AIME Problems
2024

Problems

1. Among the 900 residents of Aimeville, there are 195 who own a diamond ring, 367 who own a set of golf clubs, and 562 who own a garden spade. In addition, each of the 900 residents owns a bag of candy hearts. There are 437 residents who own exactly two of these things, and 234 residents who own exactly three of these things. Find the number of residents of Aimeville who own all four of these things.

2. A list of positive integers has the following properties:
   - The sum of the items in the list is 30.
   - The unique mode of the list is 9.
   - The median of the list is a positive integer that does not appear in the list itself.

   Find the sum of the squares of all the items in the list.

3. Find the number of ways to place a digit in each cell of a 2x3 grid so that the sum of the two numbers formed by reading left to right is 999, and the sum of the three numbers formed by reading top to bottom is 99. The grid below is an example of such an arrangement because $8 + 991 = 999$ and $9 + 9 + 81 = 99$.

0	0	8
9	9	1

4. Let x, y and z be positive real numbers that satisfy the following system of equations:

$$\log_2\left(\frac{x}{yz}\right) = \frac{1}{2}$$

$$\log_2\left(\frac{y}{xz}\right) = \frac{1}{3}$$

$$\log_2\left(\frac{z}{xy}\right) = \frac{1}{4}$$

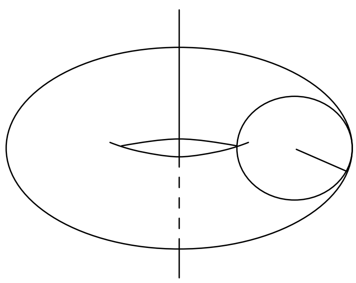
   Then the value of $\left|\log_2(x^4y^3z^2)\right|$ is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

5. Let $ABCDEF$ be a convex equilateral hexagon in which all pairs of opposite sides are parallel. The triangle whose sides are extensions of segments $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ has side lengths $200, 240,$ and 300. Find the side length of the hexagon.

6. Alice chooses a set A of positive integers. Then Bob lists all finite nonempty sets B of positive integers with the property that the maximum element of B belongs to A. Bob's list has 2024 sets. Find the sum of the elements of A.

7. Let N be the greatest four-digit integer with the property that whenever one of its digits is changed to 1, the resulting number is divisible by 7. Let Q and R be the quotient and remainder, respectively, when N is divided by 1000. Find $Q + R$.

8. Torus T is the surface produced by revolving a circle with radius 3 around an axis in the plane of the circle that is a distance 6 from the center of the circle (so like a donut). Let S be a sphere with a radius 11. When T rests on the inside of S, it is internally tangent to S along a circle with radius r_i, and when T rests on the outside of S, it is externally tangent to S along a circle with radius r_o. The difference $r_i - r_o$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



9. There are 25 indistinguishable white chips and 25 indistinguishable black chips. Find the number of ways to place some of these chips in a 5×5 grid such that

   * each cell contains at most one chip

   * all chips in the same row and all chips in the same column have the same colour

   * any additional chip placed on the grid would violate one or more of the previous two conditions.

10. Let $\triangle ABC$ have incenter I and circumcenter O with $\overline{IA} \perp \overline{OI}$, circumradius 13, and inradius 6. Find $AB \cdot AC$.

11. Find the number of triples of nonnegative integers (a, b, c) satisfying $a + b + c = 300$ and

$$a^2b + a^2c + b^2a + b^2c + c^2a + c^2b = 6{,}000{,}000.$$

12. Let $O(0,0)$, $A(\frac{1}{2}, 0)$, and $B(0, \frac{\sqrt{3}}{2})$ be points in the coordinate plane. Let $\mathcal{F}$ be the family of segments $\overline{PQ}$ of unit length lying in the first quadrant with P on the x-axis and Q on the y-axis. There is a unique point C on $\overline{AB}$, distinct from A and B, that does not belong to any segment from $\mathcal{F}$ other than $\overline{AB}$. Then $OC^2 = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

13. Let $\omega \neq 1$ be a 13th root of unity. Find the remainder when

$$\prod_{k=0}^{12} (2 - 2\omega^k + \omega^{2k})$$

    is divided by 1000.

14. Let $b \geq 2$ be an integer. Call a positive integer n b*-eautiful* if it has exactly two digits when expressed in base b, and these two digits sum to $\sqrt{n}$. For example, 81 is 13-eautiful because $81 = \underline{6}\,\underline{3}_{13}$ and $6 + 3 = \sqrt{81}$. Find the least integer $b \geq 2$ for which there are more than ten b-eautiful integers.

15. Find the number of rectangles that can be formed inside a fixed regular dodecagon (12-gon) where each side of the rectangle lies on either a side or a diagonal of the dodecagon. The diagram below shows three of those rectangles.