AIME Problems
2011

Problems

1. Gary purchased a large beverage, but only drank m/n of it, where m and n are relatively prime positive integers. If he had purchased half as much and drunk twice as much, he would have wasted only $2/9$ as much beverage. Find $m+n$.

2. On square $ABCD$, point E lies on side AD and point F lies on side BC, so that $BE = EF = FD = 30$. Find the area of the square $ABCD$.

3. The degree measures of the angles in a convex 18-sided polygon form an increasing arithmetic sequence with integer values. Find the degree measure of the smallest angle.

4. In triangle ABC, $AB = 20$ and $AC = 11$. The angle bisector of angle A intersects BC at point D, and point M is the midpoint of AD. Let P be the point of intersection of AC and the line BM. The ratio of CP to PA can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

5. The sum of the first 2011 terms of a geometric sequence is 200. The sum of the first 4022 terms is 380. Find the sum of the first 6033 terms.

6. Define an ordered quadruple of integers (a, b, c, d) as "interesting" if $1 \leq a < b < c < d \leq 10$, and $a + d > b + c$. How many interesting ordered quadruples are there?

7. Ed has five identical green marbles, and a large supply of identical red marbles. He arranges the green marbles and some of the red ones in a row and finds that the number of marbles whose right hand neighbor is the same color as themselves is equal to the number of marbles whose right hand neighbor is the other color. An example of such an arrangement is GGRRRGGRG. Let m be the maximum number of red marbles for which such an arrangement is possible, and let N be the number of ways he can arrange the $m + 5$ marbles to satisfy the requirement. Find the remainder when N is divided by 1000.

8. Let $z_1, z_2, z_3, \ldots, z_{12}$ be the 12 zeroes of the polynomial $z^{12} - 2^{36}$. For each j, let w_j be one of z_j or iz_j. Then the maximum possible value of the real part of $\sum_{j=1}^{12} w_j$ can be written as $m + \sqrt{n}$ where m and n are positive integers. Find $m + n$.

9. Let $x_1, x_2, \ldots, x_6$ be nonnegative real numbers such that $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$, and $x_1x_3x_5 + x_2x_4x_6 \geq \frac{1}{540}$. Let p and q be relatively prime positive integers such that $\frac{p}{q}$ is the maximum possible value of $x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_5x_6x_1 + x_6x_1x_2$. Find $p + q$.

10. A circle with center O has radius 25. Chord $\overline{AB}$ of length 30 and chord $\overline{CD}$ of length 14 intersect at point P. The distance between the midpoints of the two chords is 12. The quantity OP^2 can be represented as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

11. Let M_n be the $n \times n$ matrix with entries as follows: for $1 \leq i \leq n$, $m_{i,i} = 10$; for $1 \leq i \leq n-1$, $m_{i+1,i} = m_{i,i+1} = 3$; all other entries in M_n are zero. Let D_n be the determinant of matrix M_n. Then $\sum_{n=1}^{\infty} \frac{1}{8D_n+1}$ can be represented as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

    Note: The determinant of the 1×1 matrix $[a]$ is a, and the determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$; for $n \geq 2$, the determinant of an $n \times n$ matrix with first row or first column $a_1\ a_2\ a_3\ \ldots\ a_n$ is equal to $a_1C_1 - a_2C_2 + a_3C_3 - \cdots + (-1)^{n+1}a_nC_n$, where C_i is the determinant of the $(n-1) \times (n-1)$ matrix formed by eliminating the row and column containing a_i.

12. Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Let the probability that each delegate sits next to at least one delegate from another country be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

13. Point P lies on the diagonal AC of square $ABCD$ with $AP > CP$. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP, respectively. Given that $AB = 12$ and $\angle O_1PO_2 = 120°$, then $AP = \sqrt{a} + \sqrt{b}$, where a and b are positive integers. Find $a + b$.

14. There are N permutations $(a_1, a_2, \ldots, a_{30})$ of $1, 2, \ldots, 30$ such that for $m \in \{2, 3, 5\}$, m divides $a_{n+m} - a_n$ for all integers n with $1 \leq n < n + m \leq 30$. Find the remainder when N is divided by 1000.

15. Let $P(x) = x^2 - 3x - 9$. A real number x is chosen at random from the interval $5 \leq x \leq 15$. The probability that $\left\lfloor \sqrt{P(x)} \right\rfloor = \sqrt{P(\lfloor x \rfloor)}$ is equal to $\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}-d}{e}$, where a, b, c, d, and e are positive integers. Find $a + b + c + d + e$.