AMC12 Problems
2023

Problems

1. Cities A and B are 45 miles apart. Alicia lives in A and Beth lives in B. Alicia bikes towards B at 18 miles per hour. Leaving at the same time, Beth bikes toward A at 12 miles per hour. How many miles from City A will they be when they meet?

   (A) 20 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 27

2. The weight of $\frac{1}{3}$ of a large pizza together with $3\frac{1}{2}$ cups of orange slices is the same weight of $\frac{3}{4}$ of a large pizza together with $\frac{1}{2}$ cups of orange slices. A cup of orange slices weigh $\frac{1}{4}$ of a pound. What is the weight, in pounds, of a large pizza?

   (A) $1\frac{4}{5}$ **(B)** 2 **(C)** $2\frac{2}{5}$ **(D)** 3 **(E)** $3\frac{3}{5}$

3. How many positive perfect squares less than 2023 are divisible by 5?

   (A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

4. How many digits are in the base-ten representation of $8^5 \cdot 5^{10} \cdot 15^5$?

   (A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

5. Janet rolls a standard 6-sided die 4 times and keeps a running total of the numbers she rolls. What is the probability that at some point, her running total will equal 3?

   (A) $\frac{2}{9}$ **(B)** $\frac{49}{216}$ **(C)** $\frac{25}{108}$ **(D)** $\frac{17}{72}$ **(E)** $\frac{13}{54}$

6. Points A and B lie on the graph of $y = \log_2 x$. The midpoint of $\overline{AB}$ is $(6, 2)$. What is the positive difference between the x-coordinates of A and B?

   (A) $2\sqrt{11}$ **(B)** $4\sqrt{3}$ **(C)** 8 **(D)** $4\sqrt{5}$ **(E)** 9

7. A digital display shows the current date as an 8-digit integer consisting of a 4-digit year, followed by a 2-digit month, followed by a 2-digit date within the month. For example, Arbor Day this year is displayed as 20230428. For how many dates in 2023 will each digit appear an even number of times in the 8-digital display for that date?

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

8. Maureen is keeping track of the mean of her quiz scores this semester. If Maureen scores an 11 on the next quiz, her mean will increase by 1. If she scores an 11 on each of the next three quizzes, her mean will increase by 2. What is the mean of her quiz scores currently?

   (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

9. A square of area 2 is inscribed in a square of area 3, creating four congruent triangles, as shown below. What is the ratio of the shorter leg to the longer leg in the shaded right triangle?

   (A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $2 - \sqrt{3}$ **(D)** $\sqrt{3} - \sqrt{2}$ **(E)** $\sqrt{2} - 1$

10. Positive real numbers x and y satisfy $y^3 = x^2$ and $(y - x)^2 = 4y^2$. What is $x + y$?

    (A) 12 **(B)** 18 **(C)** 24 **(D)** 36 **(E)** 42

11. What is the degree measure of the acute angle formed by lines with slopes 2 and $\frac{1}{3}$?

    (A) 30 **(B)** 37.5 **(C)** 45 **(D)** 52.5 **(E)** 60

12. What is the value of

$$2^3 - 1^3 + 4^3 - 3^3 + 6^3 - 5^3 + \cdots + 18^3 - 17^3?$$

    (A) 2023 **(B)** 2679 **(C)** 2941 **(D)** 3159 **(E)** 3235

13. In a table tennis tournament every participant played every other participant exactly once. Although there were twice as many right-handed players as left-handed players, the number of games won by left-handed players was 40% more than the number of games won by right-handed players. (There were no ties and no ambidextrous players.) What is the total number of games played?

(A) 15 **(B)** 36 **(C)** 45 **(D)** 48 **(E)** 66

14. How many complex numbers satisfy the equation $z^5 = \overline{z}$, where $\overline{z}$ is the conjugate of the complex number z?

(A) 2 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 7

15. Usain is walking for exercise by zigzagging across a 100-meter by 30-meter rectangular field, beginning at point A and ending on the segment $\overline{BC}$. He wants to increase the distance walked by zigzagging as shown in the figure below ($APQRS$). What angle $\theta \angle PAB = \angle QPC = \angle RQB = \cdots$ will produce in a length that is 120 meters? (This figure is not drawn to scale. Do not assume that the zigzag path has exactly four segments as shown; there could be more or fewer.)

(A) $\arccos \frac{5}{6}$ **(B)** $\arccos \frac{4}{5}$ **(C)** $\arccos \frac{3}{10}$ **(D)** $\arcsin \frac{4}{5}$ **(E)** $\arcsin \frac{5}{6}$

16. Consider the set of complex numbers z satisfying $|1 + z + z^2| = 4$. The maximum value of the imaginary part of z can be written in the form $\frac{\sqrt{m}}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

17. Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance m with probability $\frac{1}{2^m}$. What is the probability that Flora will eventually land at 10?

(A) $\frac{5}{512}$ **(B)** $\frac{45}{1024}$ **(C)** $\frac{127}{1024}$ **(D)** $\frac{511}{1024}$ **(E)** $\frac{1}{2}$

18. Circle C_1 and C_2 each have radius 1, and the distance between their centers is $\frac{1}{2}$. Circle C_3 is the largest circle internally tangent to both C_1 and C_2. Circle C_4 is internally tangent to both C_1 and C_2 and externally tangent to C_3. What is the radius of C_4?

(A) $\frac{1}{14}$ **(B)** $\frac{1}{12}$ **(C)** $\frac{1}{10}$ **(D)** $\frac{3}{28}$ **(E)** $\frac{1}{9}$

19. What is the product of all the solutions to the equation

$$\log_{7x} 2023 \cdot \log_{289x} 2023 = \log_{2023x} 2023?$$

(A) $(\log_{2023} 7 \cdot \log_{2023} 289)^2$ **(B)** $\log_{2023} 7 \cdot \log_{2023} 289$ **(C)** 1

(D) $\log_7 2023 \cdot \log_{289} 2023$ **(E)** $(\log_7 2023 \cdot \log_{289} 2023)^2$

20. Rows 1, 2, 3, 4, and 5 of a triangular array of integers are shown below:

Each row after the first row is formed by placing a 1 at each end of the row, and each interior entry is 1 greater than the sum of the two numbers diagonally above it in the previous row. What is the units digit of the sum of the 2023 numbers in the 2023rd row?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

21. If A and B are vertices of a polyhedron, define the distance $d(A, B)$ to be the minimum number of edges of the polyhedron one must traverse in order to connect A and B. For example, if $\overline{AB}$ is an edge of the polyhedron, then $d(A, B) = 1$, but if $\overline{AC}$ and $\overline{CB}$ are edges and $\overline{AB}$ is not an edge, then $d(A, B) = 2$. Let Q, R, and S be randomly chosen distinct vertices of a regular icosahedron (regular polyhedron made up of 20 equilateral triangles). What is the probability that $d(Q, R) > d(R, S)$?

    (A) $\frac{7}{22}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{5}{12}$ **(E)** $\frac{1}{2}$

22. Let f be the unique function defined on the positive integers such that

$$\sum_{d|n} d \cdot f\left(\frac{n}{d}\right) = 1$$

for all positive integers n, where the sum is taken over all positive divisors of n. What is $f(2023)$?

(A) -1536 **(B)** 96 **(C)** 108 **(D)** 116 **(E)** 144

23. How many ordered pairs of positive real numbers (a, b) satisfy the equation

$$(1 + 2a)(2 + 2b)(2a + b) = 32ab?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** an infinite number

24. Let K be the number of sequences A_1, A_2, $\ldots$, A_n such that n is a positive integer less than or equal to 10, each A_i is a subset of $\{1, 2, 3, \ldots, 10\}$, and A_{i-1} is a subset of A_i for each i between 2 and n, inclusive. For example, $\{\}$, $\{5, 7\}$, $\{2, 5, 7\}$, $\{2, 5, 7\}$, $\{2, 5, 6, 7, 9\}$ is one such sequence, with $n = 5$.What is the remainder when K is divided by 10?

    (A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

25. There is a unique sequence of integers $a_1, a_2, \cdots a_{2023}$ such that

$$\tan 2023x = \frac{a_1 \tan x + a_3 \tan^3 x + a_5 \tan^5 x + \cdots + a_{2023} \tan^{2023} x}{1 + a_2 \tan^2 x + a_4 \tan^4 x \cdots + a_{2022} \tan^{2022} x}$$

whenever $\tan 2023x$ is defined. What is a_{2023}?

(A) -2023 **(B)** -2022 **(C)** -1 **(D)** 1 **(E)** 2023