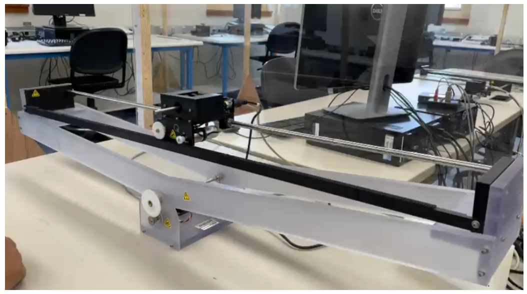
STATE FEEDBACK CONTROL OF CART SEESAW SYSTEM



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Executive Summary

In this study, we are going to balance a seesaw system in its horizontal position. The seesaw system consists of a seesaw and an electric motor driven cart moving on a track. The cart, possessing a mass, is mounted on the seesaw. The seesaw has two long arms hinged onto a support fulcrum. The seesaw rotates with respect to the pivot and an incremental encoder measures the angle of tilt. The cart moves on the track with the help of a DC motor and an optical encoder measures the position of the cart as well. The disturbance to the system is an unknown mass fixed on one end of the track. The objective is to balance the seesaw in its horizontal position (zero position), in presence of the disturbance, in minimum time and sustain the balance by moving the cart back and forth. The only input to the seesaw system is the voltage to the cart's motor and that voltage impacts the translational motion of the cart which in turn impacts the angle of the seesaw. Therefore, the process under study is not a SISO (single input, single output) system and requires the use of modern control practices. In short, we are going to design a controller to regulate the input voltage to the cart's motor so that the seesaw gets balanced around its zero-position subject to the constraints: balance time t_b< 5 sec and balance deviation index I_p < 0.02. In order to derive a mathematical model, we use Lagrangian concept to get the equations of motion of the seesaw system. The equations of motion are found to be non-linear in nature and linearized around an operating point. Then, we derive a state space model of the system using the linearized equations of motion. After the model has been validated with the practical model, we proceed to controller design. As we need to balance the seesaw in its zero position, the reference tilt angle will be zero. Hence, closed-loop full state feedback control is to be implemented so that the controller act as a regulator. The gain matrix for the full state feedback control scheme is obtained using two approaches: pole placement and LQR (Linear Quadratic Regulator). The designed controllers are simulated in MATLAB and on satisfactory performance we experimentally validate them.

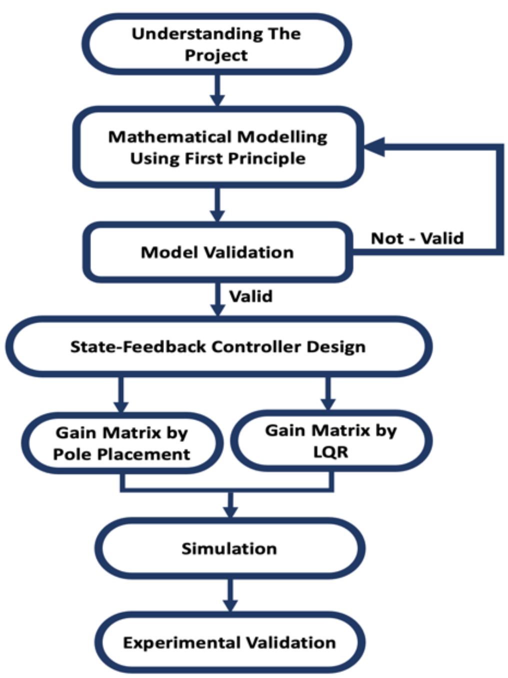


FIGURE 1: FLOW CHART OF THE PROJECT

Introduction

Overview of the Seesaw System

The seesaw system is composed of a seesaw (QUANSER SEESAW-E module) and a cart (QUANSER IP02 module) moving on a track. The schematic of the system is shown in Figure 2. The cart is mounted on the seesaw. It is equipped with a DC motor which allows the cart to move on the track by means of a gear and toothed rack. The seesaw consists of two long arms hinged onto a fulcrum and the fulcrum supports the two legs. The seesaw can tilt freely about a pivot connected to the fulcrum. The tilt angle of the seesaw around its pivot is measured using an incremental encoder. The motor-powered cart can move freely along the length of the seesaw. The only external input to the system is the cart motor's input voltage.

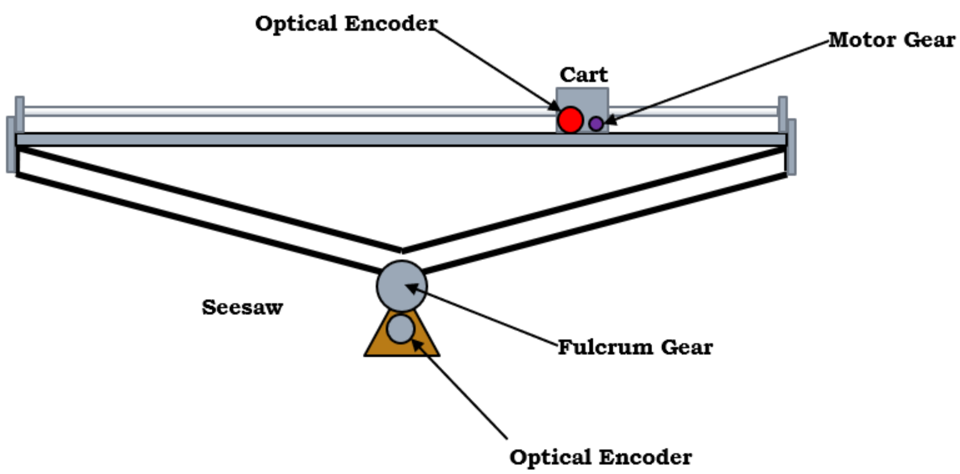
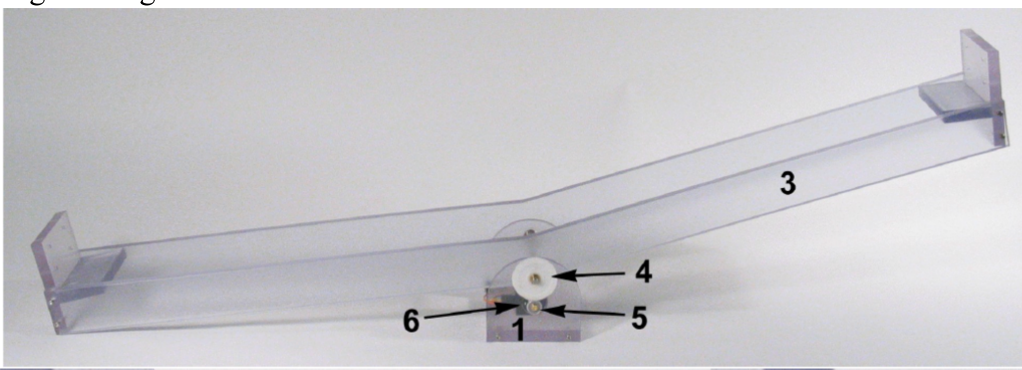


FIGURE 2: SCHEMATIC OF THE SEESAW SYSTEM

FIGURE 3: THE PHYSICAL SEESAW SYSTEM

The QUANSER SEESAW-E Module

The Quanser SEESAW-E module's design is such that it can accommodate the IP02 cart module on top of it. The seesaw has two long arms hinged to a fulcrum that supports it. The material used for the manufacturing of the seesaw is matte finished polycarbonate with machine precision. It can tilt freely about a rotation axis called pivot. The tilt angle is measured using a quadrature optical encoder through a pinion and backlash proof gear system. The powered cart is capable of moving freely along the length of the seesaw.



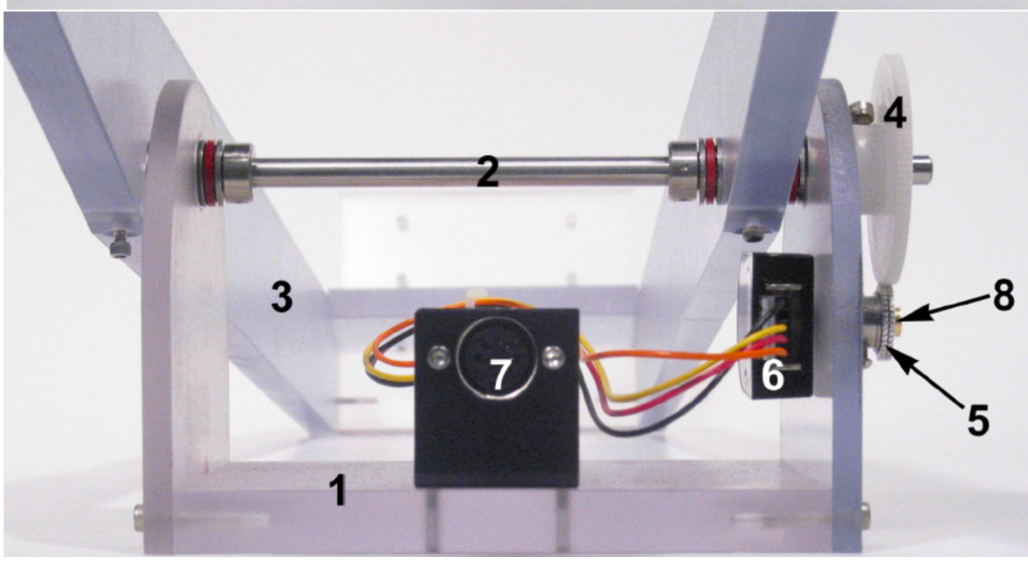


FIGURE 4: DIFFERENT PARTS OF THE SEESAW-E MODULE

ID #	Description	ID #	Description
1	Support Fulcrum	2	Pivot Axis
3	Seesaw Arm	4	Seesaw Position Pinion
5	Encoder Anti-Backlash Gear	6	Encoder
7	Encoder Connector	8	Encoder Shaft

Different parameters of the SEESAW-E module are given in table 1.

Symbol	Description	Value	Unit
M_{sw}	Mass of the one-SEESAW(-E)-plus-one-IP01-or-IP02-Track System	3.6	kg
K_{gs}	SEESAW(-E) Geartrain Gear Ratio	3	
D_T	Distance from Pivot to the IP01 or IP02 Track	0.125	m
D_c	Distance from Pivot to the Centre Of Gravity of the one-SEESAW(-E)-plus-one-IP01-or-IP02-Track System	0.058	m
J_{sw}	Moment of Inertia of the one-SEESAW(-E)-plus-one-IP01-or-IP02-Track System, about its Center Of Gravity	0.395	$kg.m^2$
B_{sw}	Viscous Damping Coefficient as seen at the Seesaw Pivot Axis	≈0	N.m.s/rad
g	Gravitational Constant on Earth	9.81	m/s^2
K_{P_SW}	SEESAW Potentiometer Sensitivity	-0.2482	rad/V
K_{E_SW}	SEESAW-E Encoder Resolution	0.0015	rad/count
$è_{range}$	SEESAW(-E) Approximative Angular Range on a Flat Surface	±11.5	°

TABLE-1 : Different parameters of the SEESAW-E module

The IP02 Cart Module

The IP02 module is a solid DC motor driven aluminium cart. The cart is capable of sliding through a stainless-steel shaft using linear bearings. The mechanism of the motion of the cart is composed of a rack and pinion. This mechanism helps in maintaining consistent and continuous motion along its track. The position of the IP02 cart is measured with a quadrature incremental encoder with its shaft meshing with the track of the cart through an additional pinion.

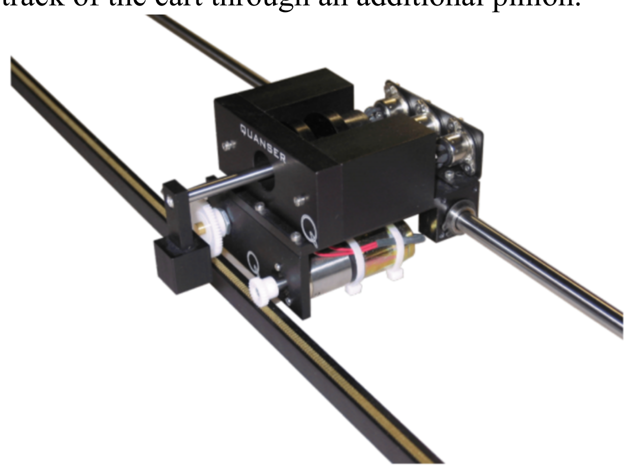


FIGURE 4: THE QUANSER IP02 MODULE

Symbol	Description	Value	Variation
V_{nom}	Motor nominal input voltage	6.0 V	
R_m	Motor armature resistance	2.6 Ω	$\pm$ 12%
L_m	Motor armature inductance	0.18 mH	
k_t	Motor current-torque constant	7.68×10^{-3} N m/A	$\pm$ 12%
k_m	Motor back-emf constant	7.68×10^{-3} V/(rad/s)	$\pm$ 12%
η_m	Motor efficiency	0.69	$\pm$ 5%
$J_{m,rotor}$	Rotor moment of inertia	3.90×10^{-7} kg $\cdot$ m^2	$\pm$ 10%
K_g	Planetary gearbox gear ratio	3.71	
η_g	Planetary geabox efficiency	0.90	$\pm$ 10%
M_c	Mass of cart	0.38 kg	
M_w	Mass of cart weight	0.37 kg	
$B_{eq,c}$	Equivalent viscous damping coefficient (Cart)	4.3 N m s/rad	
$B_{eq,c}$	Equivalent viscous damping coefficient (Cart and Weight)	5.4 N m s/rad	
L_t	Track length	0.990 m	
T_c	Cart travel	0.814 m	
P_r	Rack pitch	1.664×10^{-3} m/tooth	
r_{mp}	Motor pinion radius	6.35×10^{-3} m	
N_{mp}	Motor pinion number of teeth	24	
r_{pp}	Position pinion radius	0.01483 m	
N_{pp}	Position pinion number of teeth	56	
K_{ec}	Cart encoder resolution	2.275×10^{-5}	
K_{ep}	Pendulum encoder resolution	0.0015 rad/count	
f_{max}	Maximum input voltage frequency	50 Hz	
I_{max}	Maximum input current	1 A	
ω_{max}	Maximum motor speed	628.3 rad/s	

TABLE-2 Different parameters of the IP02 cart module

Problem Statement

The cart mounted on the seesaw can alter the centre of gravity of the seesaw system by its movement on its track and thereby capable of altering the tilt angle of the seesaw. The seesaw is said to be balanced when it stays horizontal with the angle of tilt being zero. The movement of the cart on its track can be regulated by regulating the input voltage to the DC motor of the cart. The control objective is to balance the seesaw in minimum possible time and keep the balance maintained, in presence of disturbance, by making the cart move back and forth on its track. For each tilt angle of the seesaw, there is an equilibrium position of the cart. The equilibrium position is unstable as slight disturbance can make the seesaw move away from the equilibrium. The control objective is to obtain a stabilizing controller for the seesaw such that the see saw remains horizontal and sustains the balance even with slight disturbances.

Design Objective

The design objective is to develop a stabilizing controller to balance the seesaw in horizontal position from the initial position of the seesaw system by moving the cart back and forth as necessary. The controller is supposed to manipulate the cart's motion by regulating the input voltage to the cart's DC motor. The initial position is when the cart is at one end of the track and one of the seesaw arm touches the ground. The controller should be able to reject disturbances. The unknown mass fixed on one end of the track serves as a source of disturbance. The designed controller should also work under the following two constraints:

- The balance time t_b need to be < 5 sec.
- The balance deviation index I_p need to be < 0.02

The balance time (t_b) is the time when the tilt angle of the seesaw first comes within the range of $\pm$ 0.05 rad. The balance deviation index I_p is given as,

$$I_p = \frac{1}{10}\int_{t_b}^{t_b+10} |\theta| dt \quad (\theta \text{ is the tilt angle of the seesaw})$$

Mathematical Model

The first step in designing a controller to balance the seesaw system is to obtain a mathematical model of the system which can describe the dynamics of the system reasonably well. There can be two approaches in modelling a system: 1) identifying the system based on practical data, 2) modelling the system using laws of physics (i.e., modelling from first principle). In this section, we are going to model the system using laws of physics. For modelling of the seesaw system, we divide the task in two parts: i) Modelling of the Mechanical Part, ii) Modelling of the electrical part. Then we combine the two parts to get unified model of the seesaw system.

Modeling of the Mechanical Part

The mechanical part consists of the seesaw module with the cart mounted on it. The schematic of the mechanical part is shown in figure 5.

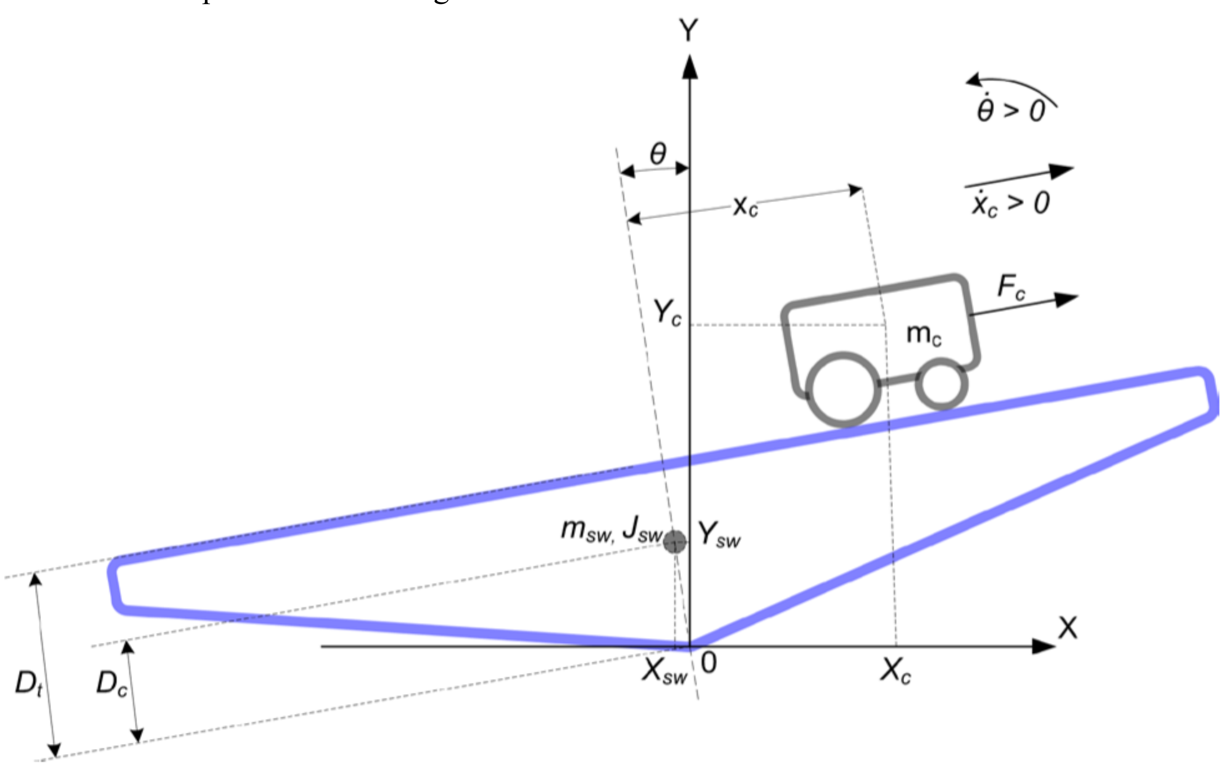


FIGURE 5: SCHEMATIC OF THE MECHANICAL PART

As shown in figure 5, the positive tilt of the seesaw is considered in the counter-clockwise direction and the positive motion of the cart is towards the right. The zero position of the seesaw is when the seesaw remains perfectly horizontal (i.e., $\theta = 0$). For brevity, the mass of the see-saw is considered to be concentrated in its centre of gravity. The input of the mechanical part is the force exerted by the cart and the outputs are the position of the cart, and the tilt angle of the seesaw.

The different parameters are as given below,

θ	$Angle\ of\ the\ seesaw$
m_c	$mass\ of\ the\ cart$

m_{sw}	$mass\ of\ cart + mass\ of\ seesaw$
x_c	$Position\ of\ cart\ with\ respect\ to\ CG\ of\ seesaw + cart$
$[X_c,\ Y_c]$	$absolute\ coordinate\ of\ CG\ of\ the\ cart$
$[X_{sw}\ ,\ Y_{sw}]$	$absolute\ coordinate\ of\ CG\ of\ seesaw + cart$
D_c	$Distance\ from\ the\ pivot\ of\ seesaw\ to\ [X_{sw}\ ,\ Y_{sw}]$
D_t	$Distance\ of\ seesaw\ from\ pivot$
J_{sw}	$Moment\ of\ inertia\ of\ seesaw + cart$
F_c	$Force\ exerted\ by\ the\ cart$

Here, the pivot of the seesaw act as the origin of the reference cartesian frame of reference. With respect to the frame of reference, the absolute coordinates of the centre of gravity of the seesaw system are given by,

$$X_{sw} = -D_c \sin(\theta)\ ,\ Y_{sw} = D_c \cos(\theta) \tag{1}$$

Similarly, the absolute coordinates of the centre of gravity of the cart are given by,

$$Xc = -D_t \sin(\theta) + x_c cos(\ \theta)$$
$$Y_c = D_t \cos(\theta) + x_c sin(\theta)$$
(2)

Therefore, from equation (2), the absolute cartesian velocity coordinates of the cart's center of gravity can be given as,

$$\dot{X_c} = -D_t \cos(\theta)\ \dot{\theta} + \ \dot{x_c} \cos(\theta) - x_c sin(\theta)\dot{\theta}$$
$$\dot{Y_c} = = -D_t \sin(\theta)\ \dot{\theta} + \ \dot{x_c} \sin(\theta) + x_c cos(\theta)\dot{\theta}$$
(3)

In order to obtain equations of motion of the seesaw system, Lagrange's method will be used. For this method, the Lagrangian of the system needs to be calculated. Lagrangian of a system is calculated through the calculation of the total kinetic and potential energies. Let us consider, the total kinetic energy (KE) of the seesaw system be T and the total potential energy (PE) of the same be V. Therefore, we can define,

$$T = Translational\ KE\ of\ the\ Cart\ (T_{c,t}) + Rotational\ KE\ of\ the\ cart(T_{c,r}) + \ Rotational\ KE\ of\ the\ seesaw(T_{ss,r})$$
$$V = PE\ of\ the\ cart\ (V_c) + PE\ of\ the\ seesaw\ (V_{ss}) \tag{4}$$

The translational kinetic energy of the cart is calculated from the linear velocity of its center of mass as (using equation (3)),

$$T_{c,t} = \frac{1}{2} m_c (\sqrt{\dot{X_c^2} \ + \dot{Y_c^2}} \ = \frac{1}{2} m_c (D_t^2 \dot{\theta^2} - 2D_t \dot{\theta} \dot{x_c} + \dot{x_c}^2 + x_c^2 \dot{\theta^2} \tag{5}$$

The rotational kinetic energy $(T_{c,r})$ of the cart is ignored as it is contributed by the dc motor with a negligible moment of inertia. The moment of inertia of the dc motor is, $J_m = 3.9x10^{-7}\ kg\ m^2$ (which is negligible).

The rotational kinetic energy of the seesaw is given as,

$$J_{sw} = \frac{1}{2} J_{sw} \dot{\theta^2} \tag{6}$$

Using equations (5) and (6), the total kinetic energy of the seesaw system,

$$T = \frac{1}{2} m_c \dot{x}_c^2 - m_c D_t \dot{\theta} \dot{x}_c + \left(\frac{1}{2} J_{sw} + \frac{1}{2} m_c D_t^2 + \frac{1}{2} m_c x_c^2\right) \dot{\theta}^2 \qquad (7)$$

The potential energies of seesaw and cart are given as,

$V_c = m_c g Y_c,\ \ V_{ss} = m_{sw} g Y_{sw}$ (Where, g is the gravitational constant) (8)

Therefore, the total potential energy of the system, using equations (1), (2), (4), and (8),

$$V = g(m_c D_t \cos\theta + m_c x_c sin\theta + m_{sw} D_c cos\theta) \qquad (9)$$

The Lagrangian of a system is defined as the difference of total kinetic energy and total potential energy of the system. For the seesaw system, the Lagrangian (L) is given as (using equation (7) and (9)),

$$L = T - V = \frac{1}{2} m_c \dot{x}_c^2 - m_c D_t \dot{\theta} \dot{x}_c + \left(\frac{1}{2} J_{sw} + \frac{1}{2} m_c D_t^2 + \frac{1}{2} m_c x_c^2\right) \dot{\theta}^2 - g(m_c D_t \cos\theta + m_c x_c sin\theta + m_{sw} D_c cos\theta) \qquad (10)$$

We use the Euler-Lagrange equation to get equations of motion defining the seesaw system. The Euler-Lagrange equation is given by,

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_1 \qquad (11.1)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_c}}\right) - \frac{\partial L}{\partial x_c} = Q_2 \qquad (11.2)$$

Where, $\boldsymbol{Q_1}$ is the generalized forces acting on the cart and $\boldsymbol{Q_2}$ is the generalized forces acting on the seesaw.

$$Q_1 = F_c - B_c \dot{x}_c\ ,\ Q_2 = -B_{sw} \dot{\theta} \qquad (12)$$

Where, B_c is the damping co-efficient of the cart and B_{sw} is the damping co-efficient of the seesaw.

From equation (11.1 and 11.2), we get the two Lagrange's equations or the equations describing the motion of the seesaw system as,

$$m_c \ddot{x}_c - m_c D_t \ddot{\theta} - m_c x_c \dot{\theta}^2 + g m_c sin\theta = F_c - B_c \dot{x}_c \qquad (13)$$

$$m_c \ddot{\theta} x_c^2 + 2 m_c \dot{x}_c \dot{\theta} + g m_c cos\theta\, x_c - m_c D_t \ddot{x}_c + (J_{sw} + m_c D_t^2)\ddot{\theta} - g(m_c D_t sin\theta - m_{sw} sin\theta) = -B_{sw} \dot{\theta} \qquad (14)$$

Electrical System Modeling

The electrical system consisting of a DC motor which generates torque, and that torque, via an output gear connected to the motor shaft, gets translated into the force exerted on the cart (F_c) through rack and pinion mechanism. The input to the electrical system is DC voltage V and the voltage is responsible for exerting the translational force on the cart. Therefore, we need to incorporate the input voltage by replacing (F_c) to the mathematical model of the seesaw system.

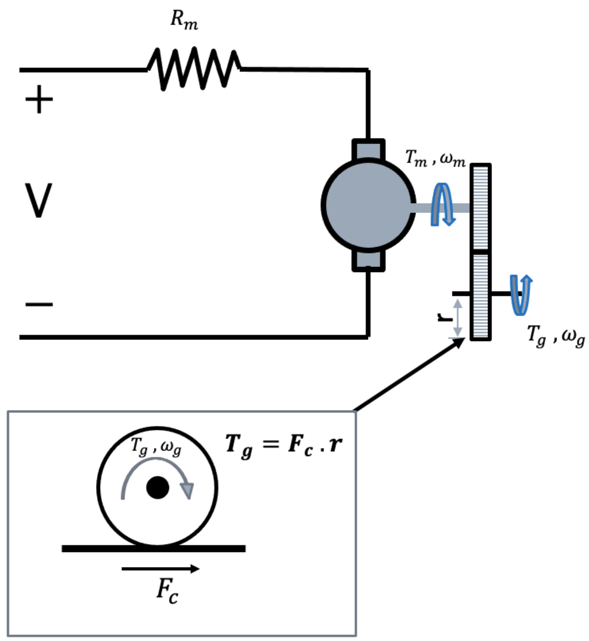


FIGURE 6: SCHEMATIC OF THE ELECTRICAL SYSTEM

Here,
$T_m = DC\ motor\ output\ torque\ at\ the\ shaft$
$R_m = DC\ motor\ armature\ resistance$
$K_b = Back\ emf\ constant$
$K_t = Motor\ Torque\ Constant$
$\omega_m = Angular\ velocity\ of\ the\ motor\ shaft$
$K_g = output\ gear\ ratio$
$\eta_g = Output\ Gear\ Efficiency$
$\eta_m = Motor\ efficiency$
$T_g = Torque\ at\ the\ output\ gear\ transferred\ to\ the\ rack$
$r = radius\ of\ the\ output\ gear,\ I = current\ flowing\ in\ the\ motor$

The electrical governing equation of the DC motor is given by,

$$V = IR_m + K_b\omega_m \tag{15}$$

The mechanical governing equation of the DC motor is given by,

$$T_m = \eta_m K_t I \tag{16}$$

Here,

$$T_g = \eta_g K_g T_m \ \Rightarrow T_m = \frac{T_g}{\eta_g K_g} \tag{17}$$

Using $T_g = F_c r$ in equation (17),

$$T_m = \frac{F_c r}{\eta_g K_g} \tag{18}$$

Also,

$$\omega_m = K_g \omega_g = K_g \frac{\dot{x}_c}{r} \tag{19}$$

Using equation (18) in equation (16) ⇒

$$I = \frac{F_c r}{\eta_m \eta_g K_t K_g} \tag{20}$$

Using equations (19) and (20) in equation (15), we get the relation of F_c and V as,

$$F_c = \frac{\eta_m \eta_g K_t K_g}{R_m r} V - \frac{\eta_m \eta_g K_t K_b K_g^2}{R_m r^2} \dot{x}_c \qquad (21)$$

Combining Electrical and Mechanical Part

For combining the electrical and mechanical part, we substitute the expression of F_c having the input voltage V into the two Lagrange's equations. The modified Lagrange's equations are given as (from equations (13) and (14)),

$$m_c \ddot{x}_c - m_c D_t \ddot{\theta} - m_c x_c \dot{\theta}^2 + m_c sin\theta + \left(\frac{\eta_m \eta_g K_t K_b K_g^2}{R_m r^2} + B_c\right) \dot{x}_c = \frac{\eta_m \eta_g K_t K_g}{R_m r} V \qquad (22)$$

$$m_c \ddot{\theta} x_c^2 + 2 m_c \dot{x}_c \dot{\theta} + g m_c cos\theta\, x_c - m_c D_t \ddot{x}_c + (J_{sw} + m_c D_t^2)\ddot{\theta} - g(m_c D_t sin\theta - m_{sw} sin\theta) + B_{sw}\dot{\theta} = 0 \qquad (23)$$

Now we solve the two Lagrange's equations (22) and (23) using second-order time derivative of the Lagrangian coordinates to get the combined equations of motion for the seesaw system as,

$$\ddot{x}_c = \frac{1}{(m_c x_c^2 + J_{sw}) m_c} [m_c^2 x_c^3 \dot{\theta}^2 + \left\{\left(\frac{\eta_m \eta_g K_t K_g}{R_m r} V - \frac{\eta_m \eta_g K_t K_b K_g^2}{R_m r^2} \dot{x}_c\right) m_c - g m_c^2 sin\theta - B_c \dot{x} m_c\right\} x_c^2 \\ + (-2 m_c^2 D_t \dot{x}_c \dot{\theta} + m_c^2 D_t^2 \dot{\theta}^2 + m_c J_{sw} \dot{\theta}^2 - m_c^2 D_t g cos\theta) x_c - (m_c D_t^2 B_c - B_c J_{sw}) \dot{x}_c - m_c D_t g\, m_{sw} D_c sin\theta] \qquad (24)$$

$$\ddot{\theta}_c = \frac{1}{(m_c x_c^2 + J_{sw})} [(D_t m_c \dot{\theta}^2 - 2 m_c \dot{x}_c \dot{\theta} - g m_c cos\theta) x_c + D_t \left(\frac{\eta_m \eta_g K_t K_g}{R_m r} V - \frac{\eta_m \eta_g K_t K_b K_g^2}{R_m r^2} \dot{x}_c\right) - D_t B_c \dot{x}_c \\ + g m_{sw} D_c sin\theta - B_{sw} \dot{\theta}] \qquad (25)$$

As we can see that the combined equations of motion of the seesaw system have translational and rotational accelerations expressed in terms of displacements and velocities. Also, the said equations are non-linear in nature. It is very much complex to analyse and implement a nonlinear model of a system. Therefore, we try to linearize the system around a set of operating points.

For linearization purpose, we take operating points as $[x_c\, , \theta, \dot{x}_c\, , \dot{\theta}] = [0,0,0,0] = z^\star$. We can consider $\ddot{x}_c$ and $\ddot{\theta}$ as nonlinear functions of $x_c\, , \theta, \dot{x}_c\, , \dot{\theta}$ as,

$$\ddot{x}_c = f(x_c, \theta, \dot{x}_c\, , \dot{\theta})$$
$$\ddot{\theta} = g(x_c\, , \theta, \dot{x}_c\, , \dot{\theta})$$

Now, we Taylor expand functions f and g around the operating points $z^\star$ and ignore the higher order terms as,

$$f(x_c, \theta, \dot{x}_c\, , \dot{\theta}) = f|_{z^*} + \frac{\partial f}{\partial x_c}|_{z^*}(x_c - 0) + \frac{\partial f}{\partial \theta}|_{z^*}(\theta - 0) + \frac{\partial f}{\partial \dot{x}_c}|_{z^*}(\dot{x}_c - 0) + \frac{\partial f}{\partial \dot{\theta}}|_{z^*}(\dot{\theta} - 0) \qquad (26)$$

$$g(x_c, \theta, \dot{x}_c\, , \dot{\theta}) = g|_{z^*} + \frac{\partial g}{\partial x_c}|_{z^*}(x_c - 0) + \frac{\partial g}{\partial \theta}|_{z^*}(\theta - 0) + \frac{\partial g}{\partial \dot{x}_c}|_{z^*}(\dot{x}_c - 0) + \frac{\partial g}{\partial \dot{\theta}}|_{z^*}(\dot{\theta} - 0) \qquad (27)$$

From, equations (26) and (27), we get the linearized equations of motion of the seesaw system as,

$$\ddot{x}_c = a_1 x_c + a_2 \theta + a_3 \dot{x}_c + a_4 \dot{\theta} + b_1 V \qquad (28)$$
$$\ddot{\theta} = a_5 x_c + a_6 \theta + a_7 \dot{x}_c + a_8 \dot{\theta} + b_2 V \qquad (29)$$

Here,

$$a_1 = -\frac{m_c D_t g}{J_{sw}}\, , \; a_2 = \frac{-g m_c R_m r^2 J_{sw} + m_c D_t R_m r^2 g m_{sw} D_c}{R_m r^2 J_{sw} m_c}$$

$$a_3 = -\frac{J_{sw} \eta_g K_g^2 \eta_m K_t K_b + J_{sw} B_c R_m r^2 + m_c D_t^2 \eta_g K_g^2 \eta_m K_t K_b + m_c D_t^2 B_c R_m r^2}{R_m r^2 J_{sw} m_c}$$

$$a_4 = -\frac{D_t B_{sw}}{J_{sw}}\ ,\ a_5 = -\frac{g m_c}{J_{sw}}\ ,\ a_6 = \frac{g m_{sw} D_c}{J_{sw}}\ ,\ a_7 = -\frac{\eta_g K_g^2 \eta_m K_t K_b D_t + B_c R_m r^2 D_t}{R_m r^2 J_{sw}}$$
$$a_8 = -\frac{B_{sw}}{J_{sw}}\ ,\ b_1 = \frac{J_{sw}\eta_g K_g \eta_m K_t r + m_c D_t^2 \eta_g K_g \eta_m K_t r}{R_m r^2 J_{sw} m_c}\ ,\ b_2 = \frac{\eta_g K_g \eta_m K_t D_t}{r R_m J_{sw}}$$

State Space Modeling of the Seesaw System

State-space modeling of dynamical systems comes under modern control theory. Modern control theory is applicable to multi-input, multi-output systems which may be linear or non-linear, time-invariant or time-variant. The seesaw system has two outputs (x_c, θ) and one input (V), therefore classical control system modeling cannot be used for the system. Therefore, we proceed with deriving state space model of the seesaw system.

The state of a dynamical system is the minimum possible set of variables, called state variables, the knowledge of which at any initial time along with the input(s), completely defines the behavior of the system at any given point of time. The state-space is the n-dimensional space consisting of axes corresponding to n-state variables. For brevity, let us consider a second order mass-spring-damper system with input T and output θ as below,

$$M\ddot{\theta} + D\dot{\theta} + k\theta = T \tag{30}$$

Let us consider two states needed of the above system as $x_1 = \theta, x_2 = \dot{\theta}$. Using (30), we can write the state equations as,

$$\dot{x_1} = x_2$$

$$\dot{x_2} = -\frac{D}{M}x_2 - \frac{k}{M}x_1 + \frac{T}{m}$$

In matrix form, the above state equations can be written as,

$$\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{D}{M} & -\frac{k}{M}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}T \tag{31}$$

If we consider, $\boldsymbol{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix}0 & 1\\ -\frac{D}{M} & -\frac{k}{M}\end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}$ and output of the system being y $=\theta = x_1$, then the state-space representation of the system can be given as,

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{B}T$$
$$y = [1\ 0]\boldsymbol{x} = \boldsymbol{Cx}$$

Sometimes, the input may directly affect the output y in some way, in that case we get another matrix called direct transition matrix **D**. For this case, the **D** matrix is zero.

For the seesaw system, we can see from equations (28) and (29) that, the variables impacting the dynamics of the system are $x_c,\ \theta, \dot{x}_c\ ,\ \dot{\theta}$. We consider, four state variables for the system as $x_1 = x_c, x_2 = \theta, x_3 = \dot{x_c}, x_4 = \dot{\theta}$. Now, we can write the state equations using equations (28) and (29) as,

$$\dot{x_1} = \dot{x_1}$$
$$\dot{x_2} = \dot{x_2}$$
$$\dot{x_3} = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + b_1 V$$
$$\dot{x_4} = a_5 x_1 + a_6 x_2 + a_7 x_3 + a_8 x_4 + b_2 V$$

With the above state equations, and with $x_1 = x_c, x_2 = \theta$ being the outputs of the system, **y** being the output vector, and writing V as u, the state space representation of the seesaw system is given as,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix} u \tag{32}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{33}$$

Writing, $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the simplified state space representation of the seesaw system is given as,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \,; \;\; \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}u \tag{34}$$

Here, **A** is the system matrix, **B** is the input matrix, **C** is the output matrix, and **D** is the output transition matrix. In this representation, the different parameters of **A** and **B** are given as follows,

$$a_1 = -\frac{m_c D_t g}{J_{sw}} \,,\; a_2 = \frac{-g m_c R_m r^2 J_{sw} + m_c D_t R_m r^2 g m_{sw} D_c}{R_m r^2 J_{sw} m_c}$$

$$a_3 = -\frac{J_{sw}\eta_g K_g^2 \eta_m K_t K_b + J_{sw} B_c R_m r^2 + m_c D_t^2 \eta_g K_g^2 \eta_m K_t K_b + m_c D_t^2 B_c R_m r^2}{R_m r^2 J_{sw} m_c}$$

$$a_4 = -\frac{D_t B_{sw}}{J_{sw}} \,,\; a_5 = -\frac{g m_c}{J_{sw}} \,,\; a_6 = \frac{g m_{sw} D_c}{J_{sw}} \,,\; a_7 = -\frac{\eta_g K_g^2 \eta_m K_t K_b D_t + B_c R_m r^2 D_t}{R_m r^2 J_{sw}}$$

$$a_8 = -\frac{B_{sw}}{J_{sw}} \,,\; b_1 = \frac{J_{sw}\eta_g K_g \eta_m K_t r + m_c D_t^2 \eta_g K_g \eta_m K_t r}{R_m r^2 J_{sw} m_c} \,,\; b_2 = \frac{\eta_g K_g \eta_m K_t D_t}{r R_m J_{sw}}$$

We have the physical parameters of the cart and the seesaw. Using the physical parameters, we can determine the matrices **A** and **B**.

Therefore, the linearized state-space model of the seesaw system is given by equation (34) with the different matrices using physical parameters obtained as,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2.33 & -9.162 & -14.016 & 0 \\ -18.63 & 5.19 & -3.23 & 0 \end{bmatrix},\; \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 1.47 \\ 0.34 \end{bmatrix},\; \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\; \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{35}$$

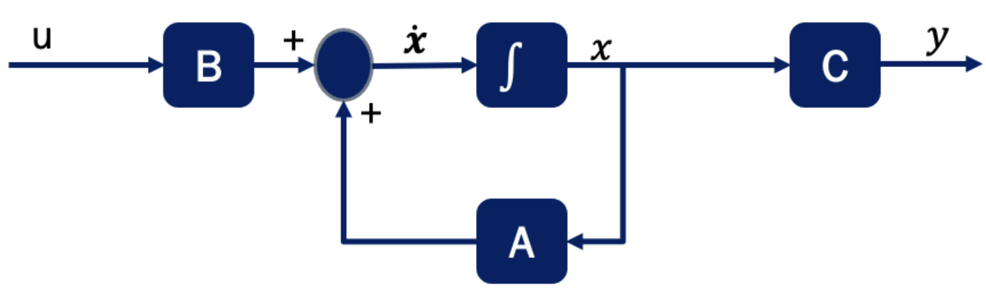


FIGURE 7: STATE SPACE BLOCK DIAGRAM REPRESENTATION OF THE SEESAW SYSTEM MODEL

Controller Design

In the previous section, we derived the mathematical model of the seesaw system using laws of physics. The model has been represented in state-space form in equation (34). Before designing a controller, we try to see the poles of the mathematical model that we have obtained. The poles or eigen values of a system in state-space form is given by the equation,

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = 0 \tag{36}$$

Where, **A** is the system matrix, and **I** is an identity matrix of the same order of **A.** For the seesaw system model, the open loop poles are at,

$$s_{1,2,3,4} = 3.1422, -1.7025 + i1.1532, \ -1.7025 - i1.1532, \ -13.5729$$

We can observer that, as per the mathematical model, the seesaw system is open loop unstable as one of the poles is on the right-half of the complex plane. If we physically observe the experimental set up, it bolsters our findings that the seesaw system is open loop unstable. The pole-zero map of the mathematical model is shown in figure 8.

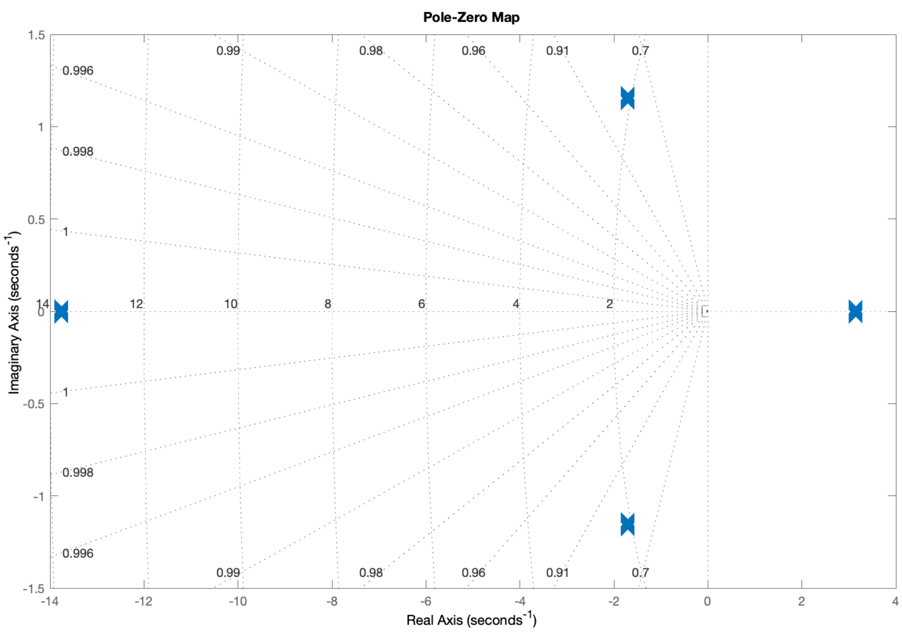


FIGURE 8: POLE-ZERO MAP OF THE MATHEMATICAL MODEL OF THE SEESAW SYSTEM

As the seesaw system is not a SISO (single input, single output) system, we cannot apply the classical control techniques like root-locus design, PID control etc. We are going to design controllers for the seesaw system using modern control methods in state-space representation.

Closed-loop control philosophy in state-space

In a general feedback control structure, the output of the dynamical system is fed back to the controller to compare it with the desired response and then to generate control input. In case of a state-space feedback control, each state variable is fed back to the system input terminal through a gain tuned for each state variable. The respective gain for each state variable can be adjusted to adjust the closed-loop poles of the system. For the seesaw system, we consider four number of state variables, hence we require four gain variables.

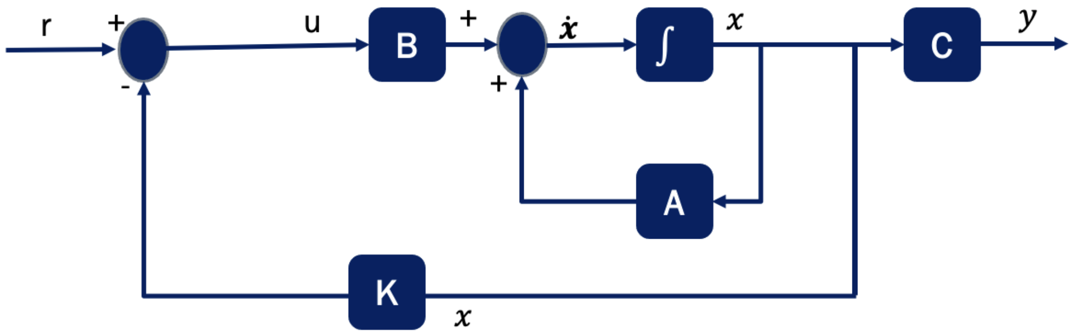


FIGURE 9: BLOCK DIAGRAM SHOWING FEEDBACK CONTROL SCHEME IN STATE SPACE

A general block diagram of a closed loop feedback control structure in state-space is shown in figure 9. Here, **r** is the reference input and **K** is the matrix containing corresponding feedback gains for each state variable. For, the seesaw system model, there are 4 state variables hence the dimension of matrix **K** is (4x1). From the block diagram, the state equations for the closed loop system can be written as,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(-\boldsymbol{K}\boldsymbol{x} + \boldsymbol{r}) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x} + \boldsymbol{B}\boldsymbol{r} \tag{37}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

From equation (37), it is clear that the system matrix for the closed loop system has become $(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})$**.** Therefore, poles of the closed-loop system are determined by the equation,

$$\det\big(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\big) = 0 \tag{38}$$

From equation (38), we can observe that the poles of the closed loop system can be adjusted by adjusting the gain matrix $\boldsymbol{K}$. Therefore, the controller design in state-feedback approach revolves around designing the gain matrix **K** for desired performance.

Another slightly different approach is to use the error between reference state variable values and actual state variable values to compute the input to the dynamic system using the gain matrix $\boldsymbol{K}$**.** This approach is called full-state feedback control. In this case, the input signal becomes, $\boldsymbol{u} = \boldsymbol{K}(\boldsymbol{r} - \boldsymbol{x}) = \boldsymbol{K}\boldsymbol{e}$.

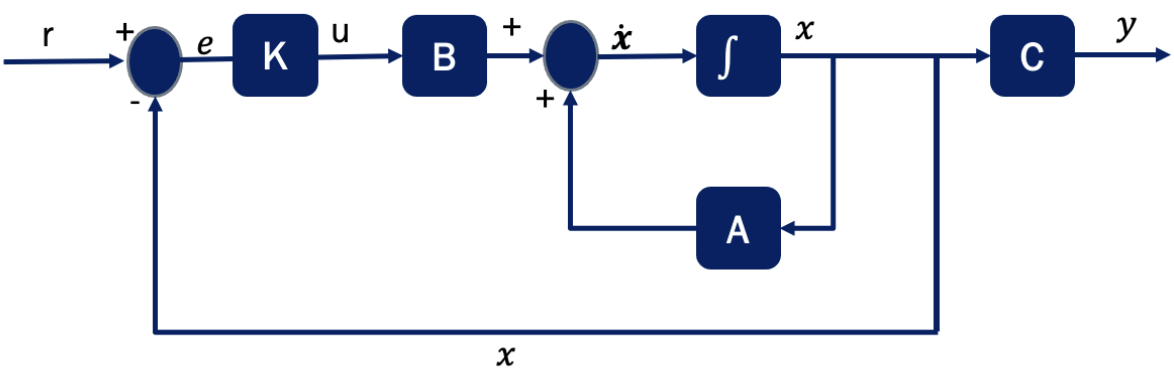


FIGURE 10: FULL-STATE FEEDBACK CONTROL SCHEME

Therefore, the state-equations for the full-state feedback control case becomes,

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x} + \boldsymbol{B}\boldsymbol{K}\boldsymbol{r} \tag{39}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

When $\mathbf{r} = \mathbf{0}$, the feedback controller is referred to as a regulator because it regulates the system around the equilibrium point $\mathbf{r} = \mathbf{0}$. For the seesaw system, we need to regulate the system around the fixed point $\theta = 0$. Therefore, full-state feedback controller acting as a regulator is the control scheme selected for the seesaw system controller design. In case of the controller acting as a regulator, the state equation is given as,

$$\begin{aligned} \dot{\boldsymbol{x}} &= (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x} \\ \boldsymbol{y} &= \boldsymbol{Cx} \end{aligned} \tag{40}$$

Therefore, in case of full-state feedback regulator, the closed-loop poles are also determined by the equation given in (38). Before we proceed to designing controllers, we try to verify that our mathematical model captures dynamics of the practical system reasonably well.

Controllability and Observability

Controllability

A system is said to be controllable at a given time if it is possible to find a control input that can transfer the system from any initial state to any other state in a finite interval of time. It is necessary for a state-space modeled system to examine the controllability. If the system model is controllable, then we can apply state-feedback control scheme to manipulate the poles of the closed-loop system to achieve desired control performance. For, a state-space represented model of a system, the controllability matrix is constructed as,

$$\boldsymbol{C} = [\boldsymbol{B}\ \boldsymbol{AB}\ \boldsymbol{A^2B}\ \ldots \boldsymbol{A^{n-1}B}] \text{ (n x n matrix)}$$

The system with controllability matrix **C** is said to be controllable if **C** is a full row rank matrix i.e., rank(**C**)= n.

For the seesaw system, the controllability matrix is found as,

$$C_{ss} = \begin{bmatrix} 0 & 1.4713 & -20.6219 & 282.4955 \\ 0 & 0.3392 & -4.7534 & 40.9745 \\ 1.4713 & -20.6219 & 282.4955 & -3.8678 \times 10^3 \\ 0.3392 & -4.7534 & 40.9745 & -553.1780 \end{bmatrix}$$

We also find that C_{ss} has a rank of 4. Therefore, the mathematical model of the seesaw system is completely state controllable.

Observability

A system is said to be observable if all the initial states can be determined from the observation of the output over a finite interval of time. The observability matrix of a state-space represented system is given as,

$$\boldsymbol{O} = [\boldsymbol{C}\ \boldsymbol{CA}\ \boldsymbol{CA^2}\ \ldots \boldsymbol{CA^{n-1}}]^{\boldsymbol{T}} \text{ (n x n matrix)}$$

If matrix O is a full row rank matrix (i.e., rank(O) = n), then the system is said to be completely observable. Observability condition is checked when we need to design observers for a system. An observer is used when all the states of the system are not measurable. For our seesaw system, all the states are measurable (θ, x_c are directly measured, and $\dot{\theta}, \dot{x_c}$ can easily be calculated in real time). Therefore, observer design is not needed for the seesaw system.

Mathematical Model Validation

The mathematical model of the seesaw system is open-loop unstable. Also, the experimental set up is not stable around the zero-tilt angle position of the seesaw in open-loop. Therefore, we try to compare the closed loop response of the mathematical model to that of the experimental set up. For this purpose, we try to obtain the gain matrix **K** by trial-and -error so that some stability is achieved around the zero-tilt position of the seesaw experimentally. We have used the gain matrix as,

$$K = [72.5807 \quad -44.0340. \quad 4.2323 \qquad -13.9846]$$

With the gain matrix above, we compare the response of the mathematical model and experimental system with same initial conditions.

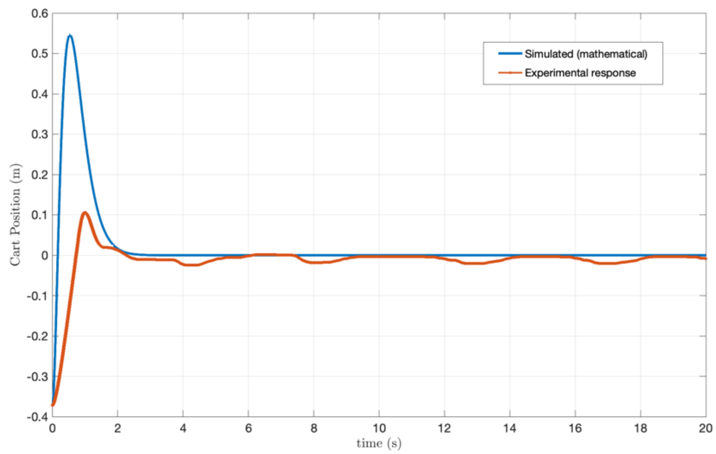


FIGURE 11: SEESAW TILT ANGLE COMPARISON BETWEEN MODEL AND PRACTICAL SYSTEM

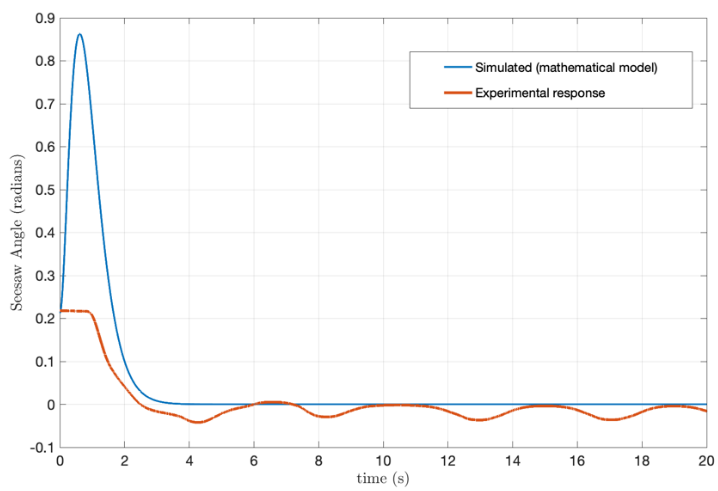


FIGURE 12: CART POSITION COMPARISON BETWEEN MODEL AND PRACTICAL SYSTEM

For comparison purpose of the practical and simulated response, we have used mean squared error. The mean squared error for any practical data **X** and its estimated values $\hat{X}$ is given as,

$$Mean\ Squared\ Error(MSE) = \frac{1}{n}\sum_{i=1}^{n}\left(X - \hat{X}\right)^2 \quad (n\ is\ the\ \ number\ of\ data\ points)$$

For the cart position and tilt angle, the mean squared error between practical and simulated response are found as,

$$MSEx_c\ (cart\ postion) = 0.0113,\ MSE_\theta\ (see - saw\ angle) = 0.0139$$

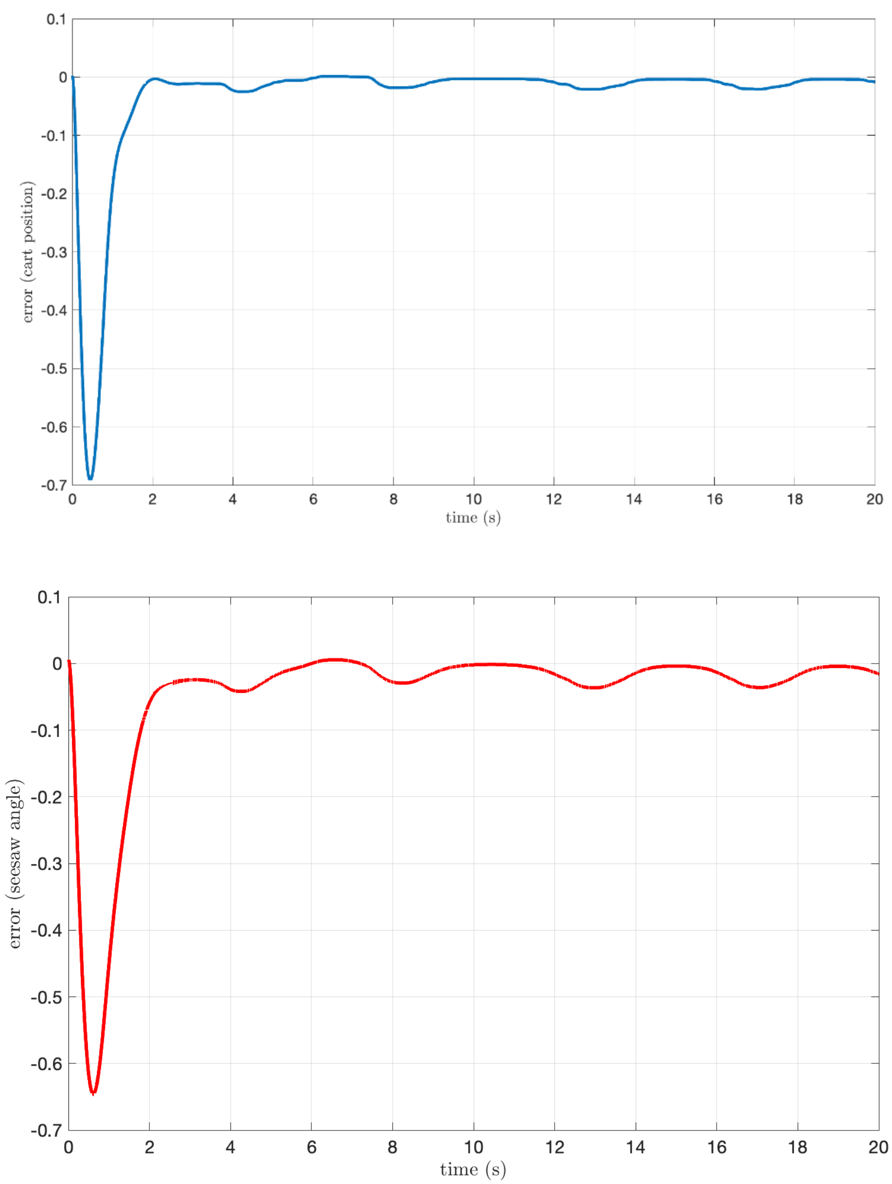


FIGURE 12: ERROR BETWEEN PRACTICAL RESPONSE AND MATHEMATICAL MODEL RESPONSE

We can see the mean squared errors are reasonably low and we can consider that our mathematical model of the seesaw fairly captures the dynamics of the seesaw system. Moreover, our mathematical model is also a linear approximation of the actual dynamics. Considering that, it compares to the practical system reasonably well. The initial spikes in the simulated response are due to the fact that there is no physical constraint on the input voltage, tilt angle, cart position, and measurement noise for the simulation environment. Therefore, the mathematical model is valid, and it will be used to design the controllers.

Controller Design by Pole Placement

The governing equation of the closed-loop full-state feedback control strategy is given in equation (39). Conditions for controller design using pole-placement are:

- All the states of the system must be measurable and available for feedback. For, the see-saw system angular position of the see-saw, and cart position are measured. The other two states (i.e., velocities) can be calculated from the position measurement. Therefore, for the see-saw system, all the states of the mathematical model are available for feedback.
- The system needs to be completely state controllable. This condition is also satisfied for the see-saw system.
- Magnitude of the gain matrix K must be physically realizable.

The key step in controller design using pole-placement for state-feedback control is to design suitable gain matrix K by placing the poles of the system in desired position. The steps involved for deciding the desired closed-loop poles are:

- Compute the open-loop poles of the system.
- Observe the behavior of the open loop poles. The pole(s) closest to the origin on the left-half of the s-plane is/are dominant poles of the open loop system. We look for dominant second order behavior for this purpose.
- Place rest of the poles such that they are much faster (i.e., real part at least 2-3 times) than the dominant open-loop pole(s).

There are various methods for obtaining the gain matrix K using pole-placement technique. For brevity, we discuss a simple method to get an idea. The characteristic equation of the closed-loop see-saw system model is given as,

$$\det\,(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BK}) = 0 \tag{41}$$

Let, $\mu_1, \mu_2, \mu_3, \mu_4$ are desired closed-loop poles. With the desired closed-loop characteristic equation will be,

$$(s - \mu_1)(s - \mu_2)(s - \mu_3)(s - \mu_4) = 0 \tag{42}$$

Both equation (41) and equation (42) should be equal in the left-hand side, which gives,

$$\det\,(s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BK}) = (s - \mu_1)(s - \mu_2)(s - \mu_3)(s - \mu_4) \tag{43}$$

By comparing coefficients of different powers of s' in equation (43), we can obtain the gain matrix K, which is the only unknown set of quantities**.**

For any dynamical system in state-space form, the gain matrix K for closed loop state feedback control can be conveniently obtained using the MATLAB by pole-placement technique. MATLAB uses the technique proposed by Kautsky, Nichols, and Dooren (1985).

The open-loop poles for the seesaw system are,

$$s_{1,2,3,4} = 3.1422, -1.7025 + i1.1532, \ -1.7025 - i1.1532, \ -13.5729$$

As we can see that the pole s_1 is in the right half of s-plane. Therefore, we would like to place it in the left-half of s-plane. We try to place s_1 so that it is over 4-times of the real parts of dominant open-loop poles s_2, s_3. Then, the dynamics of the closed-loop system is governed by s_2, s_3 which are much slower than s_1, s_4. Therefore, the desired closed-loop poles for the see-saw system, in this case, are taken as,

$$s^{\star}_{1,2,3,4} = -7, -1.7025 + i1.1532, \ -1.7025 - i1.1532, \ -13.5729$$

With the desired closed loop poles above, using MATLAB, we find the gain matrix K as,

$$K = [134.57437 \quad -76.246. \quad 12.3610 \qquad -24.2518] \tag{44}$$

We name the controller / gain matrix given in equation (44) as **SFC1.**

In order to obtain faster dynamics, we try to place the desired poles with zero complex part (critically damped) at,

$$s^{\star}_{1,2,3,4} = -10, \quad -11, \quad -7, \quad -9$$

With the desired closed loop poles above, using MATLAB, we find the gain matrix K as,

$$\boldsymbol{K} = [451.4594 \quad -449.3483 \quad -47.6357 \ -138.8892]$$

Sl.	Desired Poles	Gain (K)
SFC1	$-7, -1.7025 + i1.1532, \ -1.7025 - i1.1532, \ -13.5729$	$[134.57437 \quad -76.246. \quad 12.3610 \qquad -24.2518]$
SFC2	$-10, \quad -11, \quad -7, \quad -9,$	$[451.4594 \quad -449.3483 \quad -47.6357 \ -138.8892]$

Table-3: Gains calculated by pole-placement method

LQR Control

LQR stands for Linear Quadratic Regulator. The nomenclature revolves around the fact that the system being controlled is linear, the cost function is quadratic, and the controller is a regulator. The state-feedback strategy with reference input r = 0 is called a regulator. LQR control strategy falls under optimal control. As LQR control is optimal in nature, the design procedure involves optimization of a cost function **J**. The cost function is given as,

$$J = \int_0^{\infty} L(x,u)dt \ = \int_0^{\infty} (\boldsymbol{x^T Q x} + \boldsymbol{u^T R u} + \boldsymbol{2x^T N u})dt \tag{47}$$

Here, $\boldsymbol{Q}$ is a nxn positive semi-definite weighting matrix, $\boldsymbol{R}$ is a p x p matrix (p is the input dimension), and $\boldsymbol{N}$ is an n x p positive semi-definite matrix**.** The objective of LQR design is to minimize the cost function J using the state feedback control input $\boldsymbol{u} = \ -\boldsymbol{Kx}$**.**

The cost function J may be understood as an energy function. If we keep the cost function small, then the total energy of the system remains small. It is worthy of noting that, state vector and inputs are weighted in the expression of J. Therefore, if we keep J small, both system states and inputs will not be large. If we can minimize J, that means that the states of the system goes to zero as time approaches infinity.

Matrix $\boldsymbol{Q}$ represents the error tolerance for the system states. Matrix $\boldsymbol{R}$ represents tolerance for the input, and $\boldsymbol{N}$ defines error tolerance of the system states connected to the magnitude of the input. Selecting large $\boldsymbol{Q}$ implies that the system states must be smaller, in order to keep J small. Keeping large **R** means that the control input need to be smaller to keep J small. For our case, the matrix $\boldsymbol{N} = \boldsymbol{0}$**.** Therefore, the equation (47) becomes,

$$J = \int_0^{\infty} (\boldsymbol{x^T Q x} + \boldsymbol{u^T R u})dt \quad (48)$$

Using $\boldsymbol{u} = -\boldsymbol{Kx}$, equation (48) becomes,

$$J = \int_0^{\infty} (\boldsymbol{x^T Q x} + \boldsymbol{x^T K^T R K x})dt = \int_0^{\infty} \boldsymbol{x^T}(\boldsymbol{Q} + \boldsymbol{K^T R K})\boldsymbol{x}dt \quad (49)$$

We assume that there exists a matrix P such that,

$$\frac{d}{dx}\boldsymbol{x^T P x} = -\boldsymbol{x^T}(\boldsymbol{Q} + \boldsymbol{K^T R K})\boldsymbol{x} \quad (50)$$

From equation (48),

$$J = -\int_0^{\infty} \frac{d}{dx}\boldsymbol{x^T P x}\, dt \quad (51)$$

From equation (51) we can see that J depends on $\boldsymbol{P}$, not on **K.** If **P** is a positive definite constant matrix, then the system states go to zeros as time goes to infinity.

From equation (50),

$$\frac{d}{dx}\boldsymbol{x^T P x} + \boldsymbol{x^T Q x} + \boldsymbol{x^T K^T R K x} = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{\dot{x}^T P x} + \boldsymbol{x^T P \dot{x}} + \boldsymbol{x^T Q x} + \boldsymbol{x^T K^T R K x} = \boldsymbol{0} \quad (52)$$

The closed-loop state feedback regulator equation from (40) is,

$$\boldsymbol{\dot{x}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x} \quad (53)$$

Using equation (53) in equation (52),

$$\boldsymbol{A^T P} + \boldsymbol{PA} + \boldsymbol{K^T R K} - \boldsymbol{K^T B^T P} - \boldsymbol{PBK} + \boldsymbol{Q} = \boldsymbol{0} \quad (54)$$

If we take,

$$\boldsymbol{K} = \boldsymbol{R^{-1} B^T P} \quad (55)$$

Then equation (54) becomes,

$$\boldsymbol{A^T P} + \boldsymbol{PA} - \boldsymbol{PBR^{-1}B^T P} + \boldsymbol{Q} = \boldsymbol{0} \quad (56)$$

Equation (56) is called algebraic Riccati equation and solving this we get the matrix $\boldsymbol{P}$ and using $\boldsymbol{P}$ we can calculate the gain matrix $\boldsymbol{K}$ using equation (55).

Selection of LQR Weighting Matrices

The selection of $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{and\ N}$ can be done using three ways,

i. **Common weighting of state errors and Input Magnitudes:**

This method is the simplest of all the methods. We first use $\boldsymbol{Q} = \boldsymbol{I}, \boldsymbol{R} = \rho\boldsymbol{I}, and\ \boldsymbol{N} = 0$. Here, $\boldsymbol{I}$ is an identity matrix. In this approach the inputs and system states will be weighted equally whereas, the input magnitude to errors in the system states is weighted by ρ, such that $L = \left||x|\right|^2 + \rho\left||u|\right|^2$. If ρ is chosen as $\rho > 1$ then the input will be penalized more than the system states resulting more error in states but less controller actuation. If $\rho < 1$ then the system states will be penalized more than the input on same order of magnitude allowing less error in the system with larger input magnitudes. The use of this situation is where the input saturation is not an issue, but the error should be minimized.

ii. **Unique Weightings of State Errors and Input Magnitudes Based on State Error:**

The weighting coefficients can be chosen such that each reflects the amount of allowable error in states or allowable magnitude of inputs. We set the weighting matrices, with $\boldsymbol{N} = 0$, as diagonal matrices given as,

$$\mathbf{Q} = \begin{bmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_n \end{bmatrix}, \mathbf{R} = \rho \begin{bmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_p \end{bmatrix}$$

Here, q_i is the weighting coefficient associated with the system states x_i $(i = 1,2 \dots, n)$. Where, $q_i = \frac{1}{e_{x_i}^2}$ with e_{x_i} is the allowable error in state x_i. There can be other approaches in selecting q_i as well. ρ the weighting matrix for r_i $(i = 1,2 \dots, m)$. r_i is defined as, $r_i = \frac{1}{u_m^2}$ (u_m is maximum acceptable value of input)

iii. Unique Weightings of State Errors and Input Magnitudes Based on Output Errors

If there are specific outputs that must be kept small, and these outputs can be expressed as $y = Sx$, then the weighting matrices can be set such that,

$$\mathbf{Q} = \mathbf{S}^T\mathbf{S}, \mathbf{R} = \rho \begin{bmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_p \end{bmatrix}, \text{ and } \mathbf{N} = \mathbf{0}.$$

LQR Controller Design for the Seesaw System

Our control objective does not require to keep the errors in the states related to velocities under a certain margin. That's why we have not used the common weightings of state error approach for controller design purpose. Instead, we have used the unique weightings of state errors and input magnitudes to design LQR controller. For the see-saw system, there are four states x_1, x_2, x_3, x_4 representing cart position, see-saw angle, cart velocity, see-saw angular velocity. Also, there is only one input u. The weighting matrices are defined as,

$$\boldsymbol{Q} = \begin{bmatrix} q_1 & 0 & 1 & 0 \\ 0 & q_2 & 0 & 1 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix}, \boldsymbol{R} = \rho r_1, \boldsymbol{N} = \boldsymbol{0}$$

For the see-saw system we are more concerned about the see-saw angle compared to the cart position Hence, the error tolerance for the see-saw angle will be smaller than the cart-position. If we keep $e_{x_1} = 0.022 \; and \; e_{x_2} = 0.013$ then we get $q_1 \approx 2000$, $q_2 \approx 6000$. Therefore, we get,

$$\boldsymbol{Q} = \begin{bmatrix} 2000 & 0 & 1 & 0 \\ 0 & 6000 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As we are having only one input for the see-saw system, which is the voltage with nominal value 6 V. Therefore, in weighting matrix $\boldsymbol{R}$, $r_1 = \frac{1}{6^2} \approx 0.28$. We choose, $\rho = 35.7$ to keep a balance of the input signal, so that $\boldsymbol{R} \approx 10$. Using MATLAB, the gain matrix for the selecting weighting matrices is calculated as,

$$K = [126.0995 \quad -83.8043 \quad 12.0913 \quad -27.1509]$$

Different gains for different combinations of $\mathbf{Q}$ with $\boldsymbol{R} = 10$ are calculated as given are provided in Table-4.

Sl. No.	Q Matrix	Gain (K)
LQR1	$\begin{pmatrix} 2000 & 0 & 0 & 0 \\ 0 & 6000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$[126.0995 \quad -83.8043 \quad 12.0913 \quad -27.1509]$
LQR2	$\begin{pmatrix} 1000 & 0 & 0 & 0 \\ 0 & 4000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$[116.9656 \quad -75.5759 \quad 11.1503 \quad -24.4726]$
LQR3	$\begin{pmatrix} 1000 & 0 & 0 & 0 \\ 0 & 6000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$[124.2691 \quad -82.7519 \quad 11.9258 \quad -26.7385]$
LQR4	$\begin{pmatrix} 1000 & 0 & 0 & 0 \\ 0 & 7000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$[127.3739 \quad -85.8875 \quad -12.2566 \quad -27.7153]$
LQR5	$\begin{pmatrix} 4000 & 0 & 0 & 0 \\ 0 & 4000 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$[122.9831 \quad 79.1089 \quad 11.6992 \quad -25.8412]$

Table-4: LQR Gains for different Q

The closed loop-poles of the see-saw system model with different LQR gains is given in Table-5.

Closed Loop Poles with different LQR gains	
LQR1	[-13.6642 + 0.0000i ,-2.4125 + 2.5573i, -2.4125 - 2.5573i, -4.1086 + 0.0000i]
LQR2	[-13.7081 + 0.0000i,-2.2459 + 2.3693i,-2.2459 - 2.3693i,-3.9219 + 0.0000i]
LQR3	[-13.7064 + 0.0000i,-2.3546 + 2.5921i, -2.3546 - 2.5921i,-4.0787 + 0.0000i]
LQR4	[-13.7056 + 0.0000i,-2.4009 + 2.6813i,-2.4009 - 2.6813i, -4.1423 + 0.0000i]
LQR5	[-13.5794 + 0.0000i, -2.4345 + 2.2622i, -2.4345 - 2.2622i, -4.0167 + 0.0000i]

Table-5: Closed loop poles with different LQR gains

Therefore, the closed-loop system is stable with all the designed K values using LQR method.

Implementation /Simulation

In the controller design section, we have mentioned that the see-saw mathematical model's response is compared with practical response of the see-saw system. The comparison suggested that the mathematical model represents the practical see-saw system reasonably well. First, we simulate the open-loop see-saw model in MATLAB to validate the fact that the poles in the right half s-plane drives the model's response towards unbounded value for bounded inputs.

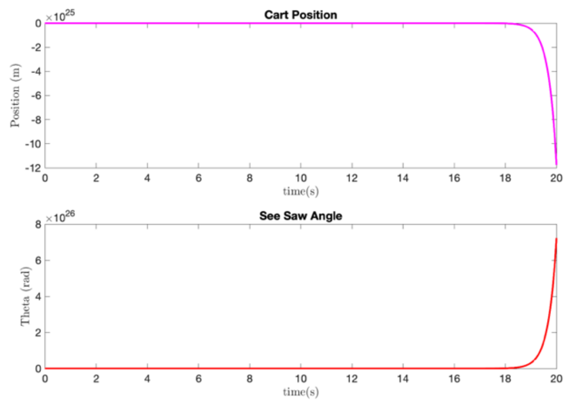


FIGURE 13: OPEN-LOOP SIMULATION OF THE SEE-SAW MATHEMATICAL MODEL

Therefore, using full-state feedback closed-loop philosophy, we designed a gain matrix K with trial and error just to stabilize the system. Then we compare the responses of the mathematical model and practical system as depicted in model validation part.

For practical implementation, we used LabView program interface that is connected through data acquisition module to the see-saw experimental system. Gain matrix is entered in the LabView interface to get the closed-loop system response. The cart position and see-saw angular position are the outputs of the system. From those outputs, we can calculate the velocities required for the state space control design.

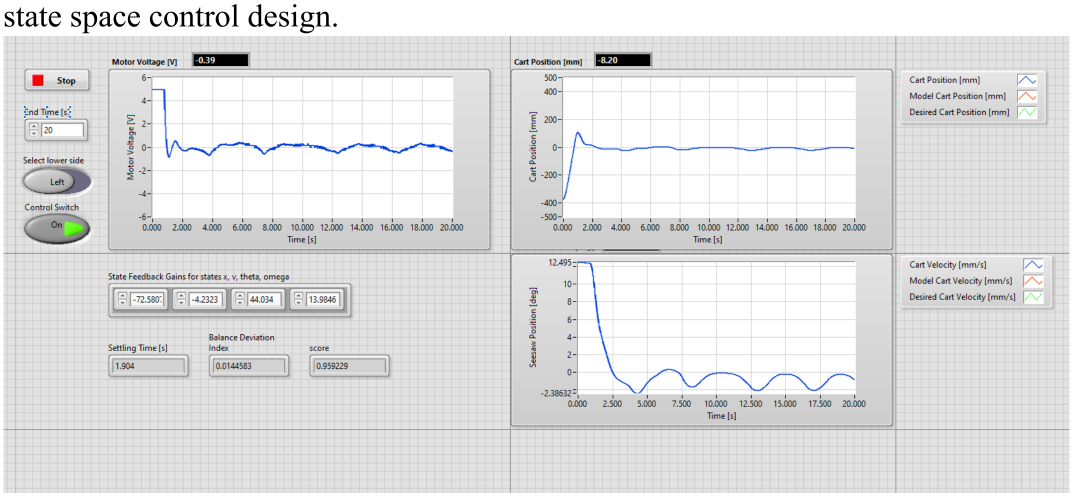


FIGURE 14: THE LABVIEW PROGRAM INTERFACE FOR PRACTICAL EXPERIMENT

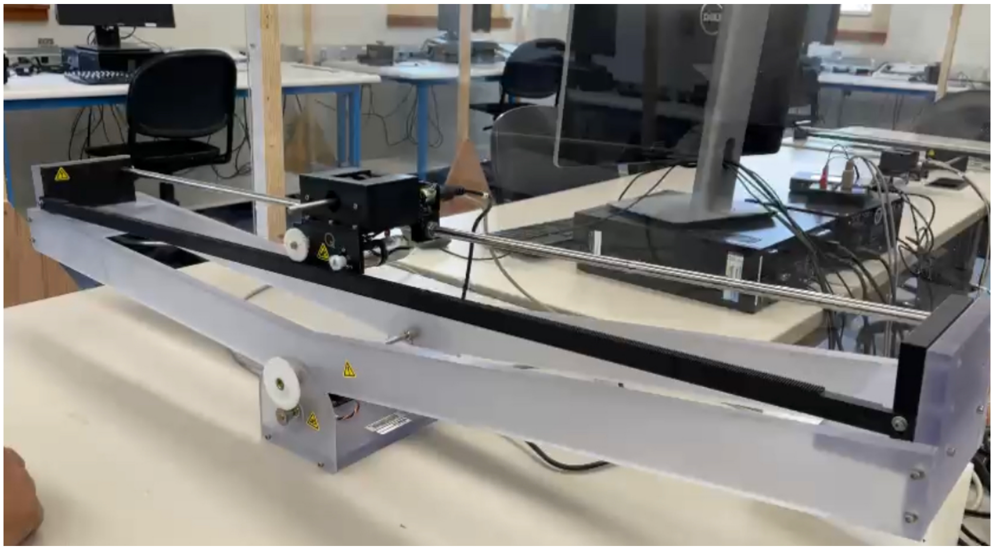


FIGURE 15: THE EXPERIMENTAL SET UP CONNECTED TO THE LABVIEW PROGRAM

Using pole-placement technique, we first tried to place the unstable pole near to the origin keeping other poles same. The desired closed-loop poles are,

$$s^{\star}_{1,2,3,4} = -7, -1.7025 + i1.1532,\ -1.7025 - i1.1532,\ -13.5729 \quad (57)$$

In MATLAB, for the above desired closed-loop poles, the mathematical model simulated response took balance time slightly greater than the required i.e. 5 sec.

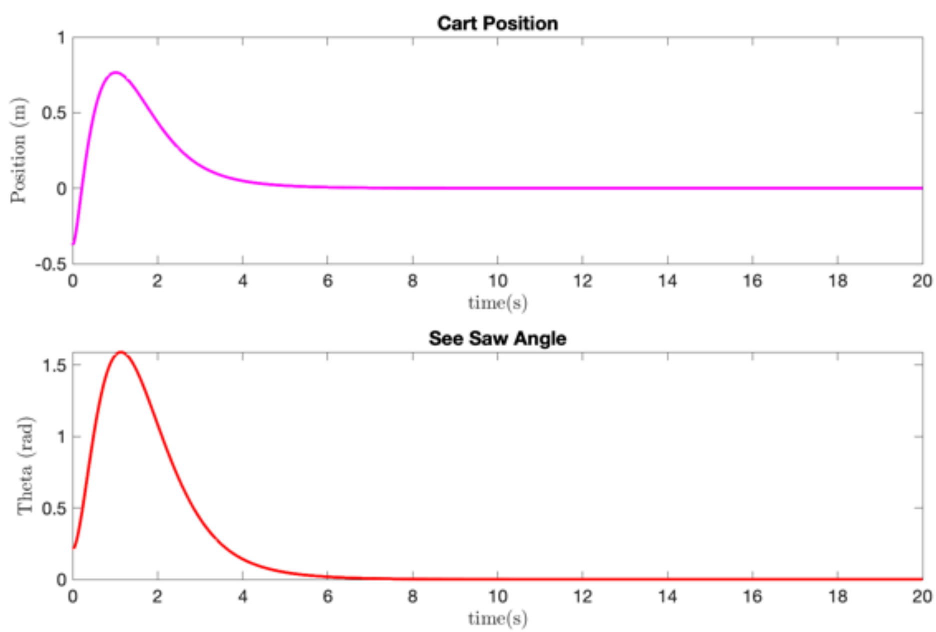


FIGURE 16: CLOSED-LOOP SIMULATION WITH DESIRED POLES GIVEN IN (57)

We placed the unstable pole further away from the origin as given in **SFC1** and **SFC2** so that the balance time becomes smaller.

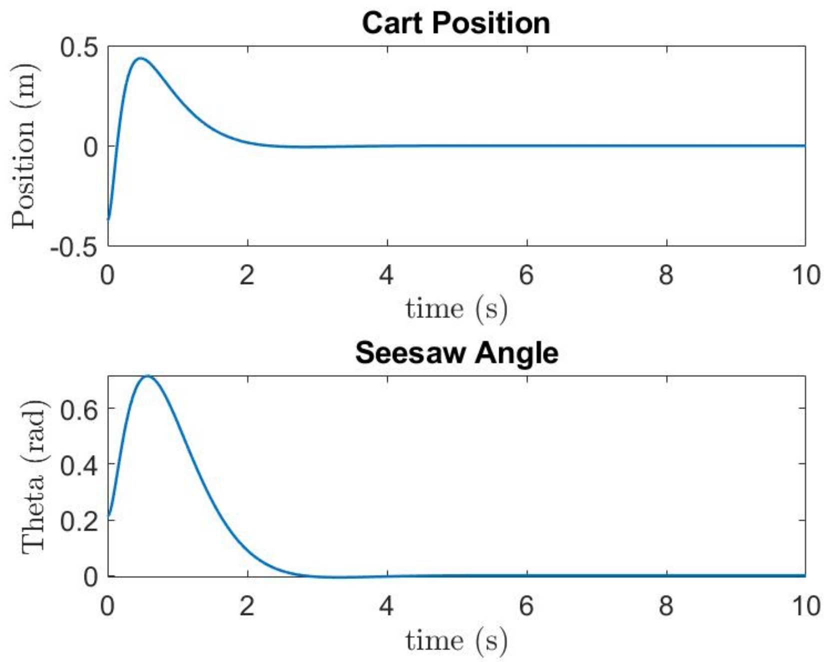


FIGURE 17: SIMULATED MODEL RESPONSE WITH GAIN SFC1

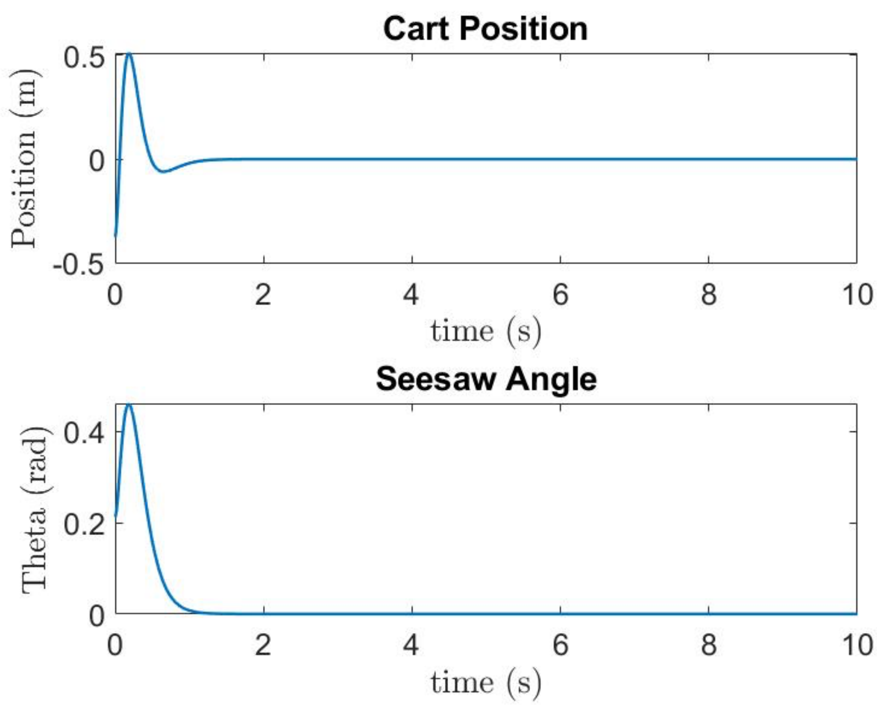


FIGURE 18: SIMULATED MODEL RESPONSE WITH GAIN SFC2

As the closed-loop system with gains given in SFC1 and SFC2 showed desired behavior. We implemented them in experimental set up.

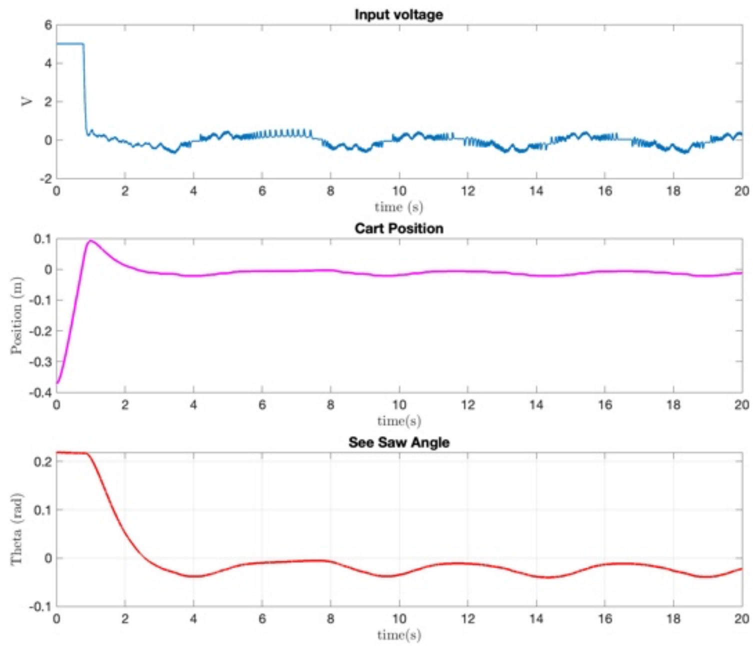


FIGURE 19: PRACTICAL SYSTEM RESPONSE WITH GAINS IN SFC1

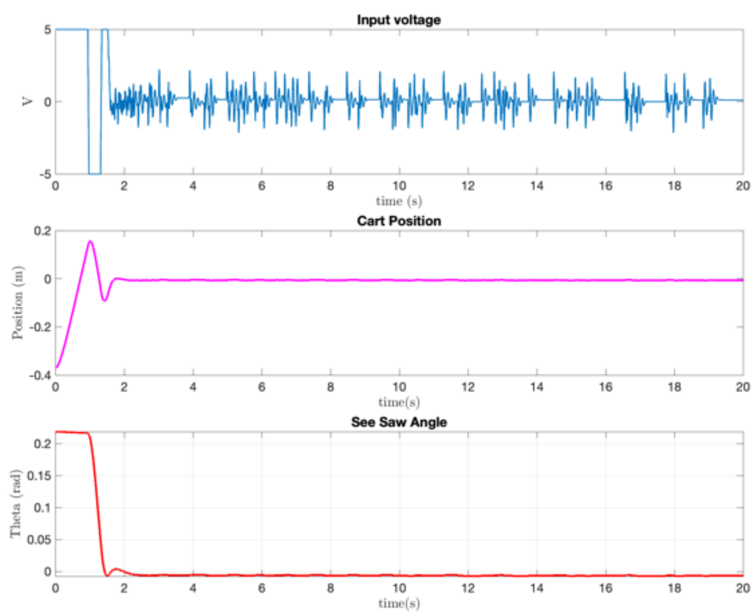


FIGURE 20: PRACTICAL SYSTEM RESPONSE WITH SFC2

For the LQR control design, we did not use common weighting of state errors as we are not concerned about the errors in the states related to linear and angular velocities. We used unique weighting of state errors and calculated the controller gain K as given in Table-5.

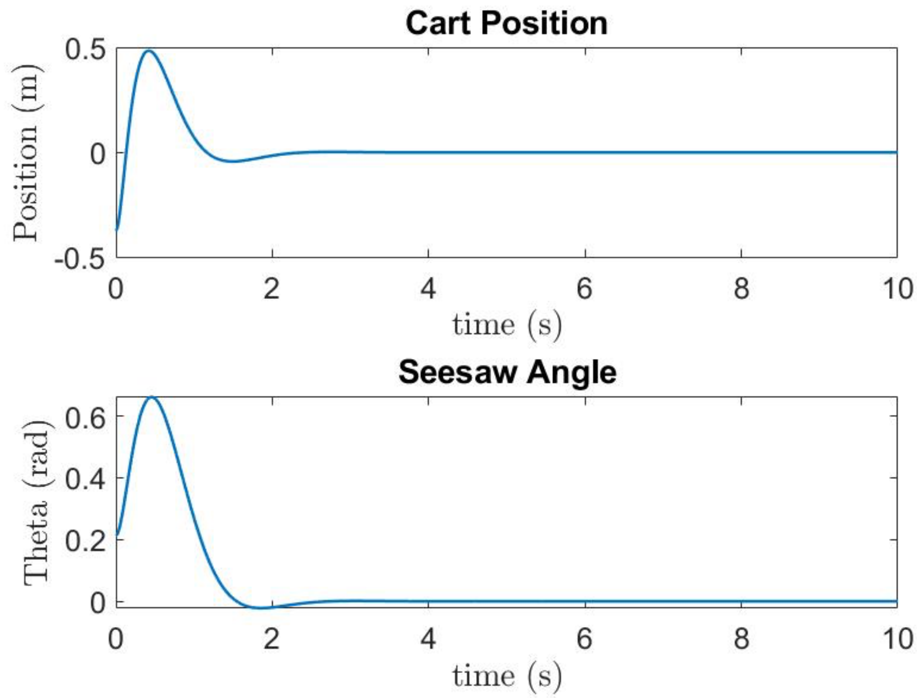


FIGURE 21: SIMULATED CLOSED LOOP RESPONSE WITH GAIN IN LQR1

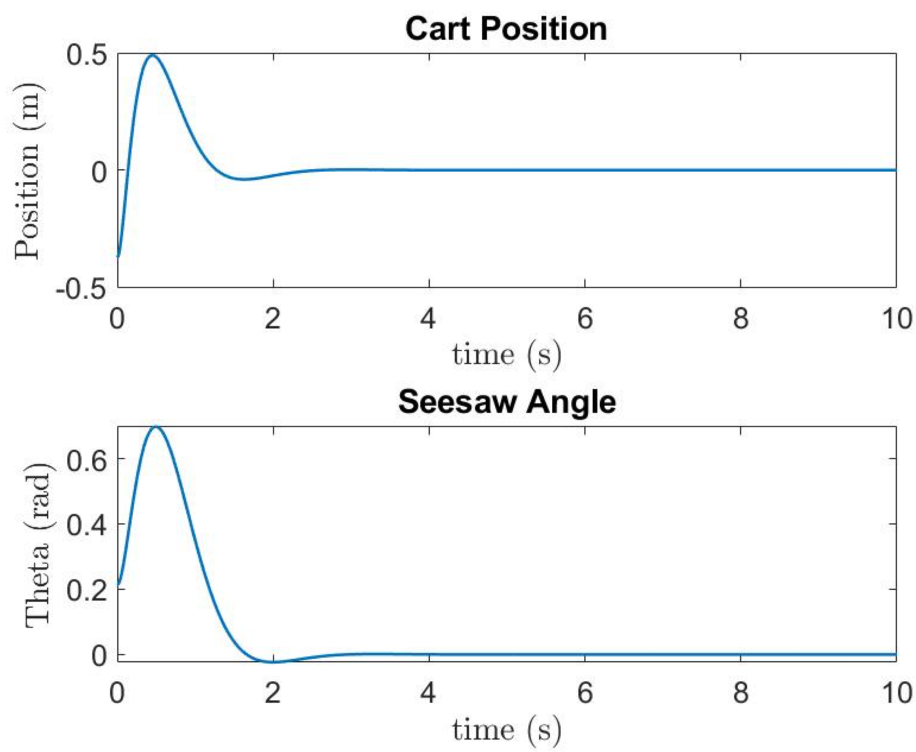


FIGURE 22: SIMULATED CLOSED LOOP RESPONSE WITH GAIN IN LQR2

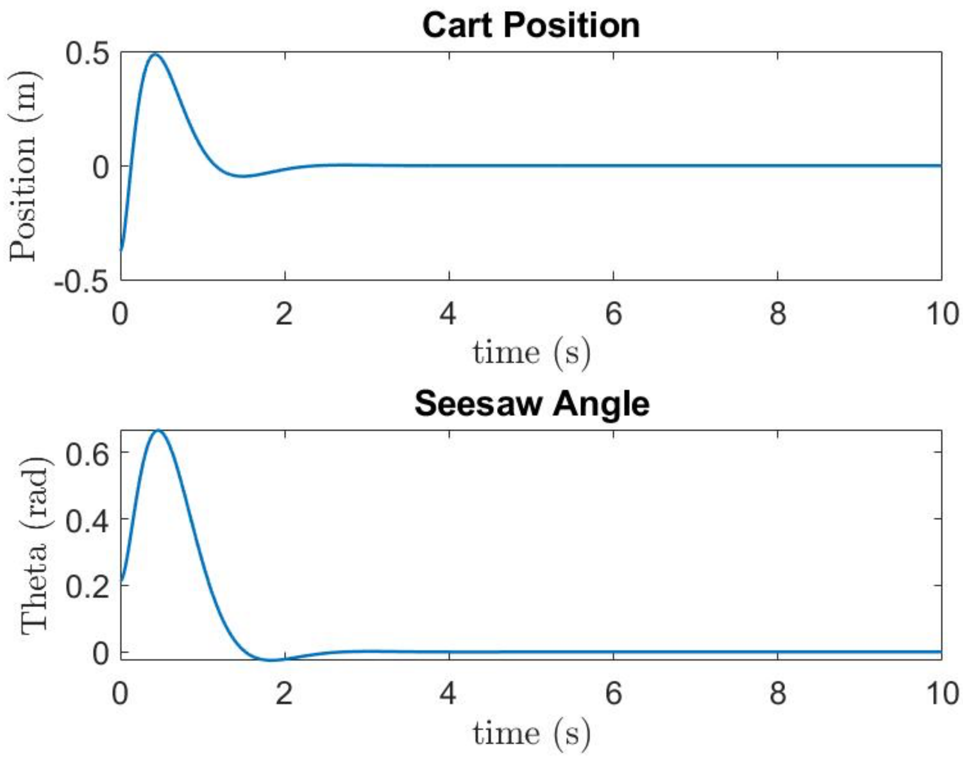


FIGURE 23: SIMULATED CLOSED LOOP RESPONSE WITH GAIN IN LQR3

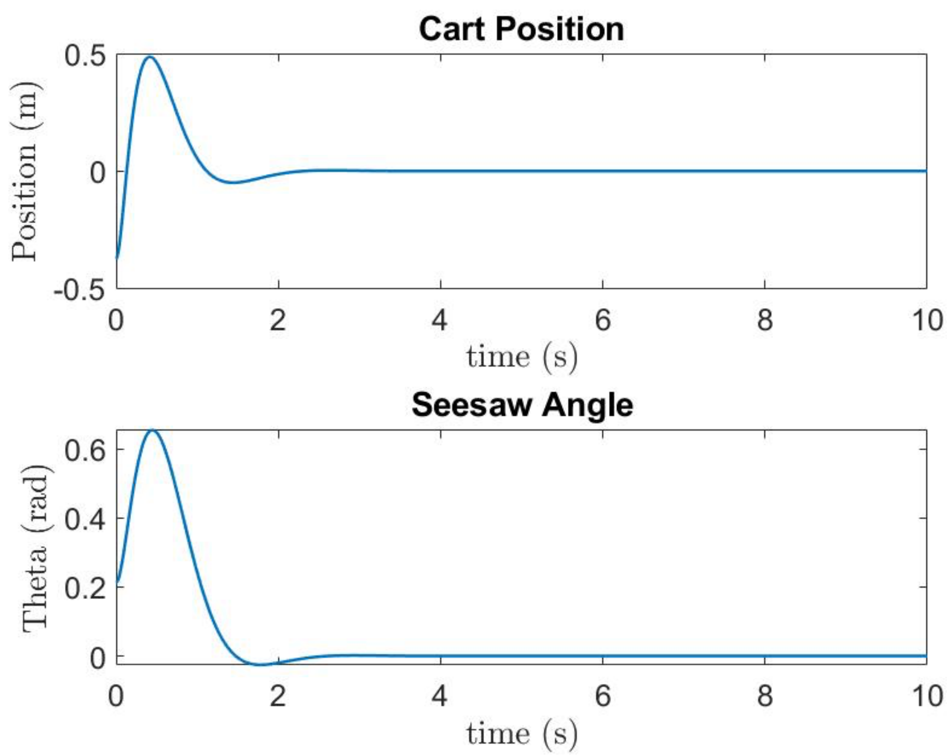


FIGURE 24: SIMULATED CLOSED LOOP RESPONSE WITH GAIN IN LQR4

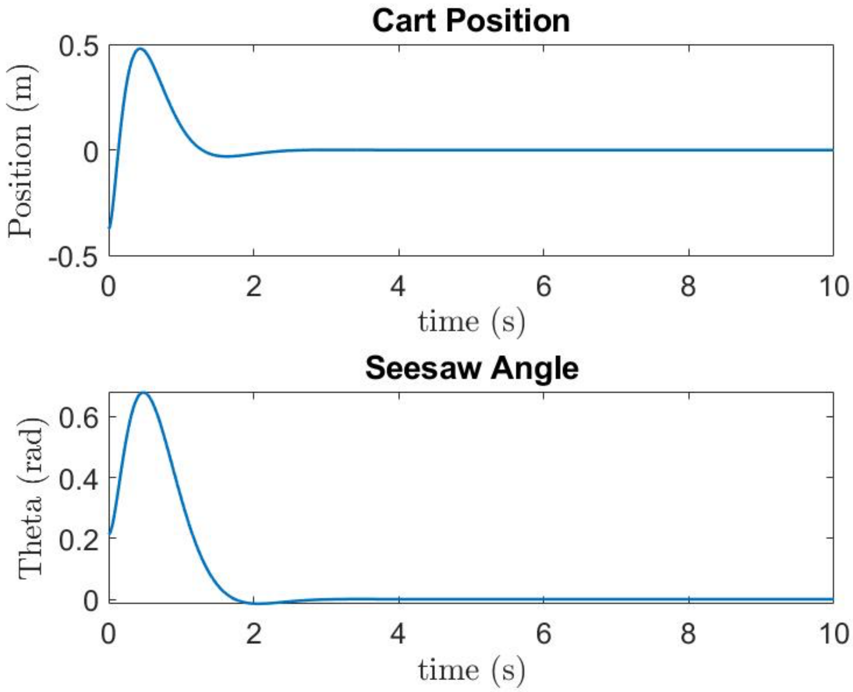


FIGURE 25: SIMULATED CLOSED LOOP RESPONSE WITH GAIN IN LQR5

As we got desired results using gains in LQR1, ……….,LQR4, LQR5, we implemented them in the practical set up.

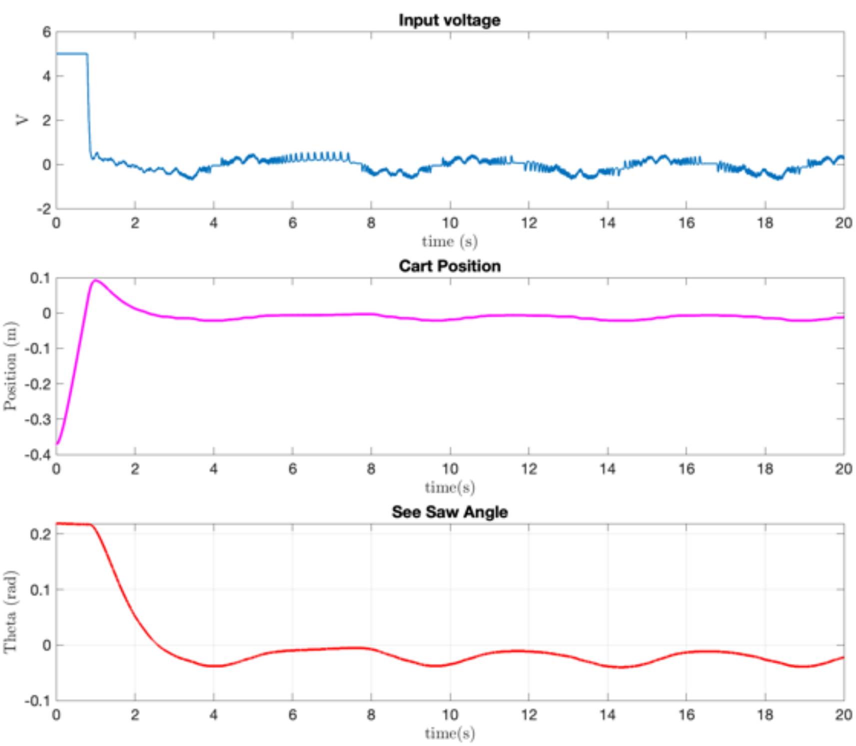


FIGURE 26: PRACTICAL CLOSED LOOP RESPONSE WITH GAINS IN LQR1

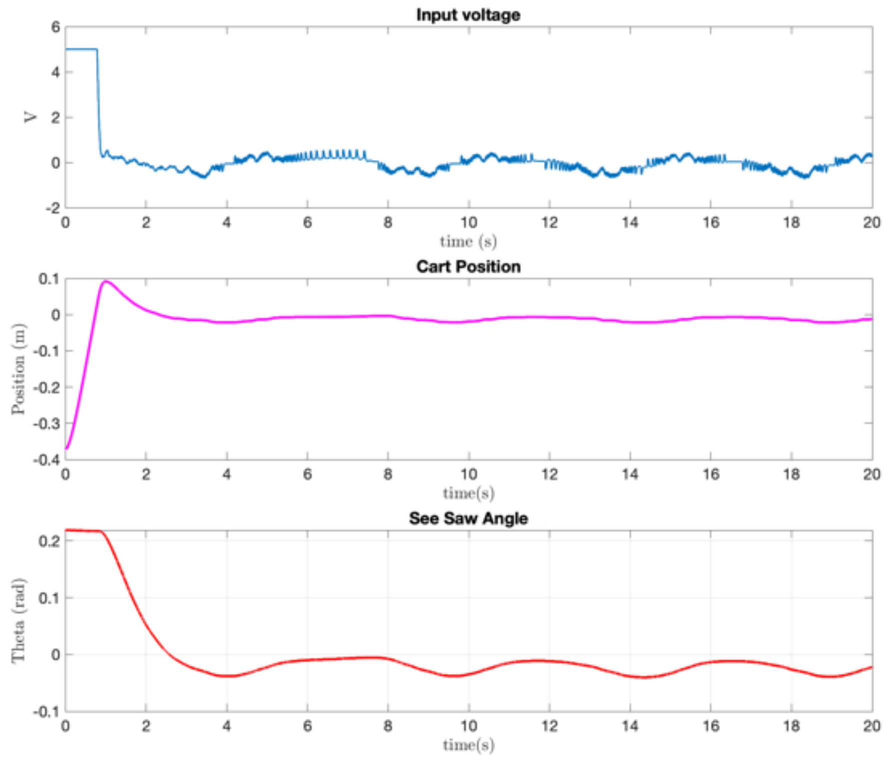


FIGURE 27: PRACTICAL CLOSED LOOP RESPONSE WITH GAINS IN LQR2

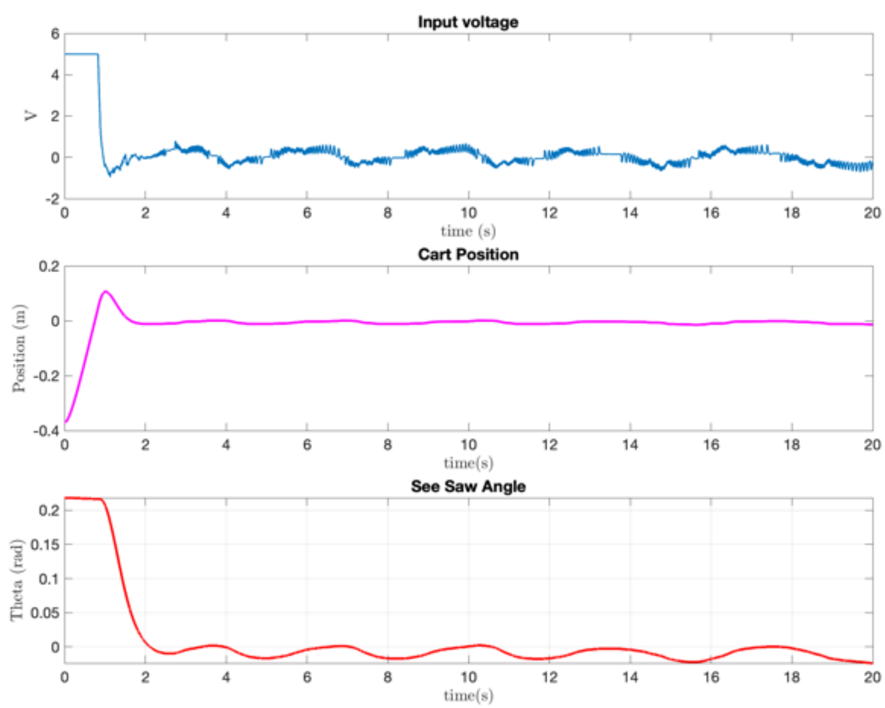


FIGURE 28: PRACTICAL CLOSED LOOP RESPONSE WITH GAINS IN LQR3

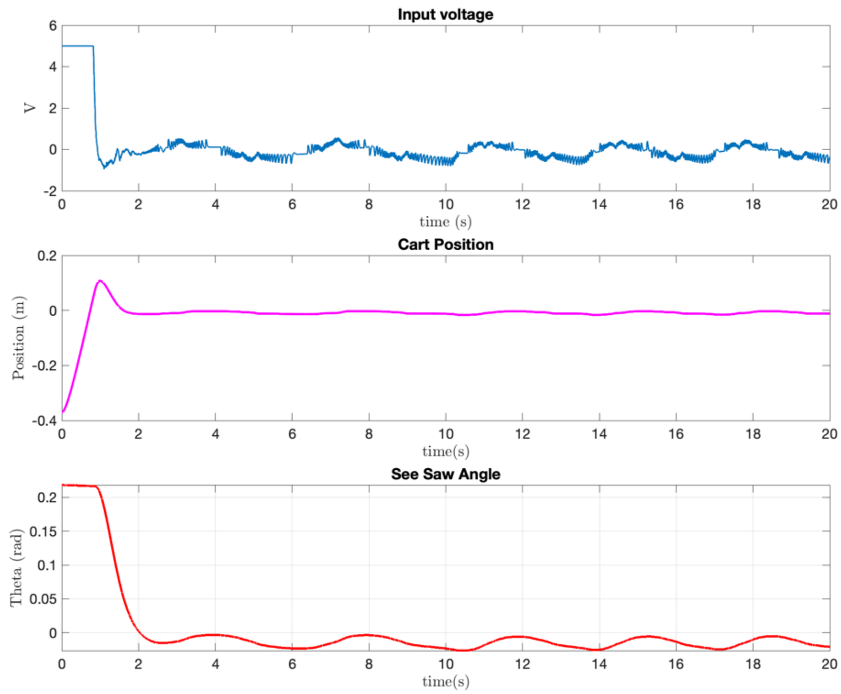


FIGURE 29: PRACTICAL CLOSED LOOP RESPONSE WITH GAINS IN LQR4

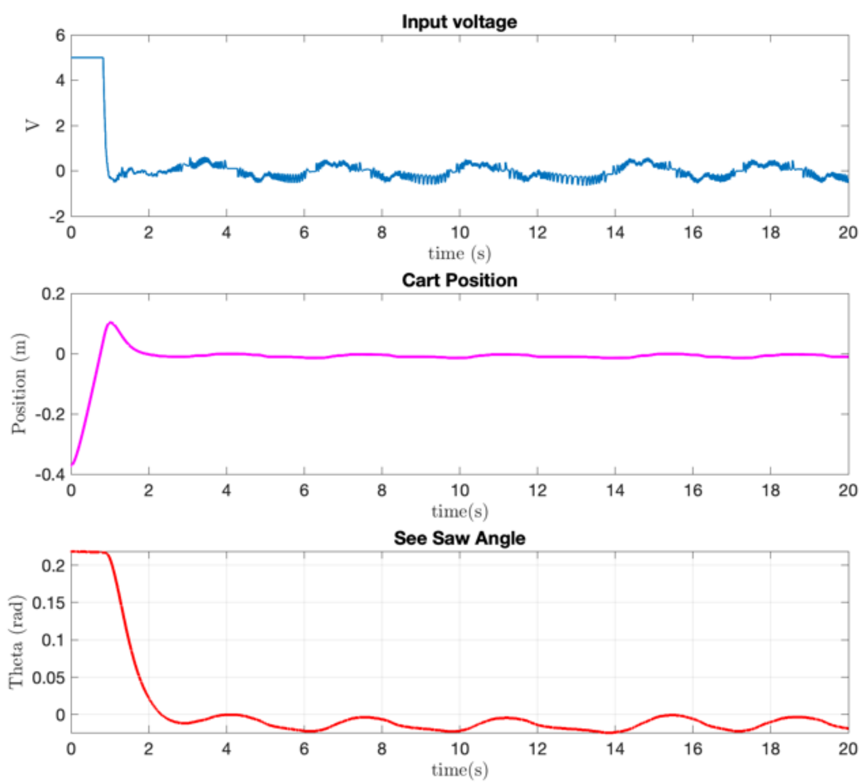


FIGURE 30: PRACTICAL CLOSED LOOP RESPONSE WITH GAINS IN LQR5

The design performances of the practical system with different gain matrices using pole-placement and LQR method are given in Table-6.

Controller	**Balance time (t_b)**	**Balance Deviation Index I_p (From LabView Program) (Rad)**	**Balance Deviation Index I_p (From LabView Program) (deg)**	**Balance Deviation Index I_p (calculated offline) (Rad)**	**Balance Deviation Index I_p (calculated offline) (Deg)**
LQR1	1.656	0.007368	0.4223694268	0.002842	0.16284
LQR2	1.734	0.007381	0.4231146497	0.0022124	0.1267
LQR3	1.642	0.008219	0.4711528662	0.00599	0.3432
LQR4	1.614	0.01426	0.8174522293	0.01254	0.71888
LQR5	1.758	0.01204	0.6901910828	0.00973	0.5577
SFC1	2.014	0.0199	1.140764331	0.01745	1.0003
SFC2	1.31	0.00583	0.3342038217	0.00514	0.2947

Table-6: Performance of different gain matrices in practical experiment.

From Table-6 we can see that with the gains in SFC2, we get the lowest balance time and lowest deviation index. Among the LQR controllers, LQR4 gives lowest balance time and LQR1 gives the lowest balance deviation index. Balance time and deviation index of all the controllers in Table-6 are well below the requirements. LQR2 gives the best result overall among all the LQR controllers considering both balance time and balance deviation index.

Conclusion

In this experiment, we have successfully done balance control of the see-saw system using full-state state-feedback control. The mathematical model derived using first principle represents the physical system reasonably well. The controllers designed with pole placement and LQR method in MATLAB using the mathematical model. First, we simulated the closed-loop see-saw system in MATLAB, then we implemented various designed controllers in the practical set up. The designed state-feedback controllers meet the design specification given below,

- The balance time t_b < 5 sec.
- The balance deviation index I_p < 0.02 (rad)

References

[1] Ogata, K., 2010. *Modern control engineering*. Prentice hall.

[2] Tserendondog, T., Ragchaa, B., Badarch, L. and Amar, B., 2016, June. State feedback control of unbalanced seesaw. In *2016 11th International Forum on Strategic Technology (IFOST)* (pp. 566-570). IEEE.

[3] Kizir, S.E.L.Ç.U.K., 2019. Real Time Full State Feedback Control of a Seesaw System Based on LQR. JOURNAL OF POLYTECHNIC-POLITEKNIK DERGISI, cilt.22, sa.4, ss.1023-1030

[3] Linear Quadratic Regulator (LQR) State Feedback Design , F.L. Lewis, https://lewisgroup.uta.edu/Lectures/lqr.pdf

[4] Nise, N.S., 2020. *Control systems engineering*. John Wiley & Sons.

[5] Lin, J., Zhan, J.H. and Chang, J., 2008. Stabilization and equilibrium control of a new pneumatic cart-seesaw system. *Robotica*, *26*(2), p.219.

[6] Tserendondog, T., Ragchaa, B., Badarch, L. and Amar, B., 2016, June. State feedback control of unbalanced seesaw. In *2016 11th International Forum on Strategic Technology (IFOST)* (pp. 566-570). IEEE.

[7] Kang, K.W., Jung, C.B. and Park, K.H., 1999. A Study on the Equilibrium Control of a Seesaw System. In *Proceedings of the KIEE Conference* (pp. 706-708). The Korean Institute of Electrical Engineers.

Appendix

MATLAB Code for Open-Loop System Simulation

```
% Enviornment Parameters
g = 9.81;

% Motor Parameters
Jm = 3.9e-7;
Bm = 0;
Km = 7.68e-3;
Kt = Km;
Rm = 2.6;
Lm = 0.18 / 1000;
em = 0.69;

% Gear Box
Kg = 3.71;
eg = 0.9;

% Pinion Parameters
Jl = 0;
Bl = 0;

% Cart Parameters
Mc = 0.37 + 0.38;
Bc = 5.4;
Beq = Bc;
rc = 6.35E-3;
Rc = 0.01483;

% Pendulum Parameters
Mp = 0.23;
Jp = 7.88E-3;
Lp = 0.6413;
lp = 0.3302;
Bp = 0.0024;

% Seesaw Parameters
Ms = 3.6;
Js = 0.395;
Bs = 0;
Dt = 0.125;
Dc = 0.058;


a31 = -(Mc*Dt*g)/Js;
a32 = -(g*Mc*Rm*(rc^2)*Js-Mc*Dt*Rm*(rc^2)*g*Ms*Dc)/(Rm*(rc^2)*Js*Mc);
a33 = -
(Js*eg*(Kg^2)*em*Kt*Km+Js*Beq*Rm*(rc^2)+Mc*(Dt^2)*eg*(Kg^2)*em*Kt*Km+Mc*(Dt^2
)*Beq*Rm*(rc^2))/(Rm*(rc^2)*Js*Mc);
a34 = -(Dt*Bs)/Js;
a41 = -(g*Mc)/Js;
a42 = (g*Ms*Dc)/Js;
a43 = -(eg*(Kg^2)*em*Kt*Km*Dt+Beq*Rm*(rc^2)*Dt)/(Rm*(rc^2)*Js);
```

```
a44 = -Bs/Js;

B = [0; 0; (Js*eg*Kg*em*Kt*rc+Mc*(Dt^2)*eg*Kg*em*Kt*rc)/(Rm*(rc^2)*Js*Mc);
(eg*Kg*em*Kt*Dt)/(rc*Rm*Js)];


A = [0 0 1 0; 0 0 0 1; a31 a32 a33 a34; a41 a42 a43 a44];

C = [1 0 0 0; 0 1 0 0];
D = 0;

% Check for controllability of the model

Crank = rank(ctrb(A,B));

if Crank == 4
    fprintf('The system is completely state controllable')
else
    fprintf('Not state controllable')
end

t = 0:0.002:20;
x0 = -0.374; theta0 = deg2rad(12.16);      % Defines initial conditions
x0 = [ x0  theta0  0  0 ];        % Creates initial conditions vector
u = zeros(length(t),1);                  % Creates a zero forcing vector

sys = ss(A,B,C,0);

ols = lsim(sys,u,t,x0);

figure(1)
subplot(2,1,1)
plot(t,ols(:,1), 'm','LineWidth', 2.5)
title('Cart Position')
xlabel('time(s)', 'interpreter', 'latex')
ylabel('Position (m)', 'interpreter', 'latex')
set(gca, 'fontsize', 15)
subplot(2,1,2)
plot(t,ols(:,2),'r','LineWidth', 2.5)
title('See Saw Angle')
xlabel('time(s)', 'interpreter', 'latex')
ylabel('Theta (rad)', 'interpreter', 'latex')
set(gca, 'fontsize', 15)
```

MATLAB Code for Pole Placement Controller Design and Simulation

```
% Enviornment Parameters
g = 9.81;

% Motor Parameters
Jm = 3.9e-7;
Bm = 0;
Km = 7.68e-3;
Kt = Km;
Rm = 2.6;
Lm = 0.18 / 1000;
em = 0.69;

% Gear Box
Kg = 3.71;
eg = 0.9;

% Pinion Parameters
Jl = 0;
Bl = 0;

% Cart Parameters
Mc = 0.37 + 0.38;
Bc = 5.4;
Beq = Bc;
rc = 6.35E-3;
Rc = 0.01483;

% Pendulum Parameters
Mp = 0.23;
Jp = 7.88E-3;
Lp = 0.6413;
lp = 0.3302;
Bp = 0.0024;

% Seesaw Parameters
Ms = 3.6;
Js = 0.395;
Bs = 0;
Dt = 0.125;
Dc = 0.058;


a31 = -(Mc*Dt*g)/Js;
a32 = -(g*Mc*Rm*(rc^2)*Js-Mc*Dt*Rm*(rc^2)*g*Ms*Dc)/(Rm*(rc^2)*Js*Mc);
a33 = -
(Js*eg*(Kg^2)*em*Kt*Km+Js*Beq*Rm*(rc^2)+Mc*(Dt^2)*eg*(Kg^2)*em*Kt*Km+Mc*(Dt^2
)*Beq*Rm*(rc^2))/(Rm*(rc^2)*Js*Mc);
a34 = -(Dt*Bs)/Js;
a41 = -(g*Mc)/Js;
a42 = (g*Ms*Dc)/Js;
a43 = -(eg*(Kg^2)*em*Kt*Km*Dt+Beq*Rm*(rc^2)*Dt)/(Rm*(rc^2)*Js);
a44 = -Bs/Js;
```

```
B = [0; 0; (Js*eg*Kg*em*Kt*rc+Mc*(Dt^2)*eg*Kg*em*Kt*rc)/(Rm*(rc^2)*Js*Mc);
(eg*Kg*em*Kt*Dt)/(rc*Rm*Js)];


A = [0 0 1 0; 0 0 0 1; a31 a32 a33 a34; a41 a42 a43 a44];



C = [1 0 0 0; 0 1 0 0];
D = 0;

% Check for controllability of the model

Crank = rank(ctrb(A,B));

if Crank == 4
    fprintf('The system is completely state controllable')
else
    fprintf('Not state controllable')
end

t = 0:0.002:20;
x0 = -0.374; theta0 = deg2rad(12.16);      % Defines initial conditions
x0 = [ x0  theta0  0  0 ];       % Creates initial conditions vector
u = zeros(length(t),1);                  % Creates a zero forcing vector

%desired closed loop poles
dp = [-7 -1.7025+i*1.1532 -1.7025-i*1.1532 -13.5729];

K = place(A,B,dp);

sys_cl = ss(A-B*K,B,C,0);

K

lsim(sys_cl,u,t,x0);
```

MATLAB Code for LQR Controller Design and Simulation

```
% Enviornment Parameters
g = 9.81;

% Motor Parameters
Jm = 3.9e-7;
Bm = 0;
Km = 7.68e-3;
Kt = Km;
Rm = 2.6;
```

```
Lm = 0.18 / 1000;
em = 0.69;

% Gear Box
Kg = 3.71;
eg = 0.9;

% Pinion Parameters
Jl = 0;
Bl = 0;

% Cart Parameters
Mc = 0.37 + 0.38;
Bc = 5.4;
Beq = Bc;
rc = 6.35E-3;
Rc = 0.01483;

% Pendulum Parameters
Mp = 0.23;
Jp = 7.88E-3;
Lp = 0.6413;
lp = 0.3302;
Bp = 0.0024;

% Seesaw Parameters
Ms = 3.6;
Js = 0.395;
Bs = 0;
Dt = 0.125;
Dc = 0.058;


a31 = -(Mc*Dt*g)/Js;
a32 = -(g*Mc*Rm*(rc^2)*Js-Mc*Dt*Rm*(rc^2)*g*Ms*Dc)/(Rm*(rc^2)*Js*Mc);
a33 = -
(Js*eg*(Kg^2)*em*Kt*Km+Js*Beq*Rm*(rc^2)+Mc*(Dt^2)*eg*(Kg^2)*em*Kt*Km+Mc*(Dt^2
)*Beq*Rm*(rc^2))/(Rm*(rc^2)*Js*Mc);
a34 = -(Dt*Bs)/Js;
a41 = -(g*Mc)/Js;
a42 = (g*Ms*Dc)/Js;
a43 = -(eg*(Kg^2)*em*Kt*Km*Dt+Beq*Rm*(rc^2)*Dt)/(Rm*(rc^2)*Js);
a44 = -Bs/Js;



B = [0; 0; (Js*eg*Kg*em*Kt*rc+Mc*(Dt^2)*eg*Kg*em*Kt*rc)/(Rm*(rc^2)*Js*Mc);
(eg*Kg*em*Kt*Dt)/(rc*Rm*Js)];



A = [0 0 1 0; 0 0 0 1; a31 a32 a33 a34; a41 a42 a43 a44];
```

```matlab
C = [1 0 0 0; 0 1 0 0];
D = 0;

% Check for controllability of the model

Crank = rank(ctrb(A,B));

if Crank == 4
    fprintf('The system is completely state controllable')
else
    fprintf('Not state controllable')
end

t = 0:0.002:20;
x0 = -0.374; theta0 = deg2rad(12.16);      % Defines initial conditions
x0 = [ x0  theta0  0  0 ];       % Creates initial conditions vector
u = zeros(length(t),1);                    % Creates a zero forcing vector

Q = [4000, 0, 0, 0; 0, 4000, 0, 0; 0, 0, 0, 0; 0, 0, 0, 0]
R = [10];
N = [0];
K = lqr(A,B,Q,R,N);

sys_cl = ss(A-B*K,B,C,0);

K

lsim(sys_cl,u,t,x0);
```

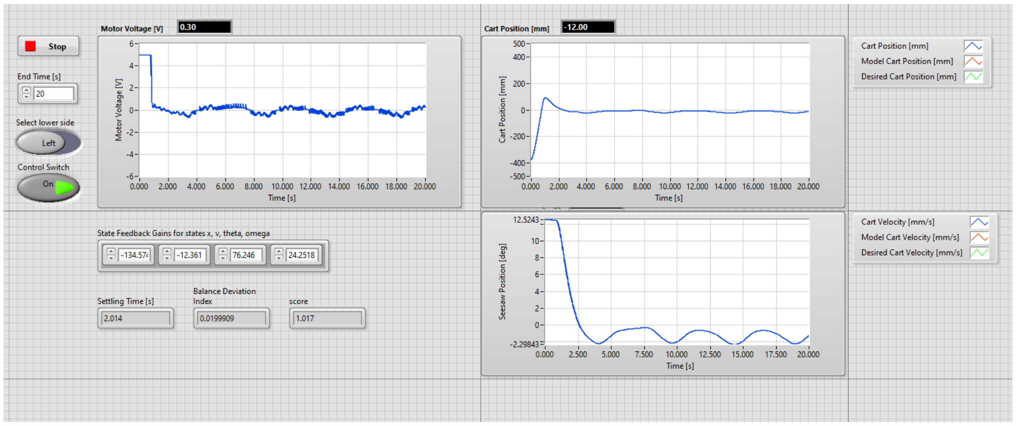
Python code to calculate balance deviation index from excel data (offline calculation)

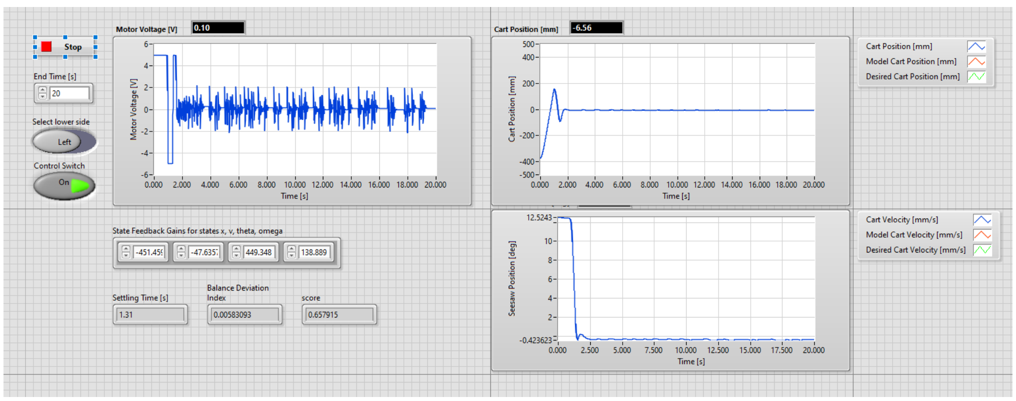
```python
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

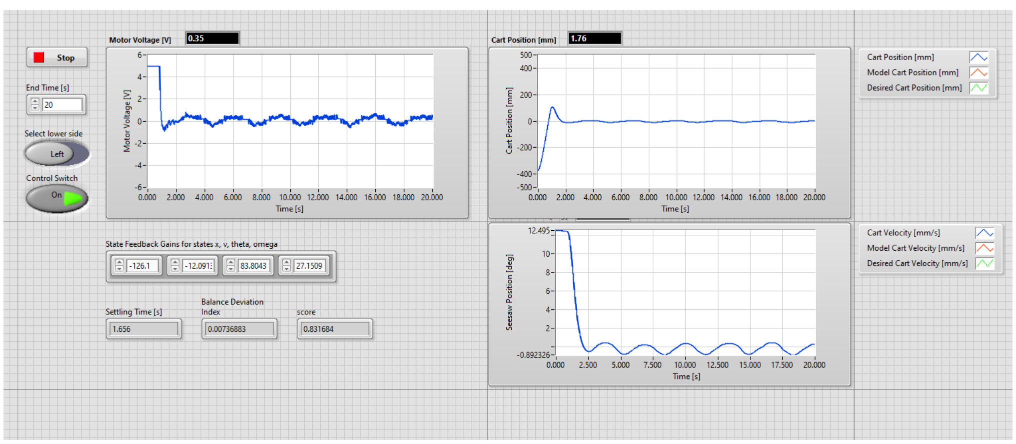
```python
#df= pd.read_csv("/content/drive/MyDrive/lqr2.csv")
df =
pd.DataFrame(pd.read_excel("/content/drive/Shareddrives/M
ME 536 Project/Final_Readings/SFC/sfc3.xlsx"))
```

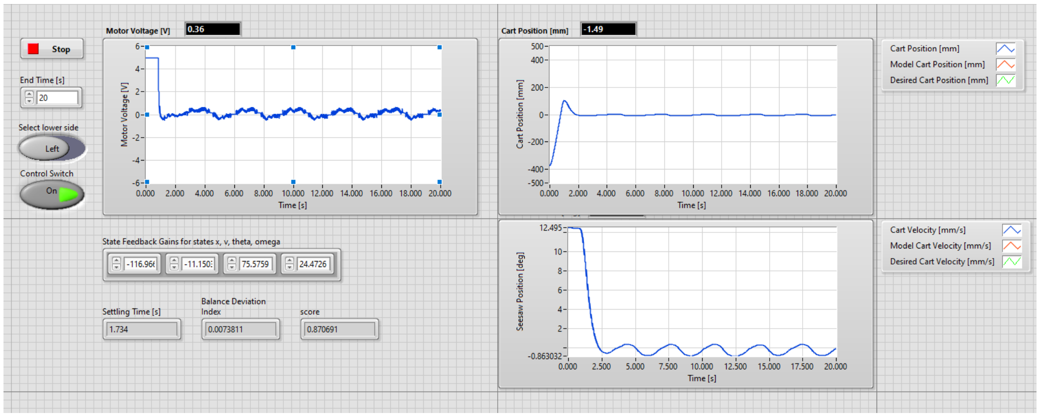
```
#df= pd.read_csv("/content/drive/Shareddrives/MME 536
Project/practical/LQR/lqr1.xlsx")
df.head()

delta=2.8
val=0
new_pos=[]
new_time=[]
#print(len(pos))
for i in range(0,len(pos)):
  if(pos[i]<=delta):
    new_pos.append(pos[i])
    new_time.append(time[i])
print(len(new_time))
x=0.002*(3.14/180)*abs((sum(new_pos[0:5000])))/10
y=(new_time[5000]-new_time[0])
print("time",y)
print("deviation index",x)
#print(new_time[2]-new_time[1])

in_deg= x*180/np.pi
print("in_deg",in_deg)
```

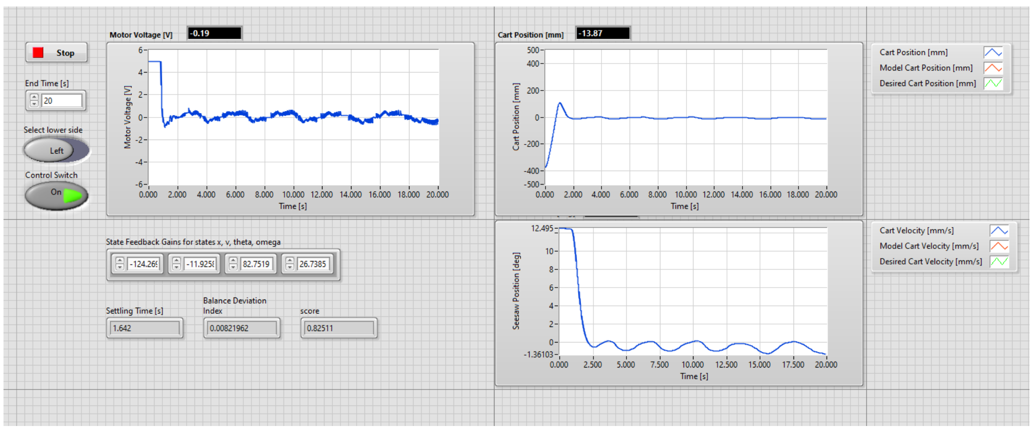
Experimental Results:

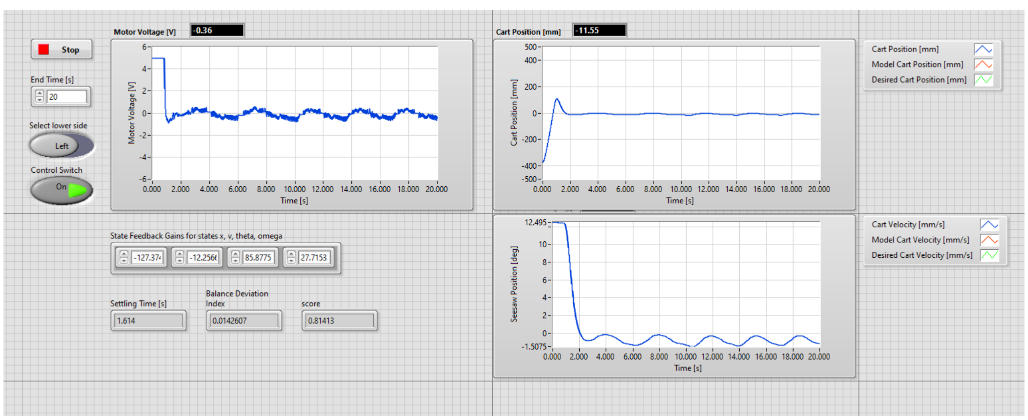

Stop
End Time [s]
20
Select lower side
Left
Control Switch
On
Motor Voltage [V]
0.30
Time [s]
Cart Position [mm]
-12.00
Cart Position [mm]
Model Cart Position [mm]
Desired Cart Position [mm]
State Feedback Gains for states x, v, theta, omega
-134.574
-12.361
76.246
24.2518
Settling Time [s]
2.014
Balance Deviation Index
0.0199909
score
1.017
Seesaw Position [deg]
12.5243
-2.29843
Cart Velocity [mm/s]
Model Cart Velocity [mm/s]
Desired Cart Velocity [mm/s]

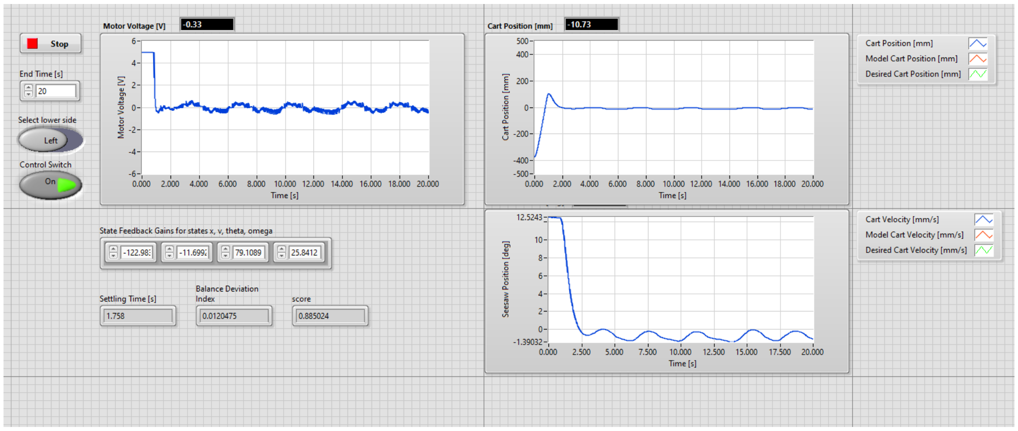

Stop
End Time [s]
20
Select lower side
Left
Control Switch
On
Motor Voltage [V]
0.10
Time [s]
Cart Position [mm]
-6.56
Cart Position [mm]
Model Cart Position [mm]
Desired Cart Position [mm]
State Feedback Gains for states x, v, theta, omega
-451.459
-47.6357
449.348
138.889
Settling Time [s]
1.31
Balance Deviation Index
0.00583093
score
0.657915
Seesaw Position [deg]
12.5243
-0.423623
Cart Velocity [mm/s]
Model Cart Velocity [mm/s]
Desired Cart Velocity [mm/s]


Stop
End Time [s]
20
Select lower side
Left
Control Switch
On
Motor Voltage [V]
0.35
Time [s]
Cart Position [mm]
1.76
Cart Position [mm]
Model Cart Position [mm]
Desired Cart Position [mm]
State Feedback Gains for states x, v, theta, omega
-126.1
-12.0913
83.8043
27.1509
Settling Time [s]
1.656
Balance Deviation Index
0.00736883
score
0.831684
Seesaw Position [deg]
12.495
-0.892326
Cart Velocity [mm/s]
Model Cart Velocity [mm/s]
Desired Cart Velocity [mm/s]


Stop
End Time [s]
20
Select lower side
Left
Control Switch
On
Motor Voltage [V]
0.36
Time [s]
Cart Position [mm]
-1.49
Cart Position [mm]
Model Cart Position [mm]
Desired Cart Position [mm]
State Feedback Gains for states x, v, theta, omega
75.5759
24.4726
Settling Time [s]
1.734
Balance Deviation Index
0.0073811
score
0.870691
Seesaw Position [deg]
Cart Velocity [mm/s]
Model Cart Velocity [mm/s]
Desired Cart Velocity [mm/s]


Stop
End Time [s]
20
Select lower side
Left
Control Switch
On
Motor Voltage [V]
-0.19
Time [s]
Cart Position [mm]
-13.87
Cart Position [mm]
Model Cart Position [mm]
Desired Cart Position [mm]
State Feedback Gains for states x, v, theta, omega
82.7519
26.7385
Settling Time [s]
1.642
Balance Deviation Index
0.00821962
score
0.82511
Seesaw Position [deg]
Cart Velocity [mm/s]
Model Cart Velocity [mm/s]
Desired Cart Velocity [mm/s]


Stop
End Time [s]
20
Select lower side
Left
Control Switch
On
Motor Voltage [V]
-0.36
Time [s]
Cart Position [mm]
-11.55
Cart Position [mm]
Model Cart Position [mm]
Desired Cart Position [mm]
State Feedback Gains for states x, v, theta, omega
85.8775
27.7153
Settling Time [s]
1.614
Balance Deviation Index
0.0142607
score
0.81413
Seesaw Position [deg]
Cart Velocity [mm/s]
Model Cart Velocity [mm/s]
Desired Cart Velocity [mm/s]



<u>*All the equations are typed and thoroughly explained in the respective explanations in the report, that's why equations are avoided in the appendix section.</u>