

CS/ECE/ME532 Assignment 3

1. *Polynomial fitting.* Suppose we observe pairs of points (a_i, b_i) , $i = 1, \dots, m$ representing measurements from a scientific experiment. The variables a_i are the experimental conditions and the b_i correspond to the measured response in each condition. We fit a degree $d < m$ polynomial to these data. In other words, we want to find the coefficients of a degree p polynomial $w(a)$ so that $w(a_i) \approx b_i$ for $i = 1, 2, \dots, m$.
 - a) Suppose $w(a)$ is a degree p polynomial. Write the general expression for $w(a_i) = b_i$.
 - b) Express the $i = 1, \dots, m$ equations as a system in matrix form $\mathbf{A}\mathbf{x} = \mathbf{d}$ while defining \mathbf{A} and \mathbf{d} . What is the form/structure of \mathbf{A} in terms of the given a_i ?
 - c) Write a script to find the least-squares model fit to the $m = 30$ data points in `polydata.mat`. Plot the points and the polynomial fits for $p = 1, 2, 3$.

2. Least Squares Approximation of Matrices.
 - a) Derive the solution to least-squares problem $\min_{\mathbf{w}} \|\mathbf{x} - \mathbf{T}\mathbf{w}\|_2^2$ when \mathbf{T} is an n -by- r matrix of orthonormal columns. Your solution should not involve a matrix inverse.
 - b) Let $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_p]$ be an n -by- p matrix. Use the least-squares problems $\min_{\mathbf{w}_i} \|\mathbf{x}_i - \mathbf{T}\mathbf{w}_i\|_2^2$ to find $\mathbf{W} = [\mathbf{w}_1 \ \mathbf{w}_2 \ \cdots \ \mathbf{w}_p]$ in the approximation $\mathbf{X} \approx \mathbf{T}\mathbf{W}$. Your solution should express \mathbf{W} as a function of \mathbf{T} and \mathbf{X} .

3. We return to the movies rating problem of Activity 5. The ratings on a scale of 1-10 are:

Movie	Jake	Jennifer	Jada	Theo	Ioan	Bo	Juanita
Star Trek	4	7	2	8	7	4	2
Pride and Prejudice	9	3	5	6	10	5	5
The Martian	4	8	3	7	6	4	1
Sense and Sensibility	9	2	6	5	9	5	4
Star Wars: Empire Strikes	4	9	2	8	7	4	1

A matrix \mathbf{X} containing this data is available in the file `movie.mat` and the csv file `movie.csv`. Our goal is to approximate \mathbf{X} using r “tastes”, the columns of \mathbf{T} , that

is, $\mathbf{X} \approx \mathbf{T}\mathbf{W}$ where \mathbf{T} is 5-by- r . You will use a Gram-Schmidt orthogonalization code to find a set of tastes that approximate the ratings. A script that implements Gram-Schmidt orthogonalization is available.

Define a 5-by- r taste matrix $\mathbf{T} = [\mathbf{t}_1 \ \mathbf{t}_2 \ \cdots \ \mathbf{t}_r]$ with orthonormal columns and the r -by-7 weight matrix

$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{17} \\ w_{21} & w_{22} & \cdots & w_{27} \\ & & \vdots & \\ w_{r1} & w_{r2} & \cdots & w_{r7} \end{bmatrix}$$

- a) In Activity 5 you found the baseline (average) rating for each friend by requiring the first basis vector in the taste matrix to be

$$\mathbf{t}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

You have noticed by now that the first vector in the Gram-Schmidt procedure is a scaled version of the first vector of the matrix, so you decide to define an augmented matrix $\tilde{\mathbf{X}} = [\mathbf{1} \ \mathbf{X}]$ where $\mathbf{1}$ is a column vector containing five unity entries. Apply a Gram-Schmidt orthogonalization code to $\tilde{\mathbf{X}}$ to find a set of orthonormal basis vectors. Is the first basis vector you obtain equal to \mathbf{t}_1 ?

- b) Use your solution to the preceding problem in this homework assignment to find the rank-1 approximation of \mathbf{X} using only \mathbf{t}_1 . That is, find \mathbf{W} so that $\mathbf{X} \approx \mathbf{t}_1\mathbf{W}$. Use \mathbf{W} to compute $\mathbf{t}_1\mathbf{W}$. This gives you each friend's baseline ratings. Also compute the residual error $\mathbf{X} - \mathbf{t}_1\mathbf{W}$.
- c) Now find a rank-2 approximation using $\mathbf{T} = [\mathbf{t}_1 \ \mathbf{t}_2]$. That is, find \mathbf{W} so that $\mathbf{X} \approx \mathbf{T}\mathbf{W}$. Use \mathbf{W} to compute $\mathbf{T}\mathbf{W}$. This gives you a rank-2 approximation to the ratings. Also compute the residual error $\mathbf{X} - \mathbf{T}\mathbf{W}$. How does \mathbf{t}_2 relate to the distinction between sci-fi and and romance movie preferences?
- d) Now find a rank-3 approximation using $\mathbf{T} = [\mathbf{t}_1 \ \mathbf{t}_2 \ \mathbf{t}_3]$. That is, find \mathbf{W} so that $\mathbf{X} \approx \mathbf{T}\mathbf{W}$. Use \mathbf{W} to compute $\mathbf{T}\mathbf{W}$. This gives you a rank-3 approximation to the ratings. Also compute the residual error $\mathbf{X} - \mathbf{T}\mathbf{W}$. Qualitatively discuss the effect of increasing the rank of the approximation on the residual error.
- e) Suppose you interchange the order of Jake and Jennifer so that Jennifer's ratings are in the first column of \mathbf{X} and Jake's ratings are in the second column. Does the

rank-2 approximation change? Why or why not? Does the rank-3 approximation change? Why or why not?

4. Let $\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

a) Is $\mathbf{Q} \succ 0$?

b) Sketch the surface $y = \mathbf{x}^T \mathbf{Q} \mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. If you find 3-D sketching too difficult, you may draw a contour map with labeled contours.

5. Suppose $\mathbf{P} \succ 0$ and $\mathbf{Q} \succ 0$ are (symmetric) positive definite $n \times n$ matrices. Prove that $\mathbf{QPQ} \succ 0$.