July 23, 2008

PROPER AND PIECEWISE PROPER FAMILIES OF REALS

VICTORIA GITMAN

Abstract. I introduced the notions of *proper* and *piecewise proper* families of reals to make progress on a long standing open question in the field of models of Peano Arithmetic [Git08]. A family of reals is *proper* if it is arithmetically closed and its quotient Boolean algebra modulo the ideal of finite sets is a proper poset. A family of reals is *piecewise proper* if it is the union of a chain of proper families of size $\leq \omega_1$.

Here, I investigate the question of the existence of proper and piecewise proper families of reals of different cardinalities. I show that it is consistent relative to ZFC to have continuum many proper families of cardinality ω_1 and continuum many piecewise proper families of cardinality ω_2.

1. Introduction

One of the central concepts in the field of models of Peano Arithmetic is the *standard system* of a model of PA. The *standard system* of a model of PA is the collection of subsets of the natural numbers that arise as intersections of the parametrically definable sets of the model with its *standard part* $\mathbb{N}$. The notion of a *Scott set* captures three key properties of standard systems.

Definition 1.1. A non-empty $\mathfrak{X} \subseteq \mathcal{P}(\mathbb{N})$ is a *Scott set* if

(1) $\mathfrak{X}$ is a Boolean algebra of sets.
(2) If $A \in \mathfrak{X}$ and B is Turing computable from A, then $B \in \mathfrak{X}$.
(3) If T is an infinite binary tree coded by a set in $\mathfrak{X}$, then $\mathfrak{X}$ has a set coding some path through T.

In 1962, Scott observed that every standard system is a Scott set and, remarkably, proved the partial converse that every *countable* Scott set is the standard system of a model of PA [Sco62]. The question of whether *every* Scott set is the standard system of a model of PA became known in the folklore as Scott's Problem. In 1982, Knight and Nadel showed that every Scott set of size ω_1 is the standard system of a model of PA [KN82]. It has proved very difficult to make further progress on Scott's Problem. Schmerl, for example, showed that the arithmetic closure of a collection of mutually Cohen generic sets (in the sense of forcing over models of PA, see [KS06]) is the standard system of a model of PA [Sch73]. My approach, following Engström [Eng04] and suggested several years earlier by Hamkins, Marker, etc., has been to use the set theoretic techniques of forcing and forcing axioms.

We can associate with every family of reals $\mathfrak{X}$, the poset $\mathfrak{X}/\text{Fin}$, which consists of the infinite sets in $\mathfrak{X}$ under the ordering of *almost inclusion*[1]. A familiar and thoroughly studied instance of this poset is $\mathcal{P}(\mathbb{N})/\text{Fin}$. Engström, in [Eng04], introduced the use of the posets $\mathfrak{X}/\text{Fin}$ for arithmetically closed families $\mathfrak{X}$ in connection with Scott's Problem. A family of reals is *arithmetically closed* if whenever A is in

[1] Using the ordering given by inclusion yields a forcing equivalent poset but the almost inclusion ordering makes the properties of the generic extension more transparent.

it and B is arithmetically definable from A, then B is also in it. The arithmetic closure of $\mathfrak{X}$ is an essential ingredient in the constructions that make the posets $\mathfrak{X}/\text{Fin}$ useful in investigating properties of uncountable models of PA (for details of the constructions, see [Git08]). A family of reals $\mathfrak{X}$ is *proper* if it is arithmetically closed and the poset $\mathfrak{X}/\text{Fin}$ is proper[2]. A family of reals is piecewise proper[3] if it is the union of a chain of proper families each of size $\leq \omega_1$. I showed in [Git08] that:

Theorem 1.2. *Assuming the Proper Forcing Axiom* (PFA)*, every proper or piecewise proper family of reals is the standard system of a model of* PA*.*

The artificial seeming restriction that piecewise proper families have to be chains of proper families of size *no larger* than ω_1 is motivated by the fact this reduces the assumption of PFA to PFA$^-$ in Theorem 1.2 for piecewise proper families of size ω_2. Unlike PFA, the axiom PFA$^-$ has no large cardinals strength. I will give an extended discussion of properness in Section 2. It is easy to see that every countable arithmetically closed family of reals is proper and $\mathcal{P}(\mathbb{N})$ is proper[4] as well (see Section 3). Every arithmetically closed family of size $\leq \omega_1$ is trivially piecewise proper since it is the union of a chain of countable arithmetically closed families. It becomes much more difficult to find instances of uncountable proper families of reals other than $\mathcal{P}(\mathbb{N})$. Also, it was not clear for a while whether there are piecewise proper families of size larger than ω_1. In Section 3, I will show:

Theorem 1.3. *If* CH *holds, then* $\mathcal{P}^V(\mathbb{N})/\text{Fin}$ *remains proper in any generic extension by a ccc poset.*

Theorem 1.4. *There is a generic extension of V that contains continuum many proper families of reals of size ω_1.*

Theorem 1.5. *There is a generic extension of V that contains continuum many piecewise proper families of reals of size ω_2.*

2. Proper Posets and the PFA

Proper posets were invented by Shelah, who sought a class of ω_1 preserving posets that would extend the ccc and countably closed classes of posets and be preserved under iterations with countable support [She98]. The Proper Forcing Axiom (PFA) was introduced by Baumgartner, who showed that it is consistent relative to the existence of a supercompact cardinal [Bau84]. This remains the only known proof of the consistency of PFA and at the least PFA implies that there is an inner model with infinitely many Woodin cardinals [Ste05]. Remarkably, PFA decides the size of the continuum. It was shown independently by Velićković [Vel92] and Todorćević [Bek91] that under PFA, the continuum is ω_2.

Recall that for a cardinal λ, the set H_λ is the collection of all sets whose transitive closure has size less than λ. Let $\mathbb{P}$ be a poset and λ be a cardinal greater than $2^{|\mathbb{P}|}$. Since we can always take an isomorphic copy of $\mathbb{P}$ on the cardinal $|\mathbb{P}|$, we can assume

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[2]Note that in [Git08] I did not include arithmetic closure in the definition of a proper family of reals. There, a family of reals was called proper whenever the quotient Boolean algebra modulo the ideal of finite sets was proper. If the family was additionally arithmetically closed, it was called *A-proper*. Since the current paper deals only with arithmetically closed families it made practical sense to include arithmetic closure in the definition.

[3]In [Git08] such families were called *A-piecewise proper*. See note above.

[4]Throughout the paper, I equate *reals* with subsets of $\mathbb{N}$.

without loss of generality that $\mathbb{P}$ and $\mathcal{P}(\mathbb{P})$ are elements of H_λ. In particular, all dense subsets of $\mathbb{P}$ are in H_λ. Let M be a countable elementary submodel of H_λ containing $\mathbb{P}$ as an element. If G is a filter on $\mathbb{P}$, we say that G is *M-generic* if for every maximal antichain $A \in M$ of $\mathbb{P}$, the intersection $G \cap A \cap M \neq \emptyset$. It must be explicitly specified what M-generic means in this context since the usual notion of generic filters makes sense only for transitive structures and M is not necessarily transitive. The definition of M-generic is closely related to the definition for transitive structures. To see this, let M^* be the Mostowski collapse of M and $\mathbb{P}^*$ be the image of $\mathbb{P}$ under the collapse. Let $G^* \subseteq \mathbb{P}^*$ be the pointwise image of $G \cap M$ under the collapse. Then G is M-generic if and only if G^* is M^*-generic for $\mathbb{P}^*$ in the usual sense.

Later we will need the following important characterization of M-generic filters.

Theorem 2.1. *If $\mathbb{P}$ is a poset in $M \prec H_\lambda$, then a V-generic filter $G \subseteq \mathbb{P}$ is M-generic if and only if $M \cap \mathrm{Ord} = M[G] \cap \mathrm{Ord}$.* (See [She98], p. 105)

Definition 2.2. Let $\mathbb{P} \in H_\lambda$ be a poset and M be an elementary submodel of H_λ containing $\mathbb{P}$. Then a condition $q \in \mathbb{P}$ is *M-generic* if and only if q forces that the generic filter is M-generic.

Definition 2.3. A poset $\mathbb{P}$ is *proper* if for every $\lambda > 2^{|\mathbb{P}|}$ and every countable $M \prec H_\lambda$ containing $\mathbb{P}$, for every $p \in \mathbb{P} \cap M$, there is an M-generic condition below p.

When proving that a poset is proper it is often easier to use the following equivalent characterization that appears in [She98] (p. 102).

Theorem 2.4. *A poset $\mathbb{P}$ is proper if there exists a $\lambda > 2^{|\mathbb{P}|}$ and a club of countable $M \prec H_\lambda$ containing $\mathbb{P}$, such that for every $p \in \mathbb{P} \cap M$, there is an M-generic condition below p.*

Definition 2.5. *The Proper Forcing Axiom* (PFA) is the assertion that for every proper poset $\mathbb{P}$ and every collection $\mathcal{D}$ of at most ω_1 many dense subsets of $\mathbb{P}$, there is a filter on $\mathbb{P}$ that meets all of them.

Definition 2.6. PFA^- is the assertion that for every proper poset $\mathbb{P}$ and every collection $\mathcal{D}$ of at most ω_1 many dense subsets of $\mathbb{P}$ of *size* ω_1, there is a filter on $\mathbb{P}$ that meets all of them.

Unlike PFA, the axiom PFA^- is equiconsistent with ZFC [She98] (p. 122).

3. Proper and Piecewise Proper Families

Let $\mathfrak{X}$ be a family of reals. Recall that the poset $\mathfrak{X}/\mathrm{Fin}$ consist of the infinite sets in $\mathfrak{X}$ ordered by almost inclusion. That is, for infinite A and B in $\mathfrak{X}$, we say that $A \leq B$ if and only if $A \subseteq_{\mathrm{Fin}} B$. For a property of posets $\mathscr{P}$, if $\mathfrak{X}$ is an arithmetically closed family of reals and $\mathfrak{X}/\mathrm{Fin}$ has $\mathscr{P}$, I will simply say that $\mathfrak{X}$ *has property $\mathscr{P}$*. Whenever a poset $\mathfrak{X}/\mathrm{Fin}$ is considered, unless stated otherwise, we will be assuming that $\mathfrak{X}$ *arithmetically closed*. For general interest, if the arithmetic closure is not necessary for the arguments, this will be pointed out. Recall that this is motivated by the requirement of arithmetic closure of $\mathfrak{X}$ in the constructions with models of PA in which $\mathfrak{X}/\mathrm{Fin}$ is used (see [Git08] for details).

The easiest way to show that a poset is proper is to show that it is ccc or countably closed. Thus, every countable arithmetically closed family is proper since it is ccc and $\mathcal{P}(\mathbb{N})$ is proper since it is countably closed. Unfortunately these two conditions do not give us any other instances of proper families.

Theorem 3.1. *Every ccc family of reals is countable.*

Proof. Let $\mathfrak{X}$ be an arithmetically closed family of reals. If x is a finite subset of $\mathbb{N}$, let $\ulcorner x \urcorner$ denote the code of x using Gödel's coding. For every $A \in \mathfrak{X}$, define an associated $A' = \{\ulcorner A \cap n \urcorner \mid n \in \mathbb{N}\}$. Clearly A' is definable from A[5], and hence in $\mathfrak{X}$. Observe that if $A \neq B$, then $|A' \cap B'| < \omega$. Hence if $A \neq B$, we get that A' and B' are incompatible in $\mathfrak{X}/\mathrm{Fin}$. It follows that $\mathscr{A} = \{A' \mid A \in \mathfrak{X}\}$ is an antichain of $\mathfrak{X}/\mathrm{Fin}$ of size $|\mathfrak{X}|$. This shows that $\mathfrak{X}/\mathrm{Fin}$ always has antichains as large as the whole poset. $\square$

Thus, the poset $\mathfrak{X}/\mathrm{Fin}$ has the worst possible chain condition, namely $|\mathfrak{X}|^+$-c.c..

Theorem 3.2. $\mathcal{P}(\mathbb{N})$ *is the only countably closed family of reals.*

Proof. Suppose $\mathfrak{X}$ is a countably closed family of reals. I will show that every $A \subseteq \mathbb{N}$ is in $\mathfrak{X}$. Let χ_A be the characteristic function of A. Define a sequence $\langle B_n \mid n \in \omega\rangle$ of subsets of $\mathbb{N}$ by $B_n = \{m \in \mathbb{N} \mid$ the n^{th} digit in the binary expansion of m is $\chi_A(n)\}$. Let $A_m = \cap_{n \leq m} B_n$ and observe that each A_m is infinite. Thus, we get a descending sequence $A_0 \geq A_1 \geq \cdots \geq A_m \geq \ldots$ of elements of $\mathfrak{X}/\mathrm{Fin}$. By countable closure, there exists $C \in \mathfrak{X}/\mathrm{Fin}$ such that $C \subseteq_{\mathrm{Fin}} A_m$ for all $m \in \mathbb{N}$, and hence $C \subseteq_{\mathrm{Fin}} B_n$ for all $n \in \mathbb{N}$. It follows that $A = \{n \in \mathbb{N} \mid \exists m \forall k \in C$ if $k > m$, then the n^{th} digit in the binary expansion of k is $1\}$. This shows that A is arithmetic in C, and hence $A \in \mathfrak{X}$ by arithmetic closure[6]. Since A was arbitrary, this concludes the proof that $\mathfrak{X} = \mathcal{P}(\mathbb{N})$. $\square$

Enayat showed in [Ena06] that ZFC proves the existence of an arithmetically closed family of size ω_1 which collapses ω_1, and hence is *not* proper. Enayat (in [Ena06]) asked whether every Borel family of reals is proper. This was recently answered negatively by Enayat and Shelah, who showed in [SE07] that there is a Borel arithmetically closed family that is *not* proper.

I will show below that it is consistent with ZFC that there are continuum many proper families of size ω_1 (Theorem 3.12) and it is consistent with ZFC that there are continuum many piecewise proper families of size ω_2 (Theorem 3.15). But first I will consider the question of when does forcing to add new reals preserve the properness of the reals of the ground model. I will show that if CH holds, forcing with a ccc poset preserves the properness of the reals of the ground model (Theorem 3.5).

Lemma 3.3 below is a standard set theoretic fact but we include the proof for completeness.

Lemma 3.3. *Suppose* $\mathbb{P}$ *is a ccc poset and* $G \subseteq \mathbb{Q}$ *is* V*-generic for a countably closed poset* $\mathbb{Q}$*. Then* $\mathbb{P}$ *remains ccc in* $V[G]$*.*

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[5]Full arithmetic closure is not needed here, it suffices to assume only that $\mathfrak{X}$ is closed under computability.

[6]The assumption that $\mathfrak{X}$ is arithmetically closed is not necessary here. It is an easy observation that any family of reals $\mathfrak{X}$ such that $\mathfrak{X}/\mathrm{Fin}$ is countably closed must already be arithmetically closed (see [Git08]).

Proof. Suppose $\mathbb{P}$ does not remain ccc in $V[G]$. Fix a $\mathbb{Q}$-name $\dot{A}$ and $r \in \mathbb{Q}$ such that $r \Vdash$ "$\dot{A}$ is a maximal antichain of $\check{\mathbb{P}}$ of size ω_1". Choose $q_0 \leq r$ and $a_0 \in \mathbb{P}$ such that $q_0 \Vdash \check{a_0} \in \dot{A}$. Suppose that we have defined $q_0 \geq q_1 \geq \cdots \geq q_\xi \geq \cdots$ for $\xi < \beta$ where β is some countable ordinal, together with a corresponding sequence $\langle a_\xi \mid \xi < \beta \rangle$ of elements of $\mathbb{P}$ such that $q_\xi \Vdash \check{a}_\xi \in \dot{A}$ and $a_{\xi_1} \neq a_{\xi_2}$ for all $\xi_1 < \xi_2$. By countable closure of $\mathbb{Q}$, we can find $q \in \mathbb{Q}$ such that $q \leq q_\xi$ for all $\xi < \beta$. Let $q_\beta \leq q$ and $a_\beta \in \mathbb{P}$ such that $q_\beta \Vdash \check{a}_\beta \in \dot{A}$ and $a_\beta \neq a_\xi$ for all $\xi < \beta$. Such a_β must exist since we assumed $r \Vdash$ "$\dot{A}$ is a maximal antichain of $\check{\mathbb{Q}}$ of size ω_1" and $q \leq r$. Thus, we can build a descending sequence $\langle q_\xi \mid \xi < \omega_1 \rangle$ of elements of $\mathbb{Q}$ and a corresponding sequence $\langle a_\xi \mid \xi < \omega_1 \rangle$ of elements of $\mathbb{P}$ such that $q_\xi \Vdash \check{a}_\xi \in \dot{A}$. But clearly $\langle a_\xi \mid \xi < \omega_1 \rangle$ is an antichain in V of size ω_1, which contradicts the assumption that $\mathbb{P}$ was ccc. $\square$

Lemma 3.4. *Let $\mathfrak{X}_0 \subseteq \mathfrak{X}_1 \subseteq \cdots \subseteq \mathfrak{X}_\xi \subseteq \cdots$ for $\xi < \omega_1$ be a continuous chain of countable families of reals and let $\mathfrak{X} = \cup_{\xi<\omega_1} \mathfrak{X}_\xi$. If M is a countable elementary substructure of some H_λ and $\langle \mathfrak{X}_\xi \mid \xi < \omega_1 \rangle \in M$, then $M \cap \mathfrak{X} = \mathfrak{X}_\alpha$ where $\alpha = \mathrm{Ord}^M \cap \omega_1$.*

Proof. Let $\alpha = \mathrm{Ord}^M \cap \omega_1$. Suppose $\xi \in \alpha$, then $\xi \in M$, and hence $\mathfrak{X}_\xi \in M$. Since $\mathfrak{X}_\xi$ is countable, it follows that $\mathfrak{X}_\xi \subseteq M$. Thus, $\mathfrak{X}_\alpha \subseteq \mathfrak{X} \cap M$. Now suppose $A \in \mathfrak{X} \cap M$, then the least ξ such that $A \in \mathfrak{X}_\xi$ is definable in H_λ. It follows that $\xi \in M$, and hence $\xi \in \alpha$. Thus, $\mathfrak{X} \cap M \subseteq \mathfrak{X}_\alpha$. $\square$

Lemma 3.4 will play a crucial role in the arguments below.

Theorem 3.5. *If* CH *holds, then $\mathcal{P}^V(\mathbb{N})/\mathrm{Fin}$ remains proper in any generic extension by a ccc poset.*

Proof. Let $\mathbb{P}$ be a ccc poset and fix a V-generic $g \subseteq \mathbb{P}$. In V, let $\mathcal{P}(\mathbb{N}) = \cup_{\xi<\omega_1} \mathfrak{X}_\xi$ where each $\mathfrak{X}_\xi$ is countable and $\mathfrak{X}_0 \subseteq \mathfrak{X}_1 \subseteq \cdots \subseteq \mathfrak{X}_\xi \subseteq \cdots$ is a continuous chain. For sufficiently large cardinals λ, it is easy to see that $H_\lambda^{V[g]} = H_\lambda[g]$. The countable elementary substructures of $H_\lambda[g]$ of the form $M[g]$ where $M \subseteq V$ and $\langle \mathfrak{X}_\xi \mid \xi < \omega_1 \rangle, \mathbb{P} \in M[g]$ form a club. So by Theorem 2.4, it suffices to find generic conditions only for these elementary substructures. Fix a countable $M[g] \prec H_\lambda[g]$ in $V[g]$ such that $\langle \mathfrak{X}_\xi \mid \xi < \omega_1 \rangle, \mathbb{P} \in M[g]$ and $M \subseteq V$. We need to prove that for every $B \in M[g] \cap \mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$, there exists $A \in \mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$ such that $A \subseteq_{\mathrm{Fin}} B$ and A is $M[g]$-generic in $V[g]$. By Lemma 3.4, $M[g] \cap \mathcal{P}^V(\mathbb{N}) = \mathfrak{X}_\alpha$ where $\alpha = \mathrm{Ord}^M \cap \omega_1$. Let $\mathcal{D} = \{\mathscr{D} \cap \mathfrak{X}_\alpha \mid \mathscr{D} \in M \text{ and } \mathscr{D} \text{ dense in } \mathcal{P}^V(\mathbb{N})/\mathrm{Fin}\}$. Observe that $\mathcal{D} \subseteq V$ and $|\mathcal{D}| = \omega$. Since $\mathbb{P}$ is ccc, we can show that there is $\mathcal{D}' \supseteq \mathcal{D}$ of size ω in V. In V, use $\mathcal{D}'$ and $\mathfrak{X}_\alpha$ to define $\mathcal{E} = \{\mathscr{D} \in \mathcal{D}' \mid \mathscr{D} \text{ dense in } \mathfrak{X}_\alpha\}$. It is clear that $\mathcal{D} \subseteq \mathcal{E}$. By the countable closure of $\mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$ in V, we can find an infinite $A \subseteq_{\mathrm{Fin}} B$ such that every $\mathscr{D} \in \mathcal{E}$ contains some C above A. It follows that A is M-generic. In fact, I will show that A is $M[g]$-generic. To verify this, we need to check that whenever $A \in G$ and $G \subseteq \mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$ is $V[g]$-generic, then $M[g][G] \cap \mathrm{Ord} = M[g] \cap \mathrm{Ord}$. Since we are forcing with $\mathbb{P} \times \mathcal{P}(\mathbb{N})/\mathrm{Fin}$, we have $M[g][G] = M[G][g]$. It is clear that $M \cap \mathrm{Ord} = M[g] \cap \mathrm{Ord}$, and so it remains to show that $M[G][g] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$. Since $A \in G$ and A is M-generic, we have that $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$. The poset $\mathbb{P}$ remains ccc in $V[G]$, by Lemma 3.3, since $\mathcal{P}(\mathbb{N})^V/\mathrm{Fin}$ is countably closed. Also we have $M[G] \prec H_\lambda[G]$, even though $M[G]$ itself may not be an element of $V[G]$. Let $\mathscr{A}$ be an antichain of $\mathbb{P}$ in $M[G]$, then $\mathscr{A} \in H_\lambda[G]$, and hence $\mathscr{A}$ has size ω. It follows that $\mathscr{A} \subseteq M[G]$. Since g is $V[G]$-generic, it must meet $\mathscr{A}$. So g is $M[G]$-generic, and hence $M[G][g] \cap \mathrm{Ord} = M[G] \cap \mathrm{Ord}$. $\square$

The consistency of having an uncountable proper family of reals other than $\mathcal{P}(\mathbb{N})$ follows immediately. For example, start in L and force to add a Cohen real. In the resulting generic extension, the reals of L will be an uncountable proper family. Contrast this with the result that there are no uncountable ccc families of reals and $\mathcal{P}(\mathbb{N})$ is the only countably closed family.

Next, I will show how to force the existence of continuum many proper families of reals. It is helpful to begin by looking at what properness translates into in the specific context of posets of the form $\mathfrak{X}/\mathrm{Fin}$.

Proposition 3.6. *Suppose $\mathfrak{X}$ is a family of reals and $\mathscr{A}$ is a countable antichain of $\mathfrak{X}/\mathrm{Fin}$. Then for $B \in \mathfrak{X}$, TFAE:*

1. *Every V-generic filter $G \subseteq \mathfrak{X}/\mathrm{Fin}$ containing B meets $\mathscr{A}$.*
2. *There exists a finite list $A_0, \ldots, A_n \in \mathscr{A}$ such that $B \subseteq_{\mathrm{Fin}} A_0 \cup \ldots \cup A_n$.*

Proof.
(2)$\Longrightarrow$(1): Suppose $B \subseteq_{\mathrm{Fin}} A_0 \cup \ldots \cup A_n$ for some $A_0, \ldots, A_n \in \mathscr{A}$. Since a V-generic filter G is an ultrafilter, one of the A_i must be in G.
(1)$\Longrightarrow$(2): Assume that every V-generic filter G containing B meets $\mathscr{A}$ and suppose toward a contradiction that (2) does not hold. Enumerate $\mathscr{A} = \{A_0, A_1, \ldots, A_n, \ldots\}$. It follows that for all $n \in \mathbb{N}$, the intersection $B \cap (\mathbb{N} - A_0) \cap \cdots \cap (\mathbb{N} - A_n)$ is infinite. Define $C = \{c_n \mid n \in \mathbb{N}\}$ such that c_0 is the least element of $B \cap (\mathbb{N} - A_0)$ and c_{n+1} is the least element of $B \cap (\mathbb{N} - A_0) \cap \cdots \cap (\mathbb{N} - A_{n+1})$ greater than c_n. Clearly $C \subseteq B$ and $C \subseteq_{\mathrm{Fin}} (\mathbb{N} - A_n)$ for all $n \in \mathbb{N}$. Let G be a V-generic filter containing C, then $B \in G$ and $(\mathbb{N} - A_n) \in G$ for all $n \in \mathbb{N}$. But this contradicts our assumption that G meets $\mathscr{A}$. $\square$

It immediately follows that:

Corollary 3.7. *A family of reals $\mathfrak{X}$ is proper if and only if there exists $\lambda > 2^{|\mathfrak{X}|}$ and a club of $M \prec H_\lambda$ containing $\mathfrak{X}$ and having the property that below every $C \in M \cap \mathfrak{X}/\mathrm{Fin}$, there is $B \in \mathfrak{X}/\mathrm{Fin}$ such that for every maximal antichain $\mathscr{A}$ of $\mathfrak{X}/\mathrm{Fin}$ in M, there are $A_0, \ldots, A_n \in \mathscr{A} \cap M$ containing B.*

The next definition is key to all the remaining arguments in the paper.

Definition 3.8. Let $\mathfrak{X}$ be a countable family of reals, let $\mathcal{D}$ be some collection of dense subsets of $\mathfrak{X}/\mathrm{Fin}$, and let $B \in \mathfrak{X}$. We say that an infinite set $A \subseteq \mathbb{N}$ is *$\langle \mathfrak{X}, \mathcal{D} \rangle$-generic below B* if $A \subseteq_{\mathrm{Fin}} B$ and for every $\mathscr{D} \in \mathcal{D}$, there is $C \in \mathscr{D}$ such that $A \subseteq_{\mathrm{Fin}} C$.

Here one should think in the context of having a countable $M \prec H_\lambda$ and a large family of reals $\mathfrak{Y} \in M$ with $\mathfrak{X} = \mathfrak{Y} \cap M$ and $\mathcal{D} = \{\mathscr{D} \cap M \mid \mathscr{D} \in M \text{ and } \mathscr{D} \text{ dense in } \mathfrak{Y}/\mathrm{Fin}\}$. The set A is thought of as coming from the large family $\mathfrak{Y}$ and the requirement for A to be $\langle \mathfrak{X}, \mathcal{D} \rangle$-generic is a generalization of the requirement to be M-generic.

Lemma 3.9. *Let $\mathfrak{X}$ be a countable family of reals and $\mathcal{D}$ be the collection of dense subsets of $\mathfrak{X}/\mathrm{Fin}$. If $B \in \mathfrak{X}/\mathrm{Fin}$ and $G \subseteq \mathfrak{X}/\mathrm{Fin}$ is a V-generic filter containing B, then in $V[G]$, there exists $A \subseteq \mathbb{N}$ that is $\langle \mathfrak{X}, \mathcal{D} \rangle$-generic below B.*

Proof. Since G is countable and directed in $V[G]$, there exists an infinite $A \subseteq \mathbb{N}$ such that $A \subseteq_{\mathrm{Fin}} C$ for all $C \in G$. It is clear that this A works. $\square$

Lemma 3.10. *Let $\mathfrak{X}_0 \subseteq \mathfrak{X}_1 \subseteq \cdots \subseteq \mathfrak{X}_\xi \subseteq \cdots$ for $\xi < \omega_1$ be a continuous chain of countable families of reals and let $\mathfrak{X} = \cup_{\xi<\omega_1} \mathfrak{X}_\xi$. Assume that for every $\xi < \omega_1$, if $B \in \mathfrak{X}_\xi$ and $\mathcal{D}$ is a countable collection of dense subsets of $\mathfrak{X}_\xi$, there is $A \in \mathfrak{X}/\mathrm{Fin}$ that is $\langle \mathfrak{X}_\xi, \mathcal{D}\rangle$-generic below B. Then $\mathfrak{X}$ is proper.*

Proof. Fix a countable $M \prec H_\lambda$ such that $\langle \mathfrak{X}_\xi \mid \xi < \omega_1\rangle \in M$. It suffices to show that generic conditions exist for such M since these form a club. By Lemma 3.4, $\mathfrak{X} \cap M = \mathfrak{X}_\alpha$ where $\alpha = \mathrm{Ord}^M \cap \omega_1$. Fix $B \in \mathfrak{X}_\alpha$ and let $\mathcal{D} = \{\mathscr{D} \cap M \mid \mathscr{D} \in M \text{ and } \mathscr{D} \text{ dense in } \mathfrak{X}/\mathrm{Fin}\}$. By hypothesis, there is $A \in \mathfrak{X}/\mathrm{Fin}$ that is $\langle \mathfrak{X}_\alpha, \mathcal{D}\rangle$-generic below B. Clearly A is M-generic. Thus, we were able to find an M-generic element below every $B \in M \cap \mathfrak{X}/\mathrm{Fin}$. $\square$

We are finally ready to show how to force to add a proper family of reals of size ω_1.

Theorem 3.11. *For every cardinal κ, there is a generic extension of V that contains a proper family of reals of size ω_1 different from $\mathcal{P}(\mathbb{N})$ and satisfies $\mathfrak{c} = \kappa$.*

Proof. First, I will describe a forcing to add a proper family of reals of size ω_1. The conclusion of the theorem will follow by combining this forcing with forcings to modify the size of the continuum.

The forcing to add a proper family of reals will be a finite support iteration $\mathbb{P}$ of length ω_1 of ccc posets, and hence ccc (see [Jec03], p. 271). The iteration $\mathbb{P}$ will add, step-by-step, a continuous chain $\mathfrak{X}_0 \subseteq \mathfrak{X}_1 \subseteq \cdots \subseteq \mathfrak{X}_\xi \subseteq \cdots$ for $\xi < \omega_1$ of countable arithmetically closed families such that $\cup_{\xi<\omega_1}\mathfrak{X}_\xi$ will have the property of Lemma 3.10. The idea will be to obtain generic elements for $\mathfrak{X}_\xi$, as in Lemma 3.9, by adding generic filters. Once $\mathfrak{X}_\xi$ has been constructed, I will force with $\mathfrak{X}_\xi/\mathrm{Fin}$ below every one of its elements cofinally often before the iteration is over. If B is an element of $\mathfrak{X}_\xi$, let $\mathfrak{X}_\xi^B/\mathrm{Fin}$ be the poset $\mathfrak{X}_\xi/\mathrm{Fin}$ below B. At each stage δ, I will force with the finite support product $\prod_{B\in\mathfrak{X}_\xi/\mathrm{Fin}} \mathfrak{X}_\xi^B/\mathrm{Fin}$ for some $\xi < \delta$. This will obtain generic elements for a collection of dense sets of $\mathfrak{X}_\xi/\mathrm{Fin}$ and these elements will get added to $\mathfrak{X}_{\delta+1}$. It will follow that the union of the $\mathfrak{X}_\xi$ has the property of Lemma 3.10.

Fix a bookkeeping function f mapping ω_1 onto ω_1 such that each ordinal appears cofinally often in the range. Let $\mathfrak{X}_0$ be any countable arithmetically closed family of reals. Each subsequent $\mathfrak{X}_\xi$ will be created in $V^{\mathbb{P}_\xi}$. For the length of the iteration, define $\dot{\mathbb{Q}}_\beta = \prod_{B\in\mathfrak{X}_\xi/\mathrm{Fin}} \mathfrak{X}_\xi^B/\mathrm{Fin}$ where $f(\beta) = \xi$. If $\delta = \beta + 1$, then $\mathbb{P}_\delta = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$. In $V[G_\delta] = V[G_\beta][H]$, we have added generic elements A^B for every $B \in \mathfrak{X}_\xi$. So let $\mathfrak{X}_{\beta+1}$ be the arithmetic closure of $\mathfrak{X}_\beta$ and all A^B. If λ is a limit, in $V[G_\lambda]$, define $\mathfrak{X}_\lambda = \cup_{\xi<\lambda}\mathfrak{X}_\xi$. At limits, use finite support.

Let $\mathfrak{X} = \cup_{\xi<\omega_1}\mathfrak{X}_\xi$. We will verify that $\mathfrak{X}$ satisfies the hypothesis of Lemma 3.10 in $V[G]$. Fix $\mathfrak{X}_\xi$, a set $B \in X_\xi$, and a countable collection $\mathcal{D}$ of dense subsets of $\mathfrak{X}_\xi/\mathrm{Fin}$. Since the poset $\mathbb{P}$ is a finite support ccc iteration and all elements of $\mathcal{D}$ are countable, they must appear at some stage α below ω_1. Since we forced with $\mathfrak{X}_\xi^B/\mathrm{Fin}$ cofinally often, we have added a $\langle \mathfrak{X}_\xi, \mathcal{D}\rangle$-generic condition below B at some stage above α.

Observe that a standard nice name counting argument shows $\mathfrak{c}^V = \mathfrak{c}^{V[G]}$ after forcing with $\mathbb{P}$.

Now, suppose that κ is a cardinal in V. If $\kappa > \mathfrak{c}$, let $\mathbb{Q}$ be the forcing to add κ many Cohen reals. If $\kappa < \mathfrak{c}$, let $\mathbb{Q}$ be the trivial forcing. Finally, if $\mathfrak{c} = \omega_1 = \kappa$, let $\mathbb{Q}$ be the forcing to add ω_2 many Cohen reals. Force with $\mathbb{Q}$ to obtain a generic extension $V[H]$. Note that in $V[H]$, $\neg$CH holds, κ is a cardinal, and $\mathfrak{c} \geq \kappa$. Next, we force with $\mathbb{P}$. Let $V[H][G]$ be the new generic extension and $\mathfrak{X} \in V[H][G]$ be the proper family of reals of size ω_1 added by $\mathbb{P}$. Recall that forcing with $\mathbb{P}$ preserves all cardinals and does not change the size of the continuum. It follows that in $V[H][G]$, $\neg$CH holds, κ is a cardinal, and $\mathfrak{c} \geq \kappa$. For the trivial reason that $\neg$CH holds, we know that $\mathfrak{X} \neq \mathcal{P}(\mathbb{N})$. Thus, if $\mathfrak{c} = \kappa$, we are done. If not, finally force to collapse continuum to κ. Since this can be done by a countably closed forcing, it will not affect the properness of $\mathfrak{X}$. $\square$

We can push this argument further by forcing to add *continuum* many proper families of reals of size ω_1.

Theorem 3.12. *For every cardinal κ, there is a generic extension of V that contains continuum many proper families of reals of size ω_1 and satisfies $\mathfrak{c} = \kappa$.*

Proof. I will first describe a forcing to add continuum many proper families of reals of size ω_1 under the assumption that the universe satisfies MA + $\neg$CH. This assumption is used to show that the forcing is ccc and therefore preserves all cardinals. The conclusion of the theorem will follow by combining this forcing with forcings that cause MA to hold and forcings that modify the size of the continuum.

The forcing to add continuum many proper families of reals of size ω_1 will be a finite support product $\mathbb{Q} = \Pi_{\xi<2^\omega}\mathbb{P}^\xi$ of posets $\mathbb{P}^\xi$ that add a proper family of reals (Theorem 3.11). Since Martin's Axiom implies that finite support products of ccc posets are ccc (see [Jec03], p. 277), the product poset $\mathbb{Q}$ is ccc. Let $G \subseteq \mathbb{Q}$ be V-generic, then each $G^\xi = G \restriction \mathbb{P}^\xi$ together with $\mathbb{P}^\xi$ can be used to build a proper family $\mathfrak{X}^\xi$ as described in Theorem 3.11. Each such $\mathfrak{X}^\xi$ will be the union of an increasing chain of countable families $\mathfrak{X}^\xi_\gamma$ for $\gamma < \omega_1$. First, I claim that all $\mathfrak{X}^\xi$ are distinct. Fixing $\alpha < \beta$, I will show that $\mathfrak{X}^\alpha \neq \mathfrak{X}^\beta$. Consider $V[G \restriction \beta + 1] = V[G \restriction \beta][G^\beta]$ a generic extension by $(\mathbb{Q} \restriction \beta) \times \mathbb{P}^\beta$. Observe that $\mathfrak{X}^\alpha$ already exists in $V[G \restriction \beta]$. Recall that to build $\mathfrak{X}^\beta$, we start with a countable family $\mathfrak{X}_0$ and, in the first step, force with a finite support product of sub-posets of $\mathfrak{X}_0/\text{Fin}$. In particular, we are forcing with $\mathfrak{X}_0/\text{Fin}$. Let g be the generic filter for $\mathfrak{X}_0/\text{Fin}$ definable from G^β. The next step in constructing $\mathfrak{X}^\beta$ chooses some $A \subseteq \mathbb{N}$ below g to end up in $\mathfrak{X}^\beta_1$. It should be clear that g is definable from A and $\mathfrak{X}_0$. Since g is $V[G \restriction \beta]$-generic, it follows that $g \notin V[G \restriction \beta]$. Thus, $A \notin V[G \restriction \beta]$, and hence $\mathfrak{X}^\beta \neq \mathfrak{X}^\alpha$. It remains to show that each $\mathfrak{X}^\alpha$ is proper in $V[G]$. Fix $\alpha < 2^\omega$ and let $V[G] = V[G \restriction \alpha][G^\alpha][G_{\text{tail}}]$ where G_{tail} is the generic for $\mathbb{Q}$ above α. By the commutativity of products, $V[G \restriction \alpha][G^\alpha][G_{\text{tail}}] = V[G \restriction \alpha][G_{\text{tail}}][G^\alpha]$ and G^α is $V[G \restriction \alpha][G_{\text{tail}}]$-generic. Fix a countable $M \prec H_\lambda^{V[G]}$ containing the sequence $\langle \mathfrak{X}^\alpha_\xi \mid \xi < \omega_1\rangle$ as an element. By Lemma 3.4, $M \cap \mathfrak{X}^\alpha$ is some $\mathfrak{X}^\alpha_\gamma$. This is the key step of the proof since it allows us to know exactly what $M \cap \mathfrak{X}^\alpha$ is, even though we know nothing about M. Let $G^\alpha_\xi = G^\alpha \restriction \mathbb{P}^\alpha_\xi$ for $\xi < \omega_1$. Let $\mathcal{D} = \{\mathscr{D} \cap M \mid \mathscr{D} \in M \text{ and } \mathscr{D} \text{ dense in } \mathfrak{X}^\alpha/\text{Fin}\}$. There must be some $\beta < \omega_1$ such that $\mathcal{D} \in V[G \restriction \alpha][G_{\text{tail}}][G^\alpha_\beta]$. By construction, there was some stage $\delta > \beta$ at which we forced with $\mathfrak{X}^\alpha_\gamma/\text{Fin}$ and added a set A below H where $G^\alpha_{\delta+1} = G^\alpha_\delta * H$.

Now observe that H is $V[G \upharpoonright \alpha][G_{\text{tail}}][G^\alpha_\delta]$-generic for $\mathfrak{X}^\alpha_\gamma/\text{Fin}$. Therefore H meets all the sets in $\mathcal{D}$. So we can conclude that A is M-generic.

A standard nice name counting argument will again show that $\mathfrak{c}^V = \mathfrak{c}^{V[G]}$.

Now, suppose that κ is a cardinal in V. If $\kappa > \mathfrak{c}$, let $\mathbb{R}$ force $\text{MA} + \mathfrak{c} = k$. If $\kappa < \mathfrak{c}$, let $\mathbb{R}$ force $\text{MA} + \kappa < \mathfrak{c}$. Finally, if $\mathfrak{c} = \omega_1 = \kappa$, let $\mathbb{R}$ force $\text{MA} + \mathfrak{c} = \omega_2$. Now proceed as in the proof of Theorem 3.11. $\square$

Similar techniques allow us to force the existence of a piecewise proper family of reals of size ω_2.

Lemma 3.13. *Let $\mathfrak{X}_0 \subseteq \mathfrak{X}_1 \subseteq \cdots \subseteq \mathfrak{X}_\xi \subseteq \cdots$ for $\xi < \omega_1$ be a continuous chain of countable families of reals and let $\mathfrak{X} = \cup_{\xi<\omega_1} \mathfrak{X}_\xi$. Assume that for every $\xi < \omega_1$, if $B \in \mathfrak{X}_\xi$ and $\mathcal{D}$ is a countable collection of dense subsets of $\mathfrak{X}_\xi$, there is $A \in \mathfrak{X}/\text{Fin}$ that is $\langle \mathfrak{X}_\xi, \mathcal{D}\rangle$-generic below B. Then $\mathfrak{X}$ is proper and $\mathfrak{X}$ remains proper after forcing with any absolutely ccc poset.*

Proof. The proof is a straightforward modification of the proof of Theorem 3.5. Let $\mathbb{P}$ be an absolutely ccc poset and $g \subseteq \mathbb{P}$ be V-generic. We need to show that $\mathfrak{X}$ is proper in $V[g]$. Fix a countable $M[g] \prec H_\lambda[g]$ in $V[g]$ such that $\langle \mathfrak{X}_\xi : \xi < \omega_1\rangle, \mathbb{P} \in M[g]$ and $M \subseteq V$. Let $\mathfrak{X}_\alpha = M \cap \mathfrak{X}$ and let $\mathcal{D} = \{\mathscr{D} \cap \mathfrak{X}_\xi \mid \mathscr{D} \in M \text{ and } \mathscr{D} \text{ dense in } \mathfrak{X}\}$. Observe that $\mathcal{D} \subseteq V$ and $|\mathcal{D}| = \omega$. Define $\mathcal{E}$ as in the proof of Theorem 3.5. Now choose $A \in \mathfrak{X}$ that is $\langle \mathfrak{X}_\alpha, \mathcal{E}\rangle$-generic in V. It follows that A is M-generic. Next proceed exactly as in the proof of Theorem 3.5, using the fact that $\mathbb{P}$ is absolutely ccc in the final stage of the argument. $\square$

Theorem 3.14. *There is a generic extension of V containing a piecewise proper family of reals of size ω_2.*

Proof. We will define a ccc forcing iteration $\mathbb{P}_{\omega_2}$ of length ω_2 to accomplish this. Let $\mathbb{P}_0$ be the forcing to add a proper family $\mathfrak{X}_0$ of size ω_1 (Theorem 3.11). At the α^{th}-stage, force with the poset to add a proper family $\mathfrak{X}_\alpha \supseteq \cup_{\beta<\alpha}\mathfrak{X}_\beta$. Observe here, that the poset from Theorem 3.11 can be very easily modified to the poset which adds a proper family extending any family of reals from the ground model. Let $G \subseteq \mathbb{P}_{\omega_2}$ be V-generic. I claim each $\mathfrak{X}_\alpha$ remains proper in $V[G]$. Fix $\mathfrak{X}_\alpha$ and factor the forcing $\mathbb{P}_{\omega_2} = \mathbb{P}_{\alpha+1} * \mathbb{P}_{\text{tail}}$. The family $\mathfrak{X}_\alpha$ is proper in $V[G_{\alpha+1}]$ and $\mathbb{P}_{\text{tail}}$ is absolutely ccc in $V[G_\alpha]$. The poset $\mathbb{P}_{\text{tail}}$ is absolutely c.c.c since the forcing to add a proper family is a finite support iteration of countable posets. By Lemma 3.13, $\mathfrak{X}_\alpha$ remains proper in $V[G_\alpha][G_{\text{tail}}]$. Thus, $\mathfrak{X} = \cup_{\alpha<\omega_2}\mathfrak{X}_\alpha$ is clearly piecewise proper. $\square$

By exactly following the proof of Theorem 3.12, we can extend Theorem 3.14 to obtain:

Theorem 3.15. *There is a generic extension of V containing continuum many piecewise proper families of reals of size ω_2.*

By Enayat's [Ena06] example of a non-proper arithmetically closed family of size ω_1, we know that there are piecewise proper families that are not proper. This follows by recalling that arithmetically closed families of size ω_1 are trivially piecewise proper. It is not clear whether every proper family has to be piecewise proper. In particular, it is not known whether $\mathcal{P}(\mathbb{N})$ is piecewise proper. Observe that the proof of Theorem 3.14 can be easily modified to force $\mathcal{P}(\mathbb{N})$ to be piecewise proper. We just need to make sure that we include every subset of $\mathbb{N}$ in the $\mathfrak{X}$ we are constructing. So at least we know that it is consistent for $\mathcal{P}(\mathbb{N})$ to be piecewise proper.

Finally, I will discuss a possible construction of proper families under PFA. The idea is, in some sense, to mimic the forcing iteration like that of Theorem 3.11 in the ground model. Unfortunately, the main problem with the construction is that it is not clear whether we are getting the whole $\mathcal{P}(\mathbb{N})$. This problem never arose in the forcing construction since we were building families of size ω_1 and knew that the continuum was larger than ω_1. I will describe the construction and a possible way of ensuring that the resulting family is not $\mathcal{P}(\mathbb{N})$.

Fix an enumeration $\{\langle A_\xi, B_\xi\rangle \mid \xi < \omega_2\}$ of $\mathcal{P}(\omega)\times\mathcal{P}(\omega)$. Also fix a bookkeeping function f from ω_2 onto ω_2 such that each element appears cofinally in the range. I will build a family $\mathfrak{X}$ of size ω_2 as the union of an increasing chain of families $\mathfrak{X}_\xi$ of size ω_1 for $\xi < \omega_2$. Start with any family $\mathfrak{X}_0$ of size ω_1. Suppose we have constructed $\mathfrak{X}_\beta$ for $\beta \leq \alpha$ and we need to construct $\mathfrak{X}_{\alpha+1}$. Consult $f(\alpha) = \gamma$ and consider the pair $\langle A_\gamma, B_\gamma\rangle$ in the enumeration of $\mathcal{P}(\omega)\times\mathcal{P}(\omega)$. First, suppose that A_γ codes a countable family $\mathfrak{Y} \subseteq \mathfrak{X}_\alpha$ and B_γ codes a countable collection $\mathcal{D}$ of dense subsets of $\mathfrak{Y}$. Let G be some filter on $\mathfrak{Y}$ meeting all sets in $\mathcal{D}$ and let $A \subseteq_{\mathrm{Fin}} C$ for all $C \in G$. Define $\mathfrak{X}_{\alpha+1}$ to be the arithmetic closure of $\mathfrak{X}_\alpha$ and A. If the pair $\langle A_\gamma, B_\gamma\rangle$ does not code such information, let $\mathfrak{X}_{\alpha+1} = \mathfrak{X}_\alpha$. At limit stages take unions.

I claim that $\mathfrak{X}$ is proper. Fix some countable $M \prec H_\lambda$ containing $\mathfrak{X}$. Let $\mathfrak{Y} = M \cap \mathfrak{X}$ and let $\mathcal{D} = \{\mathscr{D} \cap M \mid \mathscr{D} \in M \text{ and } \mathscr{D} \text{ dense in } \mathfrak{X}/\mathrm{Fin}\}$. There must be some γ such that $\langle A_\gamma, B_\gamma\rangle$ codes $\mathfrak{Y}$ and $\mathcal{D}$. Let δ such that $M\cap\mathfrak{X}$ is contained in $\mathfrak{X}_\delta$, then there must be some $\alpha > \delta$ such that $f(\alpha) = \gamma$. Thus, at stage α in the construction we considered the pair $\langle A_\gamma, B_\gamma\rangle$. Since $\alpha > \delta$, we have $M\cap\mathfrak{X} = M\cap\mathfrak{X}_\alpha$. It follows that at stage α we added an M-generic set A to $\mathfrak{X}$.

A way to prove that $\mathfrak{X} \neq \mathcal{P}(\mathbb{N})$ would be to show that some fixed set C is not in $\mathfrak{X}$. Suppose the following question had a positive answer:

Question 3.16. Let $\mathfrak{X}$ be an arithmetically closed family such that $C \notin \mathfrak{X}$ and $\mathfrak{Y} \subseteq \mathfrak{X}$ be a countable family. Is there a $\mathfrak{Y}/\mathrm{Fin}$-name $\dot{A}$ such that $1_{\mathfrak{Y}/\mathrm{Fin}} \Vdash$ "$\dot{A} \subseteq_{\mathrm{Fin}} B$ for all $B \in \dot{G}$ and $\check{C}$ is not in the arithmetic closure of $\dot{A}$ and $\check{\mathfrak{X}}$"?

Assuming that the answer to Question 3.16 is positive, let us construct a proper family $\mathfrak{X}$ in such a way that C is not in $\mathfrak{X}$. We will carry out the above construction being careful in our choice of the filters G and elements A. Start with $\mathfrak{X}_0$ that does not contain C and assume that $C \notin \mathfrak{X}_\alpha$. Suppose the pair $\langle A_\gamma, B_\gamma\rangle$ considered at stage α codes meaningful information. That is, A_γ codes a countable family $\mathfrak{Y} \subseteq \mathfrak{X}_\alpha$ and B_γ codes a countable collection $\mathcal{D}$ of dense subsets of $\mathfrak{Y}$. Choose some transitive $N \prec H_{\omega_2}$ of size ω_1 such that $\mathfrak{X}_\alpha$, $\mathfrak{Y}$, and $\mathcal{D}$ are elements of N. Since we assumed a positive answer to Question 3.16, H_{ω_2} satisfies that there exists a $\mathfrak{Y}/\mathrm{Fin}$-name $\dot{A}$ such that $1_{\mathfrak{Y}/\mathrm{Fin}} \Vdash$ "$\dot{A} \subseteq_{\mathrm{Fin}} B$ for all $B \in \dot{G}$ and $\check{C}$ is not in the arithmetic closure of $\dot{A}$ and $\check{\mathfrak{X}_\alpha}$". But then N satisfies the same statement by elementarity. Hence there is $\dot{A} \in N$ such that N satisfies $1_{\mathfrak{Y}/\mathrm{Fin}} \Vdash$ "$\dot{A} \subseteq_{\mathrm{Fin}} B$ for all $B \in \dot{G}$ and $\check{C}$ is not in the arithmetic closure of $\dot{A}$ and $\check{\mathfrak{X}_\alpha}$". Now use PFA to find an N-generic filter G for $\mathfrak{Y}/\mathrm{Fin}$. Since G is fully generic for the model N, the model $N[G]$ will satisfy that C is not in the arithmetic closure of $\mathfrak{X}_\alpha$ and $A = \dot{A}_G$. Thus, it is really true that C is not in the arithmetic closure of $\mathfrak{X}_\alpha$ and A. Since G also met all the dense sets in $\mathcal{D}$ and $A \subseteq_{\text{Fin}} B$ for all $B \in G$, we can let $\mathfrak{X}_{\alpha+1}$ be the arithmetic closure of $\mathfrak{X}_\alpha$ and A. Thus, $C \notin \mathfrak{X}_{\alpha+1}$. We can conclude that $C \notin \mathfrak{X}$.

4. Questions

The continuing project of this work will be to determine whether universes of ZFC in general contain many proper and piecewise proper families of reals. That is, it would be ideal to replace the consistency results above with theorems of ZFC.

Question 4.1. Can ZFC or ZFC + PFA prove the existence of an uncountable proper family of reals other than $\mathcal{P}(\mathbb{N})$?

Question 4.2. Can ZFC or ZFC + PFA prove the existence of a piecewise proper family of size ω_2?

Question 4.3. Is it consistent with ZFC that there are proper families of reals of size ω_2 other than $\mathcal{P}(\mathbb{N})$?

Question 4.4. What is the answer to Question 3.16?

Question 4.5. Can $\mathcal{P}(\mathbb{N})$ be non-piecewise proper?

References

[Bau84] J. E. Baumgartner. Applications of the proper forcing axiom. In K. Kunen and J. Vaughan, editors, *Handbook of Set Theoretic Topology*, pages 913–959. North–Holland, Amsterdam, 1984.

[Bek91] M. Bekkali. *Topics in set theory.* Springer–Verlag, New York, 1991.

[Ena06] A. Enayat. Uncountable expansions of the standard model of Peano arithmetic. Preprint, 2006.

[Eng04] F. Engström. *Expansions, omitting types, and standard systems.* PhD thesis, Chalmers University of Technology and Goteborg University, 2004.

[Git08] V. Gitman. Scott's problem for proper Scott sets. *The Journal of Symbolic Logic*, 73(3):845–860, 2008.

[Jec03] T. Jech. *Set Theory.* Springer Monographs in Mathematics. Springer–Verlag, New York, third edition, 2003.

[KN82] J. Knight and M. Nadel. Models of Peano arithmetic and closed ideals. *Journal of Symbolic Logic*, 47(4):833–840, 1982.

[KS06] R. Kossak and J. H. Schmerl. *The Structure of Models of Peano Arithmetic*, volume 50 of *Oxford Logic Guides.* Oxford University Press, New York, 2006.

[Sch73] J. H. Schmerl. Peano models with many generic classes. *Pacific J. Math.*, 46:523–536, 1973.

[Sco62] D. Scott. Algebras of sets binumerable in complete extensions of arithmetic. In *Proc. Sympos. Pure Math. Vol. V*, pages 117–121. American Mathematical Society, Providence, R.I., 1962.

[SE07] S. Shelah and A. Enayat. Improper borel families. 2007. Preprint.

[She98] S. Shelah. *Proper and Improper Forcing.* Perspectives in Mathematical Logic. Springer–Verlag, New York, second edition, 1998.

[Ste05] J. R. Steel. PFA implies $\mathrm{AD}^{L(\mathbb{R})}$. *J. Symbolic Logic*, 70(4):1255–1296, 2005.

[Vel92] B. Veličković. Forcing axioms and stationary sets. *Advances in Mathematics*, 94(2):256–284, 1992.

New York City College of Technology (CUNY), Mathematics, 300 Jay Street, Brooklyn, NY 11201 USA

E-mail address: vgitman@nylogic.org