Métodos Numéricos - L1

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1. Regra do produto:

$$\boxed{(f \cdot g)' = f' g + f \cdot g'}$$

Aplicando à questão:

$[\exp(t) \cdot \exp(-t)]' =$

$\exp(t)' \cdot \exp(-t) + \exp(t) \cdot \exp(-t)' =$

$e^t . e^{-t} + e^t . (-e^t) =$

$e^0 - e^0 = 0$

Isso ocorre porque $\exp(-t) \cdot \exp(t) =$ $e^{-t} \cdot e^t = e^0 = 1$, uma constante. Assim, temos que $\frac{d(1)}{dt} = 0$

2. Questão do AC1

3. Primeiros 6 meses:

$x = 100 \cdot \exp(0{,}5 \cdot 0{,}06)$

Após os primeiros 6 meses:

$x \cdot \exp(0{,}5 \cdot 0{,}1)$

Após 1 ano:

$100 \cdot \exp(0{,}5 \cdot 0{,}06) \cdot \exp(0{,}5 \cdot 0{,}1) =$

$100 \cdot \exp(0{,}03) \exp(0{,}05) =$

$100 \cdot 1{,}03045 \cdot 1{,}05127 = \underline{108{,}3287}$

Ano com taxa fixa de 8%

$100 \cdot \exp(1 . 0{,}08) =$

$100 \cdot \exp(0{,}08) = \underline{108{,}3287}$

A taxa equivalente para um ano inteiro é exatamente 8%

4. a) $\exp(\Delta t) = \sum_{n=0}^{\infty} \frac{\Delta t}{n!} = 1 + \Delta t + \frac{\Delta t^2}{2} + \frac{\Delta t^3}{6} \ldots$

como observado, $\exp(\Delta t)$ é igual a $1 + \Delta t$ somado a uma série infinita de termos positivos. Assim, $\exp(\Delta t) > 1 + \Delta t$

b) $\frac{1}{1 - \Delta t} = \sum_{n=0}^{\infty} \Delta t^n = 1 + \Delta t + \Delta t^2 + \Delta t^3$

Ao comparar a série da letra b com a série da letra A, vê-se que a diferença entre elas é a presença de um denominador positivo crescente a partir de $n=2$, fazendo $\exp(\Delta t)$ ser menor que $1/_{1-\Delta t}$

5. $\frac{dy}{dt} + \overbrace{2y}^{-ay} = q(t)$, $y(0) = 0$, $q(t) = \underline{H(t-4) - H(t-6)}$

$\int_{t-6}^{t-4} \frac{dH}{dt} \, dt$

fator integrante:

$M(t) = e^{2t}$ $\quad M(0) = e^0 = 1$

$$e^{2t} \cdot y(t) - \cancel{y(0)}^{0} = \int e^{2t} \cdot H(t-4)\, dt - \int e^{2t} \cdot H(t-6)\, dt$$

$$e^{2t} y(t) = \int_4^t e^{2x} dx - \int_6^t e^{2x} dx \qquad u = 2x,\ du = 2\,dx$$

$$e^{2t} \cdot y(t) = \frac{1}{2}(e^{2t} - e^8) \cdot H(t-4) - \frac{1}{2}(e^{2t} - e^{12}) H(t-6)$$

$$y(t) = \frac{\frac{1}{2}(e^{2t} - e^8) \cdot H(t-4)}{e^{2t}} - \frac{\frac{1}{2}(e^{2t} - e^{12}) H(t-6)}{e^{2t}}$$

$$y(t) = \frac{1}{2e^{2t}}(e^{2t} - e^8) \cdot H(t-4) - \frac{1}{2e^{2t}}(e^{2t} - e^{12}) H(t-6)$$

com $t \to \infty$

$$\underset{t \to \infty}{y(t)} = \frac{(1 - \cancel{e^{8-2t}}^{0})}{2} - \frac{(1 - \cancel{e^{12-2t}}^{0})}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

6. a) $\dfrac{dy}{dt} = (y+4)\cos t$

$$\frac{dy}{y+4} = \cos t \, dt$$

$$\int \frac{dy}{y+4} = \int \cos t \, dt \qquad \boxed{\int \frac{dy}{y+4} \quad u = y+4 \quad \frac{du}{dy} = 1 \quad du = dy \quad \int \frac{du}{u} = \ln u + c}$$

$$\ln(y+4) + c = \text{sen}\, t + c$$

$$\exp[\ln(y+4)] = \exp[\text{sen}(t) + c]$$

$$y+4 = \exp[\text{sen}(t)] \cdot \exp[c] \quad (c \to C)$$

$$\boxed{y = \exp[\text{sen}(t)] \cdot c - 4}$$

$y(0) = 1$

$$1 = \exp[\text{sen}(0)] \cdot c - 4$$

$$1 = \exp[0] \cdot c - 4$$

$$1 = c - 4$$

$$\boxed{c = 5}$$

b) $\dfrac{dy}{dt} = y \exp(t)$

$$\frac{dy}{y} = \exp(t)\, dt$$

$$\int \frac{dy}{y} = \int \exp(t)\, dt$$

$$\ln y = \exp(t) + c$$

$$e^{(\ln y)} = e^{\exp(t) + c}$$

$$y = e^{\exp(t)} \cdot e^{c}$$

$$\boxed{y = e^{\exp(t)} \cdot c}$$

$y(0) = 1$

$$1 = e^{\exp(0)} \cdot c$$

$$1 = e^{1} \cdot c$$

$$\boxed{c = 1/e}$$

7. a) $\boxed{M\,dx + N\,dy = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

$$\frac{dy}{dt} = \frac{(-3t^2 - 2y^2)}{(4ty + 6y^2)}$$

$$dy\,(4ty + 6y^2) = dt\,(-3t^2 - 2y^2)$$

$$dy\,(4yt + 6y^2) = -dt\,(3t^2 + 2y^2)$$

$$dy\,\underbrace{(4yt + 6y^2)}_{M} + dt\,\underbrace{(3t^2 + 2y^2)}_{N} = 0$$

$$\boxed{\frac{\partial M}{\partial t} = 4y \qquad \frac{\partial N}{\partial y} = 4y}$$

É exata!

$$\boxed{F(y,t) = C}$$

$$\int \frac{\partial F}{\partial y} = \int 4ty + 6y^2\, dy \to F(y,t) = 2ty^2 + 2y^3 + c(t)$$

$$\int \frac{\partial F}{\partial t} = \int 3t^2 + 2y^2\, dt \to F(y,t) = t^3 + 2y^2 t + c(y)$$

$$\boxed{F(y,t) = 2y^3 + 2y^2 t + t^3 = C}$$

b) $\dfrac{dy}{dt} = -\dfrac{[1 + y\exp(ty)]}{[2y + t\exp(ty)]}$

$$[2y + t\exp(ty)]\,dy = -dt\,[1 + y\exp(ty)]$$

$$\underbrace{[2y + t\exp(ty)]}_{M}\,dy + dt\,\underbrace{[1 + y\exp(ty)]}_{N} = 0$$

$$\frac{\partial M}{\partial t} = \exp(ty) + t\cdot\exp(ty)\cdot y = \boxed{\exp(ty)\cdot[1+ty]}$$

é exata!

$$\frac{\partial N}{\partial y} = \exp(ty) + y\exp(ty)\cdot t = \boxed{\exp(ty)\cdot[1+ty]}$$

$$\boxed{F(y,t) = C}$$

$$\int \frac{\partial F}{\partial y} = \int 2y + t\exp(ty)\,dy = y^2 + \int t\exp(\overset{u}{ty})\,dy = y^2 + \int \exp(u)\,du =$$

$du = t\,dy$

$$y^2 + \exp(u) = \boxed{y^2 + \exp(ty) + c(t)}$$

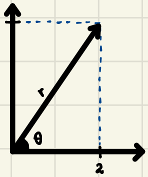
$$\int \frac{\partial F}{\partial t} = \int 1 + \exp(ty)\,dt = t + \int y\exp(\overset{u}{ty})\,dt = t + \int \exp(u)\,du = t + \exp(u) = \boxed{t + \exp(ty) + c(y)}$$

$du = y\,dt$

$$\boxed{F(y,t) = y^2 + \exp(ty) + t = C}$$

8. $\boxed{\begin{array}{l} e^{i\theta} = \cos\theta + i\,\text{sen}\,\theta \\ z = r(\cos\theta + i\,\text{sen}\,\theta) \\ z = e^{i\theta} \end{array}}$

$z = 2+3i$



$r = (9+4)^{1/2} = \sqrt{13}$

$tg\,\theta = 3/2 \to \theta = \text{arctg}\,3/2$

a) $\boxed{\begin{array}{l}\text{forma exponencial:} \\ z = \sqrt{13}\cdot\exp(i\,\text{arctg}\,3/2)\end{array}}$

$$\frac{\exp(i\omega t)}{\sqrt{13}\cdot\exp(i\,\text{arctg}\,3/2)} = \frac{e^{i\omega t}\cdot e^{-(i\,\text{arctg}\,3/2)}}{\sqrt{13}}$$

$$\frac{1}{\sqrt{13}}\exp(i\omega t - i\,\text{arctg}\,3/2) =$$

$$\frac{1}{\sqrt{13}}\cdot\exp[i(\underbrace{\omega t - \text{arctg}\,3/2}_{\theta})]$$

forma exponencial

b) forma polar

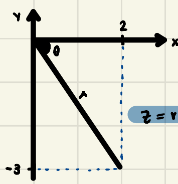
$$y = \frac{1}{\sqrt{13}}\left[\cos(\omega t - \text{arctg}\,3/2) + i\,\text{sen}(\omega t - \text{arctg}\,3/2)\right]$$

$$Re(y) = \frac{1}{\sqrt{13}}\left[\cos(\omega t - \text{arctg}\,3/2)\right]$$

$$Im(y) = \frac{1}{\sqrt{13}}\left[i\,\text{sen}(\omega t - \text{arctg}\,3/2)\right]$$

$$\frac{[2-3i][\exp(i\omega t)]}{[2-3i][2+3i]} = \frac{[2-3i][\exp(i\omega t)]}{4-9i^2} = \frac{[2-3i][\exp(i\omega t)]}{13}$$

forma exponencial de 2-3i



$r = \sqrt{13}$

$tg\,\theta = -3/2$

$z = re^{i\theta} = \sqrt{13}\cdot\exp(i\,\text{arctg}\,-3/2)$

$$\frac{\sqrt{13}\cdot\exp(i\,\text{arctg}\,-3/2)\cdot\exp(i\omega t)}{13} =$$

$$\frac{\sqrt{13}}{13}\exp[i(\overbrace{\text{arctg}\,-3/2 + \omega t}^{\theta})]$$

forma polar:

c) $\underbrace{\frac{\sqrt{13}}{13}\cos(\text{arctg}\,-3/2+\omega t)}_{\text{parte real}} + \underbrace{\frac{\sqrt{13}}{13}\,i\,\text{sen}(\text{arctg}\,-3/2+\omega t)}_{\text{parte imaginária}}$

9. $\frac{dz}{dt} - 2z = e^{i\omega t}$

Solução particular:

$z = Z\cdot e^{i\omega t}$

$\frac{dz}{dt} = Z\cdot i\omega\cdot e^{i\omega t}$

Portanto,

$Z\cdot i\omega\cdot\cancel{e^{i\omega t}} - 2Z\cdot\cancel{e^{i\omega t}} = \cancel{e^{i\omega t}}$

$Zi\omega - 2Z = 1$

$Z(i\omega - 2) = 1$

$\boxed{Z = 1/i\omega - 2}$

$z = Z\cdot e^{i\omega t}$

$z = \frac{e^{i\omega t}}{i\omega - 2}$

Pegando a parte real:

$$z = \frac{e^{i\omega t}}{i\omega - 2}\cdot\frac{i\omega+2}{i\omega+2} = \frac{e^{i\omega t}(i\omega+2)}{i^2\omega^2-4} =$$

$$\frac{e^{i\omega t}(i\omega+2)}{-\omega^2-4} = \frac{(\cos\omega t + i\,\text{sen}\,\omega t)(i\omega+2)}{-\omega^2-4} =$$

$$\frac{i\omega\cdot\cos\omega t + i^2\omega\,\text{sen}\,\omega t + 2\cos\omega t + 2i\,\text{sen}\,\omega t}{-\omega^2-4}$$

$$\boxed{\text{Assim, } Re\{z\} = \frac{2\cos\omega t - \omega\,\text{sen}\,\omega t}{-\omega^2-4}}$$

$y_m = c\exp(2t)$

$y = y_m + y_p$

$y(0) = y_m(0) + y_p(0)$

$0 = c + \frac{2}{-\omega^2-4}$

$c = 2/\omega^2+4$

$$y_p(0) = \frac{2\cos 0 - \omega\,\text{sen}\,0}{-\omega^2-4} = \frac{2}{-\omega^2-4}$$

Assim,

$$y = \frac{2\exp(2t) - 2\cos\omega t + \omega\,\text{sen}\,\omega t}{\omega^2+4}$$

10. $L\frac{dI}{dt} + RI(t) = V\exp(i\omega t)$, $Z = V/I$

$I(t) = I\exp(i\omega t)$

$\frac{dI}{dt} = Ii\omega \cdot \exp(i\omega t)$

$L[Ii\omega \cancel{e^{i\omega t}}] + R[I\cancel{e^{i\omega t}}] = V\cdot\cancel{e^{i\omega t}}$

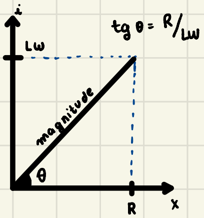
$L\cdot Ii\omega + RI = V$

$\cancel{I}(Li\omega + R) = Z\cancel{I}$

$Li\omega + R = Z$

$Z = a + bi$

$Z = \underbrace{R}_{a} + \underbrace{L\omega}_{b} i$



A magnitude vai ser $\sqrt{R^2 + L^2\omega^2}$,

enquanto o ângulo vai ser $arc\ tg\ {}^{L\omega}/_R$

Como $Z^2 = R^2 + L^2\omega^2$, tem-se que quanto maior L, maior será Z!

11. $\frac{dy}{dt} = sen(t)y + Q\,sen(t)$; $y_m(0) = 1$

$\frac{dy}{dt} - a(t)y = q(t)$

$\frac{dy}{dt} - \overbrace{sen(t)y}^{a(t)y} = \overbrace{Q\,sen(t)}^{q(t)}$

$y(t) = G(t,0)\,y(0)$

$G(t,s) = e^{\int_s^t sen(u)\,du}$

$\int_0^t sen(h)\,dh = \left[-\cos(h)\right]_s^t = -\cos t + \cos s$

$G(t,s) = e^{-\cos t + \cos s}$

$y(t) = G(t,0)\cdot y(0)$

$G(t,0) = e^{-\cos t + \cos 0} = e^{-\cos t + 1}$

$y(0) = y_p(0) + y_n(0) = 1 + 0 = 1$

$y(t) = e^{1-\cos t}$

$y_m(t) = G(t,0) = e^{-\cos t + 1}$

$y_p(t) = \int_0^t G(t,s)\cdot q(s)\,ds = \int_0^t e^{-\cos t + \cos s}\cdot Q\,sen(s)\,ds$

$Q\,e^{-\cos t}\int_0^t e^{\overbrace{\cos s}^{u}}\cdot sen(s)\,ds$ $\quad du = -sen\,s\,ds$

$-Q\cdot e^{-\cos t}\int_0^t e^u\,du = e^u = \left[e^{\cos s}\right]_0^t = e^{\cos t} - e$

$-Q\cdot e^{-\cos t}\left[e^{\cos t} - e\right]$

$y_p(t) = Q\,e^{1-\cos t} - Q$

12. $\frac{dy}{dt} = ay - by^2$, capacidade $a/b = 14\cdot10^9$ pessoas

$\frac{dy}{dt} = y(a - by)$

$\frac{dy}{dt} = a\left(1 - \frac{b}{a}y\right)y = a\left(1 - \frac{y}{a/b}\right)y$

$$\boxed{\frac{dy}{dt} = a\left(1 - \frac{y}{14\cdot10^9}\right)y}$$

soluções: $y = 0$ ou $y = a/b$

Ponto de Inflexão: 2ª derivada $= 0$

$\frac{dy}{dt} = ay - by^2$

$\frac{d}{dt}\left[\frac{dy}{dt}\right] = \frac{d}{dt}\left[ay - by^2\right]$

$(a - 2by)\left(\frac{dy}{dt}\right) = 0 \rightarrow y = a/_{2b} \rightarrow$

Ponto de inflexão:

$y = \frac{14\cdot10^9}{2} = 7\cdot10^9$

$\frac{dy}{dt} = a\left(\frac{a}{2b}\right) - b\left(\frac{a}{2b}\right)^2$

$\frac{dy}{dt} = \frac{a^2}{2b} - \frac{\cancel{b}a}{4b^{\cancel{2}}} = \frac{a^2}{2b} - \frac{a^2}{4b} = \frac{2a^2 - a^2}{4b} = \frac{a^2}{4b} = \left(\frac{a}{2b}\right)^2\cdot b = (7\cdot10^9)^2\,b = 49\cdot10^{18}\,b$

13. $\boxed{\dfrac{dy}{dt} = ay - by^m \,, \; z = y^{1-m}}$

$$\frac{dy}{dt} - ay = -by^m \; (\cdot\, y^{-m})$$

$$y^{-m}\cdot\frac{dy}{dt} - ay^{1-m} = b \qquad z = y^{1-m},\; z' = (1-m)\,y^{-m}\cdot y'$$

$$\frac{z'}{1-m} - az = b \quad (\cdot\, 1-m)$$

$\boxed{z' - (1-m)\,az = (1-m)b}$ edo linear! $[y' + P(x)y = Q(x)]$

Solução:

$$y(x) = \frac{1}{I(x)}\left(\int I(x)\,Q(x)\;dx + C\right)$$

$$I(x) = e^{\int P(x)\,dx}$$

$$I(t) = e^{-\int (1-m)a\,dt}$$

$$Q(t) = (1-m)\,b$$

Portanto,

$$z(t) = \frac{1}{e^{-(1-n)at}}\cdot\left[\int e^{-(1-n)at}\cdot -(1-m)\,b\;dt\right]$$

u, $du = -(1-n)\,az$

$$\int e^{\overbrace{-(1-n)at}^{u}}\cdot -(1-m)\,b\;dt$$

$$\frac{b}{a}\int e^{u}\,du = \frac{b}{a}\,e^{-(1-m)at} + C$$

$$z(t) = \frac{1}{\cancel{e^{-(1-n)at}}}\cdot\left[\frac{b}{a}\,\cancel{e^{-(1-m)at}} + C\right]$$

$$z(t) = b/a + C \rightarrow y^{1-n} = \frac{b}{a} + C$$

14. a) $\dfrac{dy}{dt} = 2(1-y)(1-e^y)$

$\downarrow$ $Y_0 = 1$, $\downarrow$ $Y_1 = 0$

derivamdo:

$$\frac{dF}{dy} = 2\left[(1-y)\cdot(0-e^y) + (0-1)(1-e^y)\right]$$

$$\frac{dF}{dy} = 2\left[-e^y(1-y) + (-1+e^y)\right]$$

$$\frac{dF}{dy} = 2\left[-\cancel{e^y} + e^y\cdot y - 1 + \cancel{e^y}\right]$$

$$\frac{dF}{dy} = 2(ye^y - 1)$$

Para $Y=1$ $\quad 2(e-1) > 0 \rightarrow$ Instável

Para $Y=0$ $\quad 2\cdot(-1) = -2 < 0 \rightarrow$ Estável

b) $\dfrac{dy}{dt} = (1-y^2)(4-y^2)$

$\downarrow$ $y^2 = 1$: $Y_0 = 1$, $Y_1 = -1$

$\downarrow$ $y^2 = 4$: $Y_2 = 2$, $Y_3 = -2$

derivando:

$$\frac{dF}{dy} = (-2y)(4-y^2) + (1-y^2)\cdot(-2y) =$$

$$-8y + 2y^3 + (-2y + 2y^3) =$$

$$-8y + 2y^3 - 2y + 2y^3 =$$

$$-10y + 4y^3$$

Para $Y = 1$ $\quad -10 + 4 = -6 < 0$ Estável

Para $Y = -1$ $\quad 10 - 4 = 6 > 0$ Instável

Para $Y = 2$ $\quad -20 + 32 = 12 > 0$ Instável

Para $Y = -2$ $\quad 20 - 32 = -12 < 0$ Estável