

AIME Problems  
2010

## 2010 AIME I

- Maya lists all the positive divisors of  $2010^2$ . She then randomly selects two distinct divisors from this list. Let  $p$  be the probability that exactly one of the selected divisors is a perfect square. The probability  $p$  can be expressed in the form  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
- Find the remainder when  $9 \times 99 \times 999 \times \cdots \times \underbrace{99 \cdots 9}_{999 \text{ 9's}}$  is divided by 1000.
- Suppose that  $y = \frac{3}{4}x$  and  $x^y = y^x$ . The quantity  $x + y$  can be expressed as a rational number  $\frac{r}{s}$ , where  $r$  and  $s$  are relatively prime positive integers. Find  $r + s$ .
- Jackie and Phil have two fair coins and a third coin that comes up heads with probability  $\frac{4}{7}$ . Jackie flips the three coins, and then Phil flips the three coins. Let  $\frac{m}{n}$  be the probability that Jackie gets the same number of heads as Phil, where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
- Positive integers  $a, b, c$ , and  $d$  satisfy  $a > b > c > d$ ,  $a + b + c + d = 2010$ , and  $a^2 - b^2 + c^2 - d^2 = 2010$ . Find the number of possible values of  $a$ .
- Let  $P(x)$  be a quadratic polynomial with real coefficients satisfying  $x^2 - 2x + 2 \leq P(x) \leq 2x^2 - 4x + 3$  for all real numbers  $x$ , and suppose  $P(11) = 181$ . Find  $P(16)$ .
- Define an ordered triple  $(A, B, C)$  of sets to be *minimally intersecting* if  $|A \cap B| = |B \cap C| = |C \cap A| = 1$  and  $A \cap B \cap C = \emptyset$ . For example,  $(\{1, 2\}, \{2, 3\}, \{1, 3, 4\})$  is a minimally intersecting triple. Let  $N$  be the number of minimally intersecting ordered triples of sets for which each set is a subset of  $\{1, 2, 3, 4, 5, 6, 7\}$ . Find the remainder when  $N$  is divided by 1000.  
 "Note":  $|S|$  represents the number of elements in the set  $S$ .
- For a real number  $a$ , let  $\lfloor a \rfloor$  denote the greatest integer less than or equal to  $a$ . Let  $\mathcal{R}$  denote the region in the coordinate plane consisting of points  $(x, y)$  such that  $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25$ . The region  $\mathcal{R}$  is completely contained in a disk of radius  $r$  (a disk is the union of a circle and its interior). The minimum value of  $r$  can be written as  $\frac{\sqrt{m}}{n}$ , where  $m$  and  $n$  are integers and  $m$  is not divisible by the square of any prime. Find  $m + n$ .
- Let  $(a, b, c)$  be a real solution of the system of equations  $x^3 - xyz = 2$ ,  $y^3 - xyz = 6$ ,  $z^3 - xyz = 20$ . The greatest possible value of  $a^3 + b^3 + c^3$  can be written in the form  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
- Let  $N$  be the number of ways to write 2010 in the form  $2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$ , where the  $a_i$ 's are integers, and  $0 \leq a_i \leq 99$ . An example of such a representation is  $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$ . Find  $N$ .
- Let  $\mathcal{R}$  be the region consisting of the set of points in the coordinate plane that satisfy both  $|8 - x| + y \leq 10$  and  $3y - x \geq 15$ . When  $\mathcal{R}$  is revolved around the line whose equation is  $3y - x = 15$ , the volume of the resulting solid is  $\frac{m\pi}{n\sqrt{p}}$ , where  $m, n$ , and  $p$  are positive integers,  $m$  and  $n$  are relatively prime, and  $p$  is not divisible by the square of any prime. Find  $m + n + p$ .
- Let  $m \geq 3$  be an integer and let  $S = \{3, 4, 5, \dots, m\}$ . Find the smallest value of  $m$  such that for every partition of  $S$  into two subsets, at least one of the subsets contains integers  $a, b$ , and  $c$  (not necessarily distinct) such that  $ab = c$ .  
 "Note": a partition of  $S$  is a pair of sets  $A, B$  such that  $A \cap B = \emptyset$ ,  $A \cup B = S$ .
- Rectangle  $ABCD$  and a semicircle with diameter  $AB$  are coplanar and have nonoverlapping interiors. Let  $\mathcal{R}$  denote the region enclosed by the semicircle and the rectangle. Line  $\ell$  meets the semicircle, segment  $AB$ , and segment  $CD$  at distinct points  $N, U$ , and  $T$ , respectively. Line  $\ell$  divides region  $\mathcal{R}$  into two regions with areas in the ratio 1 : 2. Suppose that  $AU = 84$ ,  $AN = 126$ , and  $UB = 168$ . Then  $DA$  can be represented as  $m\sqrt{n}$ , where  $m$  and  $n$  are positive integers and  $n$  is not divisible by the square of any prime. Find  $m + n$ .

14. For each positive integer  $n$ , let  $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$ . Find the largest value of  $n$  for which  $f(n) \leq 300$ .  
"Note:"  $\lfloor x \rfloor$  is the greatest integer less than or equal to  $x$ .
15. In  $\triangle ABC$  with  $AB = 12$ ,  $BC = 13$ , and  $AC = 15$ , let  $M$  be a point on  $\overline{AC}$  such that the incircles of  $\triangle ABM$  and  $\triangle BCM$  have equal radii. Then  $\frac{AM}{CM} = \frac{p}{q}$ , where  $p$  and  $q$  are relatively prime positive integers. Find  $p + q$ .