really must be a bijection by Σ_0-elementarity. Fix a sequence $\vec{a} = \langle a_\xi \mid \xi < \beta\rangle$ such that $\beta < \kappa$ and $a_\xi \in \bar{M}$ for every $\xi < \beta$. Let $b_\xi = f(a_\xi)$ and let $\vec{b} = \langle b_\xi \mid \xi < \beta\rangle$. The sequence $\vec{b} \in V_\kappa$, and hence $\vec{b} \in M$. Thus, $\vec{a} \in M$, and hence $\vec{a} \in \bar{M}$ since M thinks it is an imperfect κ-model. □

Definition 2.4. Suppose that M is an imperfect weak κ-model.

- We will say that $U \subseteq P^M(\kappa)$ is an *M-ultrafilter* if the structure
$$\langle M, \in, U\rangle \models \text{“}U \text{ is a uniform}^4 \text{ normal ultrafilter on } \kappa.\text{”}$$
- An M-ultrafilter U is *good* if the ultrapower of M by U is well-founded.
- An M-ultrafilter U is *countably complete* if for every sequence $\langle A_n \mid n < \omega\rangle$ with $A_n \in U$ (but the sequence itself not necessarily in M), $\bigcap_{n<\omega} A_n \neq \emptyset$.

Theorem 2.5 (Kunen [Kun70])**.** *If M is an imperfect weak κ-model and U is a countably complete M-ultrafilter, then U is good.*

By Kunen's theorem, whenever M is an imperfect weak κ-model such that $M^\omega \subseteq M$, then every M-ultrafilter is automatically good. Even when the ultrapower N of M by U is ill-founded, by our assumptions on U, $V_\kappa \cup \{V_\kappa\} \cup \{\kappa\} \subseteq N$ and $V_\kappa^M = V_\kappa^N$.[5]

Definition 2.6. Suppose that M is an imperfect weak κ-model. An M-ultrafilter U is *weakly amenable* if for all $A \in M$ with $|A|^M = \kappa$, $A \cap U \in M$.

Note that for simple models M, a weakly amenable M-ultrafilter U is fully amenable in the sense that for every $A \in M$, we have $A \cap U \in M$. A weakly amenable M-ultrafilter can be iterated to carry out the iterated ultrapower construction. Recall that in the case of say a measure U on κ (namely if κ is a measurable cardinal), we can define an Ord-length system of iterated ultrapowers of $U = U_0$. We let $j_{01} : V \to M_1$ be the ultrapower by U_0 and use j_{01} to obtain the ultrafilter for the next stage of the iteration by defining $U_1 = j_{01}(U_0)$. This gives us a general procedure for obtaining the next stage ultrafilter at the successor stages of the iteration and at limits we take a direct limit of the system of embeddings obtained thus far. By a theorem of Gaifman, all the iterated ultrapowers M_ξ are well-founded [Gai74]. We cannot apply the same procedure to obtain successor stage ultrafilters with an M-ultrafilter U because U is (in most interesting cases) not an element of M. However, given weak amenability, we can

[4]Recall that a filter on a cardinal κ is *uniform* if it contains all tail sets (α, κ) for $\alpha < \kappa$.

[5]We are assuming here that we have collapsed the well-founded part of N.

define, for example, U_1, given that $j : M \to N$ is the ultrapower embedding, by $U_1 := \{A = [f] \subseteq j(\kappa) \mid \{\alpha < \kappa \mid f(\alpha) \in U\} \in U\}$.

Weak amenability has an equivalent characterization in terms of the preservation of subsets of κ between the model and its ultrapower. An M-ultrafilter U for an imperfect weak κ-model M is weakly amenable if and only if M and its ultrapower N have the same subsets of κ. Note that this holds true regardless of whether N is well-founded or not.

Proposition 2.7. *If M is a simple imperfect weak κ-model, U is a weakly amenable M-ultrafilter, and (N, E) is the ultrapower of M by U, then $M = H^N_{\kappa^+}$.*

Proof. Let's assume that we have collapsed the well-founded part of (N, E) so that for well-founded sets, $\in = \mathrm{E}$. Clearly $M \subseteq H^N_{\kappa^+}$. So suppose that $a \,\mathrm{E}\, H^N_{\kappa^+}$ and assume without loss that a is transitive in E. Since $(N, \mathrm{E}) \models \mathrm{ZFC}^-$, N thinks that there is a bijection f between (a, E) and a $A \subseteq \kappa \times \kappa$, and it must be correct about this. Since $A \in M$, if A was ill-founded, then M would see this and hence the sequence witnessing ill-foundedness would end up in N as well. Thus, A is well-founded. We can now argue by recursion on rank that the Mostowski collapse of A in M is the Mostowski collapse of A in N, and hence $(a, \mathrm{E}) = (a, \in)$. □

The simplest characterization of a large cardinal in terms of embeddings on weak κ-models belongs to weakly compact cardinals. An inaccessible cardinal κ is weakly compact if and only if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is a good M-ultrafilter on κ. Indeed, equivalently we get that *every* weak κ-model has a countably complete M-ultrafilter. It is then natural to ask what happens if we ask that the M-ultrafilter is additionally required to be weakly amenable. The answer is that we get a much stronger large cardinal notion. But before we define this notion, let us note that the following result from [HL21].

Theorem 2.8 ([HL21])**.** *A cardinal[6] κ is completely ineffable if and only if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is a weakly amenable M-ultrafilter (potentially with an ill-founded ultrapower).*

Definition 2.9 ([GW11])**.** A cardinal κ is *1-iterable* if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is a good weakly amenable M-ultrafilter.

The 1-iterable cardinals are weakly ineffable limits of ineffable cardinals [Git11] (the extra strength comes from requiring that the weakly amenable M-ultrafilter is good). In fact, we can define the hierarchy of α-iterable cardinals (for $1 \leq \alpha \leq \omega_1$), where a cardinal κ is *α-iterable* if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is a weakly amenable M-ultrafilter with α-many well-founded iterated ultrapowers [GW11]. Unlike the situation with a measure on κ, where all the iterated ultrapowers are well-founded, an M-ultrafilter can have exactly α-many well-founded iterated ultrapowers for any $0 \leq \alpha < \omega_1$, or it must have all well-founded iterated ultrapowers since this is already implied by having the first ω_1-many be well-founded. The ω_1-iterable cardinals are slightly weaker than Ramsey cardinals, which are ω_1-iterable limits of ω_1-iterable cardinals [SW11].

[6] Note that, unlike in the characterization of a weakly compact cardinal, we don't require κ to be inaccessible. This is because inaccessibility follows from the existence of weakly amenable M-ultrafilters.

Theorem 2.10 (Mitchell [Mit79])**.** *A cardinal κ is Ramsey if and only if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is a weakly amenable countably complete M-ultrafilter on κ.*

We cannot strengthen the definitions of α-iterable cardinals or Mitchell's characterization of Ramsey cardinals analogously to the case of weakly compact cardinals because even the assumption that we have the required M-ultrafilters for κ-models instead of weak κ-models pushes up the consistency strength beyond a Ramsey cardinal. Indeed, assuming that we have a weakly amenable M-ultrafilter for every weak κ-model is inconsistent [Git11]. We can strengthen Ramsey cardinals, via the Mitchell characterization, in the following two ways.

Definition 2.11 ([Git11])**.** Suppose that κ is a cardinal.

- κ is *strongly Ramsey* if every $A \subseteq \kappa$ is an element of a κ-model M for which there is a weakly amenable M-ultrafilter.
- κ is *super Ramsey* if every $A \subseteq \kappa$ is an element of a κ-model $M \prec H_{\kappa^+}$ for which there is a weakly amenable M-ultrafilter.

Strongly Ramsey cardinals are limits of Ramsey cardinals, super Ramsey cardinals are limits of strongly Ramsey cardinals, and measurable cardinals are limits of super Ramsey cardinals [Git11]. It is natural to ask given these various notions what will happen (1) if we stratify by closure on the weak κ-model M, weakening, for instance, the κ-model assumption to just countable closure on M, and (2) if we ask for elementarity in a large H_θ. The second question does not completely make sense as stated because a weak κ-model cannot be elementary in H_θ for $\theta > \kappa^+$, but this is precisely where the weakening to imperfect weak κ-models comes into play. Combining both of these ideas, Holy and the second author introduced the α-Ramsey hierarchy strengthening the super Ramsey cardinals, while staying below a measurable cardinal [HS18].

Definition 2.12 ([HS18])**.** A cardinal κ is α-Ramsey for a regular $\omega_1 \leq \alpha \leq \kappa$ if for every $A \subseteq \kappa$ and arbitrarily large cardinals θ, there is a $<\alpha$-closed imperfect weak κ-model $M \prec H_\theta$ with $A \in M$ for which there is a weakly amenable M-ultrafilter.

The α-Ramsey cardinals have a natural game-theoretic characterization [HS18] arising out of the question of whether given a weak κ-model M, an M-ultrafilter U, and another weak κ-model N extending M, we can always find an N-ultrafilter extending U. The first author showed that this extension property is inconsistent [HS18], but a game version of it is what characterizes the α-Ramsey cardinals.

Consider the following game Ramsey$G^\theta_\alpha(\kappa)$, for regular cardinals $\omega_1 \leq \alpha \leq \kappa$ and $\theta > \kappa$, of perfect information played by two players the challenger and the judge. The challenger starts the game and plays an imperfect κ-model $M_0 \prec H_\theta$. The judge responds by playing an M_0-ultrafilter U_0. In the next step the challenger plays an imperfect κ-model $M_1 \prec H_\theta$ such that $\langle M_0, U_0\rangle \in M_1$, and the judge responds with an M_1-ultrafilter U_1 extending U_0. More generally, at stage γ, the challenger plays an imperfect κ-model $M_\gamma \prec H_\theta$ such that $\{\langle M_\xi, U_\xi\rangle \mid \xi < \gamma\} \in M_\gamma$, and the judge responds with an M_γ-ultrafilter U_γ extending $\bigcup_{\xi<\gamma} U_\gamma$. The judge wins the game if she can continue playing for α-many steps. Otherwise, the challenger wins. It is shown in [HS18] that if either of the players has a winning strategy in the game Ramsey$G^\theta_\alpha(\kappa)$, then the same player has a winning strategy in the game Ramsey$G^\rho_\alpha(\kappa)$ for any other regular cardinal $\rho > \kappa$.

Theorem 2.13. *([HS18]) A cardinal κ is α-Ramsey if and only if the challenger does not have a winning strategy in the game Ramsey$G^\theta_\alpha(\kappa)$ for some/all regular cardinals $\theta > \kappa$.*

It is shown in [HS18] that a κ-Ramsey cardinal κ is a limit of super Ramsey cardinals.

Theorem 2.14. *A strongly Ramsey cardinal is a limit of cardinals α that are $<\alpha$-Ramsey.*

Proof. Suppose that κ is strongly Ramsey. Let M be a simple κ-model for which there is a weakly amenable M-ultrafilter. Let $j : M \to N$ be the ultrapower map. We will argue that κ is $<\kappa$-Ramsey in $V_j(\kappa)^N$. Fix $\alpha < \kappa$. By Theorem 2.13, it suffices to show that the challenger does not have a winning strategy in the game Ramsey$G^{\kappa^+}_\alpha(\kappa)$. Suppose towards a contradiction that, in $V^N_{j(\kappa)}$, the challenger has a winning strategy σ in Ramsey$G^{\kappa^+}_\alpha(\kappa)$. Consider a run of the game Ramsey$G^{\kappa^+}_\alpha(\kappa)$ in which the challenger plays according to σ and the judge responds at stage γ with $U \cap M_\gamma$, which is in M by weak amenability. Note that the sequences of the judge's moves are in M (and hence in N) by the $<\kappa$-closure of M, and therefore we can apply σ to them. The entire play has length $\alpha < \kappa$ and hence is an element of N as well by closure. But clearly the judge wins this play, contradicting our assumption that σ was a winning strategy for the challenger. Now we can conclude by elementarity that, in V_κ, κ is a limit of cardinals α that are $<\alpha$-Ramsey. Moreover, V_κ is correct about this since to verify that α is β-Ramsey we only need to consider games with elementary substructures of $H_{\alpha^+} \subseteq V_\kappa$. □

For future sections, let us consider a variant game Ramsey$G^{*\theta}_\alpha(\kappa)$, where we ask the judge to play, instead of an M_γ-ultrafilter U_γ extending his moves from the previous stages, a structure $\langle N_\gamma, \in, U_\gamma\rangle$ such that N_γ is a κ-model with $P^{M_\gamma}(\kappa) \subseteq N_\gamma$ and U_γ is an N_γ-ultrafilter, keeping the requirement that the U_γ must extend. We additionally ask the challenger to ensure that the sequence of the judge's previous plays $\{\langle N_\xi, \in, U_\xi\rangle \mid \xi < \gamma\}$ is an element of M_γ. In this variant game, we are giving the judge the extra ability to ensure that certain subsets of κ make it into the final model. Essentially the same argument as for the game Ramsey$G^\theta_\alpha(\kappa)$ shows that the existence of a winning strategy for either player is independent of θ in the game Ramsey$G^{*\theta}_\alpha(\kappa)$, and moreover κ is α-Ramsey if and only if the challenger doesn't have a winning strategy in Ramsey$G^{*\theta}_\alpha(\kappa)$. It follows that the challenger has a winning strategy in the game Ramsey$G^\theta_\alpha(\kappa)$ if and only if he has a winning strategy in Ramsey$G^{*\theta}_\alpha(\kappa)$. But, in fact, this is true for the judge as well.

Proposition 2.15. *Either player has a winning strategy in the game Ramsey$G^\theta_\alpha(\kappa)$ if and only if they have a winning strategy in the game Ramsey$G^{*\theta}_\alpha(\kappa)$.*

Proof. Suppose that the judge has a winning strategy σ in the game Ramsey$G^\theta_\alpha(\kappa)$. Let's come up with a winning strategy σ^* for the judge in the game Ramsey$G^{*\theta}_\alpha(\kappa)$. If the first move of the challenger is $M_0 \prec H_\theta$, then the judge plays $\langle N_0, \in, U_0\rangle$, where U_0 is the move dictated by σ and $N_0 = H_{\kappa^+} \cap M_0$. Given that the judge continues the strategy of playing $N_\gamma = M_\gamma \cap H_{\kappa^+}$ and U_γ given by σ, the challenger will have fulfilled the requirement of having $\{\langle N_\xi, \in, U_\xi\rangle \mid \xi < \gamma\} \in M_\gamma$ because each N_ξ is definable from $P^{N_\xi}(\kappa)$, which is in turn definable from U_ξ. Clearly σ^* thus defined is a winning strategy for the judge.

Next, suppose that the judge has a winning strategy σ^* in the game $\text{Ramsey}G^{*\theta}_\alpha(\kappa)$. Let's come up with a winning strategy σ for the judge in the game $\text{Ramsey}G^\theta_\alpha(\kappa)$. If the first move of the challenger is $M_0 \prec H_\theta$ and $\langle N_0, \in, U^*_0\rangle$ is the move given by σ^* in response, the judge plays $U_0 = U^*_0 \cap P^{M_0}(\kappa)$. At each stage γ, the judge continues to play the restriction U_γ to M_γ of the filter U^*_γ given by σ^* in response to $M^*_\gamma \prec H_\theta$, which extends M_γ to include $\{\langle N_\xi, \in, U^*_\xi\rangle \mid \xi < \gamma\}$. $\square$

Above the α-Ramsey hierarchy, but still below a measurable cardinal sits a large cardinal notion introduced by Holy and Lücke [HL21].

Definition 2.16 ([HL21])**.** A cardinal κ is *locally measurable* if every $A \subseteq \kappa$ is an element of a weak κ-model M such that M has a normal ultrafilter on κ.

Notice, in particular, that $P^M(\kappa)$ must be an element of any such weak κ-model M, and so it obviously cannot be simple. Note also that U is not required to be good. It is shown in [HL21] that a measurable cardinal is a limit of locally measurable cardinals and that a locally measurable cardinal is a limit of cardinals α that are α-Ramsey.

Proposition 2.17. *If κ is locally measurable, then every $A \subseteq \kappa$ is an element of a weak κ-model M such that M has a normal ultrafilter U on κ with a well-founded ultrapower $N \subseteq M$.*

Proof. Fix $A \subseteq \kappa$. Choose any weak κ-model M such that $A \in M$ and M has a normal ultrafilter U on κ. Let $\delta = 2^\kappa$. We can assume without loss of generality that δ is the largest cardinal of M by replacing M with $H^M_{\delta^+}$ if necessary. Let $\langle N, \mathrm{E}\rangle$ be the ultrapower of M by U. The model M believes that the relation E is well-founded in the sense that it doesn't have a sequence of functions $\{f_n : \kappa \to M \mid n < \omega\}$ such that for all $n < \omega$, $\{\xi < \kappa \mid f_{n+1}(\xi) \in f_n(\xi)\} \in U$. Since models of ZFC^- can perform Mostowski collapses, it suffices to argue that every element of N is set-like. By elementarity, $[c_\delta]_U$, where c_δ is the constant function with value δ, is the largest cardinal of N, and thus, every element of N is bijective with $[c_\delta]_U$. So it actually suffices to argue that $[c_\delta]_U$ has set-many elements in N. For this, observe that $[f]_U \,\mathrm{E}\, [c_\delta]_U$ if and only if $f : \kappa \to \delta$, and the collection of such functions is a set in M. $\square$

Thus, as long as 2^κ is the largest cardinal of a weak κ-model M with a normal ultrafilter U on κ, the ultrapower of M by U is contained in M just as in the case of an ultrapower by an actual measure. In fact, it follows from the proof that this will be case whenever M has a largest cardinal γ and $P^M(\gamma) \in M$. Note that it is definitely possible for an ultrapower of a weak κ-model M by a normal ultrafilter U in M to have more ordinals than M. Suppose that κ is measurable, $2^\kappa = \kappa^+$ and $2^{\kappa^+} \gg \kappa^{++}$. Let M be the Mostowski collapse of any elementary substructure of $H_{\kappa^{+++}}$ of size κ. Let $\alpha = (\kappa^{++})^M$, which is the largest cardinal of M, and let $\bar{U}$ be a normal ultrafilter on κ in M. Then elements of $[c_\alpha]_{\bar{U}}$ are the equivalence classes $[f]_{\bar{U}}$, where $f : \kappa \to \alpha$, and this cannot be a set in M.

3. The structures $\langle M, \in, U\rangle$ as models of set theory

Suppose that M is a simple weak κ-model and U is an M-ultrafilter. By assumption, M is a model of ZFC^-, but separation and replacement can fail very badly in the structure $\langle M, \in, U\rangle$ once the model learns about the ultrafilter. Weak amenability of U is equivalent to having a minimal amount of separation for $\langle M, \in, U\rangle$. Note that for the results in this section, we do not need to assume anywhere that the ultrapower of M by U is well-founded, that is, U need not be good.

Proposition 3.1. *Suppose that M is a simple weak κ-model and U is an M-ultrafilter. Then U is weakly amenable if and only if the structure $\langle M, \in, U\rangle$ satisfies Δ_0-separation.*

Proof. Suppose that Δ_0-separation holds in the language with U. Fix $A \in M$ and observe that $U \cap A = \{x \in A \mid x \in U\}$, which indeed requires separation only for atomic formulas. Now suppose that U is weakly amenable to M. Fix $A \in M$ and a Δ_0-formula $\varphi(x, b)$ in the language with U. We need to argue that

$$\{x \in A \mid M \models \varphi(x, b)\} \in M.$$

Now observe that since $\varphi(x, b)$ is a Δ_0-formula, all quantifiers are bounded and therefore, U restricted to $\mathrm{TCl}(b)$, the transitive closure of b, suffices to interpret $\varphi(x, b)$ correctly. But this piece of U is an element of M by weak amenability. $\square$

We will see in Section 6 that a structure $\langle M, \in, U\rangle$ with a weakly amenable M-ultrafilter U need not satisfy even Δ_0-replacement.

Next, let's observe that our typical structures $\langle M, \in, U\rangle$ have a Δ_1-definable total order that from the point of view of the structure is a strong well-order. Let us say that a total order $\lhd$ on a weak κ-model M is a *strong well-order of M* if for every formula $\varphi(x, a)$ over M, the structure $\langle M, \in, \lhd\rangle$ satisfies that there is a $\lhd$-least b such that $\varphi(b, a)$. Usually, a well-order is defined to have the property that every set has a least element. This property is equivalent to our stronger requirement for a set-like order, but the orders we encounter won't necessarily be set-like. These structures $\langle M, \in, U\rangle$ will also have a Δ_1-definable truth predicate for $\langle M, \in\rangle$.

Proposition 3.2. *Suppose that M is a simple weak κ-model and U is a weakly amenable M-ultrafilter. Then the structure $\langle M, \in, U\rangle$ has a Δ_1-definable strong well-order $\lhd$ of M.*

Proof. Let $\langle N, \mathrm{E}\rangle$ be the possibly ill-founded ultrapower by U. Since U is weakly amenable, we have $M = H^N_{\kappa^+}$ and $(\kappa^+)^N = \mathrm{Ord}^M$ by Proposition 2.7.

Every set $a \in M$ can be coded in M by a subset A of κ. The code A codes a subset of $\kappa \times \kappa$, which in turn, codes the membership relation on $\mathrm{TCl}(a)$. Although, a set a can have many different codes, by elementarity (because this holds true in M of every H_α), N has a set $\mathcal{C}$ consisting of a unique code for every set in M. In fact, N has a membership relation $\mathcal{E}$ for elements of this set as well, making $\langle \mathcal{C}, \mathcal{E}\rangle$ isomorphic to $\langle M, \in\rangle$. Let $[C]_U$ be the equivalence class representing the set $\mathcal{C}$ of N. Now let's consider the complexity of a series of statements in the structure $\langle M, \in, U\rangle$.

Observe first that any $A \subseteq \kappa$ from M is represented in the ultrapower N by the equivalence class of the function f_A such that $f_A(\xi) = A \cap \xi$. To verify this, it suffices to check that $[f_A] \cap \xi = A \cap \xi$ for every $\xi < \kappa$. Since ξ and $A \cap \xi$ are in V_κ, they are represented by the constant functions $[c_\xi]$ and $[c_{A\cap\xi}]$. So we need to check that $[f_A] \cap [c_\xi] = [c_{A\cap\xi}]$ holds in the ultrapower N. By the ŁośTheorem, this holds if and only if

$$\{\alpha < \kappa \mid f_A(\alpha) \cap \xi = A \cap \xi\} \in U,$$

but this is certainly true since it holds on a tail. Let "A is a code" be the assertion that A is an element of $\mathcal{C}$ in the ultrapower N. The complexity of the statement "A is a code" in the structure $\langle M, \in, U\rangle$ is Δ_1. The Σ_1-version says that there exists a function f such that $f(\xi) = A \cap \xi$ and there exists a set $\{\alpha < \kappa \mid f(\xi) \in C(\xi)\}$ and this set is in U, and the Π_1-version says that for every function f such that $f(\xi) = A \cap \xi$ and for every set equal to $\{\alpha < \kappa \mid f(\xi) \in C(\xi)\}$, the set is in U. Next, let "A is the code for a" be the assertion that "A is a code" and A codes a as explained above. Let's argue that the complexity of the assertion "A is the code of a" is Δ_1 in the structure $\langle M, \in, U\rangle$. We just need to argue that we can say that A codes a in a Δ_1-way. The Σ_1-assertion is that there are sets $\bar{a}$ and π such that $\bar{a} = \mathrm{TCl}(a)$ and π is an isomorphism between $\bar{a}$ and A ($\bar{a} = \mathrm{TCl}(a)$ is Δ_1). The Π_1-assertion is that for every set $\bar{a}$, B, and π, if $\bar{a} = \mathrm{TCl}(a)$ and π is an isomorphism between $\bar{a}$ and B, then $A = B$.

Finally, let $[W]$ represent the equivalence class of a well-order $\mathcal{W}$ of $\mathcal{C}$ in the ultrapower N. Given a and b in M, we define that $a \lhd b$ whenever there are codes A and B of a and b respectively such that A is below B according to $\mathcal{W}$. The assertion that A is below B according to $\mathcal{W}$ translates to saying that $[f_A]$ is below $[f_B]$ according to $[W]$. With our preliminary analysis this is clearly a Δ_1-assertion in the language with U, using the Łoś-Theorem.

$\square$

If $\{\alpha < \kappa \mid M \models 2^\alpha = \alpha^+\} \in U$, then the ultrapower N has a well-order of M of order-type κ^+, which must then be an actual well-order, since $(\kappa^+)^N$ is well-founded. Also, if the M-ultrafilter U is good, and hence, N is well-founded, then its well-order of M is an actual well-order. Otherwise, it is possible that N's well-order of M is ill-founded externally.

Proposition 3.3. *Suppose that M is a simple weak κ-model and U is a weakly amenable M-ultrafilter. A truth predicate $\mathrm{Tr}_M(\varphi, x)$ for $\langle M, \in\rangle$ is Δ_1-definable in the structure $\langle M, \in, U\rangle$.*

Proof. Let N be the (possibly ill-founded) ultrapower by U. Following the notation of the proof of Proposition 3.2, the model N has a truth predicate $\mathcal{T}$ for the structure $\langle \mathcal{C}, \mathcal{E}\rangle$. Thus, we have that $M \models \varphi(a)$ whenever A is the code for a and $\langle \varphi, A\rangle \in \mathcal{T}$. Let $[T]$ represent $\mathcal{T}$ in the ultrapower N. Then $\langle \varphi, A\rangle \in \mathcal{T}$ if and only if

$$\{\alpha < \kappa \mid \langle \varphi, f_A(\alpha)\rangle \in T(\alpha)\} \in U.$$

$\square$

Proposition 3.4. *Suppose that M is a simple weak κ-model and U is a weakly amenable M-ultrafilter. Then for every $n \in \omega$, $\langle M, \in, U\rangle$ has a Σ_n-definable truth predicate $\mathrm{Tr}^U_n(\varphi, x)$ for Σ_n-formulas in the language with U.*

Proof. Given a Δ_0-formula $\varphi(x, a)$ in the language with U, observe that we only need $U_a := U \cap \mathrm{TCl}(a)$ to evaluate it. By the weak amenability of U, we have that $U_a \in M$ for every $a \in M$. Thus, a truth predicate for a Δ_0-formula $\varphi(x, a)$ can be defined as usual with U_a as a parameter. This means we should require that a set u is a part of the witnessing sequence for truth and the extra step in the definition needs to check that $u = U \cap \mathrm{TCl}(a)$, but that is a Δ_0-assertion. Once we have truth for Δ_0-formulas, the rest follows by induction on complexity of formulas because we have only finitely many quantifiers. $\square$

Let ZFC^-_n be the theory where the separation and collection schemes are restricted to Σ_n-formulas.

In the complexity calculations that follow, we will repeatedly use the observation that Σ_n-collection implies a normal form theorem for Σ_n-assertions, namely that every assertion with a Σ_n-alternation of unbounded quantifiers is equivalent to a Σ_n-assertion.

In the next proposition, we will show that the structures $\langle M, \in, U\rangle$ satisfying enough collection have strong reflection properties.

Proposition 3.5. *Suppose that M is a simple weak κ-model and U is an M-ultrafilter. If $\langle M, \in, U\rangle \models \mathrm{ZFC}^-_{n+1}$, for $n \geq 1$, then for every $A \in M$, there is a κ-model $\bar{M} \in M$ with $A \in M$ such that*

$$\bar{M} \prec M \text{ and } \langle \bar{M}, \in, U \cap \bar{M}\rangle \prec_{\Sigma_n} \langle M, \in, U\rangle.$$

Proof. Assume that Σ_{n+1}-collection and separation hold in the structure $\langle M, \in, U\rangle$. First, let's argue that for every set $X \in M$, there exists a unique set $X^* \in M$ of $\lhd$-least witnesses for Σ_n-assertions $\theta(x, a)$ with $a \in X$.

Let C be a collecting set for the assertion

$$\forall a \in X \, \forall \theta \in \omega \, \exists y \, \varphi(a, \theta, y),$$

where $\varphi(a, \theta, y)$ is the assertion

$$\left(y = \emptyset \wedge \forall z \, \neg \mathrm{Tr}^U_n(\theta, \langle z, a\rangle)\right) \vee \left(\mathrm{Tr}^U_n(\theta, \langle y, a\rangle) \wedge \forall z (z \lhd y \to \neg \mathrm{Tr}^U_n(\theta, \langle z, a\rangle))\right).$$

The complexity of the assertion $\varphi(a, \theta, y)$ is clearly at most Σ_{n+1} (it is a disjunction of a Π_n and a Σ_n-formula), and so we can apply Σ_{n+1}-collection.[7] Now using Σ_{n+1}-separation on C, we can obtain the desired unique set X^*.

Observe that the assertion that N is a κ-model is Δ_1. The assertion that N is a weak κ-model is Δ_1 because you just have to say that it is transitive, has κ, V_κ as elements, and satisfies ZFC^- (this last gives the unbounded quantifier). The assertion that the model is closed under $<\kappa$-sequences is clearly Π_1, but in fact it is also Σ_1. The Σ_1-assertion is "there is a bijection $f : V_\kappa \to N$ such that for every $g : \xi \to V_\kappa$ in V_κ, there exists $x \in N$ such that $x = f \circ g$." Next, observe that the assertion that N is the $\lhd$-least κ-model extending a set X is Π_1. Finally, observe that the assertion that $\langle N, \in\rangle \prec \langle M, \in\rangle$ is Δ_1 using the Δ_1-definable truth predicate $\mathrm{Tr}_M(\varphi, x)$ for $\langle M, \in\rangle$.

Now consider the assertion $\psi(s, \lambda)$, which says that s is a sequence of length λ such that:

(1) $s_0 = \{A\}$
(2) $s_\delta = \bigcup_{\xi<\delta} s_\xi$ for limit ordinals δ,
(3) if ξ is an even successor ordinal, then $s_\xi = s^*_{\xi-1}$, the unique closure under existential witnesses,
(4) if ξ is an odd successor ordinal, then s_ξ is the $\lhd$-least κ-model extending $s_{\xi-1}$.

To ensure that the κ-model in the odd successor stages steps exists, we apply $\lhd$ to the assertion that for every set, there is a κ-model containing it. The sequences s are coherent by uniqueness. So by Σ_{n+1}-collection, there is a set $S \in M$, collecting the sequences s for every $\xi < \kappa$. Then by Σ_{n+1}-separation, we can pick out the sequences and union them to obtain a Σ_n-elementary substructure $\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec$

[7] Note this calculation would not work if we allowed $n = 0$.

$\langle M, \in, U\rangle$. Since, in M, $\bar{M}$ is a κ-length union of κ-models, it is itself a κ-model. Finally, $\bar{M}$ really is a κ-model by Proposition 2.3. □

Question 3.6. Can we reduce the hypothesis in Proposition 3.5 to ZFC^-_n?

We can get an almost converse to Proposition 3.5.

Proposition 3.7. *Suppose that M is a simple weak κ-model, U is a weakly amenable M-ultrafilter, and $n \geq 1$. If for every $A \in M$, there is $\bar{M} \in M$ with $A \in \bar{M}$ such that*

$$\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec_{\Sigma_n} \langle M, \in, U\rangle,$$

then $\langle M, \in, U\rangle \models \mathrm{ZFC}^-_n$.

Proof. Suppose that

$$\langle M, \in, U\rangle \models \forall x \in a\, \exists y\, \varphi(x, y, a)$$

for a Σ_n-formula $\varphi(x, y, a)$. Let $a \in \bar{M}$ such that

$$\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec_{\Sigma_n} \langle M, \in, U\rangle.$$

Fix $b \in a$. By assumption,

$$\langle M, \in, U\rangle \models \exists y\, \varphi(b, y, a).$$

Then, by elementarity, $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \exists y\, \varphi(b, y, a)$.[8] Thus,

$$\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \forall x \in a\, \exists y\, \varphi(x, y, a),$$

and so $\bar{M}$ is the required collecting set for φ. Now fix $a \in M$ and a Σ_n-formula $\psi(x, b)$ in the language with U. Let $a, b \in \bar{N}$ such that

$$\langle \bar{N}, \in, U \cap \bar{N}\rangle \prec_{\Sigma_n} \langle M, \in, U\rangle.$$

The structure $\langle \bar{N}, \in, U \cap \bar{N}\rangle$ reflects $\langle M, \in, U\rangle$ for the formula $\psi(x, b)$. By weak amenability of U, we have $U \cap \bar{N} \in M$ and therefore

$$\{x \in a \mid \langle M, \in, U\rangle \models \psi(x, b)\} = \{x \in a \mid \langle \bar{N}, \in, U \cap \bar{N}\rangle \models \psi(x, b)\},$$

and the right-hand side set exists by comprehension in M. □

Proposition 3.8. *Suppose that M is a simple weak κ-model, $n \geq 1$, and U is an M-ultrafilter such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-_{n+1}$. If $\bar{M} \in M$ is such that*

$$\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec_{\Sigma_{n+1}} \langle M, \in, U\rangle,$$

then $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \mathrm{ZFC}^-_n$.

Proof. First, let's verify Σ_n-collection. Suppose that

$$\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \forall x \in a\, \exists y\, \varphi(x, y, a),$$

for a Σ_n-formula $\varphi(x, y, a)$. Indeed, we can assume without loss of generality that $\varphi(x, y, a)$ is Π_{n-1}. By Σ_{n+1}-elementarity, it follows that

$$\langle M, \in, U\rangle \models \forall x \in a\, \exists y\, \varphi(x, y, a).^9$$

By Σ_n-collection, $\langle M, \in, U\rangle \models \psi(a)$, where $\psi(a) := \exists z \forall x \in a\, \exists y \in z\, \varphi(x, y)$. If $n = 1$, then $\psi(a)$ is a Σ_1-formula. So by elementarity, $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \psi(a)$. If $n > 1$, we can assume that we already verified Σ_{n-1}-collection in $\langle \bar{M}, \in, U \cap \bar{M}\rangle$. Using Σ_{n-1}-collection in $\langle M, \in, U\rangle$, $\psi(a)$ is equivalent to a Σ_n-assertion $\phi(a)$. So

[8] Note that this step would not work if we allowed $n = 0$.

[9] Note that this step would not work if we allowed $n = 0$.

by elementarity, $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \phi(a)$, and since we assumed Σ_{n-1}-collection, $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \psi(a)$.

Next, let's verify Σ_n-separation. Fix a set $b \in \bar{M}$ and a Σ_n-formula $\varphi(x, a)$ with $a \in \bar{M}$. By Σ_n-separation, $\langle M, \in, U\rangle \models \psi(a, b)$, where

$$\psi(a, b) := \exists x\, (\forall y \in x\, (y \in b \wedge \varphi(y, a) \wedge \forall y \in b\, (\varphi(y, a) \to y \in x))\,.$$

In $\langle M, \in, U\rangle$, using Σ_{n+1}-collection, this formula is equivalent to a Σ_{n+1}-assertion $\phi(a, b)$. Thus, $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \phi(a, b)$, and hence also $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \psi(a, b)$. □

Question 3.9. Can we reduce the hypothesis in Proposition 3.8 to ZFC^-_n?

Suppose that M is a weak κ-model and U is a weakly amenable M-ultrafilter. Let N be the ultrapower of M by U and let $W = \{[f] \subseteq [c_\kappa] \mid \{\alpha < \kappa \mid f(\alpha) \in U\} \in U\}$ be the N-ultrafilter on $[c_\kappa]$ derived from U. We know that the Łoś-Theorem holds for the ultrapower in the language $\in$. We will say that the Łoś-Theorem *holds* for an assertion $\varphi(x)$ in the language with a predicate for the ultrafilter if $A = \{\alpha < \kappa \mid \langle M, \in, U\rangle \models \varphi(f(\alpha))\} \in M$ for every $f : \kappa \to M$ from M and $\langle N, \in, W\rangle \models \varphi([f])$ if and only if $A \in U$.

Proposition 3.10. *If $\langle M, \in, U\rangle \models \mathrm{ZFC}^-_n$, then the Łoś-Theorem holds for Σ_n and Π_n-assertions in the language with a predicate for the ultrafilter.*

Proof. First, let's argue that the extended Łoś-Theorem is true for all Δ_0-assertions. It is true for atomic formulas by the definition of W. The conjunctions case is clear. So let's suppose that the assertion is true for some Δ_0-formula $\varphi(x)$ and argue that it is true for $\neg\varphi(x)$. By our assumption, for every function $f : \kappa \to M$ from M, the set $A_f = \{\alpha < \kappa \mid \langle M, \in, U\rangle \models \varphi(f(\alpha))\} \in M$. Thus, its complement $B_f = \{\alpha < \kappa \mid \langle M, \in, U\rangle \models \neg\varphi(f(\alpha))\}$ is in M as well. Now we have that $B_f \in U$ if and only if $A_f \notin U$ if and only if $\langle N, \in, W\rangle$ does not satisfy $\varphi([f])$ if and only if $\langle N, \in, W\rangle \models \neg\varphi([f])$.

So suppose now that the assertion holds for a Δ_0-formula $\varphi(x, y)$. First, observe that for functions $f : \kappa \to M$ and $g : \kappa \to M$ from M, the set

$$A_{f,g} = \{\alpha < \kappa \mid \langle M, \in, U\rangle \models \exists x \in g(\alpha)\, \varphi(x, f(\alpha))\}$$

is in M by Δ_0-separation because by Δ_0-collection the range of g is bounded in M. Suppose that

$$\langle N, \in, W\rangle \models \exists x \in [g]\, \varphi(x, [f]).$$

Then there is $[h]$ such that $\langle N, \in, W\rangle \models \varphi([h], [f]) \wedge [h] \in [g]$ and so by our assumption, the set

$$\{\alpha < \kappa \mid \varphi(h(\alpha), f(\alpha)) \wedge h(\alpha) \in g(\alpha)\} \in U,$$

and hence $A_{f,g} \in U$. In the other direction, suppose that $A_{f,g} \in U$. We can construct the desired witness function $[h]$ now using Δ_0-collection.

Observe, next, that the negation case holds for any formula provided that we have the inductive assumption for that formula, and the existential quantifier case works exactly as the bounded existential quantifier case provided that we have enough separation and collection.

□

4. Baby measurable cardinals

In this section, we will define the baby measurable cardinals, whose properties we will study in this article. Baby measurable cardinals and their variations were introduced by Bovykin and McKenzie in [BM]. They used baby measurable cardinals to obtain the following result.

Theorem 4.1 ([BM]). *The following theories are equiconsistent.*

(1) ZFC *together with the scheme consisting of assertions for every $n < \omega$:*

There exists an n-baby measurable cardinal κ such that $V_\kappa \prec_{\Sigma_n} V$.

(2) NFUM

The theory NFUM is a natural strengthening of the theory NFU (New Foundations with Urelements). We introduce additional notions related to their baby measurable cardinals and make slight modifications to their original definitions.

Definition 4.2. Suppose that κ is a cardinal.

(1) κ is *very weakly n-baby measurable* if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is an M-ultrafilter such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-_n$.
(2) κ is *weakly n-baby measurable* if we further require in the definition of very weakly n-baby measurable that U is good.
(3) κ is *very weakly baby measurable* if every $A \subseteq \kappa$ is an element of a weak κ-model M for which there is an M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$.
(4) κ is *weakly baby measurable* if we further required in the definition of very weakly baby measurable that U is good.
(5) κ is *n-baby measurable* if we further require in the definition of very weakly baby measurable that M is a κ-model.[10]
(6) κ is *weakly α-game baby measurable* for a regular cardinal $\omega_1 \leq \alpha \leq \kappa$ if for every $A \subseteq \kappa$ and arbitrarily large θ there is a $<\alpha$-closed imperfect weak κ-model $M \prec H_\theta$ with $A \in M$ for which there is an M-ultrafilter U such that $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$.
(7) κ is *baby measurable* if we further require in the definition of very weakly baby measurable that M is a κ-model.
(8) κ is *α-game baby measurable* for a regular cardinal $\omega_1 \leq \alpha \leq \kappa$ if for every $A \subseteq \kappa$ and arbitrarily large θ there is a $<\alpha$-closed imperfect weak κ-model model $M \prec H_\theta$ with $A \in M$ for which there is a weakly amenable M-ultrafilter U such that for every $B \subseteq \kappa$ with $B \in M$, there is a κ-model $B \in \bar{M} \in M$ such that $\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec \langle H^M_{\kappa^+}, \in, U\rangle$.
(9) κ is *strongly α-game baby measurable* for a regular $\omega_1 \leq \alpha \leq \kappa$ if for every $A \subseteq \kappa$ and arbitrarily large θ there is a $<\alpha$-closed imperfect weak κ-model $M \prec H_\theta$ with $A \in M$ for which there is a weakly amenable M-ultrafilter U such that for every $B \in M$, there is an imperfect κ-model $B \in \bar{M} \in M$ such that $\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec \langle M, \in, U\rangle$.

[10] Bovykin and McKenzie had a slightly different definition of n-baby measurable cardinals. They defined them to have the property that every $A \subseteq \kappa$ is an element of a κ-model M for which there is a weakly amenable M-ultrafilter U and a weakly amenable global well-order $<$ such that for every $B \in M$, there is a κ-model $B \in \bar{M} \in M$ with $\langle \bar{M}, \in, U \cap \bar{M}, < \cap \bar{M}\rangle \prec_{\Sigma_n} \langle M, \in, U, <\rangle$. Since we can always assume that the models involved are simple, by our Proposition 3.2 and Propositions 3.5 for $n \geq 1$, our notion of $n+1$-baby measurability implies their notion of n-baby measurability.

Observe that if $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$, then U is fully amenable to M. Observe also that, excluding the game baby measurable cardinals, we can always assume that the weak κ-models involved in the definitions above are simple.

The very weakly (n-)baby measurable cardinals are supposed to be analogues of completely ineffable cardinals, but with stronger M-ultrafilters. The weakly (n-)baby measurable cardinals are supposed to be analogues of 1-iterable cardinals, but with stronger M-ultrafilters. Note that the M-ultrafilter in the definition of very weakly baby measurable cardinals need not be good, but the notion will still turn out to be quite strong. The (n-)baby measurable cardinals are supposed to be analogues of strongly Ramsey cardinals, but with stronger M-ultrafilters. The weakly α-game baby measurable cardinals are supposed to be analogues of the α-Ramsey cardinals, but with stronger M-ultrafilters. The α-game baby measurable cardinals strengthen the weak notion by requiring full reflection in the structure $\langle H_{\kappa^+}, \in, U\rangle$ instead of Σ_n-reflection which follows from assuming that $\langle H_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$. Finally, the strongly α-game baby measurable cardinals strengthen the reflection property to the entire structure $\langle M, \in, U\rangle$. As the names suggest, the game baby measurable cardinals will have equivalent game-theoretic definitions similar to that of α-Ramsey cardinals.

5. Games and baby measurable cardinals

In this section, we will describe the games associated to the game baby measurable cardinals.

Consider the following game $\mathrm{weak}G^\theta_\alpha(\kappa)$, for regular cardinals α and θ such that $\omega_1 \leq \alpha \leq \kappa$ and $\theta > \kappa$, of perfect information played by two players, the challenger and the judge. The game proceeds for α-many steps. The challenger starts the game and plays an imperfect κ-model $M_0 \prec H_\theta$. The judge responds by playing a structure $\langle N_0, \in, U_0\rangle$, where $P^{M_0}(\kappa) \subseteq N_0$ is a κ-model and U_0 is an N_0-ultrafilter. At stage γ, the challenger plays an imperfect κ-model $M_\gamma \prec H_\theta$ such that

$$\{\langle N_\xi, \in, U_\xi\rangle \mid \xi < \gamma\} \in M_\gamma,$$

and the judge responds with a structure $\langle N_\gamma, \in, U_\gamma\rangle$ such that $P^{M_\gamma}(\kappa) \subseteq N_\gamma$ is a κ-model and U_γ is an N_γ-ultrafilter extending $\bigcup_{\xi<\gamma} U_\gamma$. After α-many steps, let $M = \bigcup_{\xi<\alpha} M_\xi$ and $U = \bigcup_{\xi<\alpha} U_\xi$. The judge wins the game if she was able to play for α-many steps such that at the end $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$. Otherwise, the challenger wins.

Note that this game is played precisely as the game $\mathrm{Ramsey}G^{*\theta}_\alpha(\kappa)$, but, while in that game in order to win, the judge needed to ensure that U is an M-ultrafilter, here the judge needs to ensure that $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$.

Let's argue that $H^M_{\kappa^+} = \bigcup_{\xi<\alpha} N_\xi$ is the union of the judge's moves. If $A \in H^M_{\kappa^+}$, then A can be coded by a subset of κ in some M_ξ, and hence $A \in N_\xi$. If $A \in N_\xi$, then $A \in H_{\kappa^+}$, and M knows this because $M \prec H_\theta$.

Next, consider the following game $G^\theta_\alpha(\kappa)$, for regular cardinals α and θ such that $\omega_1 \leq \alpha \leq \kappa$ and $\theta > \kappa$, of perfect information played by two players, the challenger and the judge. The game proceeds for α-many steps. The challenger starts the game and plays an imperfect κ-model $M_0 \prec H_\theta$. The judge responds by playing a structure $\langle N_0, \in, U_0\rangle$, where $P^{M_0}(\kappa) \subseteq N_0$ is a κ-model and U_0 is an N_0-ultrafilter. At stage γ, the challenger plays an imperfect κ-model $M_\gamma \prec H_\theta$ such that

$$\{\langle N_\xi, \in, U_\xi\rangle \mid \xi < \gamma\} \in M_\gamma.$$

Letting $\bar{N}_\gamma = \bigcup_{\xi<\gamma} N_\xi$ and $\bar{U}_\gamma = \bigcup_{\xi<\gamma} U_\xi$, the judge responds with a structure $\langle \bar{N}_\gamma, \in, \bar{U}_\gamma\rangle \prec \langle N_\gamma, \in, U_\gamma\rangle$, where $P^{M_\gamma}(\kappa) \subseteq N_\gamma$ is a κ-model. After α-many steps, let $M = \bigcup_{\xi<\alpha} M_\xi$ and $U = \bigcup_{\xi<\alpha} U_\xi$. The judge wins the game if she was able to play for α-many steps. Otherwise, the challenger wins. As before, observe that $H^M_{\kappa^+} = \bigcup_{\xi<\alpha} N_\xi$.

Note that the only difference between the games weak$G^\theta_\alpha(\kappa)$ and $G^\theta_\alpha(\kappa)$ is that in the later the judge is required to maintain elementarity between her moves.

Finally, consider the following game strong$G^\theta_\alpha(\kappa)$, for regular cardinals α and θ such that $\omega_1 \leq \alpha \leq \kappa$ and $\theta > \kappa$, of perfect information played by two players the challenger and the judge. The game proceeds for α-many steps. The challenger starts the game and plays an imperfect κ-model $M_0 \prec H_\theta$. The judge responds by playing a structure $\langle N_0, \in, U_0\rangle$ such that $M_0 \in N_0 \prec H_\theta$ is an imperfect κ-model and U_0 an N_0-ultrafilter. At stage γ, the challenger plays an imperfect κ-model $M_\gamma \prec H_\theta$ such that $\{\langle N_\xi, \in, U_\xi\rangle \mid \xi < \gamma\} \in M_\gamma$. Letting $\bar{N}_\gamma = \bigcup_{\xi<\gamma} N_\xi$ and $\bar{U}_\gamma = \bigcup_{\xi<\gamma} U_\xi$, the judge responds with a structure $\langle \bar{N}_\gamma, \in, \bar{U}_\gamma\rangle \prec \langle N_\gamma, \in, U_\gamma\rangle$, where $M_\gamma \in N_\gamma \prec H_\theta$ and U_γ is an N_γ-ultrafilter. After α-many steps, let $M = \bigcup_{\xi<\alpha} M_\xi = \bigcup_{\xi<\kappa} N_\xi$ and $U = \bigcup_{\xi<\alpha} U_\xi$. The judge wins the game if she was able to play for α-many steps. Otherwise, the challenger wins.

6. The hierarchy

In this section, we will show where the various baby measurable cardinals fit into the large cardinal hierarchy. First, we show that surprisingly the weakly n-baby measurable and the n-baby measurable hierarchies are intertwined.

Proposition 6.1. *A very weakly $n+2$-baby measurable κ is n-baby measurable and a limit of n-baby measurable cardinals.*

Proof. Fix $A \subseteq \kappa$ and choose a weak κ-model M, with $A \in M$, for which there is an M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-_{n+2}$. By Proposition 3.5, we can find a κ-model $\bar{M} \in M$ such that $A \in \bar{M}$ and $\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec_{\Sigma_{n+1}} \langle M, \in, U\rangle$. By Proposition 3.8, $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \mathrm{ZFC}^-_n$, verifying that κ is n-baby measurable. Also, the same argument shows that κ is n-baby measurable in the ultrapower N of M by U, and hence κ must be a limit of n-baby measurable cardinals. □

Next, we show that consistently very weakly 0-baby measurable cardinals, having just the additional assumption of Δ_0-collection, are stronger than Ramsey cardinals, and hence already outside of L.

Theorem 6.2. *Suppose that κ is very weakly 0-baby measurable and the* GCH *holds below κ, then κ is ω_1-iterable and a limit of Ramsey cardinals.*

Proof. Suppose that M is a simple weak κ-model and U is an M-ultrafilter such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-_0$. Let N be the ultrapower of M by U. First, observe that Δ_0-collection implies Σ_1-collection, so we can actually assume that Σ_1-collection holds. Next, observe that since the GCH holds below κ, $N \models 2^\kappa = \kappa^+$. Thus, by our earlier observations, the Δ_1-definable relation $\lhd$ has order-type $(\kappa^+)^N = \mathrm{Ord}^M$. Thus, M has a Δ_1-definable bijection $F : \mathrm{Ord}^M \to M$.

Consider the assertion $\psi(\bar{M}, \bar{N})$ in two variables $\bar{M}$ and $\bar{N}$ expressing that:

(1) $\bar{M}$ and $\bar{N}$ are weak κ-models.
(2) $\bar{M} \prec \bar{N}$,

(3) $\bar{M} \cap U \in \bar{N}$,
(4) $\bar{N}$ is $\lhd$-least with the above 3 properties.

To ensure that the $\lhd$-least model $\bar{N}$ exists, we apply $\lhd$ to the assertion that for every κ-model and every set, there is a κ-model elementarily extending it and containing that set. Let's analyze the complexity of ψ. Let $\mathrm{Tr}_{\bar{M}}(\theta, a)$ be the Δ_1-formula expressing that $\bar{M} \models \theta(a)$ and similarly for $\mathrm{Tr}_{\bar{N}}(\theta, a)$. The assertion (1) about $\bar{M}$ consists of the Δ_0-statement that $\bar{M}$ is transitive and $\kappa, V_\kappa \in \bar{M}$, and the assertion

$$\forall \theta \in \mathrm{ZFC}^- \ \mathrm{Tr}_{\bar{M}}(\theta, \emptyset).$$

Since $\mathrm{Tr}_{\bar{M}}(\theta, \emptyset)$ has a Σ_1 form, the assertion $\forall \theta \in \mathrm{ZFC}^- \ \mathrm{Tr}_{\bar{M}}(\theta, \emptyset)$ is equivalent to a Σ_1-formula by Σ_1-collection, and, also, it obviously has a Π_1 form. Next, the assertion that $\bar{M} \prec \bar{N}$ says

$$\forall \theta \in \omega \, \forall a \in \bar{M} \ \mathrm{Tr}_{\bar{M}}(\theta, a) \to \mathrm{Tr}_{\bar{N}}(\theta, a).$$

Again, by Σ_1-collection, this is a Σ_1-formula, and it has a Π_1 form. Assertion (4) is the most complicated. It says

$$\exists \alpha \left(F(\alpha) = \bar{N} \wedge \forall \beta \in \alpha \left(\exists z \left(F(\beta) = z \wedge (\neg(1) \vee \neg(2) \vee \neg(3) \right) \right) \right).$$

Again, by Σ_1-collection, this is equivalent to a Σ_1-formula.

Fix $A \subseteq \kappa$ in M_0. Fix a weak κ-model $M_0 \prec M$ with $M_0 \in M$ and $A \in M_0$. Such an M_0 exists because $M = H^N_{\kappa^+}$, and so in N, we can construct elementary substructures of M. Now consider the assertion $\varphi(s, n)$ expressing that s is a sequence of length n such that $s_0 = \langle M_0, U \cap M_0 \rangle$, and

$$\forall i < n \, s_i = \langle M_i, U \cap M_i \rangle \wedge \psi(M_i, M_{i+1}).$$

By Σ_1-collection, this is a Σ_1-formula. Note that if $\phi(s, n)$ and $\phi(t, m)$ holds with $n < m$, then $t_i = s_i$ for $i < n$, so that the sequences extend each other.

It is clearly the case that $\langle M, \in, U \rangle$ satisfies $\forall n < \omega \, \exists s \, \varphi(s, n)$ because we can obtain such $M_i \prec M$ in the ultrapower N. Let S be the collecting set for this assertion. We cannot pick the sequences s out of S because we do not have enough separation.

With the separation we have available, we can pick out sequences of the form $\langle \langle N_0, U \cap N_0 \rangle, \ldots, \langle N_{n-1}, U \cap N_{n-1} \rangle \rangle$ such that $A \in N_0$, each N_i is a weak κ-model, $U \cap N_i \in N_{i+1}$, and $N_i \prec N_{i+1}$. Note that we can reduce the complexity of the assertion $N_i \prec N_{i+1}$ from Δ_1 to Δ_0 by first collecting all the necessary truth witnessing sequences for models in S into a set. Let T be the set of such sequences and note that T is a tree under the extension ordering. Externally, the tree T has an infinite branch, namely the one arising from the sequences

$$\langle \langle M_0, U \cap M_0 \rangle, \ldots, \langle M_{n-1}, U \cap M_{n-1} \rangle \rangle.$$

Thus, M also has a branch through T by absoluteness. But this branch witnesses that there is a weak κ-model $\bar{M} \in M$ with $A \in \bar{M}$ for which $\bar{U} = U \cap \bar{M}$ is a weakly amenable M-ultrafilter. Observe that the model M believes that $\bar{U}$ is countably complete since U is an M-ultrafilter. First, it follows that $\bar{U}$ is fully iterable because the iteration can be constructed in the well-founded model M. Also, N satisfies that κ is a Ramsey cardinal and hence by elementarity, κ is a limit of Ramsey cardinals. $\square$

Corollary 6.3. *Very weakly* 0*-baby measurable cardinals are not consistent with* L.

For the proof of Theorem 6.2, we only really needed that the GCH held at κ in the ultrapower N of M by U, but this already implies that the GCH must hold at least cofinally below κ by elementarity. Another strategy would be to ask whether there is a well-order W of M of order-type Ord^M such that the expanded structure $\langle M, \in, U, W\rangle$ still satisfies ZFC_0^-. For instance, every model of $\langle M, \in, R\rangle \models \mathrm{ZFC}$ (where R is some additional, say unary, relation) can be expanded to a model $\langle M, \in, R, W\rangle \models \mathrm{ZFC}$ where W is a well-order of M of order-type Ord^M. The well-order W is added by forcing with the class forcing $\mathrm{Add}(\mathrm{Ord}, 1)$ over the GBC-structure whose first-order part is M and whose second-order part consists of classes generated by R. The argument that the forcing preserves GBC relies on having full collection in the original structure and having just Σ_1-collection does not appear to suffice to argue that Σ_1-collection is preserved to the forcing extension.

Question 6.4. Can we remove the GCH assumption from the hypothesis of Theorem 6.2?

Question 6.5. Are very weakly 0-baby measurable cardinals stronger than strongly Ramsey cardinals?

Adding one more level of separation on the structures $\langle M, \in, U\rangle$ pushes the large cardinal strength beyond κ-Ramsey cardinals κ.

Theorem 6.6. *A very weakly 1-baby measurable cardinal is ω_1-iterable and a limit of cardinals α that are α-Ramsey.*

Proof. Suppose that $\langle M, \in, U\rangle \models \mathrm{ZFC}_1^-$ where M is a simple weak κ-model and U is an M-ultrafilter, and let N be the ultrapower of M by U. We will argue that κ is κ-Ramsey in N. Suppose not, then the challenger has a winning strategy σ in the game $\mathrm{Ramsey}G_\kappa^{\kappa^+}(\kappa)$. The element σ of N is represented by some equivalence class $[S]$. Note that σ is a map from sequences of elements of M to elements of M, and so we can assume without loss of generality that moves in the game are the coding subsets of κ (elements of $\mathcal{C}$ as defined in the proof of Proposition 3.2). In M, we will use U to play against σ and build a run of the game $\mathrm{Ramsey}G_\kappa^{\kappa^+}(\kappa)$ won by the judge, contradicting that σ was a winning strategy. For every $\gamma < \kappa$, using collection and separation in $\langle M, \in, U\rangle$, we can build a sequence s of length γ such that for $\xi < \gamma$, s_ξ is the response of σ to the judge playing pieces of U. Using collection and separation again, we can build such a sequence of length κ witnessing that the judge has a winning run of the game against σ.

It remains to argue that Σ_1-collection and Σ_1-separation suffices to obtain such sequences. Consider the assertion $\varphi(s, \lambda)$ expressing that s is a sequence of length λ and for all $\delta < \lambda$ there exist x and y such that y is a sequence of length δ and for all $\xi < \delta$ $y_\xi = U \cap s_\xi$ and (the code of) $(y, s_{\delta+1})$ is an element of $[S]$ in the ultrapower. The assertion about being an element of $[S]$ in the ultrapower translates to asking whether the set $\{\alpha < \kappa \mid f_A(\xi) \in S(\xi)\}$ is an element of U, where A is the code for $(y, s_{\delta+1})$ and $f_A(\xi) = A \cap \xi$ for every $\xi < \kappa$. Using Σ_1-collection, this is clearly a Σ_1-assertion.

$\square$

If we additionally assume that the GCH holds below κ, then we can improve ω_1-iterable to strongly Ramsey.

Theorem 6.7. *Suppose that κ is very weakly 1-baby measurable and the* GCH *holds below κ, then κ is strongly Ramsey.*

Proof. Fix $A \subseteq \kappa$, and let M be a weak κ-model with $A \in M$ for which there is an M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}_1^-$. Using the arguments in the proof of Theorem 6.2, observe that Σ_1-collection suffices to build unique sequences $\{M_\xi \mid \xi < \beta\}$ for $\beta < \kappa$ of κ-models M_ξ such that $M_\xi \cap U_\xi \in M_{\xi+1}$, and $A \in M_0$. Σ_1-separation then suffices to pick out elements of the sequence. So we can union them up to get a κ-model $\bar{M} \in M$ such that $\bar{U} = U \cap \bar{M}$ is weakly amenable for $\bar{M}$. Thus, $\bar{M}$ and $\bar{U}$ will witness strong Ramseyness for A. □

Corollary 6.8. *Suppose that κ is very weakly 2-baby measurable, then κ is strongly Ramsey.*

Proof. With Σ_2-separation, we can no longer need the well-order to have order-type ORD^M. □

Theorem 6.9. *A very weakly baby measurable cardinal κ is an n-baby measurable limit of n-baby measurable cardinals for every $n < \omega$.*

Proof. Fix $A \subseteq \kappa$, and let M be a weak κ-model with $A \in M$ for which there is an M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$. Fix $n < \omega$. In M, there is a κ-model $\bar{M}$, with $A \in \bar{M}$, such that $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \mathrm{ZFC}_n^-$. The ultrapower $\bar{N}$ of $\bar{M}$ by U must be well-founded since the model M thinks that U is countably complete, which implies that $\bar{N}$ must be well-founded in M, but M being transitive is correct about well-foundedness. Finally, clearly κ is n-baby measurable in N for every $n < \omega$, and therefore κ is a limit of such cardinals. □

Observe that a locally measurable cardinal is trivially very weakly baby measurable because if M is a weak κ-model which thinks it has a normal ultrafilter U on κ, then $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$. It follows, by Corollary 6.8, that a locally measurable cardinal is strongly Ramsey.

Theorem 6.10. *A weakly baby measurable cardinal κ is a limit of very weakly baby measurable cardinals.*

Proof. Let M be a simple weak κ-model for which there is a good M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$. Let N be the ultrapower of M by U. We will argue that κ is very weakly baby measurable in N. In N, fix $A \subseteq \kappa$, and consider the tree T whose elements on level n are sequences $\langle\langle M_0, \in, U_0\rangle, \ldots, \langle M_{n-1}, \in, U_{n-1}\rangle\rangle$ such that

- $A \in M_0$,
- $\langle M_i, \in, U_i\rangle \models \mathrm{ZFC}_i^-$,
- $\langle M_i, \in, U_i\rangle \prec_{\Sigma_i} \langle M_j, \in, U_j\rangle$.

The tree is ill-founded in V as witnessed by elementary substructures

$$\langle N, \in, U \cap N\rangle \prec_{\Sigma_i} \langle M, \in, U\rangle$$

from Proposition 3.5. Thus, N has a branch through the tree as well. The union of models on the branch gives a structure $\langle \bar{M}, \in, W\rangle \models \mathrm{ZFC}^-$ such that $\bar{M}$ is a weak κ-model and W is a $\bar{M}$-ultrafilter. Thus, κ is very weakly baby measurable in N, and hence κ is a limit of these cardinals by elementarity. □

Proposition 6.11. *If κ is baby measurable, then it is a limit of weakly baby measurable cardinals.*

Proof. Fix a simple κ-model M with an M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$, and let N be the ultrapower of M by U. We will argue that κ is weakly baby measurable in N. Fix $A \in M$. Let M_0 be a κ-model in M with $A \in M_0$ such that $\langle M_0, \in, U \cap M_0\rangle \prec_{\Sigma_0} \langle M, \in, U\rangle$. Let $M_0 \in M_1$ be a κ-model in M such that $\langle M_1, \in, U \cap M_1\rangle \prec_{\Sigma_1} \langle M, \in, U\rangle$. In this manner, choose M_n for every $n < \omega$. Let $\bar{M} = \bigcup_{n<\omega} M_n$. It follows that $\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec \langle M, \in, U\rangle$, and so in particular, $\langle \bar{M}, \in, U \cap \bar{M}\rangle \models \mathrm{ZFC}^-$. But since M is a κ-model, we get that $\bar{M} \in M$. Thus, κ is weakly baby measurable in N, and hence κ is a limit of these cardinals by elementarity. □

Next, let's look at the game baby measurable cardinals.

Suppose that α is a regular cardinal such that $\omega_1 \leq \alpha \leq \kappa$ and observe that a strongly α-game baby measurable cardinal κ is α-game baby measurable, and an α-game baby measurable cardinal is weakly α-game baby measurable. Given imperfect weak κ-models M and $\bar{M}$ such that $\langle \bar{M}, \in, U \cap \bar{M}\rangle \prec \langle M, \in, U\rangle$, we get $\langle H^{\bar{M}}_{\kappa^+}, \in, U\rangle \prec \langle H^M_{\kappa^+}, \in, U\rangle$. Also, if the $H^M_{\kappa^+}$ is a union of an elementary chain of models, then it satisfies ZFC^- by Proposition 3.7.

Let's argue that in the definition of the game baby measurable cardinals, we can strengthen the assumption that every subset of κ can be put into the required imperfect κ-model to the show that, in fact, we can put every set into such a model.

Proposition 6.12. *If κ is weakly α-game baby measurable, then for every set A, there is a cardinal θ and an imperfect weak κ-model $M \prec H_\theta$ with $A \in M$ for which there is an M-ultrafilter U such that $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$. The same strengthening applies to both α-game baby measurable cardinals and strongly α-game baby measurable cardinals.*

Proof. Suppose that κ is weakly α-game baby measurable and fix a set A. Suppose towards a contradiction that there is a cardinal θ such that $A \in H_\theta$ for which there is no required imperfect weak κ-model containing A and elementary in H_θ. Choose any large enough H_δ, which (1) sees that there is such a counter example pair θ and A and (2) for which there is some imperfect κ-model $M \prec H_\delta$ satisfying the requirements of weak α-game baby measurability for δ. By elementarity, M has some counterexample pair $\bar{\theta}$ and $\bar{A}$. Thus, $\bar{M} = M \cap H_{\bar{\theta}} \prec M$ and $\bar{A} \in \bar{M}$, but this contradicts that $\bar{\theta}$ and $\bar{A}$ were a counterexample pair. The proof obviously generalizes to the other two game baby measurable types of cardinals. □

Indeed, the above proof shows that in the definition of the game baby measurable cardinals it suffices to assume that for sufficiently large θ, there is a single imperfect κ-model M satisfying the requirements.

It is easy to show that the weakest game baby measurable cardinal sits above a weakly baby measurable cardinal. We will see later on that a baby measurable cardinal κ is a limit of cardinals α that are weakly $<\alpha$-game baby measurable, similarly to how a strongly Ramsey cardinal κ is a limit of cardinals α that are $<\alpha$-Ramsey. Again, similarly to how a κ-Ramsey cardinal κ is a limit of strongly Ramsey cardinals, a weakly κ-game baby measurable cardinal κ will be a limit of baby measurable cardinals.

Proposition 6.13. *A weakly ω_1-game baby measurable cardinal κ is a limit of weakly baby measurable cardinals.*

Proof. Using that κ is weakly ω_1-game baby measurable, we let $M \prec H_{\kappa^+}$ be a weak κ-model for which there is a weakly amenable M-ultrafilter U such that the structure $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$. Now observe that κ is weakly baby measurable in the ultrapower N of M by U because, since being weakly baby measurable is a property of H_{κ^+}, M already satisfies that κ is weakly baby measurable. □

Proposition 6.14. *The existence of a winning strategy for either player in the game weak*$G^\theta_\alpha(\kappa)$ *is independent of* θ. *The result also holds for the game* $G^\theta_\alpha(\kappa)$.

Proof. We will prove the result for the weaker game weak$G^\theta_\alpha(\kappa)$ because the proof for $G^\theta_\alpha(\kappa)$ is nearly identical.

Fix cardinals $\theta, \rho > \kappa$. Let's argue that if either player has a winning strategy in the game weak$G^\theta_\alpha(\kappa)$, then they have a winning strategy in the game weak$G^\rho_\alpha(\kappa)$.

Suppose that the challenger has a winning strategy σ in the game weak$G^\rho_\alpha(\kappa)$. Let τ be the following strategy for the challenger in the game weak$G^\theta_\alpha(\kappa)$. Let M_0 be the first move of the challenger in σ. The first move of the challenger in τ is going to be an imperfect κ-model $\bar{M}_0 \prec H_\theta$ such that $P^{M_0}(\kappa) \subseteq \bar{M}_0$. At stage γ in the game, τ needs to respond to the moves $\{\langle N_\xi, \in, U_\xi\rangle \mid \xi < \gamma\}$ of the judge. Note that as long as we continue to choose $\bar{M}_\xi$ for $\xi < \gamma$ such that $P^{M_\xi}(\kappa) \subseteq \bar{M}_\xi$, where M_ξ is the move given by σ, then any move $\langle N_\xi, \in, U\rangle$ of the judge in the game weak$G^\theta_\alpha(\kappa)$ will also be a valid move in the game weak$G^\rho_\alpha(\kappa)$. So if M_γ is the response of σ to the moves $\{\langle N_\xi, \in, U_\xi\rangle \mid \xi < \gamma\}$ of the judge in the game weak$G^\rho_\alpha(\kappa)$, then τ will tell the challenger to respond with $\bar{M}_\gamma \prec H_\theta$ such that $P^{M_\gamma}(\kappa) \subseteq \bar{M}_\gamma$. Suppose the judge can win a run of the game weak$G^\theta_\alpha(\kappa)$. The run of the game gives models $\langle \bar{M}_\xi, \in, U_\xi\rangle$ for $\xi < \alpha$. Let $\bar{M} = \bigcup_{\xi<\kappa} \bar{M}_\xi$ and $U = \bigcup_{\xi<\kappa} U_\xi$. Since the judge wins, we have $\langle H^{\bar{M}}_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$. By construction of τ, the models $\bar{M}_\xi$ were chosen based on the moves M_ξ dictated by σ. Let But now it follows that the judge would win against the moves M_ξ by playing $\langle N_\xi, \in, U_\xi\rangle$ because $H^M_{\kappa^+} = H^{\bar{M}}_{\kappa^+} = \bigcup_{\xi<\kappa} N_\xi$.

Next, suppose that the judge has a winning strategy σ in the game weak$G^\rho_\alpha(\kappa)$. Let τ be the following strategy for the judge in the game weak$G^\theta_\alpha(\kappa)$. Let $M_0 \prec H_\theta$ be the first move of the challenger. Let $\bar{M}_0 \prec H_\rho$ be such that $P^{M_0}(\kappa) \subseteq \bar{M}_0$. Now let τ respond with the structure $\langle N_0, \in, U_0\rangle$ that is the response of σ to $\bar{M}_0$. Given a play $\{M_\xi \mid \xi < \gamma\}$ by the challenger, τ will tell the judge to respond with the response of σ to the play $\langle \bar{M}_\xi \mid \xi < \gamma\rangle$ such that $\bar{M}_\xi \prec H_\rho$ and $P^{M_\xi}(\kappa) \subseteq \bar{M}_\xi$. Now observe that if the challenger wins a play against τ with the moves $\{M_\xi \mid \xi < \alpha\}$, then the challenger would also win with the moves $\{\bar{M}_\xi \mid \xi < \alpha\}$ against σ because $H^M_{\kappa^+} = H^{\bar{M}}_{\kappa^+} = \bigcup_{\xi<\kappa} N_\xi$ for $M = \bigcup_{\xi<\alpha} M_\xi$ and $\bar{M} = \bigcup_{\xi<\alpha} \bar{M}_\xi$. □

Theorem 6.15. *A cardinal* κ *is weakly* α*-game baby measurable if and only if the challenger doesn't have a winning strategy in the game weak*$G^\theta_\alpha(\kappa)$ *for some/all cardinals* θ. *A cardinal* κ *is* α*-game baby measurable if and only if the challenger doesn't have a winning strategy in the game* $G^\theta_\alpha(\kappa)$ *for some/all cardinals* θ.

Proof. Again, we will only prove the result about the weak game baby measurable cardinals because the other proof is nearly identical.

Suppose that the challenger doesn't have a winning strategy in the game weak$G^\theta_\alpha(\kappa)$ for some fixed regular cardinal θ. Fix $A \subseteq \kappa$. In particular, starting with an imperfect κ-model $M_0 \prec H_\theta$ with $A \in M_0$ is not a winning strategy for the challenger, and so the judge wins some run of the game, where the challenger starts with such an $M_0 \prec H_\theta$. Let M the union of the challenger's moves in this run of the game and

U be the union of the ultrafilters played by the judge. Since the game was played for α-many steps, the model in each step was closed under $<\kappa$-sequences and α is a regular cardinal, the union model M is closed under $<\alpha$-sequences. Finally, since the judge wins, we have $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$.

In the other direction, suppose that κ is weakly α-game baby measurable. Suppose towards a contradiction that the challenger has a winning strategy σ in the game $\mathrm{weak}G^{\kappa^+}_\alpha(\kappa)$. It is not difficult to calculate that $\sigma \in H_\theta$ for $\theta = (2^\kappa)^+$. So fix some imperfect κ-model $M \prec H_\theta$ closed under $<\alpha$-sequences with $\sigma \in M$ for which there is an M-ultrafilter U such that $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$. We will use M and U to play against σ and win, thereby showing that it couldn't have been a winning strategy. Let M_0 be the first move of the challenger according to σ, and observe that, by elementarity, we have $M_0 \in M$. Since $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$, there is a structure $\langle N_0, \in, U\rangle \prec_{\Sigma_0} \langle H^M_{\kappa^+}, \in, U\rangle$ in M with $M_0 \in N_0$. We will have the judge play $\langle N_0, \in, U\rangle$. Now since $\langle N_0, \in, U\rangle \in M$, the response of the challenger to M_1 according to σ must be in M as well. So we will have the judge play some $\langle N_1, \in, U\rangle \prec_{\Sigma_1} \langle H^M_{\kappa^+}, \in, U\rangle$ in M with $M_1 \in N_1$. We will continue having the judge play like that so at n-th successor ordinals, the judge plays Σ_n-elementary substructures of $\langle M, \in, U\rangle$. Since M is closed under $<\alpha$-sequences, for $\gamma < \alpha$-steps of the game, M will always have the sequence of the judge's moves. Thus, the judge can continue to play for α-many steps. Let $N = \bigcup_{\xi<\alpha} N_\xi$ and $\bar{U} = U \restriction N$. By construction, $\langle N, \in, U\rangle \prec \langle H^M_{\kappa^+}, \in, U\rangle$ and therefore $\langle N, \in, U\rangle \models \mathrm{ZFC}^-$. As we already observed, N is precisely the H_{κ^+} of $M = \bigcup_{\xi<\alpha} M_\xi$, the union of the moves of the challenger. Thus, we have shown that the judge can win against σ, contradicting that σ was a winning strategy. □

It follows from the proof that κ is weakly α-game baby measurable if and only if every $A \in H_{(2^\kappa)^+}$ is an element of a $<\alpha$-closed imperfect weak κ-model $M \prec H_{(2^\kappa)^+}$, with $A \in M$, for which there is an M-ultrafilter such that $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$. An analogous result holds for the α-game baby measurable cardinals.

Theorem 6.16. *A baby measurable cardinal is a limit of cardinals α that are weakly $<\alpha$-game baby measurable. A weakly κ-game baby measurable cardinal κ is a limit of baby measurable cardinals.*

Proof. Suppose that κ is baby measurable and fix a simple κ-model M for which there is an M-ultrafilter with $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$. Let N be the ultrapower of M by U. We will argue that κ is weakly $<\kappa$-game baby measurable in $V^N_{j(\kappa)}$. Suppose towards a contradiction that κ is not weakly α-game baby measurable in $V^N_{j(\kappa)}$ for some $\alpha < \kappa$. It follows that in $V^N_{j(\kappa)}$, the challenger has a winning strategy in the game $\mathrm{weak}G^{\kappa^+}_\alpha(\kappa)$. We will use U to play against σ and argue that the resulting run of the game is in N. Using the same argument as in the proof of Theorem 6.15, it suffices to observe that M has all the required sequences by $<\kappa$-closure. Finally, we use Proposition 6.14 to argue that V_κ must be correct about a cardinal α being $<\alpha$-game baby measurable.

For the second part, observe that κ is at least baby measurable. Now, fix an imperfect κ-model $M \prec H_\theta$ for some large θ such that $\langle H^M_{\kappa^+}, \in, U\rangle \models \mathrm{ZFC}^-$, and let N be the ultrapower of M by U. We will argue that κ is baby measurable in N. Observe that being baby measurable is a property of H_{κ^+} and therefore since $M \prec H_\theta$, M satisfies that κ is baby measurable. Since $H^M_{\kappa^+} = H^N_{\kappa^+}$ by weak amenability, N also satisfies that κ is baby measurable. □

Next, we show that the (weakly) α-game baby measurable cardinals form a hierarchy.

Theorem 6.17. *Suppose that $\alpha < \beta \leq \kappa$ are regular cardinals. Then every weakly β-game baby measurable cardinal is a limit of cardinals $\delta > \alpha$ that are α-game baby measurable. An analogous result holds for β-game measurable cardinals.*

Proof. The argument is again essentially the same as in the proof of Theorem 6.15, where we use the β-closure on the imperfect weak κ-model given by the definition of a weakly β-game measurable cardinal to argue that the challenger cannot have a winning strategy in the ultrapower since we can use the ultrafilter to play against it. □

Next, we show that the smallest game baby measurable cardinal is already stronger than the strongest weakly game baby measurable cardinal.

Theorem 6.18. *An ω_1-game baby measurable cardinal is a limit of cardinals α that are weakly α-game baby measurable.*

Proof. Let κ be ω_1-game baby measurable. Fix a simple weak κ-model M for which there is an M-ultrafilter U such that for every $A \in M$ there is a κ-model $\bar{M} \in M$ such that $\langle \bar{M}, \in, U \cap \bar{M} \rangle \prec \langle M, \in, U \rangle$. The existence of such models follows easily from our assumption. In particular, $\langle \bar{M}, \in, U \cap \bar{M} \cap U \rangle \models \mathrm{ZFC}^-$. Suppose that κ is not weakly κ-game baby measurable in the ultrapower N of M by U. This means that in N, the challenger has a winning strategy σ in the game weak$G^{\kappa^+}_\kappa(\kappa)$. Using a function representing σ and applying the Łoś-Theorem, we get that σ is a definable (with parameters) class in $\langle M, \in, U \rangle$, which can see that σ is a winning strategy. By elementarity, some $\langle \bar{M}, \in, U \cap \bar{M} \rangle$ (which has the defining parameters for σ) must also satisfy that σ is a winning strategy. Since $\langle \bar{M}, \in, U \cap \bar{M} \rangle$ is elementary in $\langle M, \in, U \rangle$, it will be correct about the moves dictated by σ. Consider a play by the judge of length κ, where at n-successor ordinal stages, she plays Σ_n-elementary substructures of $\langle \bar{M}, \in, U \cap \bar{M} \rangle$ from $\bar{M}$, which exist there since it is a model of ZFC^-. The challenger, playing according to σ, will always respond with an element of $\bar{M}$ since by induction we can use the closure of $\bar{M}$ to argue that the sequence of the previous moves is an element of $\bar{M}$. The κ-length run of the game will not be in $\bar{M}$, but it will be in M because $\langle \bar{M}, \in, U \cap \bar{M} \rangle$ is an element of M and so M can see the entire construction. The union of the judge's moves will be a structure $\langle M^*, \in, U \cap M^* \rangle \prec \langle \bar{M}, \in, U \cap \bar{M} \rangle \prec \langle M, \in, U \rangle$, and thus a winning run of the game for the judge in M, which is the desired contradiction. □

Next, we prove some game theoretic equivalences for the strongly α-game baby measurable cardinals. These don't appear to be as robust as the other game baby measurable cardinals.

Theorem 6.19. *A cardinal κ is strongly α-game baby measurable if and only if the challenger doesn't have a winning strategy in the game strong$G^\theta_\alpha(\kappa)$ for any θ.*

Proof. The proof is nearly identical to the proof of Theorem 6.15. For the forward direction, we use a winning run of the game of the judge to build the required imperfect weak κ-model. In the back direction, we assume toward a contradiction that there is θ such that the challenger has winning strategy σ in the game strong$G^\theta_\alpha(\kappa)$. Let $\bar\theta$ be large enough such that $\sigma \in H_{\bar\theta}$ and let $M \prec H_{\bar\theta}$, with $\sigma, \theta \in M$, be an imperfect weak κ-model satisfying the hypothesis of a strongly α-game baby measurable cardinal. Suppose that $M_0 \prec H_\theta$ is the first move of the challenger in σ. Let $\bar M_0 \in M$ be an imperfect κ-model such that $\langle \bar M_0, \in, U\rangle \prec \langle M, \in, U\rangle$. Let $N_0 = M_0 \cap H_\theta$. The judge will respond with $\langle N_0, \in, U \cap M_0\rangle$, and the judge will continue responding in this fashion for κ-many steps, producing a winning run of the game against σ. Note that if $\langle M_0, \in, U \cap M_0\rangle \prec \langle M_1, \in, U \cap M_1\rangle$, then $\langle M_0 \cap H_\theta, \in, U \cap M_0\rangle \prec \langle M_1 \cap H_\theta, \in, U \cap M_1\rangle$. □

It is not clear whether the winning strategies for the players in the game strong$G^\theta_\alpha(\kappa)$ are independent of θ. We only see how to show the following.

Proposition 6.20. *If the challenger has a winning strategy in the game strong$G^\theta_\alpha(\kappa)$ and $\rho > \theta$, then the challenger has a winning strategy in the game strong$G^\rho_\alpha(\kappa)$. If the judge has a winning strategy in the game strong$G^\rho_\alpha(\kappa)$ and $\rho > \theta$, then the judge has a winning strategy in the game strong$G^\theta_\alpha(\kappa)$.*

Proof. Suppose the challenger has a winning strategy σ in the game strong$G^\theta_\alpha(\kappa)$ and $\rho > \theta$. We will build a strategy τ for the challenger in the game strong$G^\rho_\alpha(\kappa)$. Let $M_0 \prec H_\theta$ be the first move of the challenger in σ. Let $\theta, M_0 \in \bar M_0 \prec H_\rho$ be the first move of the challenger in τ. Suppose that the judge responds with a structure $\langle \bar N_0, \in, U_0\rangle$ with $\bar M_0 \in N_0$. It follows that $\theta \in \bar N_0$, and so $\bar N_0 \cap H_\theta = N_0 \prec H_\theta$. Let M_1 be the response of the challenger to $\langle N_0, \in, U_0\rangle$ according to σ. Then we let $M_1 \in \bar M_1 \prec H_\rho$ be the response of the challenger in τ. Suppose that the judge responds with a structure $\langle \bar N_0, \in, U_0\rangle \prec \langle \bar N_1, \in, U_1\rangle$, and observe that then $\bar N_1 \cap H_\theta = N_1 \prec H_\theta$ and $\langle N_0, \in, U_0\rangle \prec \langle N_1, \in, U_1\rangle$. Thus, we can continue to define τ in this fashion. Clearly if the judge could win against τ in strong$G^\rho_\alpha(\kappa)$, the judge could also win against σ in strong$G^\theta_\alpha(\kappa)$. So τ must be a winning strategy for the challenger in strong$G^\rho_\alpha(\kappa)$.

Next, suppose that the judge has a winning strategy σ in the game strong$G^\rho_\alpha(\kappa)$ and $\rho > \theta$. We will build a strategy τ for the judge in the game strong$G^\theta_\alpha(\kappa)$. Let $M_0 \prec H_\theta$ be the first move of the challenger. Let $\theta, M_0 \in \bar M_0 \prec H_\rho$ be the first move of the challenger in the parallel game strong$G^\rho_\alpha(\kappa)$ that the judge will play to determine the strategy in strong$G^\theta_\alpha(\kappa)$. The strategy σ gives a reply $\langle \bar N_0, \in, U_0\rangle$ to $\bar M_0$. The strategy τ will then tell the judge to reply to M_0 with $\langle N_0, \in, U_0\rangle$, where $N_0 = \bar N_0 \cap H_\theta$. We will continue to build τ in this fashion and observe that τ must be winning for the judge because σ was. □

Question 6.21. Is the existence of a winning strategy in the game strong$G^\theta_\alpha(\kappa)$ independent of θ?

Theorem 6.22. *Suppose that κ is locally measurable.*

(1) *κ is a limit of cardinals α that are α-game baby measurable.*
(2) *V_κ has a proper class of cardinals α that are strongly α-game baby measurable.*

Proof. Suppose that κ is locally measurable. Let M be a weak κ-model which thinks it has a normal ultrafilter U on κ. By Proposition 2.17, we can assume that the ultrapower N of M by U is contained in M. By standard ultrapower arguments, $N^\kappa \subseteq N$ in M. We will argue that κ is strongly κ-game baby measurable in

$\bar{N} = V^N_{j(\kappa)} \in M$. If not, then, by Theorem 6.19, in $\bar{N}$, there is a cardinal θ such that the challenger has a winning strategy σ in the game strong$G^\theta_\kappa(\kappa)$. Next, observe that, for every set $A \in H^{\bar{N}}_\theta$, M can construct imperfect κ-models $A \in N_0$ such that $\langle N_0, \in, U \cap N_0\rangle \prec \langle H^{\bar{N}}_\theta, \in, U\rangle$. By the closure of N, the structures $\langle N_0, \in, U \cap N_0\rangle$ will be elements of $\bar{N}$. This means we can construct an elementary chain of such substructures to play against σ. The run of the game will be in $\bar{N}$ again by closure.

Thus, by elementarity, V_κ has a proper class of cardinals α that are strongly α-game baby measurable. Although, we cannot conclude that each such α is actually strongly α-game baby measurable because we haven't verified the property for cofinally many cardinal θ, we do know that each such α is α-game baby measurable. □

7. Indestructibility

In this section, we provide a few basic indestructibility results for very weakly baby measurable, weakly baby measurable, and baby measurable cardinals. More specifically, we show that these large cardinals κ are indestructible by small forcing and can be made indestructible by the forcing $\mathrm{Add}(\kappa, 1)$, adding a Cohen subset to κ.

The indestructibility arguments will use properties of class forcing over models of second-order set theory. A model of second-order set theory is a triple $\mathcal{M} = \langle M, \in, \mathcal{C}\rangle$, where M consists of the sets of the model and $\mathcal{C}$ consists of the classes. The second-order theory GBc^- consists of the axioms ZFC^- for sets, the extensionality axiom for classes, the class replacement axiom asserting that every class function restricted to a set is a set, and the first-order comprehension scheme asserting that every first-order formula defines a class. The theory GBC^- is the theory GBc^- together with the assertion that there is a class well-order of sets of order-type Ord. Given a class partial order $\mathbb{P} \in \mathcal{C}$, we say that a filter $G \subseteq \mathbb{P}$ is $\mathcal{M}$-*generic* for $\mathbb{P}$ if it meets every dense subclass of $\mathbb{P}$ from $\mathcal{C}$. Given an $\mathcal{M}$-generic filter G, the forcing extension of $\mathcal{M}$ by G is the structure $\langle M[G], \in, \mathcal{C}[G]\rangle$, where $M[G]$ consists of the interpretation of $\mathbb{P}$-names by G and $\mathcal{C}[G]$ consists of the interpretation of class $\mathbb{P}$-names by G, where a *class* $\mathbb{P}$-*name* is any class whose elements are pairs of the form $\langle p, \sigma\rangle$ with $p \in \mathbb{P}$ and σ a $\mathbb{P}$-name. If $\mathcal{M}$ is ill-founded, we can still form the generic extension by taking instead of interpretations of names, the structure whose elements are equivalence classes of (class) $\mathbb{P}$-names moded out by the filter G. Although class forcing may not always preserve replacement to the forcing extension, pretame partial orders preserve the theories GBc^- and GBC^-. See [HKS18] for the definition and results on pretameness, and see [AGng] for details on models of second-order set theory and class forcing.

Theorem 7.1. *Very weakly baby measurable, weakly baby measurable, and baby measurable cardinals are indestructible by small forcing.*

Proof. Suppose that κ is weakly baby measurable, $\mathbb{P} \in V_\kappa$ is a forcing notion, and $g \subseteq \mathbb{P}$ is V-generic. Fix $A \subseteq \kappa$ in $V[g]$ and let $\dot{A}$ be a nice $\mathbb{P}$-name such that $\dot{A}_g = A$. Since $\mathbb{P} \in V_\kappa$, $\dot{A}$ is a subset of V_κ as well and hence we can put it into a simple weak κ-model M for which there is a good M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$. Let $\mathcal{C}$ be the classes of M generated by U, so that the second-order structure $\langle M, \in, \mathcal{C}\rangle \models \mathrm{GBc}^-$. Since $\mathbb{P}$ is a set forcing in M, it is, in particular, trivially pretame. Since pretame forcing preserves GBc^-, we have that the second-order structure $\langle M[g], \in, \mathcal{C}[g]\rangle \models \mathrm{GBc}^-$ as well. Let W the $M[g]$-ultrafilter generated by U in $M[g]$. Clearly $W \in \mathcal{C}[g]$, and so it follows that $\langle M[g], \in, W\rangle \models \mathrm{ZFC}^-$. Also, clearly W is good because we can lift the ultrapower embedding $j : M \to N$ to $j : M[g] \to N[g]$ and the $M[g]$-ultrafilter generated from the lift is precisely W. Since $A \in M[g]$, the structure $\langle M[g], \in, W\rangle$ witnesses weak baby measurability for A.

The case of very weakly baby measurable cardinals is even easier because it suffices to note that W is definable in $\langle M[g], \in, \mathcal{C}[g]\rangle$.

For the case of baby measurable cardinals it suffices to observe that a forcing extension $M[g]$ of a κ-model M by $\mathbb{P}$ is again a κ-model in $V[g]$. □

Theorem 7.2. *Very weakly baby measurable cardinals, weakly baby measurable cardinals, and baby measurable cardinals κ can be made indestructible by the forcing* $\mathrm{Add}(\kappa, 1)$.

Proof. Suppose that κ is very weakly baby measurable. First, we will argue that every $A \subseteq \kappa$ is an element of a simple weak κ-model M for which we can find an M-ultrafilter U and a global well-order $\lhd'$ of order-type Ord^M such that the structure

$$\langle M, \in, U, \lhd'\rangle \models \mathrm{ZFC}^-.$$

The point here is that $\lhd'$ has order-type Ord^M, which might not be the case for the well-order $\lhd$ definable from U.

So suppose that M is a simple weak κ-model with an M-ultrafilter U such that $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$. Let $\mathcal{C}$ be the classes of M generated by U, so that the second-order structure $\mathcal{M} = \langle M, \in, \mathcal{C}\rangle \models \mathrm{GBc}^-$. Consider the class forcing $\mathrm{Add}(\mathrm{Ord}, 1)^M$ over $\langle M, \in, \mathcal{C}\rangle$. Using that $\langle M, \in, \mathcal{C}\rangle$ has a global well-order $\lhd$, it can be easily shown that $\mathrm{Add}(\mathrm{Ord}, 1)^M$ is set-distributive. Set-distributivity implies pretameness and that the forcing does not add sets. Let $G \subseteq \mathrm{Add}(\mathrm{Ord}, 1)^M$ be $\mathcal{M}$-generic. Since no sets are added, the forcing extension $\mathcal{M}[G] = \langle M, \in, \mathcal{C}[G]\rangle$, and it is not difficult to see that G adds a global well-order $\lhd'$ of order-type Ord^M. But now we need to show that V has an $\mathcal{M}$-generic filter for $\mathrm{Add}(\mathrm{Ord}, 1)^M$.

First, suppose that M is a κ-model. Since $\mathcal{C}$ has size κ, V has an enumeration $\langle D_\xi \mid \xi < \kappa\rangle$ of the dense subclasses of $\mathrm{Add}(\mathrm{Ord}, 1)^M$ that are in $\mathcal{C}$. Using that the forcing $\mathrm{Add}(\mathrm{Ord}, 1)^M$ is $<\kappa$-closed in M, we can build an $\mathcal{M}$-generic filter for $\mathrm{Add}(\mathrm{Ord}, 1)^M$ by meeting the dense subclasses D_ξ for $\xi < \kappa$ in κ-many steps.

The argument is more involved if M is just a weak κ-model. Here we need techniques developed in [GJ]. In this case, we may need to replace M by a smaller model $\bar{M}$ containing A. Since $\langle M, \in, U\rangle \models \mathrm{ZFC}^-$, we can construct a chain of structures $\langle M_n, \in, U_n\rangle$ for $n < \omega$ such that

(1) $\langle M_n, \in, U_n\rangle \prec_{\Sigma_n} \langle M, \in, U\rangle$,
(2) $\langle M_n, \in, U_n\rangle \in M_{n+1}$,
(3) $A \in M_0$,
(4) M_n is a κ-model.

Let $\bar{M} = \bigcup_{n<\omega} M_n$ and let $\bar{U} = \bigcup_{n<\omega} U_n$. Clearly

$$\langle \bar{M}, \in, \bar{U}\rangle \prec \langle M, \in, U\rangle,$$

and hence $\langle \bar{M}, \in, \bar{U}\rangle \models \mathrm{ZFC}^-$. Let $\bar{\mathcal{M}} = \langle \bar{M}, \in, \mathcal{C}\rangle$, where $\mathcal{C}$ are the classes generated by U. Now we will build an $\bar{\mathcal{M}}$-generic filter for $\mathrm{Add}(\mathrm{Ord}, 1)^{\bar{M}}$ using that $\bar{M}$ is a countable union of κ-models. We construct the filter G in ω-many step.

At step 0, working inside $\bar{\mathcal{M}}$, we enumerate all the Σ_0-definable in $\langle \mathcal{M}, \in, U\rangle$, with parameters from M_0, dense subclasses of $\mathrm{Add}(\mathrm{Ord}, 1)^{\bar{M}}$ in a κ-sequence and build a κ-length sequence of conditions meeting them inside M_0 using that M_0 is a κ-model. Since $\mathrm{Add}(\mathrm{Ord}, 1)^{\bar{M}}$ is $\leq \kappa$-closed in $\bar{M}$, $\bar{M}$ has an element p_0 above the sequence we just built. Next, we choose some M_{i_1} which has p_0 as an element and meet all the Σ_1-definable in $\langle M, \in, U\rangle$, with parameters from M_{i_1}, dense subclasses of $\mathrm{Add}(\mathrm{Ord}, 1)^{\bar{M}}$ with conditions above p_0. In this way in ω-many steps, we construct the sequence $\langle p_n \mid n < \omega\rangle$ which clearly generates an $\bar{\mathcal{M}}$-generic filter for $\mathrm{Add}(\mathrm{Ord}, 1)^{\bar{M}}$.

Now we can proceed with the forcing indestructibility argument. Let $\mathbb{P}_\kappa$ be the κ-length Easton-support iteration forcing with $\mathrm{Add}(\alpha, 1)$ at cardinal stages, and let $G * g$ be V-generic for $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$. With some slight renamings, we can assume that the poset $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$ is a subset of V_κ. Every subset $A \subseteq \kappa$ in $V[G * g]$ has a nice-$\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$-name $\dot{A}$ in V, which can therefore be put into a weak κ-model M for which there is an M-ultrafilter U and a global well-order $\lhd'$ of order-type Ord^M such that $\langle M, \in, U, \lhd'\rangle \models \mathrm{ZFC}^-$. Let $\mathcal{C}$ be the classes of M generated by U and $\lhd'$. Since the forcing $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$ is a set and hence pretame, we have that the second-order structure

$$\langle M[G * g], \in, \mathcal{C}[G * g]\rangle \models \mathrm{GBC}^-.$$

Let Ult be the (pre-collapsed) ultrapower of M by U that is definable in $\langle M, \in, U, \lhd'\rangle$ and let $\Psi : M \to \mathrm{Ult}$ be the ultrapower map, which is also definable there. Note that $\langle M, \in, U, \lhd'\rangle$ can pick out a unique element of each equivalence class using its global well-order. The model $\mathcal{M}[G] = \langle M[G * g], \in, \mathcal{C}[G * g]\rangle$ has the classes M and Ult. Using $G*g$, we can define the model $\mathrm{Ult}[G*g]$ inside $\langle M[G*g], \in, \mathcal{C}[G*g]\rangle$. We can think of elements of $\mathrm{Ult}[G * g]$ as equivalence classes $[\tau]$ of $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1)$-names from Ult, where we have that τ is equivalent to σ whenever there is $p \in G * g$ such that $p \Vdash \tau = \sigma$. The entire construction is definable in $\mathcal{M}[G][g]$. We will lift the ultrapower embedding Ψ of M by U to an ultrapower embedding of $M[G * g]$ inside the structure $\langle M[G * g], \in, \mathcal{C}[G * g]\rangle$.

First, we lift Ψ to $M[G]$. Using the standard lifting criterion for lifting elementary embeddings to a forcing extension, we need to build an Ult-generic filter H for the poset $\Psi(\mathbb{P}_\kappa)$ with $\Psi\,"\,G \subseteq H$. The poset $\Psi(\mathbb{P}_\kappa)$ factors as $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, 1) * \mathbb{P}_{\mathrm{tail}}$ (we will associate the initial segment of the ultrapower that is isomorphic to M with M itself to simplify notation). We use $G * g$ for the initial segment of the forcing, thereby trivially satisfying the requirement that $\Psi\,"\,G = G$ will be contained in the filter we end up building. So it remains to find an $\mathrm{Ult}[G*g]$-generic filter G_{tail} for the tail forcing $\mathbb{P}_{\mathrm{tail}}$, which is $<\kappa^+$-closed there. The model $\langle M[G*g], \in, \mathcal{C}[G*g]\rangle$ has a well-order $\lhd''$ of order-type Ord^M of $M[G * g]$, constructed by naturally extending $\lhd'$. Using the extended well-order $\lhd''$, in $\langle M[G * g], \in, \mathcal{C}[G * g]\rangle$, we can enumerate all the dense subsets of $\Psi(\mathbb{P}_\kappa)$ in $\mathrm{Ult}[G*g]$ in a class sequence of length Ord^M. Every initial segment of this enumeration has length some ordinal $\alpha \in M$. Note next that every sequence of elements of Ult of order-type α must be an element of Ult because it is an ultrapower. The closure also transfers to the pairs $M[G]$ and $\mathrm{Ult}[G]$, as well as $M[G * g]$ and $\mathrm{Ult}[G * g]$ (for details of closure arguments, see [GJ]). Thus, as we diagonalize against the sequence of dense sets inside $\langle M[G * g], \in, \mathcal{C}[G * g]\rangle$, every initial segment of our choices of elements from the dense sets is going to be a sequence in $\mathrm{Ult}[G * g]$ and therefore since $\Psi(\mathbb{P}_\kappa)$ is $<\kappa^+$-closed in $\mathrm{Ult}[G * g]$, we can find an element below the diagonalization sequence. Thus, we can continue using replacement in our structure to define the generic filter. Note that the generic filter G_{tail} we construct in this manner is a class in $\mathcal{M}[G * g]$.

Once we have the generic filter G_{tail}, we can repeat the process to define an $\text{Ult}[G{*}g][G_{\text{tail}}]$-generic filter for the image $\Psi(\text{Add}(\kappa,1)) = \text{Add}(\Psi(\kappa),1)$ of $\text{Add}(\kappa,1)$ under the lifted ultrapower map Ψ. Thus, we can lift the ultrapower embedding Ψ to $M[G * g]$ and use the ultrapower map Ψ to define the $M[G * g]$-ultrafilter W extending U. Since W was defined inside the structure $\langle M[G * g], \in, \mathcal{C}[G * g]\rangle$, we have that $\langle M[G * g], \in, W\rangle \models \text{ZFC}^-$. The lifted ultrapower map verifies that W is good.

The case of very weakly baby measurable cardinals proceeds identically to above because we never used the well-foundedness of the ultrapower in our construction.

For the case of baby measurable cardinals it suffices to observe that a forcing extension $M[G][g]$ of a κ-model M by $\mathbb{P}_\kappa * \text{Add}(\kappa,1)$ is again a κ-model in $V[G][g]$ (for details, see [GJ]).

□

Corollary 7.3. *Very weakly baby measurable cardinals, weakly baby measurable cardinals, and baby measurable cardinals κ can be made indestructible by the forcing* $\text{Add}(\kappa,\theta)$ *for any cardinal θ.*

Proof. Suppose that $G \subseteq \text{Add}(\kappa,\theta)$ is V-generic. Observe that any $A \subseteq \kappa \in V[G]$ has an $\text{Add}(\kappa,\theta)$-name $\dot{A}$ in V that uses at most κ-many coordinates in θ. Thus, we can view $\dot{A}$ as an $\text{Add}(\kappa,1)$-name. Hence, it suffices to show that these cardinals can be made indestructible by $\text{Add}(\kappa,1)$. □

It follows that we can make the GCH fail at these cardinals. Also, a similar lifting argument can be used to show that the GCH-forcing preserves these cardinals.

8. A detour into second-order set theory

Models of ZFC^- with a largest cardinal κ that is inaccessible are bi-interpretable with models of the second-order set theory Kelley-Morse strengthened by a choice principle for classes. Let ZFC^-_I be the theory asserting that ZFC^- holds, that there is a largest cardinal κ, and that κ is inaccessible ($P(\alpha)$ exists and $2^\alpha < \kappa$ for all $\alpha < \kappa$). The assumption that κ is inaccessible implies, in particular, that V_κ exists, and that $V_\kappa \models \text{ZFC}$. Recall that Kelley-Morse (KM) is a second-order set theory whose axioms consist of GBC together with the full comprehension scheme asserting for every second-order formula that it defines a class. We can further strengthen KM by adding various very useful choice principles for classes. Let the *choice scheme* (CC) be the scheme which asserts, for every second-order formula $\varphi(x, X, A)$, that if for every set x, there is a class X witnessing φ, then there is a single class Y collecting witnesses for every set x on its slices $Y_x = \{y \mid \langle x,y\rangle \in Y\}$.

Marek showed that Kelley-Morse together with the choice scheme ($\text{KM}+\text{CC}$) is bi-interpretable with ZFC^-_I [Mar73]. Given a model $\mathcal{V} = \langle V, \in, \mathcal{C}\rangle \models \text{KM}+\text{CC}$, we obtain the corresponding model $M_{\mathcal{V}} \models \text{ZFC}^-_I$ by taking all the well-founded extensional relations in $\mathcal{C}$, modulo isomorphism, with the natural membership relation that results from viewing these relations as transitive sets. We get that $V_\kappa^{M_{\mathcal{V}}} \cong V$ and $P(V_\kappa)^{M_{\mathcal{V}}} \cong \mathcal{C}$. In the other direction, given any model $M \models \text{ZFC}^-_I$, we obtain the corresponding model $\mathcal{V} = \langle V_\kappa^M, \in, P(V_\kappa)^M\rangle \models \text{KM}+\text{CC}$. Moreover, what gives us bi-interpretability is that the ZFC_I^--model $M_{\mathcal{V}}$ corresponding to $\mathcal{V}$ is precisely M.

Let ZFC_U^- be the theory in the language with an additional unary predicate U consisting of ZFC_I^- in the extended language together with the assertion that U is a uniform normal ultrafilter on the largest cardinal κ.

On the second-order side, let KM_U be the theory in the language of second-order set theory with an additional unary predicate U on classes consisting of KM in the extended language together with the assertion that U is a uniform normal ultrafilter on Ord. Let $\mathcal{V} = \langle V, \in, \mathcal{C}, U\rangle \models \mathrm{KM}_U$. Consider the ultrapower structure $\langle Ult, \mathrm{E}\rangle$ consisting of the equivalence classes of class functions $F : \mathrm{Ord} \to V$ from $\mathcal{C}$ modulo U. It is not difficult to see that the Łoś-Theorem holds for the structures $\langle V, \in\rangle$ and $\langle \mathrm{Ult}, \mathrm{E}\rangle$: $\langle \mathrm{Ult}, \mathrm{E}\rangle \models \varphi([F])$ if and only if $\{\alpha \mid \langle V, \in\rangle \models \varphi(F(\alpha)\} \in U$. It follows that the universe V is isomorphic to a rank-initial segment Ult_κ of Ult consisting of equivalence classes of constant functions $C_a : \mathrm{Ord} \to V$ such that $C_a(\alpha) = a$ for all α, and that $\mathrm{Ult}_{\kappa+1}$ consists precisely of the classes $\mathcal{C}$ via this isomorphism. Since by the Łoś-Theorem, $\langle \mathrm{Ult}, \mathrm{E}\rangle \models \mathrm{ZFC}$, it has a well-ordering of $\mathrm{Ult}_{\kappa+1}$. Now, essentially by the argument of Proposition 3.2, we can conclude that $\mathcal{V}$ has a definable well-ordering of its classes. The existence of a definable well-ordering of classes is much stronger than the choice scheme, which clearly follows from it.

Same arguments as described above show that KM_U and ZFC_U^- are bi-interpretable. Therefore we should view ZFC_U^- as essentially being a strong second-order set theory asserting that Ord is measurable.

Our analysis of very weakly measurable cardinals sheds light on the structure in models of KM_U. Suppose $\mathcal{V} = \langle V, \in, \mathcal{C}, U\rangle \models \mathrm{KM}_U$.

(1) (large cardinals) V has a proper class of κ-Ramsey cardinals.
(2) (global well-order of classes) $\mathcal{V}$ has a definable global well-order of $\mathcal{C}$.
(3) (truth predicate) $\mathcal{V}$ has a truth predicate for $\langle \mathcal{V}, \in, \mathcal{C}\rangle$.
(4) (large cardinal properties of Ord) Ord is ineffably Ramsey in $\mathcal{V}$: every class function $F : [\mathrm{Ord}]^{<\omega} \to 2$ has a stationary homogeneous class.
(5) (strong reflection) For every class $A \in \mathcal{C}$ and $n < \omega$ there is a sub-collection $A \in \bar{\mathcal{C}}$ of $\mathcal{C}$ such that $\langle V, \in, \bar{\mathcal{C}}, U \cap \bar{\mathcal{C}}\rangle \prec_{\Sigma_n} \langle V, \in, \mathcal{C}, U\rangle$ and $\bar{\mathcal{C}}$ is coded by a single class in $\mathcal{C}$ (there is a class C such that the slices $C_\xi = \{a \in V \mid \langle \xi, a\rangle \in C\}$ are precisely the elements of $\bar{\mathcal{C}}$).

References

[AGng] Carolin Antos and Victoria Gitman. Modern class forcing. In *Research Trends in Contemporary Logic*. College Publications, Forthcoming.

[BM] Andrey Bovykin and Zachiri McKenzie. Ramsey-like cardinals that characterize the exact consistency strength of nfum. *Manuscript*.

[Gai74] Haim Gaifman. Elementary embeddings of models of set-theory and certain subtheories. In *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967)*, pages 33–101. Amer. Math. Soc., Providence R.I., 1974.

[GHJ16] Victoria Gitman, Joel David Hamkins, and Thomas A. Johnstone. What is the theory ZFC without power set? *MLQ Math. Log. Q.*, 62(4-5):391–406, 2016.

[Git11] Victoria Gitman. Ramsey-like cardinals. *The Journal of Symbolic Logic*, 76(2):519–540, 2011.

[GJ] Victoria Gitman and Thomas A. Johnstone. Indestructibility for Ramsey and Ramsey-like cardinals. *Manuscript*.

[GW11] Victoria Gitman and Philip D. Welch. Ramsey-like cardinals II. *The Journal of Symbolic Logic*, 76(2):541–560, 2011.

[HKS18] Peter Holy, Regula Krapf, and Philipp Schlicht. Characterizations of pretameness and the Ord-cc. *Ann. Pure Appl. Logic*, 169(8):775–802, 2018.

[HL21] Peter Holy and Philipp Lücke. Small models, large cardinals, and induced ideals. *Ann. Pure Appl. Logic*, 172(2):Paper No. 102889, 50, 2021.

[HS18] Peter Holy and Philipp Schlicht. A hierarchy of Ramsey-like cardinals. *Fund. Math.*, 242(1):49–74, 2018.

[Kun70] Kenneth Kunen. Some applications of iterated ultrapowers in set theory. *Ann. Math. Logic*, 1:179–227, 1970.

[Mar73] W. Marek. On the metamathematics of impredicative set theory. *Dissertationes Math. (Rozprawy Mat.)*, 98:40, 1973.

[Mit79] William Mitchell. Ramsey cardinals and constructibility. *J. Symbolic Logic*, 44(2):260–266, 1979.

[SW11] Ian Sharpe and Philip Welch. Greatly Erdős cardinals with some generalizations to the Chang and Ramsey properties. *Ann. Pure Appl. Logic*, 162(11):863–902, 2011.

[Zar82] Andrzej Zarach. Unions of ZF^--models which are themselves ZF^--models. In *Logic Colloquium '80 (Prague, 1980)*, volume 108 of *Stud. Logic Foundations Math.*, pages 315–342. North-Holland, Amsterdam, 1982.