1) a) $A = \begin{array}{c} \begin{bmatrix} 0.92 & 0.92 \\ -0.92 & 0.92 \\ 0.92 & -0.92 \\ -0.92 & -0.92 \end{bmatrix} \\ \quad a_1 \qquad a_2 \end{array}$

¿ $a_1 = \alpha a_2$? $\Rightarrow$ $0.92 = \alpha(0.92) \Rightarrow \alpha = 1$

$-0.92 = 0.92\alpha \Rightarrow -0.92 = 0.92$ ABS!

$\therefore$ columns of A are linearly independent.

b) $A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \\ a_1 \quad a_2 \quad a_3 \end{array}$

$\alpha a_1 + \beta a_2 + \gamma a_3 = 0$

$$\begin{cases} \alpha + \beta + \gamma = 0 & \text{I} \\ -\alpha + \beta - \gamma = 0 & \text{II} \\ \alpha - \beta - \gamma = 0 & \text{III} \end{cases}$$

I + II $\quad 2\beta = 0 \Rightarrow \beta = 0$ IV

IV in III $\quad \alpha = \gamma$ V

V in I $\quad 2\alpha = 0 \Rightarrow \alpha = 0 \Rightarrow \gamma = 0$

$\therefore$ columns of A are linearly independent.

c) $A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix} \\ a_1 \quad a_2 \quad a_3 \end{array}$

$\alpha a_1 + \beta a_2 + \gamma a_3 = 0$ $\quad \begin{cases} \alpha + 2\beta + 2\gamma = 0 & \text{I} \\ 3\alpha + 4\beta + 5\gamma = 0 & \text{II} \\ 5\alpha + 6\beta + 8\gamma = 0 & \text{III} \end{cases}$

2 I − II $\quad -\alpha - \gamma = 0 \Rightarrow \alpha = -\gamma$ IV

IV in III $\quad -5\gamma + 6\beta + 8\gamma = 0 \Rightarrow 6\beta = -3\gamma \Rightarrow -2\beta = \gamma$ V

IV, V in II $\quad 6\beta + 4\beta - 10\beta = 0$ ✓

$a_1 + \frac{a_2}{2} = a_3 \Rightarrow$ columns of A are not linearly dependent.

d) $A = \begin{array}{c} \begin{bmatrix} 5 & 2 \\ -5 & 2 \\ 5 & -2 \end{bmatrix} \\ a_1 \quad a_2 \end{array}$

$\alpha a_1 + \beta a_2 = 0 \Leftrightarrow \begin{cases} 5\alpha + 2\beta = 0 & \text{I} \\ -5\alpha + 2\beta = 0 & \text{II} \\ 5\alpha - 2\beta = 0 & \text{III} \end{cases}$ $\quad$ I + II $\quad 4\beta = 0 \Rightarrow \beta = 0$ IV

IV in III $\quad -5\alpha = 0 \Rightarrow \alpha = 0$

$\therefore$ columns of A linearly dependent $\Rightarrow$ rank(A) = 2

e) $A^t A w = a$

¿$A^tA > 0$? $\Leftrightarrow$ A^tA is full rank

$A^tA = \begin{bmatrix} 75 & -10 \\ -10 & 12 \end{bmatrix} \Rightarrow$ Rank $(A^tA) = 2 \Rightarrow A^tA$ is full rank $\Rightarrow A^tA > 0$

$\Rightarrow$ As $A^tA > 0 \Rightarrow$ there is a unique solution.

② a) $f(x) = \|x\|_a + \|x\|_b$ is a norm on $\mathbb{R}^n$

Properties:

1) $\|\underline{x}\| \geq 0$ for all $\underline{x}$
2) $\|\underline{x}\| = 0$ if and only if $\underline{x} = \underline{0}$
3) $\|b\underline{x}\| = |b|\|\underline{x}\|$ for all $b \in \mathbb{R}$, $\underline{x} \in \mathbb{R}^n$
4) Triangle inequality $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

1) $f(x) = \underbrace{\|x\|_a}_{\geq 0} + \underbrace{\|x\|_b}_{\geq 0} \geq 0$ ✓

2) If $x = 0 \Rightarrow \|x\|_a = 0$ and $\|x\|_b = 0 \Rightarrow f(x) = 0$ ✓

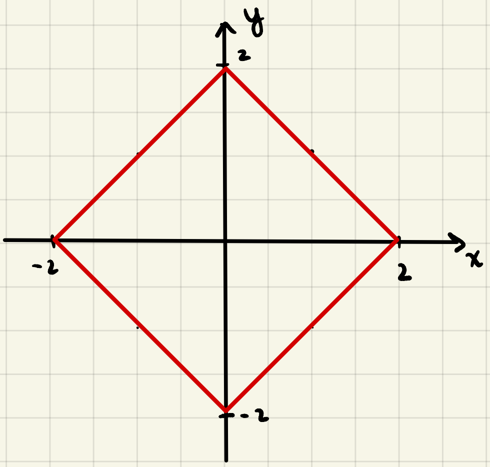
3) $f(bx) = \|bx\|_a + \|bx\|_b = |b|\|x\|_a + |b|\|x\|_b = |b|(\|x\|_a + \|x\|_b)$

4) $f(x+y) = \|x+y\|_a + \|x+y\|_b \leq \|x\|_a + \|y\|_a + \|x\|_b + \|y\|_b = (\|x\|_a + \|x\|_b) + (\|y\|_a + \|y\|_b)$

$\Rightarrow$ by transitivity $\Rightarrow f(x+y) = \|x+y\|_a + \|x+y\|_b \leq (\|x\|_a + \|x\|_b) + (\|y\|_a + \|y\|_b)$

$\therefore$ $f(x) = \|x\|_a + \|x\|_b$ is a norm on $\mathbb{R}^n$

b) $f(x) = \|x\|_1 + \|x\|_\infty$



b) The norm ball in this case is $\|\boldsymbol{x}\|_1 + \|\boldsymbol{x}\|_\infty = 1$. In two-dimensional space with $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ we may rewrite the norm ball condition as

$$|x_1| + |x_2| + \max\{|x_1|, |x_2|\} = 1$$

To understand what this looks like, we will consider special cases involving different relationships between x_1 and x_2.

i. Suppose $x_2 = 0$. Then we have $|x_1| + |x_1| = 1$ or $x_1 = \pm 1/2$.
ii. Suppose $x_1 = 0$. Then we have $|x_2| + |x_2| = 1$ or $x_2 = \pm 1/2$. Thus we know the intersections of the ball with the axes.
iii. Suppose $|x_1| = |x_2|$. Then we have $|x_1| + |x_1| + |x_1| = 1$ which implies $x_1 = \pm 1/3$ and $x_2 = \pm 1/3$.
iv. Suppose $|x_2| < |x_1|$. Then the ball is $|x_1| + |x_2| + |x_1| = 1$ which implies $|x_2| = -2|x_1| + 1$. The sign of the slope and intercept change with the sign of x_1, x_2.
v. The final condition is $|x_2| > |x_1|$. In this case we have $|x_1| + |x_2| + |x_2| = 1$ which implies $|x_2| = -1/2|x_1| + 1$. The sign of the slope and intercept change with the sign of x_1, x_2.

Given these conditions we may graph the norm ball on a quadrant-by-quadrant basis, as shown below.