AMC 12 Problems
2021

2021 AMC 12B

1. How many integer values of x satisfy $|x| < 3\pi$?

   (A) 9 **(B)** 10 **(C)** 18 **(D)** 19 **(E)** 20

2. At a math contest, 57 students are wearing blue shirts, and another 75 students are wearing yellow shirts. The 132 students are assigned into 66 pairs. In exactly 23 of these pairs, both students are wearing blue shirts. In how many pairs are both students wearing yellow shirts?

   (A) 23 **(B)** 32 **(C)** 37 **(D)** 41 **(E)** 64

3. Suppose

   $$2 + \frac{1}{1 + \frac{1}{2 + \frac{2}{3+x}}} = \frac{144}{53}.$$

   What is the value of x?

   (A) $\frac{3}{4}$ **(B)** $\frac{7}{8}$ **(C)** $\frac{14}{15}$ **(D)** $\frac{37}{38}$ **(E)** $\frac{52}{53}$

4. Ms. Blackwell gives an exam to two classes. The mean of the scores of the students in the morning class is 84, and the afternoon class's mean score is 70. The ratio of the number of students in the morning class to the number of students in the afternoon class is $\frac{3}{4}$. What is the mean of the score of all the students?

   (A) 74 **(B)** 75 **(C)** 76 **(D)** 77 **(E)** 78

5. The point $P(a, b)$ in the xy-plane is first rotated counterclockwise by $90°$ around the point $(1, 5)$ and then reflected about the line $y = -x$. The image of P after these two transformations is at $(-6, 3)$. What is $b - a$?

   (A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

6. An inverted cone with base radius 12cm and height 18cm is full of water. The water is poured into a tall cylinder whose horizontal base has a radius of 24cm. What is the height in centimeters of the water in the cylinder?

   (A) 1.5 **(B)** 3 **(C)** 4 **(D)** 4.5 **(E)** 6

7. Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

   (A) $1 : 16$ **(B)** $1 : 15$ **(C)** $1 : 14$ **(D)** $1 : 8$ **(E)** $1 : 3$

8. Three equally spaced parallel lines intersect a circle, creating three chords of lengths $38, 38$, and 34. What is the distance between two adjacent parallel lines?

   (A) $5\frac{1}{2}$ **(B)** 6 **(C)** $6\frac{1}{2}$ **(D)** 7 **(E)** $7\frac{1}{2}$

9. What is the value of

   $$\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}?$$

   (A) 0 **(B)** 1 **(C)** $\frac{5}{4}$ **(D)** 2 **(E)** $\log_2 5$

10. Two distinct numbers are selected from the set $\{1, 2, 3, 4, \ldots, 36, 37\}$ so that the sum of the remaining 35 numbers is the product of these two numbers. What is the difference of these two numbers?

    (A) 5 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

11. Triangle ABC has $AB = 13, BC = 14$ and $AC = 15$. Let P be the point on $\overline{AC}$ such that $PC = 10$. There are exactly two points D and E on line BP such that quadrilaterals $ABCD$ and $ABCE$ are trapezoids. What is the distance DE?

    (A) $\frac{42}{5}$ **(B)** $6\sqrt{2}$ **(C)** $\frac{84}{5}$ **(D)** $12\sqrt{2}$ **(E)** 18

12. Suppose that S is a finite set of positive integers. If the greatest integer in S is removed from S, then the average value (arithmetic mean) of the integers remaining is 32. If the least integer in S is also removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S. What is the average value of all the integers in the set S?

(A) 36.2 **(B)** 36.4 **(C)** 36.6 **(D)** 36.8 **(E)** 37

13. How many values of θ in the interval $0 < \theta \le 2\pi$ satisfy

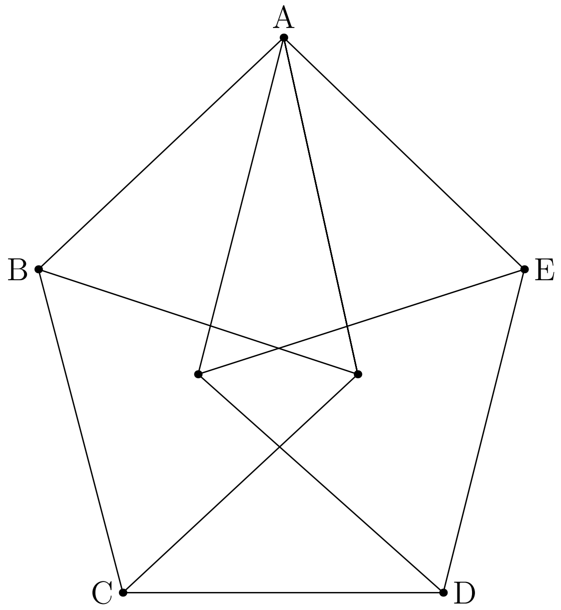
$$1 - 3\sin\theta + 5\cos 3\theta = 0?$$

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

14. Let $ABCD$ be a rectangle and let $\overline{DM}$ be a segment perpendicular to the plane of $ABCD$. Suppose that $\overline{DM}$ has integer length, and the lengths of $\overline{MA}, \overline{MC}$, and $\overline{MB}$ are consecutive odd positive integers (in this order). What is the volume of pyramid $MABCD$?

(A) $24\sqrt{5}$ **(B)** 60 **(C)** $28\sqrt{5}$ **(D)** 66 **(E)** $8\sqrt{70}$

15. The figure is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m} + \sqrt{n}$, where m and n are positive integers. What is $m + n$?

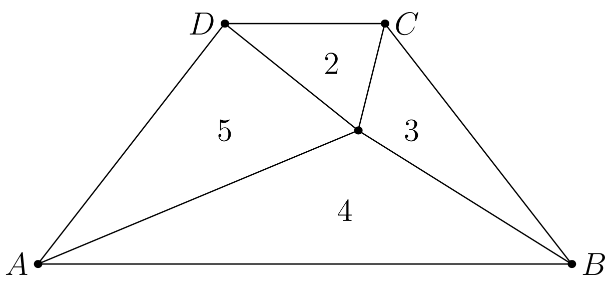


(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

16. Let $g(x)$ be a polynomial with leading coefficient 1, whose three roots are the reciprocals of the three roots of $f(x) = x^3 + ax^2 + bx + c$, where $1 < a < b < c$. What is $g(1)$ in terms of a, b, and c?

(A) $\frac{1+a+b+c}{c}$ **(B)** $1 + a + b + c$ **(C)** $\frac{1+a+b+c}{c^2}$ **(D)** $\frac{a+b+c}{c^2}$ **(E)** $\frac{1+a+b+c}{a+b+c}$

17. Let $ABCD$ be an isosceles trapezoid having parallel bases $\overline{AB}$ and $\overline{CD}$ with $AB > CD$. Line segments from a point inside $ABCD$ to the vertices divide the trapezoid into four triangles whose areas are $2, 3, 4$, and 5 starting with the triangle with base $\overline{CD}$ and moving clockwise as shown in the diagram below. What is the ratio $\frac{AB}{CD}$?



(A) 3 **(B)** $2+\sqrt{2}$ **(C)** $1+\sqrt{6}$ **(D)** $2\sqrt{3}$ **(E)** $3\sqrt{2}$

18. Let z be a complex number satisfying $12|z|^2 = 2|z+2|^2 + |z^2+1|^2 + 31$. What is the value of $z + \frac{6}{z}$?

    (A) -2 **(B)** -1 **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 4

19. Two fair dice, each with at least 6 faces are rolled. On each face of each die is printed a distinct integer from 1 to the number of faces on that die, inclusive. The probability of rolling a sum of 7 is $\frac{3}{4}$ of the probability of rolling a sum of 10, and the probability of rolling a sum of 12 is $\frac{1}{12}$. What is the least possible number of faces on the two dice combined?

    (A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

20. Let $Q(z)$ and $R(z)$ be the unique polynomials such that

    $$z^{2021} + 1 = (z^2 + z + 1)Q(z) + R(z)$$

    and the degree of R is less than 2. What is $R(z)$?

    (A) $-z$ **(B)** -1 **(C)** 2021 **(D)** $z+1$ **(E)** $2z+1$

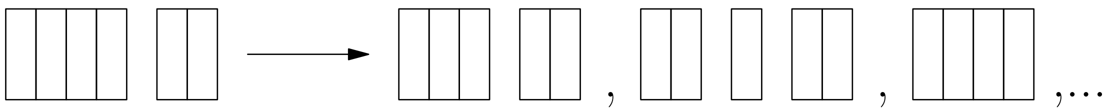
21. Let S be the sum of all positive real numbers x for which

    $$x^{2^{\sqrt{2}}} = \sqrt{2}^{2^x}.$$

    Which of the following statements is true?

    (A) $S < \sqrt{2}$ **(B)** $S = \sqrt{2}$ **(C)** $\sqrt{2} < S < 2$ **(D)** $2 \le S < 6$ **(E)** $S \ge 6$

22. Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one "wall" among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move: $(3, 2), (2, 1, 2), (4), (4, 1), (2, 2)$, or $(1, 1, 2)$.

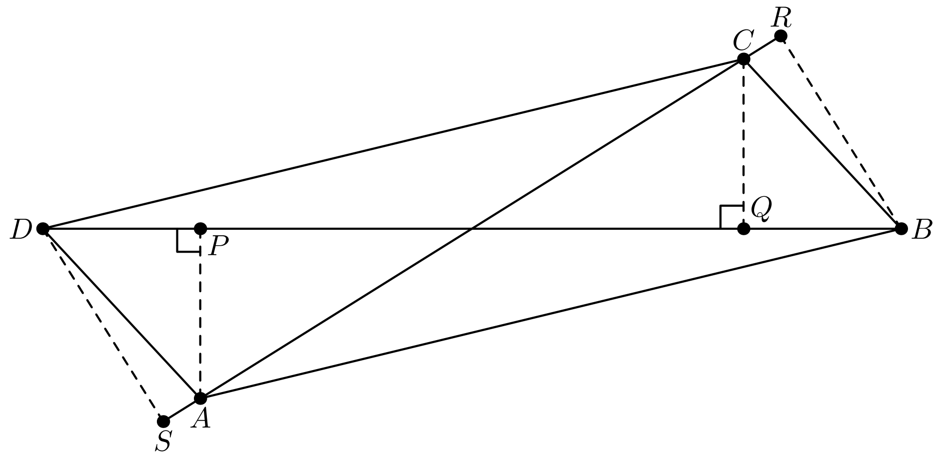
    Arjun plays first, and the player who removes the last brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

    (A) $(6, 1, 1)$ **(B)** $(6, 2, 1)$ **(C)** $(6, 2, 2)$ **(D)** $(6, 3, 1)$ **(E)** $(6, 3, 2)$

23. Three balls are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability that it is tossed into bin i is 2^{-i} for $i = 1, 2, 3, ...$ More than one ball is allowed in each bin. The probability that the balls end up evenly spaced in distinct bins is $\frac{p}{q}$, where p and q are relatively prime positive integers. (For example, the balls are evenly spaced if they are tossed into bins $3, 17$, and 10.) What is $p + q$?

    (A) 55 **(B)** 56 **(C)** 57 **(D)** 58 **(E)** 59

24. Let $ABCD$ be a parallelogram with area 15. Points P and Q are the projections of A and C, respectively, onto the line BD; and points R and S are the projections of B and D, respectively, onto the line AC. See the figure, which also shows the relative locations of these points.



Suppose $PQ = 6$ and $RS = 8$, and let d denote the length of $\overline{BD}$, the longer diagonal of $ABCD$. Then d^2 can be written in the form $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

(A) 81 **(B)** 89 **(C)** 97 **(D)** 105 **(E)** 113

25. Let S be the set of lattice points in the coordinate plane, both of whose coordinates are integers between 1 and 30, inclusive. Exactly 300 points in S lie on or below a line with equation $y = mx$. The possible values of m lie in an interval of length $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a + b$?

(A) 31 **(B)** 47 **(C)** 62 **(D)** 72 **(E)** 85