Applications of Concentration of Measure

Gregory Baimetov, Jiuzhou Fu, Ansel Goh, Victor Krassovsky, Anthony Xing
Mentored by Emily Casey, Andrea Ottolini, and Hadrian Quan

Department of Mathematics, University of Washington – Seattle

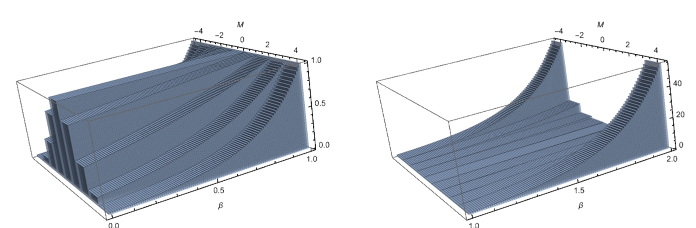
DEPARTMENT OF MATHEMATICS
UNIVERSITY *of* WASHINGTON

Motivation

Consider at time 0, N individuals take one of two opinions $x_i \in \{+1, -1\}$. At each subsequent time, one individual i changes their opinion with probability

$$p_{\text{flip}} = \exp\left(\frac{-2\beta|M^{(i)}|}{N}\right), \qquad M^{(i)} \equiv \sum_{j=1}^{N} x_j - x_i$$

where $M^{(i)}$ is the "opinion imbalance". We are interested in studying the steady state that arises from this process, which is given by $P_{N,\beta}(X) = \frac{e^{\frac{\beta}{N}\sum_{(i,j)} x_i x_j}}{Z_n(\beta)}$. For $N = 2n$, we can instead consider X distributed on $\{-1,1\}^n$ by $P\left(\sum X_i = M\right)\frac{I(M,\beta)}{Z_n(\beta)}$ which is computed in the later section. Let us consider a simple example $N = 2n = 4$:

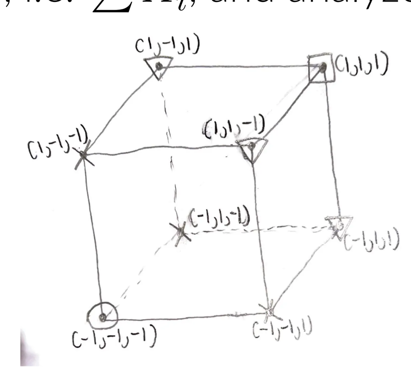


(a) $0 \le \beta \le 1$ (b) $1 \le \beta \le 1.6$

Figure 1. The 3D Plot of $I(M, \beta)$ under $n = 2$. This shows how measures are distributed on different M under different β. Notice M here must be integers from -4 to 4. The β increment is 0.01.

Curie-Weiss Model

Given a distribution of X on the vertices of a hypercube, i.e. $\{-1,1\}^n$, we want to find the distribution of the sum of X's coordinates, i.e. $\sum X_i$, and analyze how it behaves as n grows large.



Here, we have that X is distributed on $\{-1,1\}^n$, with the distribution

$$P(X) = \frac{e^{\frac{\beta}{2n}(\sum x_i)^2}}{Z_n(\beta)}, \tag{1}$$

where β is a parameter we could vary to change the distribution.

We found that the distribution of $\sum X_i$ here is

$$P\left(\sum X_i = M\right) = \frac{e^{\frac{\beta}{2n}(M)^2}}{Z_n(\beta)}\binom{n}{\frac{n+M}{2}}. \tag{2}$$

We then used Stirling's Formula to approximate the distribution of $\sum X_i$ with a continuous form:

$$P\left(\sum X_i = M\right) \approx \frac{e^{-\frac{1}{2n}M^2(1-\beta)}}{\tilde{Z}_n(\beta)}. \tag{3}$$

Using this approximation, for $\beta < 1$ and as n grows large, we found that:

$$\tilde{Z}_n(\beta) \approx 2\pi\sqrt{\frac{n}{1-\beta}}$$

In addition, we found that $P(|\sum X_i| > 0) \to 0$, in other words, the sum of the coordinates of X concentrate at 0 as n grows large for $\beta < 1$.
For $\beta > 1$, we interestingly found that $P(\sum X_i = M)$ concentrates around both 0 and 1 as n grows large.

Simulation

We can run a Monte Carlo experiment to confirm our analytic results. The experiment would simulate a group of individuals changing their opinions over some interval of time. The results are below.

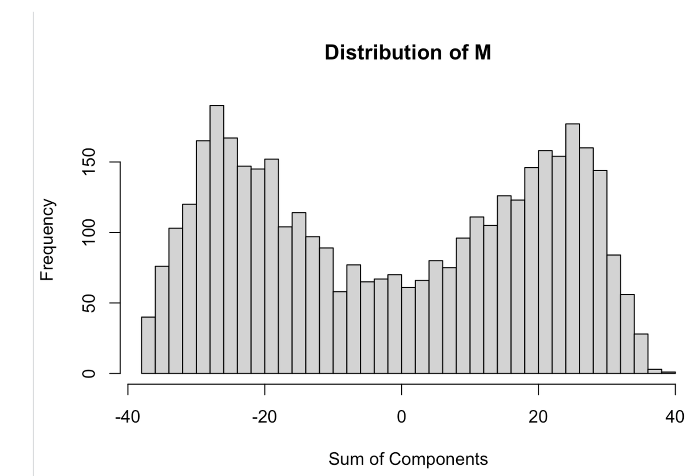


Figure 2. Exps=4000, Individuals=40, Iterations=30, $\beta = 3$

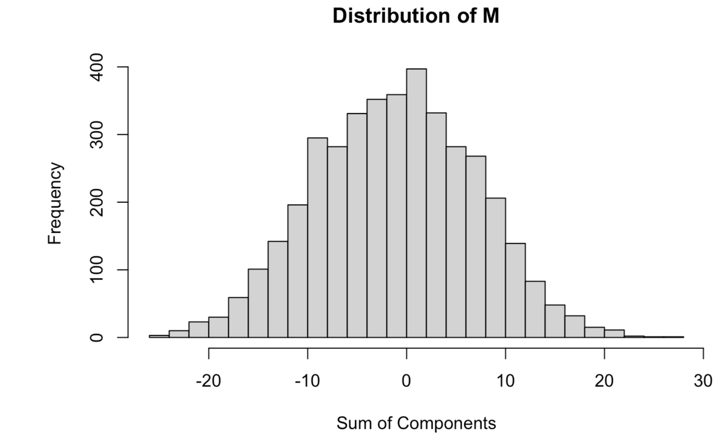


Figure 3. Exps=4000, Individuals=40, Iterations=30, $\beta = 0.5$

Sphere Case

Now let us shift from the hypercube to a sphere. Consider the probability measure given by

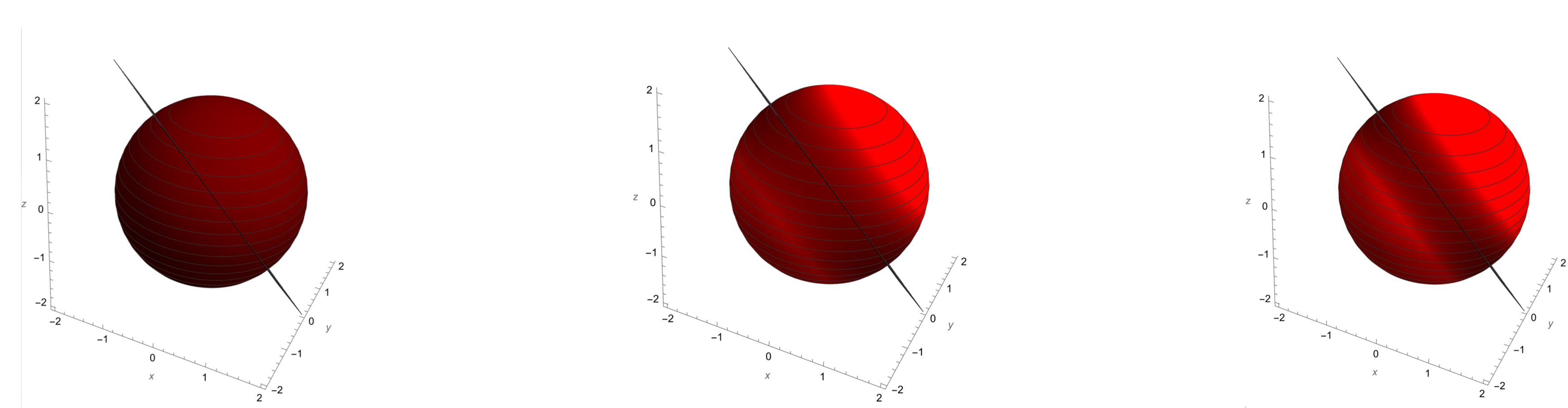
$$P(\mathbf{x}) = \frac{e^{\frac{\beta}{n}(\sum x_i)^2}}{Z_n(\beta)}$$

Consider taking intersections of planes with equations $\sum x_i = M$. For a uniform measure, we would expect the measure of the intersection to concentrate at its equator for $n > 3$. However, with the given distribution, as β increases, the probability measures are more and more "concentrated" at the poles. We found that the product of these measures concentrates somewhere between the poles and the equator, depending on β.

The intersection would be a $(n-3)$-dimensional sphere with radius $R_{\mathbb{S}^{n-3}} = \sqrt{n - \frac{M^2}{n}}$.

$$P\left(\sum x_i = M\right) = \frac{e^{\frac{\beta}{n}M^2}}{Z_n(\beta)}\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\left(n - \frac{M^2}{n}\right)^{\frac{n-3}{2}}\frac{n}{M} = \frac{e^{\frac{\beta}{n}M^2}\left(n - \frac{M^2}{n}\right)^{\frac{n-3}{2}}\frac{n}{M}}{\tilde{Z}_n(\beta)}. \tag{4}$$

By our computation, the threshold β_0 depends on n and $\beta_0 \le \frac{1}{2}$ for all n.



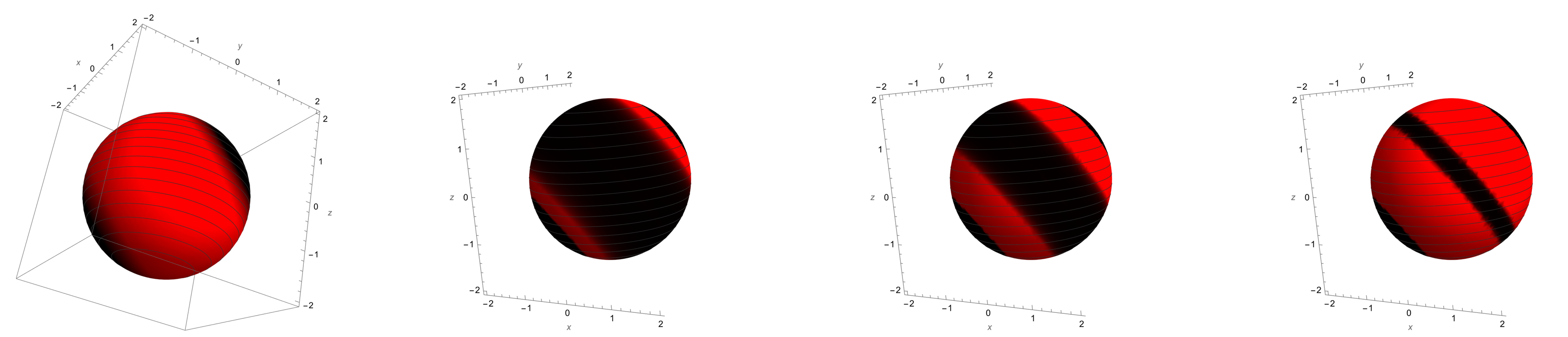
(a) $\beta = 0$ (b) $\beta = 1$ (c) $\beta = 2$

Figure 4. Illustrative example of the given measure distribution for $\mathbb{S}^2$ in $\mathbb{R}^3$. Brighter red means more measure

Using a change of variables, $x_i = \sqrt{n}z_i$, we can rewrite our probability as

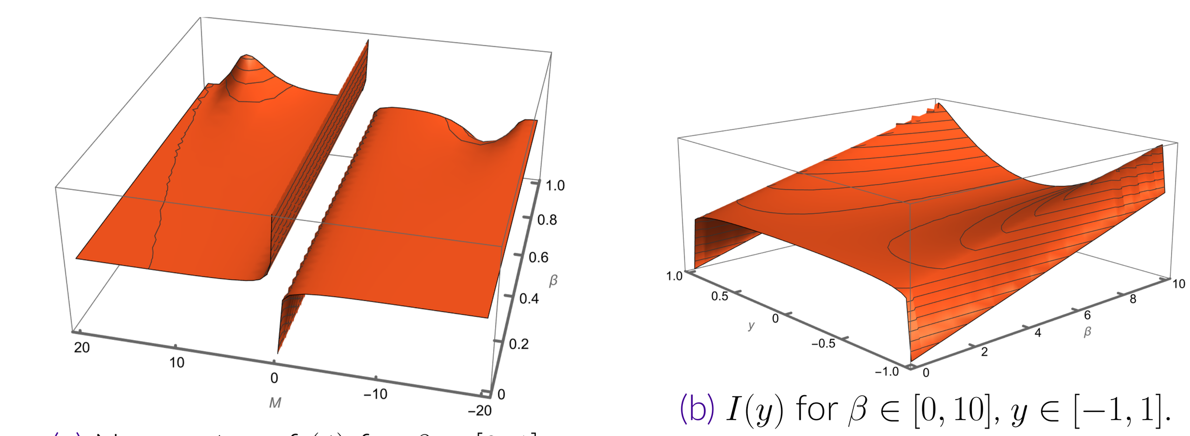
$$P\left(\sum z_i = \frac{M}{\sqrt{n}}\right) = C(n,\beta)e^{n(\beta y^2 + \frac{1}{2}\log(1-y^2)) + C(y)} \qquad \text{where} \qquad y = \frac{M}{n}.$$

Let $I_\beta(y) = \beta y^2 + \frac{1}{2}\log(1-y^2)$.



(a) (b) (c) (d)

Figure 5. Illustrative example of the measure $P\left(\sum x_i = M\right)$ as β increases for $\mathbb{S}^2$ in $\mathbb{R}^3$. Brighter red means more measure.



(a) Numerator of (4) for $\beta \in [0,1]$, $M \in [-n = -20, n = 20]$. (b) $I(y)$ for $\beta \in [0,10]$, $y \in [-1,1]$.

Figure 6. Illustrative graphs of the numerator of (4) and $I(y)$ at $n = 20$. Brighter red means more measure. Not to scale.

Concentration of Maximum of Components

Another question we might ask is, given the distribution before, where does the maximum of the components concentrate? We have

$$P(\max\{X_i\} = k) = \int P(\max\{X_i\} = k | \sum X_i = M)P(\sum X_i = M)dM \tag{5}$$

We have shown that for large n, $P(\sum X_i = M)$ concentrates at 1 or 2 places and is close to 0 elsewhere, so letting $y = \frac{m}{n}$ this integral is mostly contributed to by X when $P(\frac{\sum X_i}{n} = y)$ is maximal. When $\beta > 1$, we have found that this is when $y = \sqrt{1 - \frac{1}{\beta}}$, and when $\beta < 1$, this is when $y = 0$.

The primary related work in this area is from Chatterjee who provides us with a formula for determining $P(\max\{X_i\} = k | \frac{\sum X_i}{n} = y)$. If $b = \frac{\beta}{\beta - 1}$. Chatterjee tells us

- If $1 < b < 2$, then $P(\max\{X_i\} \ge C\log n | \frac{\sum X_i}{n} = y) \to 0$
- If $b > 2$, then $\max\{X_i\} = \sqrt{n(b-2)}$

References

[1] Sourav Chatterjee.
A note about the uniform distribution on the intersection of a simplex and a sphere, 2016.

[2] A Montanari.
Introduction and the curie-weiss model.