

AMC12 Problems  
2022

# Problems

1. What is the value of

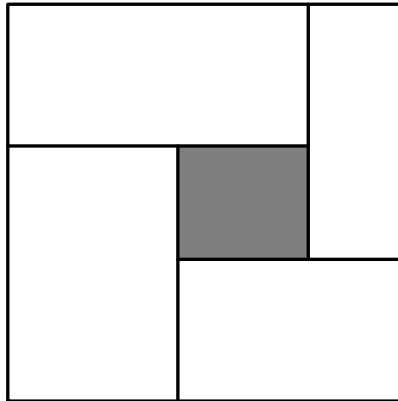
$$3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{3}}}$$

- (A)  $\frac{31}{10}$     (B)  $\frac{49}{15}$     (C)  $\frac{33}{10}$     (D)  $\frac{109}{33}$     (E)  $\frac{15}{4}$

2. The sum of three numbers is 96. The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?

- (A) 1    (B) 2    (C) 3    (D) 4    (E) 5

3. Five rectangles,  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$ , are arranged in a square as shown below. These rectangles have dimensions  $1 \times 6$ ,  $2 \times 4$ ,  $5 \times 6$ ,  $2 \times 7$ , and  $2 \times 3$ , respectively. (The figure is not drawn to scale.) Which of the five rectangles is the shaded one in the middle?



- (A)  $A$     (B)  $B$     (C)  $C$     (D)  $D$     (E)  $E$

4. The least common multiple of a positive integer  $n$  and 18 is 180, and the greatest common divisor of  $n$  and 45 is 15. What is the sum of the digits of  $n$ ?

- (A) 3    (B) 6    (C) 8    (D) 9    (E) 12

5. The taxicab distance between points  $(x_1, y_1)$  and  $(x_2, y_2)$  in the coordinate plane is given by

$$|x_1 - x_2| + |y_1 - y_2|.$$

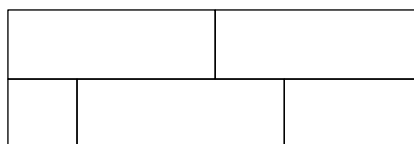
For how many points  $P$  with integer coordinates is the taxicab distance between  $P$  and the origin less than or equal to 20?

- (A) 441    (B) 761    (C) 841    (D) 921    (E) 924

6. A data set consists of 6 (not distinct) positive integers: 1, 7, 5, 2, 5, and  $X$ . The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all possible values of  $X$ ?

- (A) 10    (B) 26    (C) 32    (D) 36    (E) 40

7. A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?



- (A) 120    (B) 270    (C) 360    (D) 540    (E) 720

8. The infinite product

$$\sqrt[3]{10} \cdot \sqrt[3]{\sqrt[3]{10}} \cdot \sqrt[3]{\sqrt[3]{\sqrt[3]{10}}} \cdots$$

evaluates to a real number. What is that number?

- (A)  $\sqrt{10}$  (B)  $\sqrt[3]{100}$  (C)  $\sqrt[4]{1000}$  (D) 10 (E)  $10\sqrt[3]{10}$

9. On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

- (A) 7 (B) 12 (C) 21 (D) 27 (E) 31

10. How many ways are there to split the integers 1 through 14 into 7 pairs such that in each pair, the greater number is at least 2 times the lesser number?

- (A) 108 (B) 120 (C) 126 (D) 132 (E) 144

11. What is the product of all real numbers  $x$  such that the distance on the number line between  $\log_6 x$  and  $\log_6 9$  is twice the distance on the number line between  $\log_6 10$  and 1?

- (A) 10 (B) 18 (C) 25 (D) 36 (E) 81

12. Let  $M$  be the midpoint of  $\overline{AB}$  in regular tetrahedron  $ABCD$ . What is  $\cos(\angle CMD)$ ?

- (A)  $\frac{1}{4}$  (B)  $\frac{1}{3}$  (C)  $\frac{2}{5}$  (D)  $\frac{1}{2}$  (E)  $\frac{\sqrt{3}}{2}$

13. Let  $\mathcal{R}$  be the region in the complex plane consisting of all complex numbers  $z$  that can be written as the sum of complex numbers  $z_1$  and  $z_2$ , where  $z_1$  lies on the segment with endpoints 3 and  $4i$ , and  $z_2$  has magnitude at most 1. What integer is closest to the area of  $\mathcal{R}$ ?

- (A) 13 (B) 14 (C) 15 (D) 16 (E) 17

14. What is the value of

$$(\log 5)^3 + (\log 20)^3 + (\log 8)(\log 0.25)$$

where  $\log$  denotes the base-ten logarithm?

- (A)  $\frac{3}{2}$  (B)  $\frac{7}{4}$  (C) 2 (D)  $\frac{9}{4}$  (E) 3

15. The roots of the polynomial  $10x^3 - 39x^2 + 29x - 6$  are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

- (A)  $\frac{24}{5}$  (B)  $\frac{42}{5}$  (C)  $\frac{81}{5}$  (D) 30 (E) 48

16. A triangular number is a positive integer that can be expressed in the form  $t_n = 1 + 2 + 3 + \cdots + n$ , for some positive integer  $n$ . The three smallest triangular numbers that are also perfect squares are  $t_1 = 1 = 1^2$ ,  $t_8 = 36 = 6^2$ , and  $t_{49} = 1225 = 35^2$ . What is the sum of the digits of the fourth smallest triangular number that is also a perfect square?

- (A) 6 (B) 9 (C) 12 (D) 18 (E) 27

17. Suppose  $a$  is a real number such that the equation

$$a \cdot (\sin x + \sin(2x)) = \sin(3x)$$

has more than one solution in the interval  $(0, \pi)$ . The set of all such  $a$  that can be written in the form

$$(p, q) \cup (q, r),$$

where  $p, q$ , and  $r$  are real numbers with  $p < q < r$ . What is  $p + q + r$ ?

- (A)  $-4$     (B)  $-1$     (C)  $0$     (D)  $1$     (E)  $4$

18. Let  $T_k$  be the transformation of the coordinate plane that first rotates the plane  $k$  degrees counterclockwise around the origin and then reflects the plane across the  $y$ -axis. What is the least positive integer  $n$  such that performing the sequence of transformations  $T_1, T_2, T_3, \dots, T_n$  returns the point  $(1, 0)$  back to itself?  
(A) 359    (B) 360    (C) 719    (D) 720    (E) 721

19. Suppose that 13 cards numbered  $1, 2, 3, \dots, 13$  are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards 1, 2, 3 are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, 7, 8, 9, 10 on the fourth pass, and 11, 12, 13 on the fifth pass. For how many of the  $13!$  possible orderings of the cards will the 13 cards be picked up in exactly two passes?



- (A) 4082    (B) 4095    (C) 4096    (D) 8178    (E) 8191

20. Isosceles trapezoid  $ABCD$  has parallel sides  $\overline{AD}$  and  $\overline{BC}$ , with  $BC < AD$  and  $AB = CD$ . There is a point  $P$  in the plane such that  $PA = 1, PB = 2, PC = 3$ , and  $PD = 4$ . What is  $\frac{BC}{AD}$ ?  
(A)  $\frac{1}{4}$     (B)  $\frac{1}{3}$     (C)  $\frac{1}{2}$     (D)  $\frac{2}{3}$     (E)  $\frac{3}{4}$

21. Let

$$P(x) = x^{2022} + x^{1011} + 1.$$

Which of the following polynomials is a factor of  $P(x)$ ?

- (A)  $x^2 - x + 1$     (B)  $x^2 + x + 1$     (C)  $x^4 + 1$     (D)  $x^6 - x^3 + 1$     (E)  $x^6 + x^3 + 1$

22. Let  $c$  be a real number, and let  $z_1$  and  $z_2$  be the two complex numbers satisfying the equation  $z^2 - cz + 10 = 0$ . Points  $z_1, z_2, \frac{1}{z_1}$ , and  $\frac{1}{z_2}$  are the vertices of (convex) quadrilateral  $\mathcal{Q}$  in the complex plane. When the area of  $\mathcal{Q}$  obtains its maximum possible value,  $c$  is closest to which of the following?  
(A) 4.5    (B) 5    (C) 5.5    (D) 6    (E) 6.5

23. Let  $h_n$  and  $k_n$  be the unique relatively prime positive integers such that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = \frac{h_n}{k_n}.$$

Let  $L_n$  denote the least common multiple of the numbers  $1, 2, 3, \dots, n$ . For how many integers with  $1 \leq n \leq 22$  is  $k_n < L_n$ ?

- (A) 0    (B) 3    (C) 7    (D) 8    (E) 10

24. How many strings of length 5 formed from the digits 0, 1, 2, 3, 4 are there such that for each  $j \in \{1, 2, 3, 4\}$ , at least  $j$  of the digits are less than  $j$ ? (For example, 02214 satisfies this condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)  
(A) 500    (B) 625    (C) 1089    (D) 1199    (E) 1296

25. A circle with integer radius  $r$  is centered at  $(r, r)$ . Distinct line segments of length  $c_i$  connect points  $(0, a_i)$  to  $(b_i, 0)$  for  $1 \leq i \leq 14$  and are tangent to the circle, where  $a_i, b_i$ , and  $c_i$  are all positive integers and  $c_1 \leq c_2 \leq \dots \leq c_{14}$ . What is the ratio  $\frac{c_{14}}{c_1}$  for the least possible value of  $r$ ?  
(A)  $\frac{21}{5}$     (B)  $\frac{85}{13}$     (C) 7    (D)  $\frac{39}{5}$     (E) 17