

AIME Problems  
2019

# Problems

1. Consider the integer

$$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99 \dots 99}_{321 \text{ digits}}.$$

Find the sum of the digits of  $N$ .

2. Jenn randomly chooses a number  $J$  from  $1, 2, 3, \dots, 19, 20$ . Bela then randomly chooses a number  $B$  from  $1, 2, 3, \dots, 19, 20$  distinct from  $J$ . The value of  $B - J$  is at least 2 with a probability that can be expressed in the form  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
3. In  $\triangle PQR$ ,  $PR = 15$ ,  $QR = 20$ , and  $PQ = 25$ . Points  $A$  and  $B$  lie on  $\overline{PQ}$ , points  $C$  and  $D$  lie on  $\overline{QR}$ , and points  $E$  and  $F$  lie on  $\overline{PR}$ , with  $PA = QB = QC = RD = RE = PF = 5$ . Find the area of hexagon  $ABCDEF$ .
4. A soccer team has 22 available players. A fixed set of 11 players starts the game, while the other 11 are available as substitutes. During the game, the coach may make as many as 3 substitutions, where any one of the 11 players in the game is replaced by one of the substitutes. No player removed from the game may reenter the game, although a substitute entering the game may be replaced later. No two substitutions can happen at the same time. The players involved and the order of the substitutions matter. Let  $n$  be the number of ways the coach can make substitutions during the game (including the possibility of making no substitutions). Find the remainder when  $n$  is divided by 1000.
5. A moving particle starts at the point  $(4, 4)$  and moves until it hits one of the coordinate axes for the first time. When the particle is at the point  $(a, b)$ , it moves at random to one of the points  $(a - 1, b)$ ,  $(a, b - 1)$ , or  $(a - 1, b - 1)$ , each with probability  $\frac{1}{3}$ , independently of its previous moves. The probability that it will hit the coordinate axes at  $(0, 0)$  is  $\frac{m}{3^n}$ , where  $m$  and  $n$  are positive integers, and  $m$  is not divisible by 3. Find  $m + n$ .
6. In convex quadrilateral  $KLMN$ , side  $\overline{MN}$  is perpendicular to diagonal  $\overline{KM}$ , side  $\overline{KL}$  is perpendicular to diagonal  $\overline{LN}$ ,  $MN = 65$ , and  $KL = 28$ . The line through  $L$  perpendicular to side  $\overline{KN}$  intersects diagonal  $\overline{KM}$  at  $O$  with  $KO = 8$ . Find  $MO$ .
7. There are positive integers  $x$  and  $y$  that satisfy the system of equations

$$\log_{10} x + 2 \log_{10}(\gcd(x, y)) = 60$$

$$\log_{10} y + 2 \log_{10}(\text{lcm}(x, y)) = 570.$$

Let  $m$  be the number of (not necessarily distinct) prime factors in the prime factorization of  $x$ , and let  $n$  be the number of (not necessarily distinct) prime factors in the prime factorization of  $y$ . Find  $3m + 2n$ .

8. Let  $x$  be a real number such that  $\sin^{10} x + \cos^{10} x = \frac{11}{36}$ . Then  $\sin^{12} x + \cos^{12} x = \frac{m}{n}$  where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
9. Let  $\tau(n)$  denote the number of positive integer divisors of  $n$ . Find the sum of the six least positive integers  $n$  that are solutions to  $\tau(n) + \tau(n + 1) = 7$ .
10. For distinct complex numbers  $z_1, z_2, \dots, z_{673}$ , the polynomial

$$(x - z_1)^3(x - z_2)^3 \cdots (x - z_{673})^3$$

can be expressed as  $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$ , where  $g(x)$  is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1 \leq j < k \leq 673} z_j z_k \right|$$

can be expressed in the form  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

11. In  $\triangle ABC$ , the sides have integer lengths and  $AB = AC$ . Circle  $\omega$  has its center at the incenter of  $\triangle ABC$ . An "excircle" of  $\triangle ABC$  is a circle in the exterior of  $\triangle ABC$  that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to  $\overline{BC}$  is internally tangent to  $\omega$ , and the other two excircles are both externally tangent to  $\omega$ . Find the minimum possible value of the perimeter of  $\triangle ABC$ .
  12. Given  $f(z) = z^2 - 19z$ , there are complex numbers  $z$  with the property that  $z$ ,  $f(z)$ , and  $f(f(z))$  are the vertices of a right triangle in the complex plane with a right angle at  $f(z)$ . There are positive integers  $m$  and  $n$  such that one such value of  $z$  is  $m + \sqrt{n} + 11i$ . Find  $m + n$ .
  13. Triangle  $ABC$  has side lengths  $AB = 4$ ,  $BC = 5$ , and  $CA = 6$ . Points  $D$  and  $E$  are on ray  $AB$  with  $AB < AD < AE$ . The point  $F \neq C$  is a point of intersection of the circumcircles of  $\triangle ACD$  and  $\triangle EBC$  satisfying  $DF = 2$  and  $EF = 7$ . Then  $BE$  can be expressed as  $\frac{a+b\sqrt{c}}{d}$ , where  $a$ ,  $b$ ,  $c$ , and  $d$  are positive integers such that  $a$  and  $d$  are relatively prime, and  $c$  is not divisible by the square of any prime. Find  $a + b + c + d$ .
  14. Find the least odd prime factor of  $2019^8 + 1$ .
  15. Let  $\overline{AB}$  be a chord of a circle  $\omega$ , and let  $P$  be a point on the chord  $\overline{AB}$ . Circle  $\omega_1$  passes through  $A$  and  $P$  and is internally tangent to  $\omega$ . Circle  $\omega_2$  passes through  $B$  and  $P$  and is internally tangent to  $\omega$ . Circles  $\omega_1$  and  $\omega_2$  intersect at points  $P$  and  $Q$ . Line  $PQ$  intersects  $\omega$  at  $X$  and  $Y$ . Assume that  $AP = 5$ ,  $PB = 3$ ,  $XY = 11$ , and  $PQ^2 = \frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
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