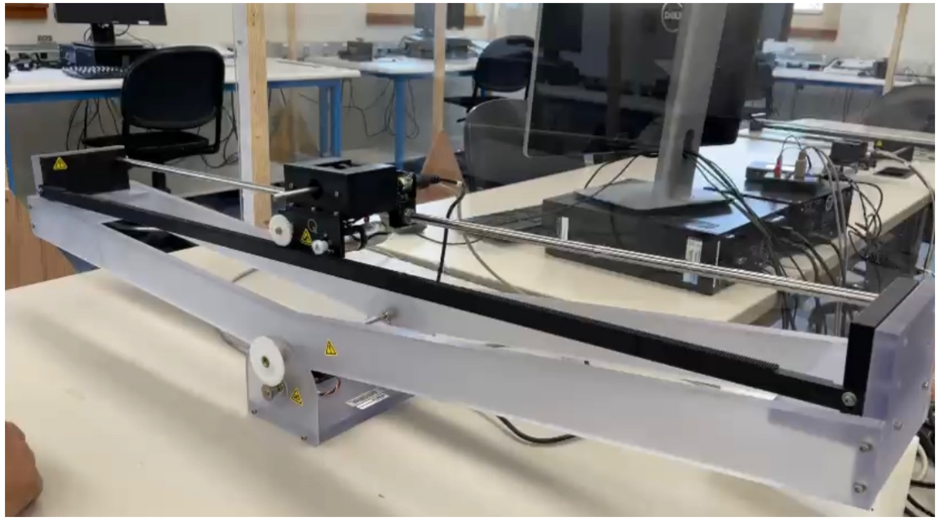
State-Feedback Balance Control of Cart See-Saw System



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Problem Statement

The seesaw system consists of a seesaw and an electric motor driven cart mounted on it. The only input to the seesaw system is the voltage to the cart's motor. In order to balance the see-saw in its horizontal position, a stabilizing controller is to be developed. The controller need to achieve the balance by regulating the input voltage to the cart's motor.

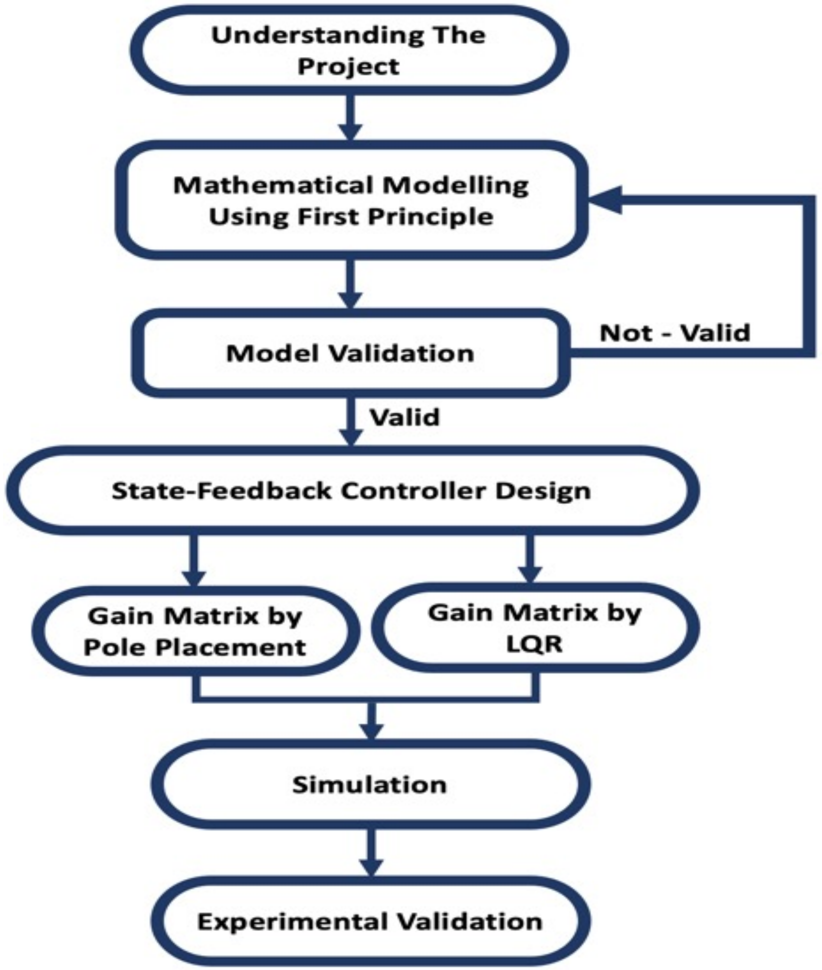
The designed controller should also work under the following two constraints:

- The balance time t_b need to be < 5 sec.
- The balance deviation index I_p need to be < 0.02

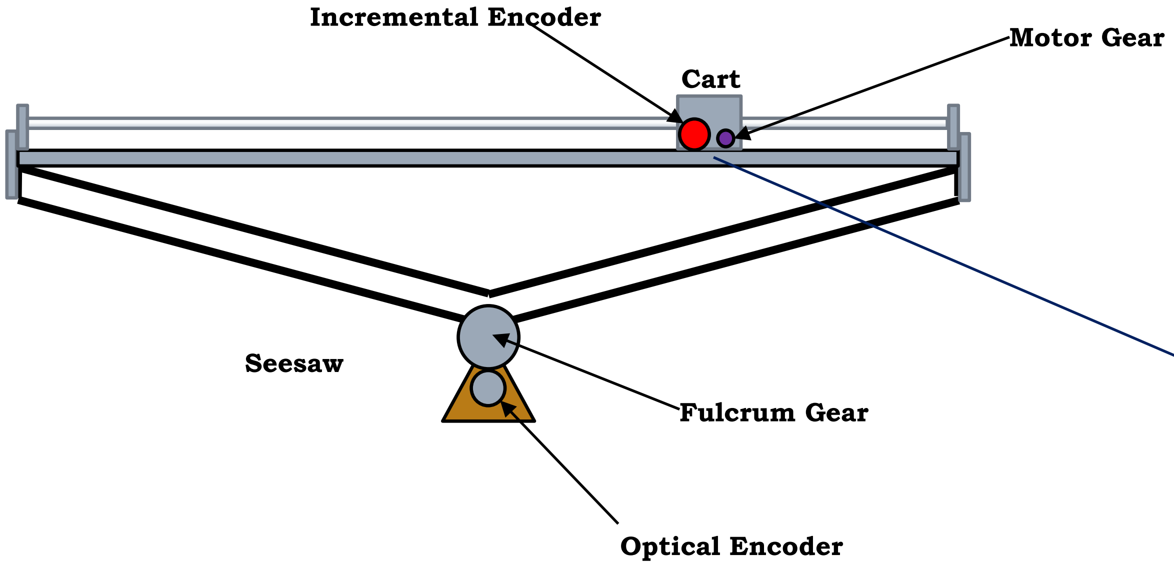
The balance time (t_b) is the time when the tilt angle of the seesaw first comes within the range of $\pm 0.05\, rad$. The balance deviation index I_p is given as,

$$I_p = \frac{1}{10}\int_{t_b}^{t_b+10} |\theta| dt \quad (\theta\ is\ the\ tilt\ angle\ of\ the\ seesaw)$$

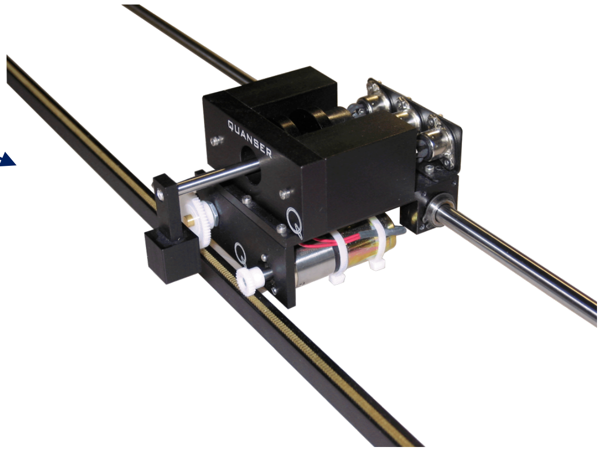
Flow Chart of the Project



The seesaw system is composed of a seesaw (QUANSER SEESAW-E module) and a cart (QUANSER IP02 module)

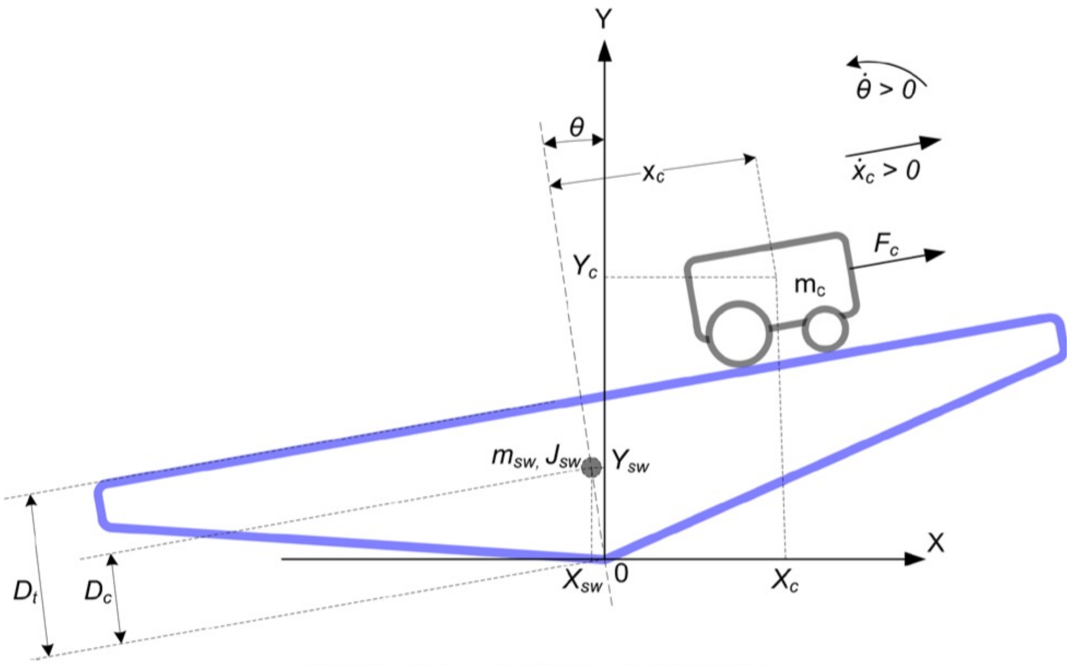


QUANSER SEESAW-E Module



IP02 unit (Electrical System)

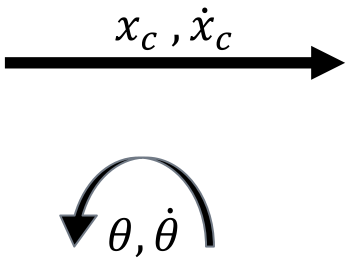
SCHEMATIC OF THE MECHANICAL SYSTEM



θ	*Angle of the seesaw*
m_c	*mass of the cart*
m_{sw}	*mass of cart + mass of seesaw*
x_c	*Position of cart with respect to CG of seesaw + cart*
$[X_c, Y_c]$	*absolute coordinate of CG of the cart*
$[X_{sw}, Y_{sw}]$	*absolute coordinate of CG of seesaw + cart*
D_c	*Distance from the pivot of seesaw to* $[X_{sw}, Y_{sw}]$
D_t	*Distance of seesaw from pivot*
J_{sw}	*Moment of inertia of seesaw + cart*
F_c	*Force exerted by the cart*

*The mass of the see-saw system is considered to be concentrated in its center of gravity

KINEMATICS OF THE MECHANICAL SYSTEM



$$T = T_{c,t} + T_{c,r} + T_{ss,r}$$

$$V = V_C + V_{SS}$$

$$T = \frac{1}{2} m_c \dot{x}_c^2 - m_c D_t \dot{\theta} \dot{x}_c + \left(\frac{1}{2} J_{sw} + \frac{1}{2} m_c D_t^2 + \frac{1}{2} m_c x_c^2 \right) \dot{\theta}^2$$

$$V = g(m_c D_t \cos\theta + m_c x_c sin\theta + m_{sw} D_c cos\theta)$$

Where,

$T = Total\ kinetic\ energy$

$V = Total\ Potential\ Energy$

$T_{c,t} = Translational\ Kinetic\ Energy\ of\ cart$

$T_{c,r} = Rotational\ Kinetic\ Energy\ of\ cart$

$T_{ss,r} = Rotational\ Kinetic\ Energy\ of\ Seesaw$

$V_c = Potential\ Energy\ of\ Cart$

$V_{ss} = Potential\ Energy\ of\ the\ Seesaw$

$* T_{c,r}$ *is neglected due to very small inertia of the DC motor* $(J_m = 3.9x10^{-7}\ kg\ m^2)$

LAGRANGIAN AND EQUATIONS OF MOTION

$\underline{Lagrangian\ of\ the\ system}$

$$\boldsymbol{L = T - V}$$

$\underline{\boldsymbol{Euler\ Lagarange\ Equation\ is\ given\ by}};$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_1 \quad \dots (1)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_c}}\right) - \frac{\partial L}{\partial x_c} = Q_2 \quad \dots (2)$$

$$Where, \qquad Q_1 = F_c - B_c\dot{x}_c, Q_2 = -B_{sw}\dot{\theta}$$

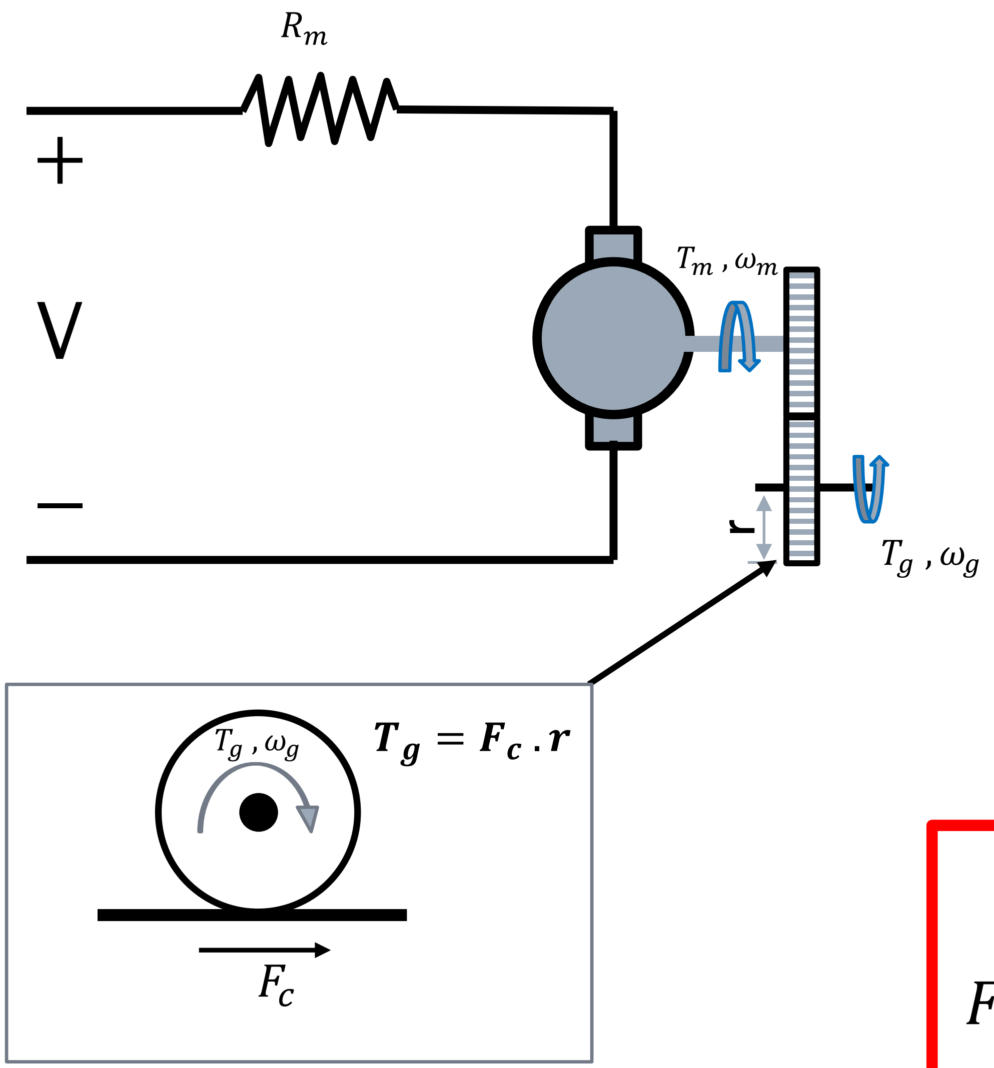
$(1) \Rightarrow m_c\ddot{x}_c - m_cD_t\ddot{\theta} - m_cx_c\dot{\theta}^2 + gm_csin\theta = F_c - B_c\dot{x}_c, Q_2$ $\qquad (1^{st}\ Lagrange's\ Equation)$

$(2) \Rightarrow m_c\ddot{\theta}x_c^2 + 2m_c\dot{x}_c\dot{\theta} + gm_ccos\theta\ x_c - m_cD_t\ddot{x}_c + (J_{sw} + m_cD_t^2)\ddot{\theta} - g(m_cD_tsin\theta - m_{sw}sin\theta) = -B_{sw}\dot{\theta}$ $\qquad (2^{nd}\ Lagrange's\ Equation)$

$\boldsymbol{Where},$

$\boldsymbol{Q_1}$	$Generalized\ forces\ acting\ on\ cart$
$\boldsymbol{Q_2}$	$Generalized\ forces\ acting\ on\ the\ seesaw$
$\boldsymbol{B_c}$	$Damping\ constant\ of\ the\ cart$
$\boldsymbol{B_{sw}}$	$Damping\ of\ seesaw$

ELECTRICAL SYSTEM MODELLING



The equations governing the DC Motor are ;

$$V = IR_m + K_b\omega_m \qquad\qquad T_m = \eta_m K_t I$$

Where,

T_m *DC motor torque at the shaft*

R_m = *Motor Armature Resistance*

K_b = *Back emf constant*

K_t = *Motor Torque Constant*

$$\omega_m = K_g\omega_g = K_g\frac{\dot{x}_c}{r}$$

$$T_g = \eta_g K_g T_m$$

K_g = *Output Gear Ratio,* η_g = *Output Gear Efficiency*

η_m = *Motor Efficiency*

Relationship of F_c & V

$$F_c = \frac{\eta_m\eta_g K_t K_g}{R_m r} V - \frac{\eta_m\eta_g K_t K_b K_g^2}{R_m r^2}\dot{x}_c$$

COMPLETE SYSTEM MODELLING

$Differential\ equations\ describing\ the\ dynamic\ of\ the\ complete\ system\ are;$

$$(1)\quad m_c\ddot{x}_c - m_cD_t\ddot{\theta} - m_cx_c\dot{\theta}^2 + m_c sin\theta + \left(\frac{\eta_m\eta_gK_tK_bK_g^2}{R_mr^2} + B_c\right)\dot{x}_c = \frac{\eta_m\eta_gK_tK_g}{R_mr}V$$

$$(2)\quad m_c\ddot{\theta}x_c^2 + 2m_c\dot{x}_c\dot{\theta} + gm_c cos\theta\, x_c - m_cD_t\ddot{x}_c + (J_{sw} + m_cD_t^2)\ddot{\theta} - g(m_cD_t sin\theta - m_{sw}sin\theta) + B_{sw}\dot{\theta} = 0$$

$\boldsymbol{Accelerations\ in\ terms\ of\ displacements\ and\ velocities}$

$$\ddot{x}_c = \frac{1}{(m_cx_c^2 + J_{sw})m_c}\Big[m_c^2x_c^3\dot{\theta}^2 + \left\{\left(\frac{\eta_m\eta_gK_tK_g}{R_mr}V - \frac{\eta_m\eta_gK_tK_bK_g^2}{R_mr^2}\dot{x}_c\right)m_c - gm_c^2 sin\theta - B_c\dot{x}m_c\right\}x_c^2 \\ +\left(-2m_c^2D_t\dot{x}_c\dot{\theta} + m_c^2D_t^2\dot{\theta}^2 + m_cJ_{sw}\dot{\theta}^2 - m_c^2D_tg cos\theta\right)x_c - (m_cD_t^2B_c - B_cJ_{sw})\dot{x}_c - m_cD_tg\, m_{sw}D_c sin\theta\Big] \tag{24}$$

$$\ddot{\theta}_c = \frac{1}{(m_cx_c^2 + J_{sw})}\Big[\left(D_tm_c\dot{\theta}^2 - 2m_c\dot{x}_c\dot{\theta} - gm_c cos\theta\right)x_c + D_t\left(\frac{\eta_m\eta_gK_tK_g}{R_mr}V - \frac{\eta_m\eta_gK_tK_bK_g^2}{R_mr^2}\dot{x}_c\right) - D_tB_c\dot{x}_c \\ + gm_{sw}D_c sin\theta - B_{sw}\dot{\theta}\Big] \tag{25}$$

SYSTEM LINEARIZATION

$$\begin{cases} \ddot{x}_c = f\left(x_c, \theta, \dot{x}_c, \dot{\theta}\right) \\ \ddot{\theta} = g\left(x_c, \theta, \dot{x}_c, \dot{\theta}\right) \end{cases}$$ *Non-linear functions*

$We\ take\ the\ operating\ points, [x_c, \theta, \dot{x}_c, \dot{\theta}] = [0,0,0,0] = z^\star$

$Now, we\ Taylor\ expand\ f\ and\ g\ around\ z^*\ as\ below;$

$$f\left(x_c, \theta, \dot{x}_c, \dot{\theta}\right) = f|_{z^*} + \frac{\partial f}{\partial x_c}|_{z^*}(x_c - 0) + \frac{\partial f}{\partial \theta}|_{z^*}(\theta - 0) + \frac{\partial f}{\partial \dot{x}_c}|_{z^*}(\dot{x}_c - 0) + \frac{\partial f}{\partial \dot{\theta}}|_{z^*}\left(\dot{\theta} - 0\right) + H.O.T.s$$

$*\ Similarly\ we\ can\ expand\ function\ g$

$Therefore, the\ linearized\ equations\ are:$

$$\ddot{x}_c = a_1 x_c + a_2 \theta + a_3 \dot{x}_c + a_4 \dot{\theta} + b_1 V \quad (26)$$

$$\ddot{\theta} = a_5 x_c + a_6 \theta + a_7 \dot{x}_c + a_8 \dot{\theta} + b_2 V \quad (27)$$

$$a_1 = -\frac{m_c D_t g}{J_{sw}}, a_2 = \frac{-g m_c R_m r^2 J_{sw} + m_c D_t R_m r^2 g m_{sw} D_c}{R_m r^2 J_{sw} m_c}$$

$$a_3 = -\frac{J_{sw}\eta_g K_g^2 \eta_m K_t K_b + J_{sw} B_c R_m r^2 + m_c D_t^2 \eta_g K_g^2 \eta_m K_t K_b + m_c D_t^2 B_c R_m r^2}{R_m r^2 J_{sw} m_c}$$

$$a_4 = -\frac{D_t B_{sw}}{J_{sw}}, a_5 = -\frac{g m_c}{J_{sw}}, a_6 = \frac{g m_{sw} D_c}{J_{sw}}, a_7 = -\frac{\eta_g K_g^2 \eta_m K_t K_b D_t + B_c R_m r^2 D_t}{R_m r^2 J_{sw}}$$

$$a_8 = -\frac{B_{sw}}{J_{sw}}, b_1 = \frac{J_{sw}\eta_g K_g \eta_m K_t r + m_c D_t^2 \eta_g K_g \eta_m K_t r}{R_m r^2 J_{sw} m_c}, b_2 = \frac{\eta_g K_g \eta_m K_t D_t}{r R_m J_{sw}}$$

System Parameters

Symbol	*Description*	*Value*	*Unit*
M_{sw}	Mass of the one-SEESAW(-E)-plus-one-IP01-or-IP02-Track System	3.6	kg
K_{gs}	SEESAW(-E) Geartrain Gear Ratio	3	
D_T	Distance from Pivot to the IP01 or IP02 Track	0.125	m
D_c	Distance from Pivot to the Centre Of Gravity of the one-SEESAW(-E)-plus-one-IP01-or-IP02-Track System	0.058	m
J_{sw}	Moment of Inertia of the one-SEESAW(-E)-plus-one-IP01-or-IP02-Track System, about its Center Of Gravity	0.395	$kg.m^2$
B_{sw}	Viscous Damping Coefficient as seen at the Seesaw Pivot Axis	≈0	N.m.s/rad
g	Gravitational Constant on Earth	9.81	m/s^2
K_{P_SW}	SEESAW Potentiometer Sensitivity	-0.2482	rad/V
K_{E_SW}	SEESAW-E Encoder Resolution	0.0015	rad/count
$\grave{e}_{range}$	SEESAW(-E) Approximative Angular Range on a Flat Surface	±11.5	º

TABLE-2 Different parameters of the IP02 cart module

Symbol	Description	Value	Variation
V_{nom}	Motor nominal input voltage	6.0 V	
R_m	Motor armature resistance	2.6 Ω	$\pm$ 12%
L_m	Motor armature inductance	0.18 mH	
k_t	Motor current-torque constant	7.68×10^{-3} N m/A	$\pm$ 12%
k_m	Motor back-emf constant	7.68×10^{-3} V/(rad/s)	$\pm$ 12%
η_m	Motor efficiency	0.69	$\pm$ 5%
$J_{m,rotor}$	Rotor moment of inertia	3.90×10^{-7} kg $\cdot$ m^2	$\pm$ 10%
K_g	Planetary gearbox gear ratio	3.71	
η_g	Planetary geabox efficiency	0.90	$\pm$ 10%
M_c	Mass of cart	0.38 kg	
M_w	Mass of cart weight	0.37 kg	
$B_{eq,c}$	Equivalent viscous damping coefficient (Cart)	4.3 N m s/rad	
$B_{eq,c}$	Equivalent viscous damping coefficient (Cart and Weight)	5.4 N m s/rad	
L_t	Track length	0.990 m	
T_c	Cart travel	0.814 m	
P_r	Rack pitch	1.664×10^{-3} m/tooth	
r_{mp}	Motor pinion radius	6.35×10^{-3} m	
N_{mp}	Motor pinion number of teeth	24	
r_{pp}	Position pinion radius	0.01483 m	
N_{pp}	Position pinion number of teeth	56	
K_{ec}	Cart encoder resolution	2.275×10^{-5}	
K_{ep}	Pendulum encoder resolution	0.0015 rad/count	
f_{max}	Maximum input voltage frequency	50 Hz	
I_{max}	Maximum input current	1 A	
ω_{max}	Maximum motor speed	628.3 rad/s	

TABLE-2 Different parameters of the IP02 cart module

STATE SPACE REPRESENTATION

The state of a dynamical system is the minimum possible set of variables, called state variables, the knowledge of which at any initial time along with the input(s) at any given time completely defines the behavior of the system at that given time.

The state-space is the n-dimensional space consisting of axes corresponding to n-state variables

Let us, consider a second order mass-spring-damper system with input T and output θ as below

$$M\ddot{\theta} + D\dot{\theta} + k\theta = T$$

Let us, consider two states needed of the above system as $x_1 = \theta, x_2 = \dot{\theta}$. We can write the state equations as,

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -\frac{D}{M}x_2 - \frac{k}{M}x_1 + \frac{T}{m}$$

In matrix form, the above state equations can be written as,

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{D}{M} & -\frac{k}{M}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}T$$

If we consider, $\boldsymbol{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix}0 & 1\\ -\frac{D}{M} & -\frac{k}{M}\end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}$ and output of the system being y =$\theta = x_1$

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}T$$

Sometimes, the input may directly affect the output y in some way, in that case we get another matrix called direct transition matrix **D**. For this case, the **D** matrix is zero

STATE SPACE REPRESENTATION

For the seesaw system, the variables impacting the dynamics of the system are $x_c, \theta, \dot{x}_c, \dot{\theta}$

We consider, four state variables for the system as $x_1 = x_c, x_2 = \theta, x_3 = \dot{x_c}, x_4 = \dot{\theta}$

Using (26) & (27), we can write the state equations,

$$\dot{x_1} = \dot{x_1}$$

$$\dot{x_2} = \dot{x_2}$$

$$\dot{x_3} = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + b_1V$$

$$\dot{x_4} = a_5x_1 + a_6x_2 + a_7x_3 + a_8x_4 + b_2V$$

With $x_1 = x_c, x_2 = \theta$ being the outputs of the system, **y** being the output vector, and writing input V as u,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \qquad (26)$$

Equation (26) gives the simplified state-space model of the see-saw system,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u\,; \quad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}u$$

With, $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}$, $\boldsymbol{x} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Here, **A** is the system matrix, **B** is the input matrix, **C** is the output matrix, and **D** is the output transition matrix

Different parameters of **A** and **B** are given as follows,

$$a_1 = -\frac{m_c D_t g}{J_{sw}}\ , a_2 = \frac{-g m_c R_m r^2 J_{sw} + m_c D_t R_m r^2 g m_{sw} D_c}{R_m r^2 J_{sw} m_c}$$

$$a_3 = -\frac{J_{sw}\eta_g K_g^2 \eta_m K_t K_b + J_{sw} B_c R_m r^2 + m_c D_t^2 \eta_g K_g^2 \eta_m K_t K_b + m_c D_t^2 B_c R_m r^2}{R_m r^2 J_{sw} m_c}$$

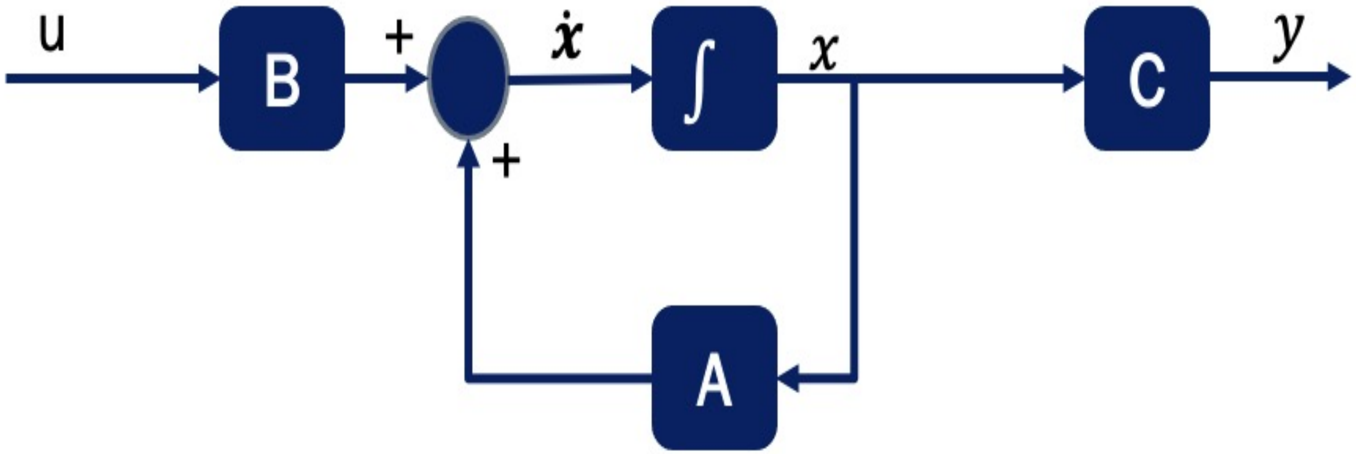
$$a_4 = -\frac{D_t B_{sw}}{J_{sw}}\ , a_5 = -\frac{g m_c}{J_{sw}}\ , a_6 = \frac{g m_{sw} D_c}{J_{sw}}\ , a_7 = -\frac{\eta_g K_g^2 \eta_m K_t K_b D_t + B_c R_m r^2 D_t}{R_m r^2 J_{sw}}$$

$$a_8 = -\frac{B_{sw}}{J_{sw}}\ , b_1 = \frac{J_{sw}\eta_g K_g \eta_m K_t r + m_c D_t^2 \eta_g K_g \eta_m K_t r}{R_m r^2 J_{sw} m_c}\ , b_2 = \frac{\eta_g K_g \eta_m K_t D_t}{r R_m J_{sw}}$$

STATE SPACE REPRESENTATION

Using the physical parameters of the see-saw system, the state matrices are calculated as,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2.33 & -9.162 & -14.016 & 0 \\ -18.63 & 5.19 & -3.23 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 1.47 \\ 0.34 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Block diagram representation of the see-saw system state-space model

Open-loop System Model Analysis

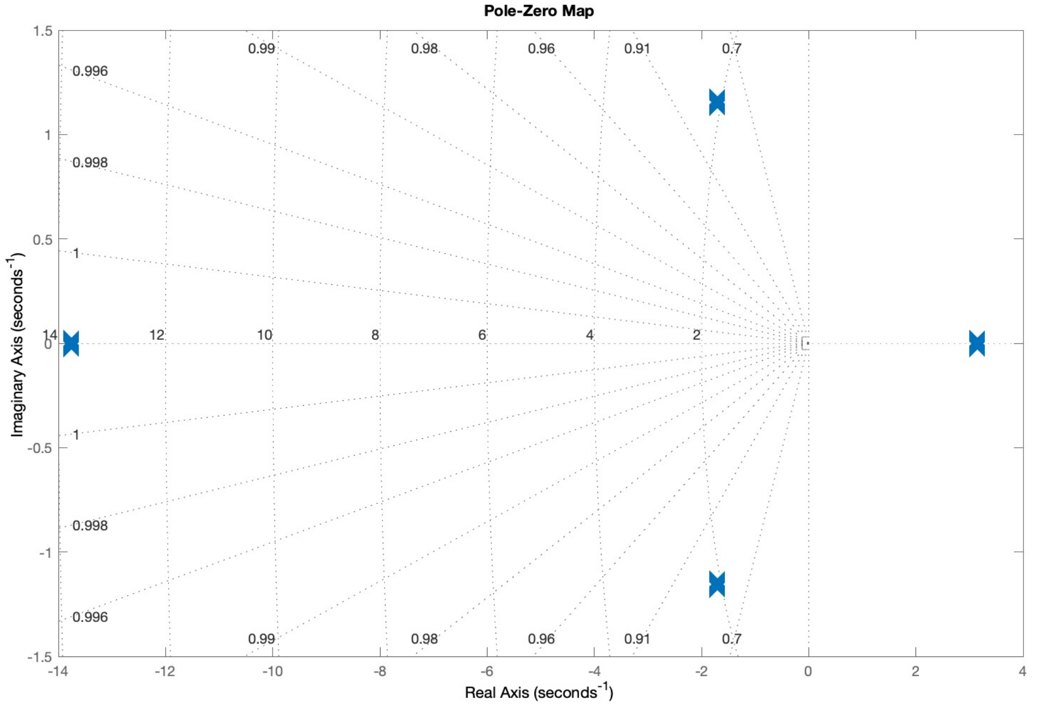
The poles or eigen values of a system in state-space form is given as,

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = 0$$

Where, **A** is the system matrix, and **I** is an identity matrix of the same order of **A**

For the seesaw system model, the open loop poles are at,

$$s_{1,2,3,4} = 3.1422, -1.7025 + i1.1532, -1.7025 - i1.1532, -13.5729$$



We can observer that, as per the mathematical model, the seesaw system is open loop unstable as one of the poles is on the right-half of the s-plane

CONTROLLABILITY

Definition:

$A\ System\ is\ controllable\ if\ there\ consists\ a\ control\ input, that\ can\ transform\ any\ state\ of\ the\ system\ to\ any\ other\ state\ in\ finite\ time.$

$Controllability\ matrix\ for\ a\ linear\ system\ is;$

$$C = [B\ AB\ A^2B\ \ \ldots A^{n-1}B]\ (A \Rightarrow n{\times}n\ matrix)$$

$For\ our\ cart\ seesaw\ mathematical\ model, the\ controllability\ matrix\ is\ given\ by;$

$$C = \begin{bmatrix} 0 & 1.4713 & -20.6219 & 282.4955 \\ 0 & 0.3392 & -4.7534 & 40.9745 \\ 1.4713 & -20.6219 & 282.4955 & -3.8678{\times}10^3 \\ 0.3392 & -4.7534 & 40.9745 & -553.1780 \end{bmatrix}$$

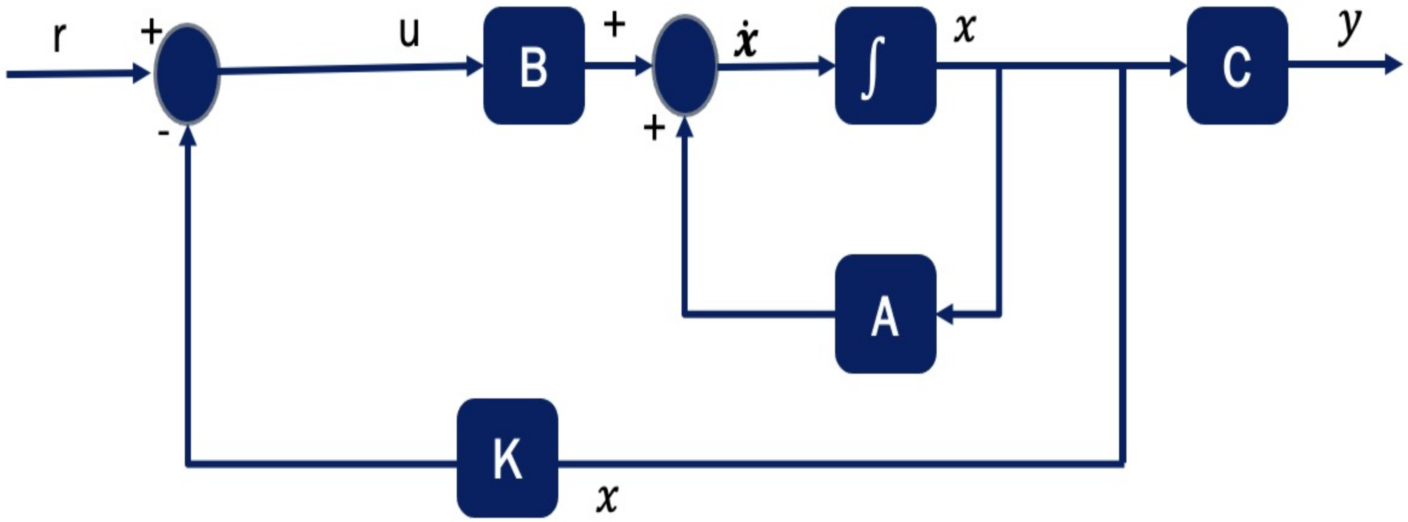
$Rank(C) = 4 \Rightarrow The\ system\ is\ state\ controllable.$

State-Feedback Control Philosophy

Each state variable is fed back to the system input terminal through a gain tuned for each state variable

The respective gain for each state variable can be adjusted to adjust the closed-loop poles of the system

For the seesaw system, we consider four number of state variables, hence we require four gain variables



Block diagram representation state-feedback control

Here, **r** is the reference input and **K** is the matrix containing corresponding feedback gains for each state variable

FULL-STATE FEEDBACK CONTROL

Here, The error between reference state variable values and actual state variable values to compute the control input to the dynamical system using the gain matrix $\boldsymbol{K}$

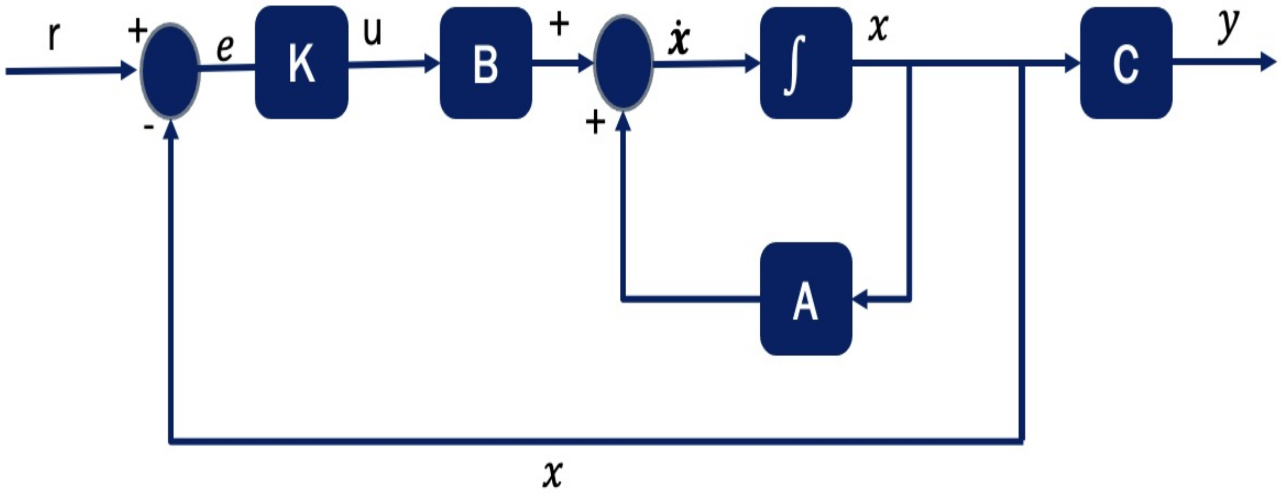
<u>*The equations for full – state feedback control are*:</u>

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x} + \boldsymbol{BKr}$$
$$\boldsymbol{y} = \boldsymbol{Cx}$$

Where $\boldsymbol{K}$ *is the gain matrix.*

When, $\mathrm{r} = 0$, the controller act as a regulator to regulate the system around $\mathrm{r} = 0$

Governing equation becomes: $\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x}$

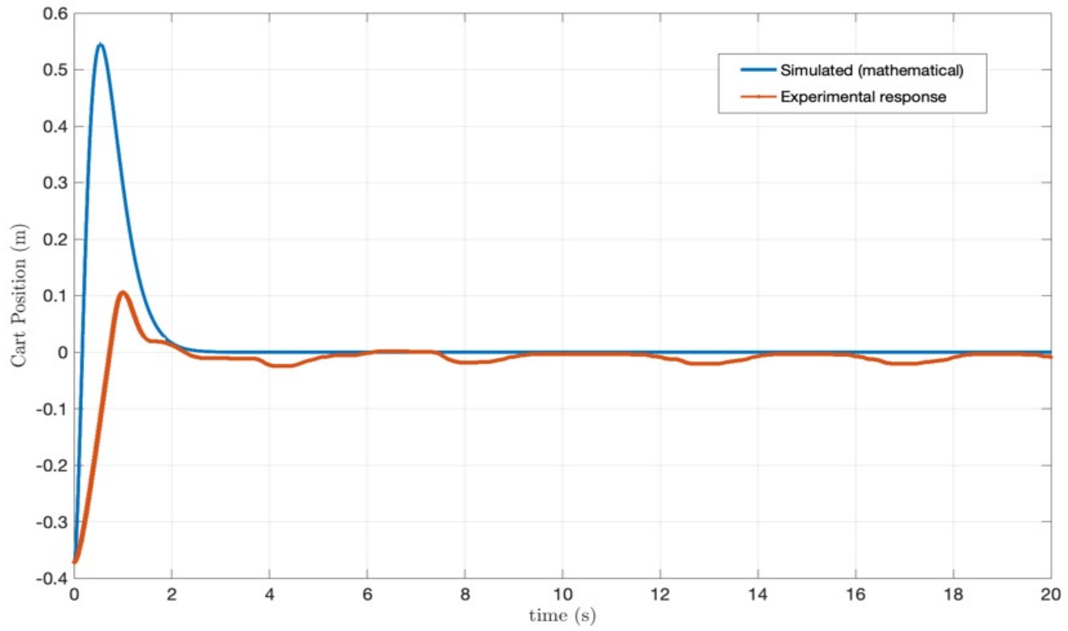


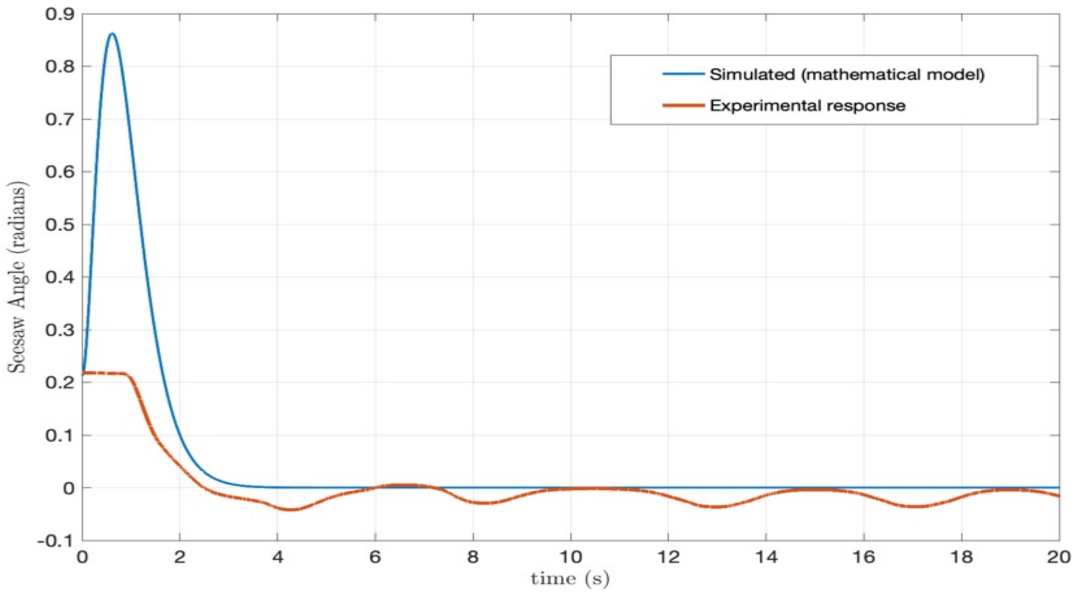
The poles of the closed-loop system ae given by,

$$\det\big(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BK})\big) = 0$$

Validation of Mathematical Model

- *K is used with trial and error as* $K = [72.5807 \quad -44.0340. \quad 4.2323 \qquad -13.9846]$
- The gain matrix is used in both simulation and practical system
- Closed-loop comparison is done as the system is open-loop unstable with full-state feedback control
- Both simulated and practical system responses are compared





$* \, It \; is \; worth \; noting \; that \; practical \; system \; has \; maximum \; voltage, see - saw \; angle, and \; cart \; postion \; limit$

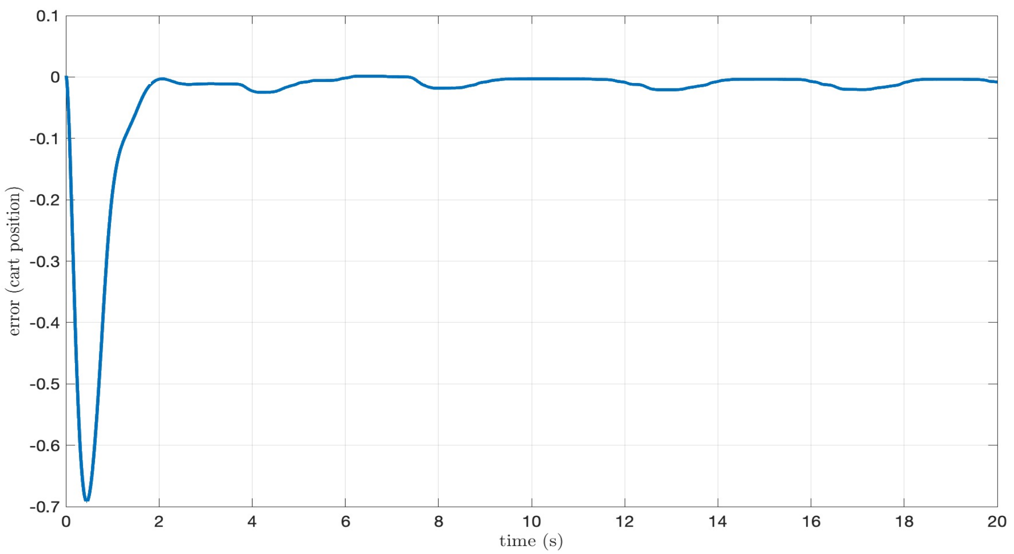
Validation of Mathematical Model

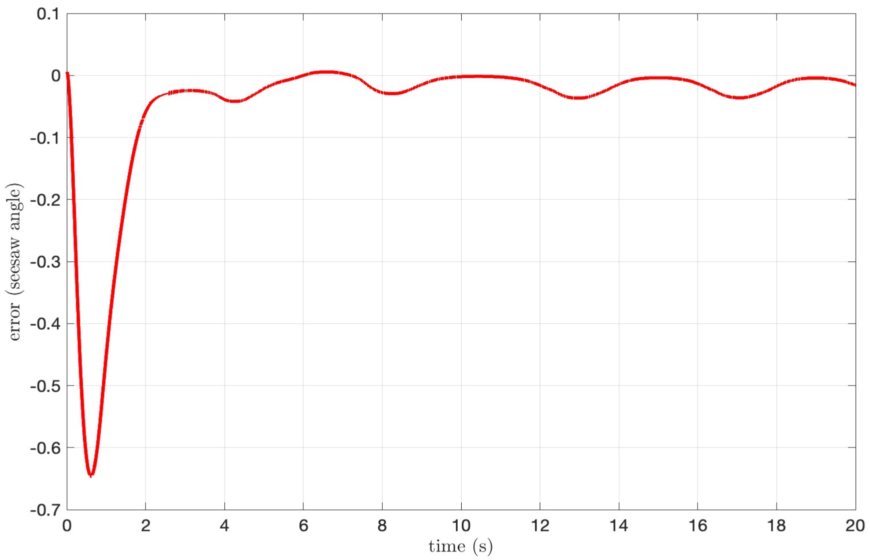
$$Mean\ Squared\ Error(MSE) = \frac{1}{n}\sum_{i=1}^{n}\left(X - \hat{X}\right)^2$$

$$(Where, n\ is\ number\ of\ data\ points, X\ is\ practical\ data, \hat{X}\ is\ simulation\ data)$$

$$MSEx_c\ (cart\ postion) = 0.0113, MSE_\theta\ (see - saw\ angle) = 0.0139$$

$MSEs\ for\ both\ cart\ position\ and\ see - saw\ angle\ are\ reasonable\ to\ consider\ our\ mathematical\ model\ valid.$





Controller Design Via Pole Placement

Controller design using pole-placement for state-feedback control is to design suitable gain matrix **K** by placing the poles of the system in desired position.

The steps involved for deciding the desired closed-loop poles are:

- Compute the open-loop poles of the system.
- Observe the behavior of the open loop poles. The pole(s) closest to the origin on the left-half of the s-plane is/are dominant poles of the open loop system. We look for dominant second order behavior for this purpose.
- Place rest of the poles such that they are much faster (i.e., real part at least 2-3 times) than the dominant open-loop pole(s).

Controller Design Via Pole Placement

The open-loop poles for the seesaw system are at,

$$s_{1,2,3,4} = 3.1422, -1.7025 + i1.1532, -1.7025 - i1.1532, -13.5729$$

The pole s_1 is in the right half of s-plane

We try to place s_1 so that it is over 4-times of the real parts of dominant open-loop poles s_2, s_3. Then, the dynamics of the closed-loop system is governed by s_2, s_3 which are much slower than s_1, s_4. Therefore, the desired closed-loop poles for the see-saw system, in this case, are taken as,

$$s^{\star}_{1,2,3,4} = -7, -1.7025 + i1.1532, -1.7025 - i1.1532, -13.5729$$

With the desired closed loop poles above, using MATLAB, we find the gain matrix K as,

$$K = [134.57437 \; -76.246. \quad 12.3610 \; -24.2518]$$

With the dominant pole real part known, settling time for the system,

$$T_s = \frac{4}{-\zeta\omega_n} = \frac{4}{1.7025} \approx 2.34\, sec < t_b$$

*We call this K as **SFC1**

Controller Design Via Pole Placement

In order to obtain faster dynamics, we try to place the desired poles with zero complex part (critically damped) at,

$$s^{\star}_{1,2,3,4} = -10, \quad -11, \quad -7, \qquad -9,$$

With the desired closed loop poles above, using MATLAB, we find the gain matrix K as,

$$K = [451.4594 \quad -449.3483 \quad -47.6357 \quad -138.8892]$$

Sl.	Desired Poles	Gain (K)
SFC1	$\mathbf{-7, -1.7025 + i1.1532, -1.7025 - i1.1532, -13.5729}$	$\mathbf{[134.57437 \quad -76.246. \quad 12.3610 \qquad -24.2518]}$
SFC2	$\mathbf{-10, \quad -11, \quad -7, \qquad -9,}$	$\mathbf{[451.4594 \quad -449.3483 \quad -47.6357 \quad -138.8892]}$

LQR Controller Design

LQR stands for Linear Quadratic Regulator

The nomenclature revolves around the fact that the system being controlled is linear, the cost function is quadratic, and the controller is a regulator

LQR control is optimal in nature, the design procedure involves optimization of a cost function **J**.

$$J = \int_0^\infty L(x, u)dt = \int_0^\infty \left(\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u} + 2\boldsymbol{x}^T\boldsymbol{N}\boldsymbol{u}\right)dt \qquad (27)$$

Here, $\boldsymbol{Q}$ is a nxn positive semi-definite weighting matrix, $\boldsymbol{R}$ is a *pxp* (p is the input dimension), and $\boldsymbol{N}$ is an *nxp* positive semi-definite matrix.

Matrix $\boldsymbol{Q}$ represents the error tolerance for the system states. Matrix $\boldsymbol{R}$ represents tolerance for the input, and $\boldsymbol{N}$ defines error tolerance of the system states connected to the magnitude of the input

If we take,

$$\boldsymbol{K} = \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}$$

(**P** is a positive definite constant matrix)

We calculate **P** from the algebraic Riccati equation given as,

$$\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{Q} = 0$$

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LQR Controller Design

Methods used for selecting $\boldsymbol{Q}$ & $\boldsymbol{R}$:
Unique Weightings of State Errors and Input Magnitudes Based on State Error

Keeping large **R** means that the control input need to be smaller to keep J small

Selecting large $\boldsymbol{Q}$ implies that the system states must be smaller, in order to keep J small

$$\mathbf{Q} = \begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{bmatrix}, \mathbf{R} = \rho \begin{bmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_p \end{bmatrix}, \text{ and}$$

q_i is the weighting coefficient associated with the system states x_i $(i = 1,2,34)$

$$q_i = \frac{1}{e_{x_i}^2}$$

(e_{x_i} is the allowable error in state x_i)

ρ the weighting matrix for input $r_i (i = 1)$.

LQR Controller Design

For the see-saw system we are more concerned about the see-saw angle compared to the cart position.

Hence, the error tolerance for the see-saw angle will be smaller than the cart-position

Also, we are not concerned about the cart velocity and see-saw angular velocity, so we keep $q_3 = q_4 = 0$

If we keep $e_1 = 0.022 \ and \ e_2 = 0.013$ then we get $q_1 \approx 2000, q_2 \approx 6000$, we get,

$$\boldsymbol{Q} = \begin{bmatrix} 2000 & 0 & 0 & 0 \\ 0 & 6000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We take $\mathbf{R} = \boldsymbol{\rho r_1} = \mathbf{10}$, as we want a balanced control input.

We get the gain matrix using MATLAB with LQR for this case as,

$$K = [126.0995 \quad -83.8043 \quad 12.0913 \quad -27.1509]$$

LQR Controller Design

We use different state error tolerances to get different LQR gain **K**

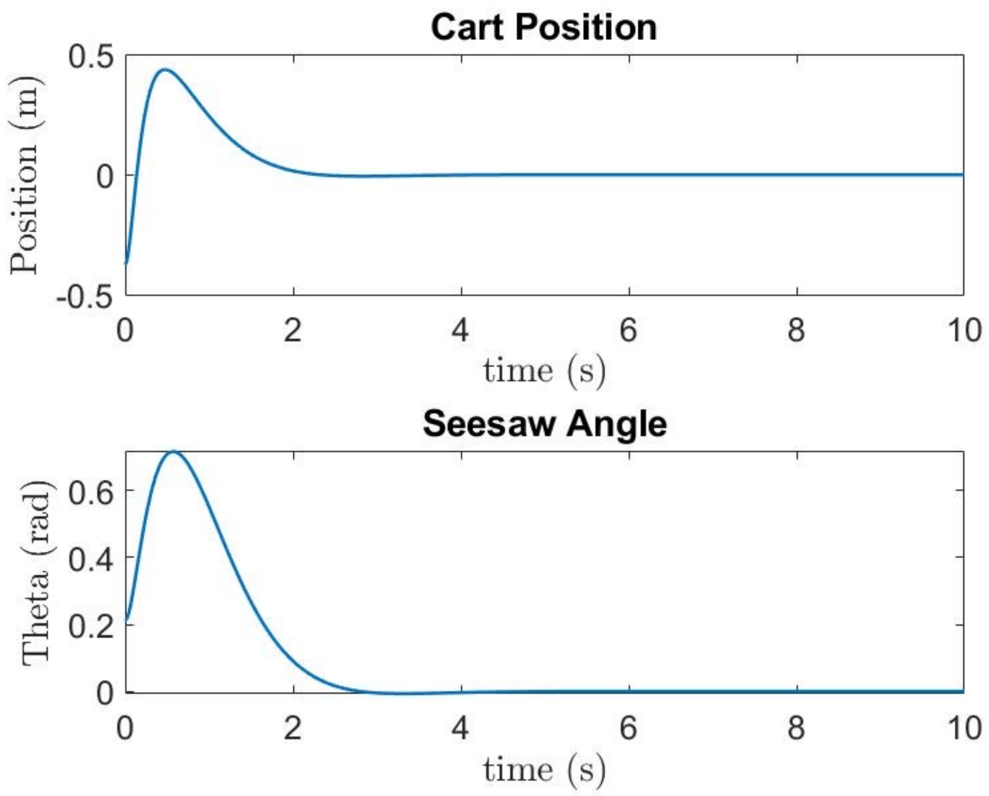
With R = 10		
Q Matrix	Gain (K)	
$\begin{pmatrix} 2000 & 0 & 0 & 0 \\ 0 & 6000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$[126.0995 \quad -83.8043 \quad 12.0913 \quad -27.1509]$	LQR1
$\begin{pmatrix} 1000 & 0 & 0 & 0 \\ 0 & 4000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$[116.9656 \quad -75.5759 \quad 11.1503 \quad -24.4726]$	LQR2
$\begin{pmatrix} 1000 & 0 & 0 & 0 \\ 0 & 6000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$[124.2691 \quad -82.7519 \quad 11.9258 \quad -26.7385]$	LQR3
$\begin{pmatrix} 1000 & 0 & 0 & 0 \\ 0 & 7000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$[127.3739 \quad -85.8875 \quad -12.2566 \quad -27.7153]$	LQR4
$\begin{pmatrix} 4000 & 0 & 0 & 0 \\ 0 & 4000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$[122.9831 \quad 79.1089 \quad 11.6992 \quad -25.8412]$	LQR5

LQR Controller Design

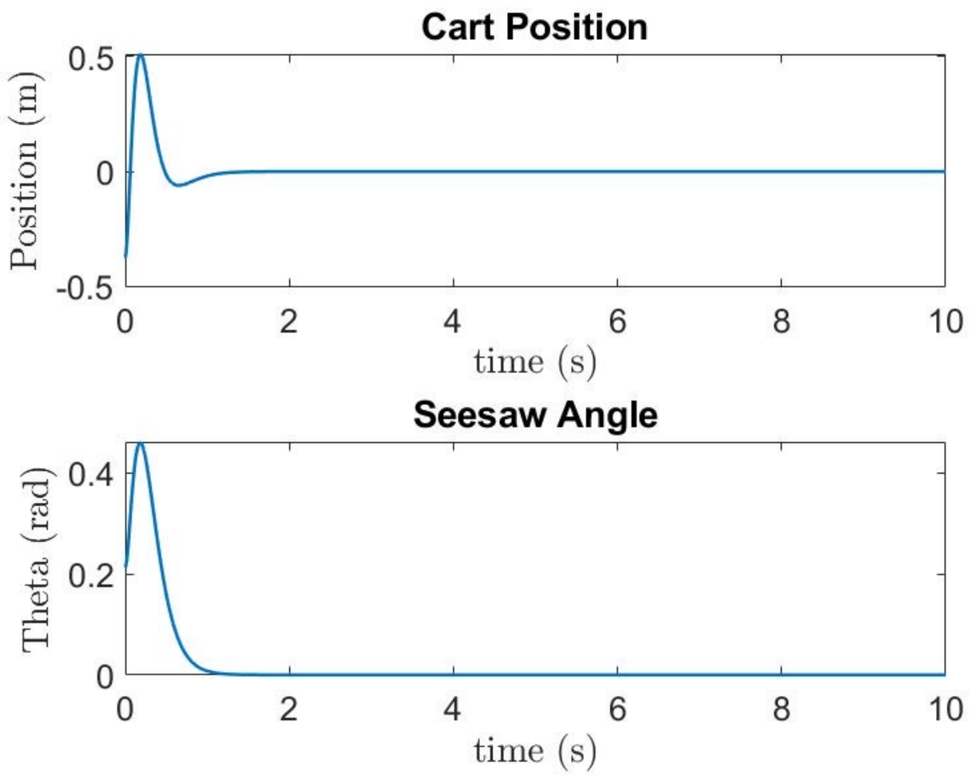
Closed Loop Poles with different LQR gains	
LQR1	[-13.6642 + 0.0000i ,-2.4125 + 2.5573i, -2.4125 - 2.5573i, -4.1086 + 0.0000i]
LQR2	[-13.7081 + 0.0000i,-2.2459 + 2.3693i,-2.2459 - 2.3693i,-3.9219 + 0.0000i]
LQR3	[-13.7064 + 0.0000i,-2.3546 + 2.5921i, -2.3546 - 2.5921i,-4.0787 + 0.0000i]
LQR4	[-13.7056 + 0.0000i,-2.4009 + 2.6813i,-2.4009 - 2.6813i, -4.1423 + 0.0000i]
LQR5	[-13.5794 + 0.0000i, -2.4345 + 2.2622i, -2.4345 - 2.2622i, -4.0167 + 0.0000i]

MATLAB Simulation Results

We simulate the state-space model of the see-saw with the designed full-state feedback gains with pole-placement and LQR approach in MATLAB to get the responses.

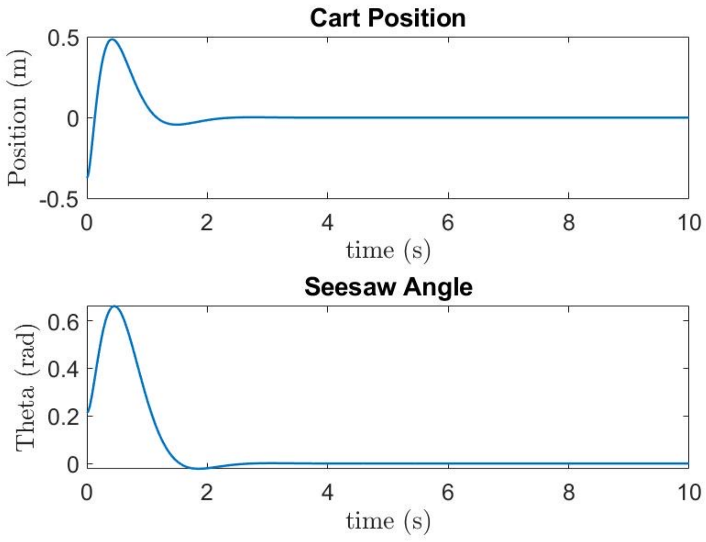


SFC1

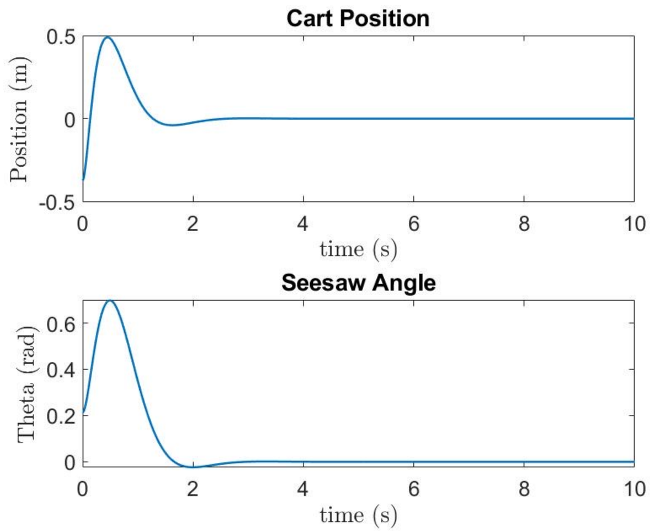


SFC2

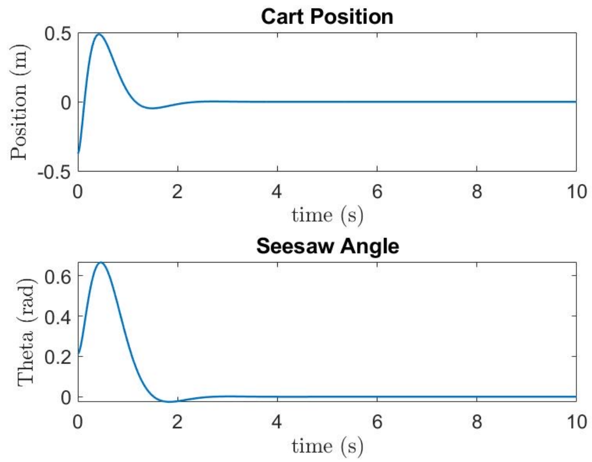
MATLAB Simulation Results



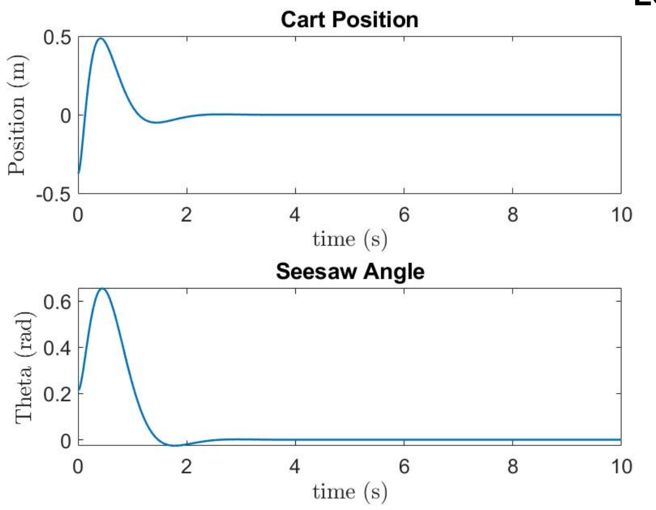
LQR1



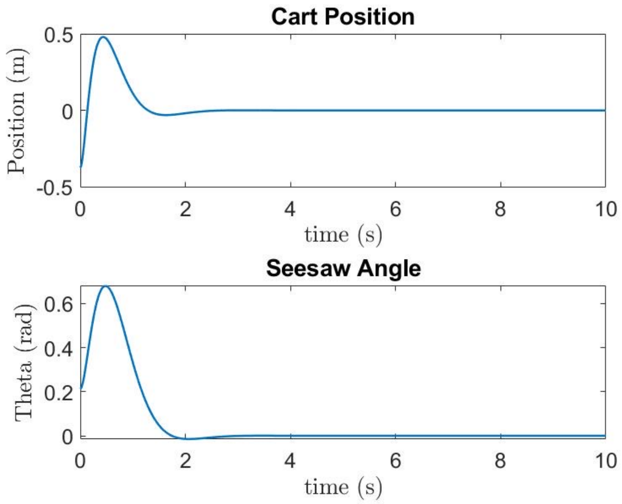
LQR2



LQR3



LQR4

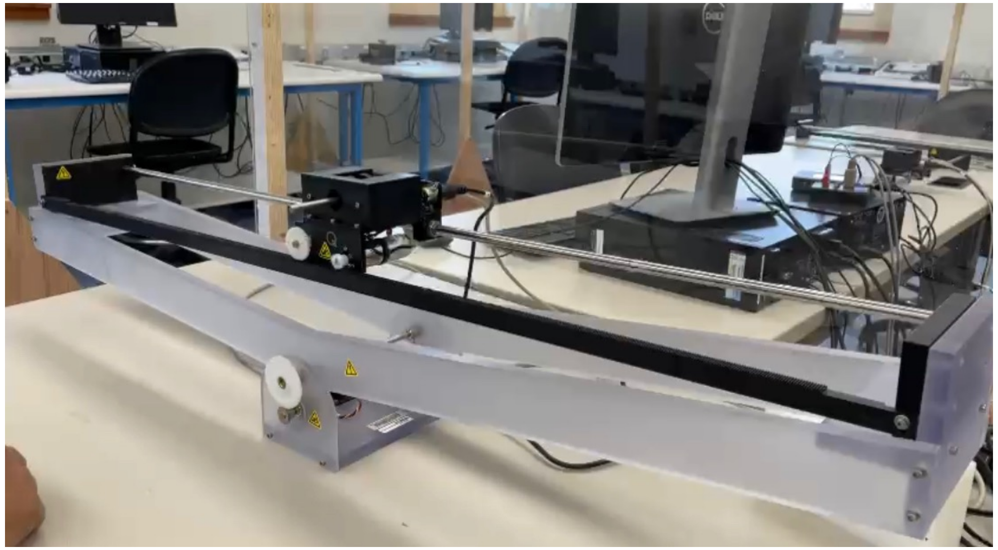


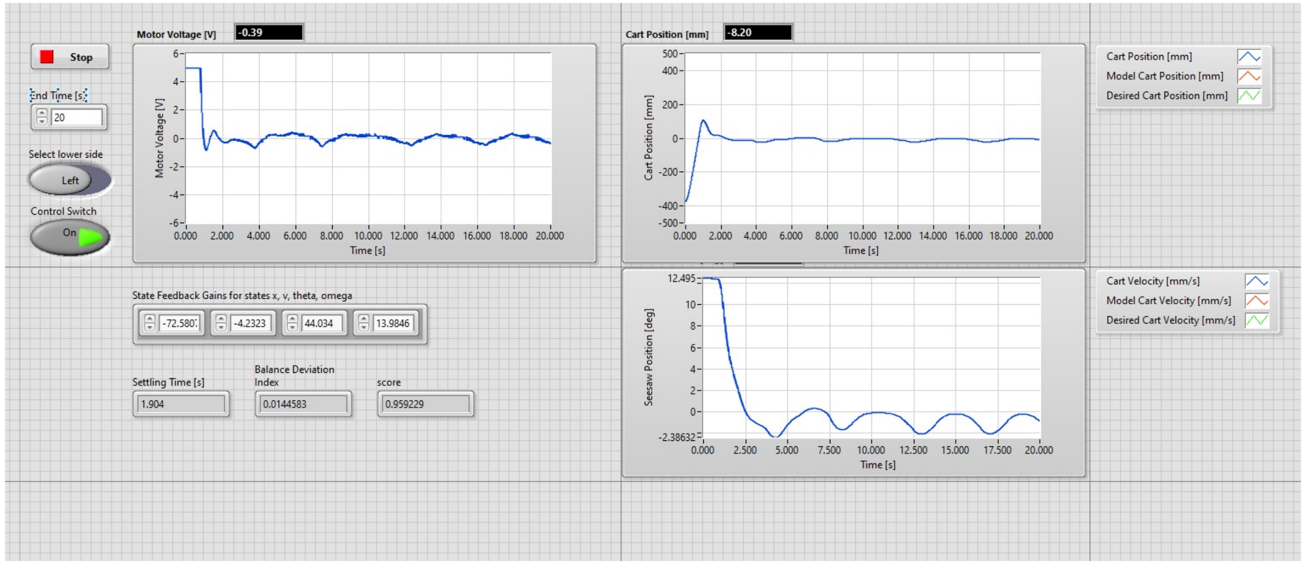
LQR5

Implementation in Practical Set Up

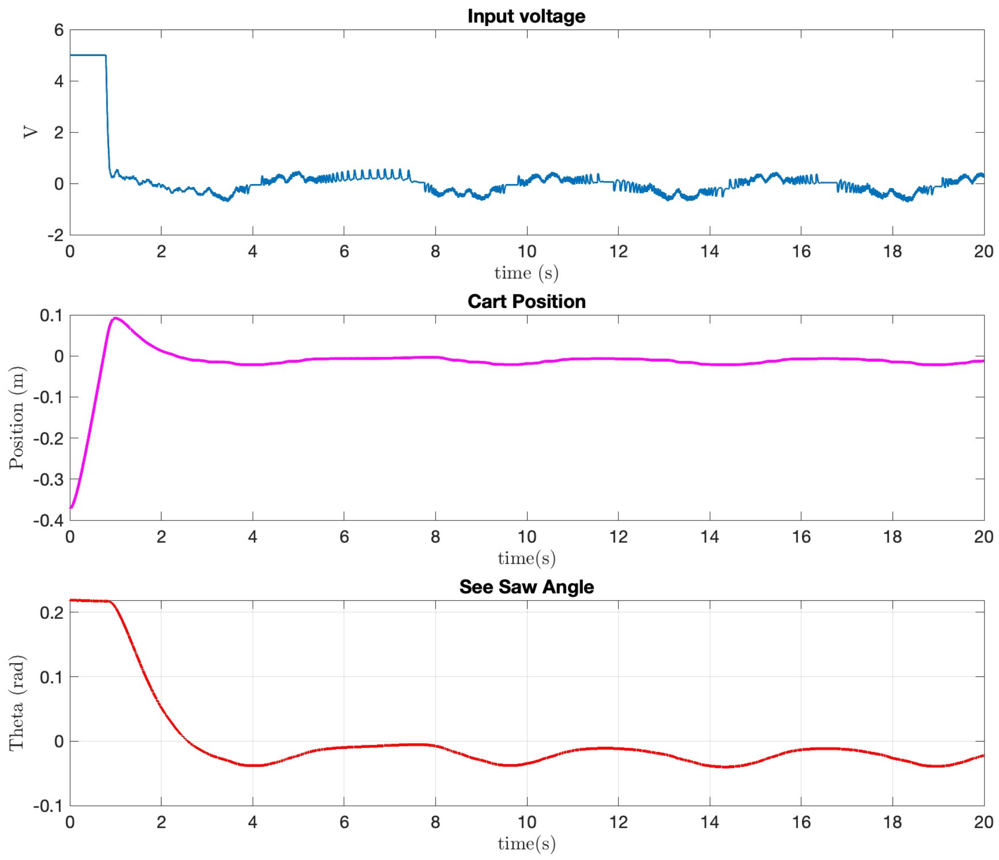
The practical experiment is done using LabView 2017 program provided.

Gain matrix Ks are entered in the LabView interface.

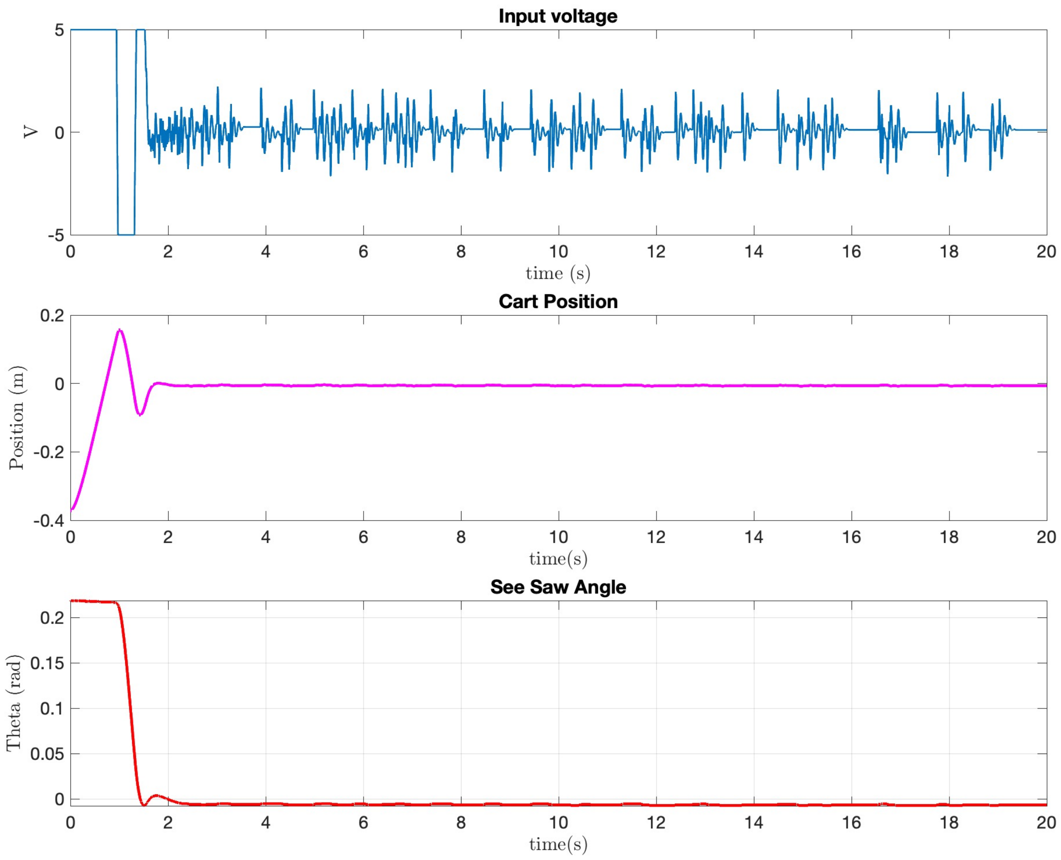




Practical Response

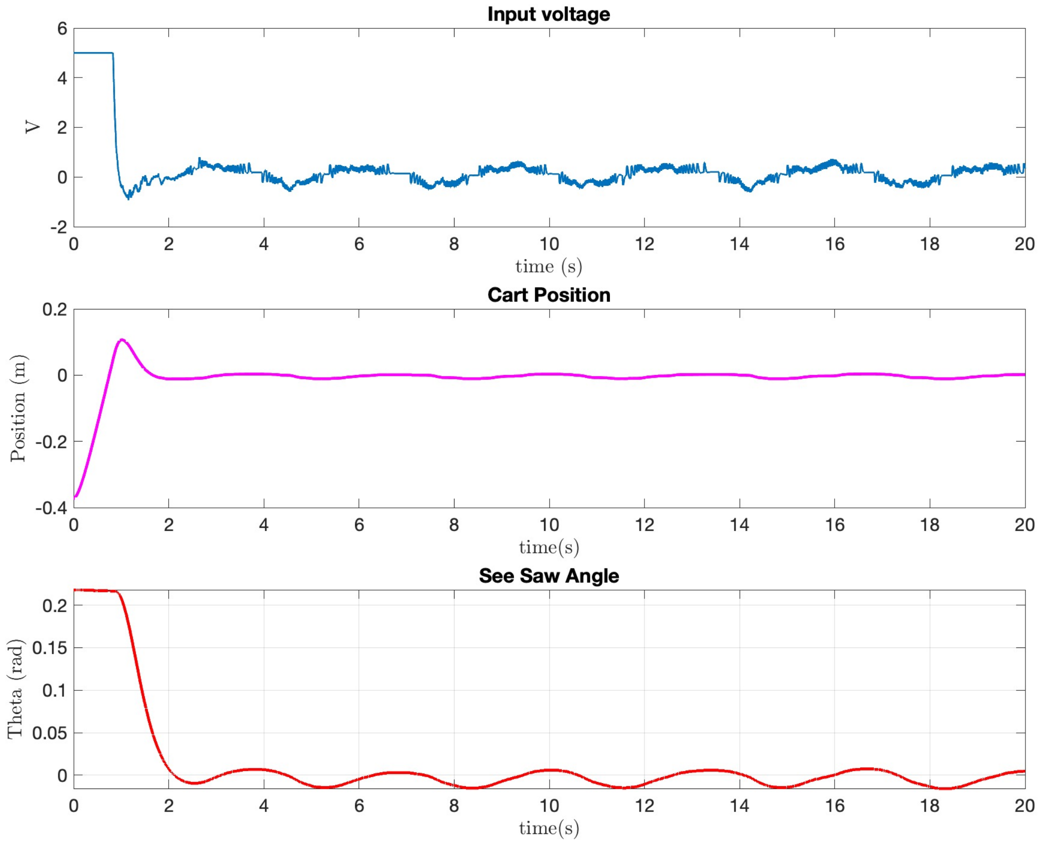


SFC1

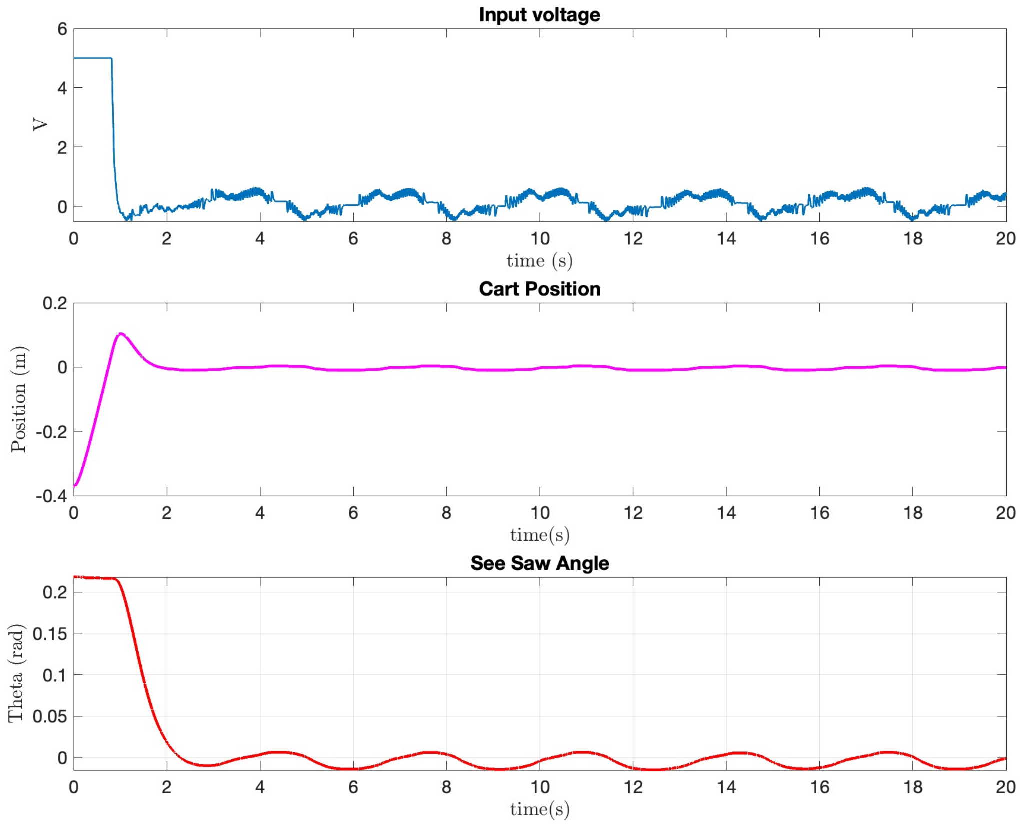


SFC2

Practical Response

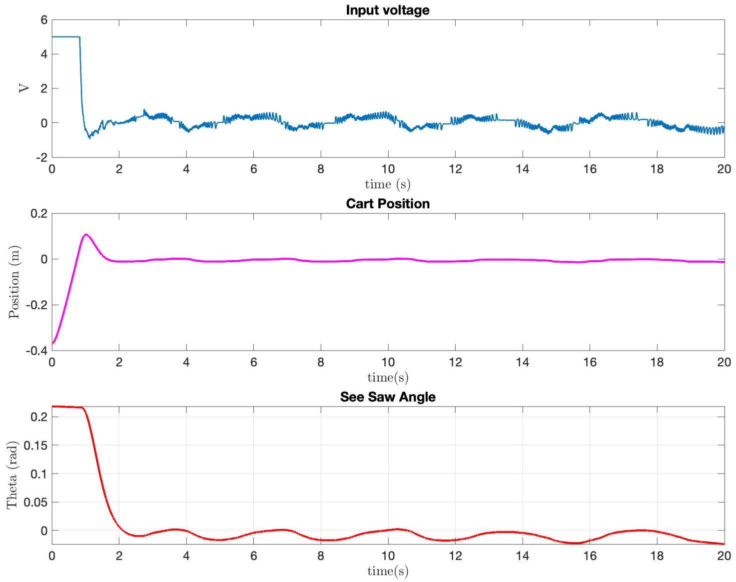


LQR1

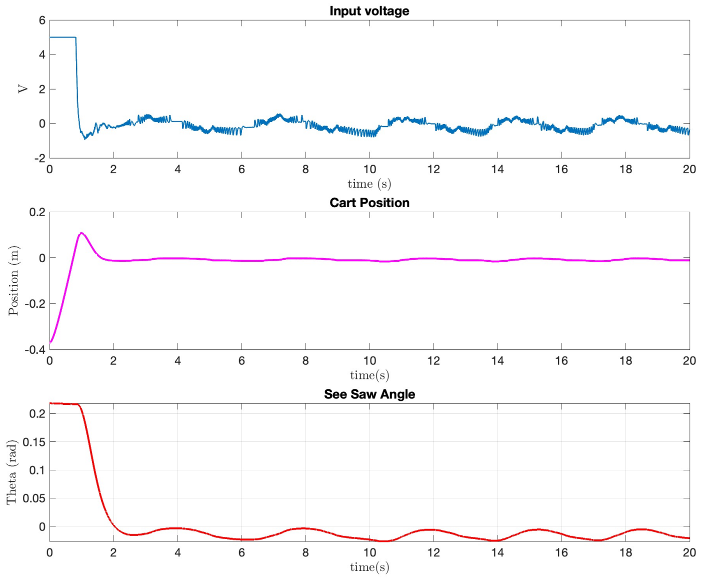


LQR2

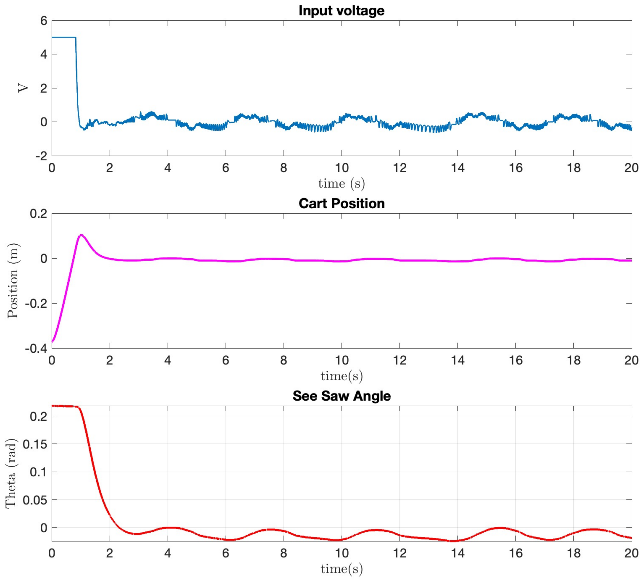
Practical Response



LQR3



LQR4



LQR5

Results

Controller	Balance time (t_b)	Balance Deviation Index I_p (From LabView Program) (Rad)	Balance Deviation Index I_p (From LabView Program) (deg)	Balance Deviation Index I_p (From our code) (Rad)	Balance Deviation Index I_p (From our code) (deg)
LQR1	1.656	0.007368	0.4223694268	0.002842	0.16284
LQR2	1.734	0.007381	0.4231146497	0.0022124	0.1267
LQR3	1.642	0.008219	0.4711528662	0.00599	0.3432
LQR4	1.614	0.01426	0.8174522293	0.01254	0.71888
LQR5	1.758	0.01204	0.6901910828	0.00973	0.5577
SFC1	2.014	0.0199	1.140764331	0.01745	1.0003
SFC2	1.31	0.00583	0.3342038217	0.00514	0.2947

Conclusion

We have successfully mathematically modeled the seesaw system and designed full-state feedback controller using pole-placement and LQR methods. Out of all the methods, the following gain matrix calculated with pole-placement gives the lowest balance time and balance deviation index,

$$\boldsymbol{K} = [\mathbf{451.4594} \quad \mathbf{-449.3483} \quad \mathbf{-47.6357} \quad \mathbf{-\,138.8892}]$$

Among the LQR based Ks, the best performance is obtained with,

$$\boldsymbol{K} =. [\mathbf{116.9656} \quad \mathbf{-75.5759} \quad \mathbf{11.1503} \quad \mathbf{-\,24.4726}]$$

For all the design cases, the performance obtained in the practical system is found to be well within the design requirements i.e.,

- The balance time $t_b < 5$ sec.
- The balance deviation index $I_p < 0.02$ (rad)

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[[1] Ogata, K., 2010. *Modern control engineering*. Prentice hall.

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Thank you