AIME Problems
2016

2016 AIME II

1. Initially Alex, Betty, and Charlie had a total of 444 peanuts. Charlie had the most peanuts, and Alex had the least. The three numbers of peanuts that each person had formed a geometric progression. Alex eats 5 of his peanuts, Betty eats 9 of her peanuts, and Charlie eats 25 of his peanuts. Now the three numbers of peanuts each person has forms an arithmetic progression. Find the number of peanuts Alex had initially.

2. There is a 40% chance of rain on Saturday and a 30% chance of rain on Sunday. However, it is twice as likely to rain on Sunday if it rains on Saturday than if it does not rain on Saturday. The probability that it rains at least one day this weekend is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

3. Let x, y, and z be real numbers satisfying the system

$$\log_2(xyz - 3 + \log_5 x) = 5,$$
$$\log_3(xyz - 3 + \log_5 y) = 4,$$
$$\log_4(xyz - 3 + \log_5 z) = 4.$$

   Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$.

4. An $a \times b \times c$ rectangular box is built from $a \cdot b \cdot c$ unit cubes. Each unit cube is colored red, green, or yellow. Each of the a layers of size $1 \times b \times c$ parallel to the $(b \times c)$ faces of the box contains exactly 9 red cubes, exactly 12 green cubes, and some yellow cubes. Each of the b layers of size $a \times 1 \times c$ parallel to the $(a \times c)$ faces of the box contains exactly 20 green cubes, exactly 25 yellow cubes, and some red cubes. Find the smallest possible volume of the box.

5. Triangle ABC_0 has a right angle at C_0. Its side lengths are pairwise relatively prime positive integers, and its perimeter is p. Let C_1 be the foot of the altitude to $\overline{AB}$, and for $n \geq 2$, let C_n be the foot of the altitude to $\overline{C_{n-2}B}$ in $\triangle C_{n-2}C_{n-1}B$. The sum $\sum_{n=2}^{\infty} C_{n-2}C_{n-1} = 6p$. Find p.

6. For polynomial $P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2$, define $Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i$. Then $\sum_{i=0}^{50} |a_i| = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

7. Squares $ABCD$ and $EFGH$ have a common center and $\overline{AB} \| \overline{EF}$. The area of $ABCD$ is 2016, and the area of $EFGH$ is a smaller positive integer. Square $IJKL$ is constructed so that each of its vertices lies on a side of $ABCD$ and each vertex of $EFGH$ lies on a side of $IJKL$. Find the difference between the largest and smallest positive integer values for the area of $IJKL$.

8. Find the number of sets $\{a, b, c\}$ of three distinct positive integers with the property that the product of a, b, and c is equal to the product of $11, 21, 31, 41, 51$, and 61.

9. The sequences of positive integers $1, a_2, a_3, \ldots$ and $1, b_2, b_3, \ldots$ are an increasing arithmetic sequence and an increasing geometric sequence, respectively. Let $c_n = a_n + b_n$. There is an integer k such that $c_{k-1} = 100$ and $c_{k+1} = 1000$. Find c_k.

10. Triangle ABC is inscribed in circle ω. Points P and Q are on side $\overline{AB}$ with $AP < AQ$. Rays CP and CQ meet ω again at S and T (other than C), respectively. If $AP = 4, PQ = 3, QB = 6, BT = 5$, and $AS = 7$, then $ST = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. For positive integers N and k, define N to be k-nice if there exists a positive integer a such that a^k has exactly N positive divisors. Find the number of positive integers less than 1000 that are neither 7-nice nor 8-nice.

12. The figure below shows a ring made of six small sections which you are to paint on a wall. You have four paint colors available and you will paint each of the six sections a solid color. Find the number of ways you can choose to paint the sections if no two adjacent sections can be painted with the same color.
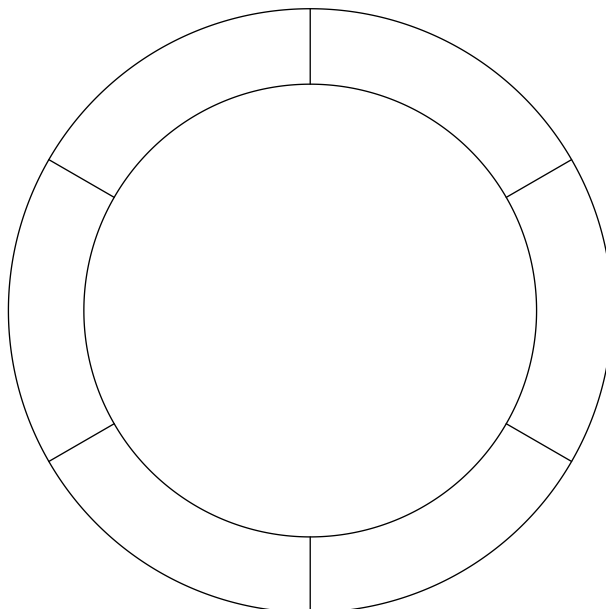
13. Beatrix is going to place six rooks on a 6×6 chessboard where both the rows and columns are labeled 1 to 6; the rooks are placed so that no two rooks are in the same row or the same column. The "value" of a square is the sum of its row number and column number. The "score" of an arrangement of rooks is the least value of any occupied square. The average score over all valid configurations is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

14. Equilateral $\triangle ABC$ has side length 600. Points P and Q lie outside the plane of $\triangle ABC$ and are on opposite sides of the plane. Furthermore, $PA = PB = PC$, and $QA = QB = QC$, and the planes of $\triangle PAB$ and $\triangle QAB$ form a 120° dihedral angle (the angle between the two planes). There is a point O whose distance from each of A, B, C, P, and Q is d. Find d.

15. For $1 \le i \le 215$ let $a_i = \dfrac{1}{2^i}$ and $a_{216} = \dfrac{1}{2^{215}}$. Let $x_1, x_2, ..., x_{216}$ be positive real numbers such that $\sum_{i=1}^{216} x_i = 1$ and $\sum_{1 \le i < j \le 216} x_i x_j = \dfrac{107}{215} + \sum_{i=1}^{216} \dfrac{a_i x_i^2}{2(1 - a_i)}$. The maximum possible value of $x_2 = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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