AMC12 Problems
2022

Problems

1. What is the value of
$$3+\frac{1}{3+\frac{1}{3+\frac{1}{3}}}?$$
(A) $\frac{31}{10}$ **(B)** $\frac{49}{15}$ **(C)** $\frac{33}{10}$ **(D)** $\frac{109}{33}$ **(E)** $\frac{15}{4}$

2. The sum of three numbers is 96. The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?
(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

3. Five rectangles, A, B, C, D, and E, are arranged in a square as shown below. These rectangles have dimensions 1×6, 2×4, 5×6, 2×7, and 2×3, respectively. (The figure is not drawn to scale.) Which of the five rectangles is the shaded one in the middle?
(A) A **(B)** B **(C)** C **(D)** D **(E)** E

4. The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n?
(A) 3 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

5. The ¡em¿taxicab distance¡/em¿ between points (x_1, y_1) and (x_2, y_2) in the coordinate plane is given by
$$|x_1 - x_2| + |y_1 - y_2|.$$
For how many points P with integer coordinates is the taxicab distance between P and the origin less than or equal to 20?
(A) 441 **(B)** 761 **(C)** 841 **(D)** 921 **(E)** 924

6. A data set consists of 6 (not distinct) positive integers: 1, 7, 5, 2, 5, and X. The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all possible values of X?
(A) 10 **(B)** 26 **(C)** 32 **(D)** 36 **(E)** 40

7. A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?
(A) 120 **(B)** 270 **(C)** 360 **(D)** 540 **(E)** 720

8. The infinite product
$$\sqrt[3]{10}\cdot\sqrt[3]{\sqrt[3]{10}}\cdot\sqrt[3]{\sqrt[3]{\sqrt[3]{10}}}\cdots$$
evaluates to a real number. What is that number?
(A) $\sqrt{10}$ **(B)** $\sqrt[3]{100}$ **(C)** $\sqrt[4]{1000}$ **(D)** 10 **(E)** $10\sqrt[3]{10}$

9. On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

(A) 7 **(B)** 12 **(C)** 21 **(D)** 27 **(E)** 31

10. How many ways are there to split the integers 1 through 14 into 7 pairs such that in each pair, the greater number is at least 2 times the lesser number?

(A) 108 **(B)** 120 **(C)** 126 **(D)** 132 **(E)** 144

11. What is the product of all real numbers x such that the distance on the number line between $\log_6 x$ and $\log_6 9$ is twice the distance on the number line between $\log_6 10$ and 1?

(A) 10 **(B)** 18 **(C)** 25 **(D)** 36 **(E)** 81

12. Let M be the midpoint of $\overline{AB}$ in regular tetrahedron $ABCD$. What is $\cos(\angle CMD)$?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{\sqrt{3}}{2}$

13. Let $\mathcal{R}$ be the region in the complex plane consisting of all complex numbers z that can be written as the sum of complex numbers z_1 and z_2, where z_1 lies on the segment with endpoints 3 and $4i$, and z_2 has magnitude at most 1. What integer is closest to the area of $\mathcal{R}$?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

14. What is the value of

$$(\log 5)^3 + (\log 20)^3 + (\log 8)(\log 0.25)$$

where log denotes the base-ten logarithm?

(A) $\frac{3}{2}$ **(B)** $\frac{7}{4}$ **(C)** 2 **(D)** $\frac{9}{4}$ **(E)** 3

15. The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

(A) $\frac{24}{5}$ **(B)** $\frac{42}{5}$ **(C)** $\frac{81}{5}$ **(D)** 30 **(E)** 48

16. A triangular number is a positive integer that can be expressed in the form $t_n = 1 + 2 + 3 + \cdots + n$, for some positive integer n. The three smallest triangular numbers that are also perfect squares are $t_1 = 1 = 1^2, t_8 = 36 = 6^2$, and $t_{49} = 1225 = 35^2$. What is the sum of the digits of the fourth smallest triangular number that is also a perfect square?

(A) 6 **(B)** 9 **(C)** 12 **(D)** 18 **(E)** 27

17. Suppose a is a real number such that the equation

$$a \cdot (\sin x + \sin (2x)) = \sin (3x)$$

has more than one solution in the interval $(0, \pi)$. The set of all such a that can be written in the form

$$(p, q) \cup (q, r),$$

where p, q, and r are real numbers with $p < q < r$. What is $p + q + r$?

(A) -4 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 4

18. Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y-axis. What is the least positive integer n such that performing the sequence of transformations $T_1, T_2, T_3, \ldots, T_n$ returns the point $(1, 0)$ back to itself?

(A) 359 **(B)** 360 **(C)** 719 **(D)** 720 **(E)** 721

19. Suppose that 13 cards numbered $1, 2, 3, \ldots, 13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards $1, 2, 3$ are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, $7, 8, 9, 10$ on the fourth pass, and $11, 12, 13$ on the fifth pass. For how many of the 13! possible orderings of the cards will the 13 cards be picked up in exactly two passes?

(A) 4082 **(B)** 4095 **(C)** 4096 **(D)** 8178 **(E)** 8191

20. Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1, PB = 2, PC = 3$, and $PD = 4$. What is $\frac{BC}{AD}$?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

21. Let

$$P(x) = x^{2022} + x^{1011} + 1.$$

Which of the following polynomials is a factor of $P(x)$?

(A) $x^2 - x + 1$ **(B)** $x^2 + x + 1$ **(C)** $x^4 + 1$ **(D)** $x^6 - x^3 + 1$ **(E)** $x^6 + x^3 + 1$

22. Let c be a real number, and let z_1 and z_2 be the two complex numbers satisfying the equation $z^2 - cz + 10 = 0$. Points z_1, z_2, $\frac{1}{z_1}$, and $\frac{1}{z_2}$ are the vertices of (convex) quadrilateral $\mathcal{Q}$ in the complex plane. When the area of $\mathcal{Q}$ obtains its maximum possible value, c is closest to which of the following?

(A) 4.5 **(B)** 5 **(C)** 5.5 **(D)** 6 **(E)** 6.5

23. Let h_n and k_n be the unique relatively prime positive integers such that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{h_n}{k_n}.$$

Let L_n denote the least common multiple of the numbers $1, 2, 3, \ldots, n$. For how many integers with $1 \le n \le 22$ is $k_n < L_n$?

(A) 0 **(B)** 3 **(C)** 7 **(D)** 8 **(E)** 10

24. How many strings of length 5 formed from the digits 0, 1, 2, 3, 4 are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j? (For example, 02214 satisfies this condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

(A) 500 **(B)** 625 **(C)** 1089 **(D)** 1199 **(E)** 1296

25. A circle with integer radius r is centered at (r, r). Distinct line segments of length c_i connect points $(0, a_i)$ to $(b_i, 0)$ for $1 \le i \le 14$ and are tangent to the circle, where a_i, b_i, and c_i are all positive integers and $c_1 \le c_2 \le \cdots \le c_{14}$. What is the ratio $\frac{c_{14}}{c_1}$ for the least possible value of r?

(A) $\frac{21}{5}$ **(B)** $\frac{85}{13}$ **(C)** 7 **(D)** $\frac{39}{5}$ **(E)** 17