# AIME Problems
# 2010

# Problems

1. Maya lists all the positive divisors of $2010^2$. She then randomly selects two distinct divisors from this list. Let $p$ be the probability that exactly one of the selected divisors is a perfect square. The probability $p$ can be expressed in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

2. Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}}$ is divided by 1000.

3. Suppose that $y = \frac{3}{4}x$ and $x^y = y^x$. The quantity $x + y$ can be expressed as a rational number $\frac{r}{s}$, where $r$ and $s$ are relatively prime positive integers. Find $r + s$.

4. Jackie and Phil have two fair coins and a third coin that comes up heads with probability $\frac{4}{7}$. Jackie flips the three coins, and then Phil flips the three coins. Let $\frac{m}{n}$ be the probability that Jackie gets the same number of heads as Phil, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

5. Positive integers $a$, $b$, $c$, and $d$ satisfy $a > b > c > d$, $a + b + c + d = 2010$, and $a^2 - b^2 + c^2 - d^2 = 2010$. Find the number of possible values of $a$.

6. Let $P(x)$ be a quadratic polynomial with real coefficients satisfying $x^2 - 2x + 2 \le P(x) \le 2x^2 - 4x + 3$ for all real numbers $x$, and suppose $P(11) = 181$. Find $P(16)$.

7. Define an ordered triple $(A, B, C)$ of sets to be *minimally intersecting* if $|A \cap B| = |B \cap C| = |C \cap A| = 1$ and $A \cap B \cap C = \emptyset$. For example, $(\{1, 2\}, \{2, 3\}, \{1, 3, 4\})$ is a minimally intersecting triple. Let $N$ be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. Find the remainder when $N$ is divided by 1000.

   "'Note"': $|S|$ represents the number of elements in the set $S$.

8. For a real number $a$, let $\lfloor a \rfloor$ denote the greatest integer less than or equal to $a$. Let $\mathcal{R}$ denote the region in the coordinate plane consisting of points $(x, y)$ such that $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25$. The region $\mathcal{R}$ is completely contained in a disk of radius $r$ (a disk is the union of a circle and its interior). The minimum value of $r$ can be written as $\frac{\sqrt{m}}{n}$, where $m$ and $n$ are integers and $m$ is not divisible by the square of any prime. Find $m + n$.

9. Let $(a, b, c)$ be a real solution of the system of equations $x^3 - xyz = 2$, $y^3 - xyz = 6$, $z^3 - xyz = 20$. The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

10. Let $N$ be the number of ways to write 2010 in the form $2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$, where the $a_i$'s are integers, and $0 \le a_i \le 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find $N$.

11. Let $\mathcal{R}$ be the region consisting of the set of points in the coordinate plane that satisfy both $|8 - x| + y \le 10$ and $3y - x \ge 15$. When $\mathcal{R}$ is revolved around the line whose equation is $3y - x = 15$, the volume of the resulting solid is $\frac{m\pi}{n\sqrt{p}}$, where $m$, $n$, and $p$ are positive integers, $m$ and $n$ are relatively prime, and $p$ is not divisible by the square of any prime. Find $m + n + p$.

12. Let $m \ge 3$ be an integer and let $S = \{3, 4, 5, \ldots, m\}$. Find the smallest value of $m$ such that for every partition of $S$ into two subsets, at least one of the subsets contains integers $a$, $b$, and $c$ (not necessarily distinct) such that $ab = c$.

    "'Note"': a partition of $S$ is a pair of sets $A$, $B$ such that $A \cap B = \emptyset$, $A \cup B = S$.

13. Rectangle $ABCD$ and a semicircle with diameter $AB$ are coplanar and have nonoverlapping interiors. Let $\mathcal{R}$ denote the region enclosed by the semicircle and the rectangle. Line $\ell$ meets the semicircle, segment $AB$, and segment $CD$ at distinct points $N$, $U$, and $T$, respectively. Line $\ell$ divides region $\mathcal{R}$ into two regions with areas in the ratio $1 : 2$. Suppose that $AU = 84$, $AN = 126$, and $UB = 168$. Then $DA$ can be represented as $m\sqrt{n}$, where $m$ and $n$ are positive integers and $n$ is not divisible by the square of any prime. Find $m + n$.

14. For each positive integer $n$, let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of $n$ for which $f(n) \le 300$.
"'Note:"' $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.

15. In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let $M$ be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Then $\frac{AM}{CM} = \frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$.