Beam Splitter Input-Output Relations

The beam splitter has played numerous roles in many aspects of optics. For example, in quantum information the beam splitter plays essential roles in teleportation, bell measurements, entanglement and in fundamental studies of the photon.

Electric fields E_1 and E_2 enter input ports 1 and 2, respectively. Field 1 evolves as $E_1 \rightarrow TE_3 + RE_4$, where T, R are the transmission and reflection coefficients for the beam splitter. Note that $|T|^2$ is the transmitted intensity. Similarly, $E_2 \rightarrow RE_3 + TE_4$. The transformation matrix is then given by

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \rightarrow \begin{pmatrix} T & R \\ R & T \end{pmatrix} \begin{pmatrix} E_3 \\ E_4 \end{pmatrix} \tag{1}$$

The elements of the beam splitter transformation matrix B are determined using the assumption that the beamsplitter is lossless. While a beamsplitter is never lossless, it is a good approximation for most applications. A lossless device implies that the transformation matrix B is unitary, which means that $B^{-1}B = B^{\dagger}B = 1 \Rightarrow B^{-1} = B^{\dagger}$. Recall that the matrix elements of $B^{\dagger}_{i,j} = B^{*}_{j,i}$. Hence, we arrive at

$$\begin{pmatrix} T & R \\ R & T \end{pmatrix} \begin{pmatrix} T^* & R^* \\ R^* & T^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2}$$

which yields

$$|T|^2 + |R|^2 = 1 \tag{3}$$

$$R^*T + RT^* = 0. \tag{4}$$

The transmission and reflection coefficients are complex numbers $T = |T|e^{i\theta}$ and $R = |R|e^{i\phi}$. For simplicity, let $\theta = 0$. Eqn. (4) then becomes $2TRCos\phi = 0$, which is satisfied by $\phi = \pi/2$. For a 50/50 beam splitter (meaning 50% reflection and transmission) the complex amplitude is then $1/\sqrt{2}$. The 50/50 beam splitter matrix is then given by

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{5}$$

Problem: prove to yourself that this matrix is unitary. Is this solution unique? In other words, other than a global phase, are there other unitary matrices which satisfy the specified assumptions?

SINGLE PHOTON INPUT

Now suppose one photon is in the input of port 1 of the input ports of the beam splitter. This is written as

$$\hat{a_1}^\dagger|0\rangle = |1\rangle_1|0\rangle_2 \tag{6}$$

where the last ket denotes that the vacuum state enters the other port of the beam splitter. Assuming a 50/50 beam splitter, then after the beam splitter the state is written as

$$|1\rangle_1|0\rangle_2 = \frac{1}{\sqrt{2}}|1\rangle_3|0\rangle_4 + \frac{i}{\sqrt{2}}|0\rangle_3|1\rangle_4 \tag{7}$$

This state is entangled, although one cannot measure the entanglement since the single photon is entangled along with the vacuum. More fundamentally, we can read this as the single photon has amplitudes in two different locations. The expectation value of the number operator in output mode 3 is then

$$\langle \hat{N}_3 \rangle = \left(\frac{1}{\sqrt{2}}{}_3\langle 1|_4\langle 0| - \frac{i}{\sqrt{2}}{}_3\langle 0|_4\langle 1| \right) \hat{a}_3^\dagger \hat{a}_3 \left(\frac{1}{\sqrt{2}}|1\rangle_3|0\rangle_4 + \frac{i}{\sqrt{2}}|0\rangle_3|1\rangle_4 \right) = \frac{1}{2} \tag{8}$$

Similarly, the expectation value for number operator in mode 4 is the same for a 50/50 beam splitter. This simply implies that the average number of photons in either one of the output ports is 50%, as expected. However, the expectation value

$$\langle \hat{N}_3 \hat{N}_4 \rangle = \left(\frac{1}{\sqrt{2}}{}_3\langle 1|_4\langle 0| - \frac{i}{\sqrt{2}}{}_3\langle 0|_4\langle 1| \right) \hat{a}_3^\dagger \hat{a}_3 \hat{a}_4^\dagger \hat{a}_4 \left(\frac{1}{\sqrt{2}}|1\rangle_3|0\rangle_4 + \frac{i}{\sqrt{2}}|0\rangle_3|1\rangle_4 \right) = 0 \tag{9}$$

This is a measure of the degree of second order coherence. This is simply a statement that a photon cannot be measured in two places simultaneously.

The expectation value of the electric field is

$$\langle \hat{E}_3(\chi) \rangle = \left(\frac{1}{\sqrt{2}}{}_3\langle 1|_4\langle 0| - \frac{i}{\sqrt{2}}{}_3\langle 0|_4\langle 1| \right) (\frac{1}{2}\hat{a}_3 e^{-i\chi} + \frac{1}{2}\hat{a}_3^\dagger e^{i\chi}) \left(\frac{1}{\sqrt{2}}|1\rangle_3|0\rangle_4 + \frac{i}{\sqrt{2}}|0\rangle_3|1\rangle_4 \right) = 0 \tag{10}$$

This result is expected for an individual arm for the mean field. The electric field variance is then found to be

$$
\begin{aligned}
\langle \hat{E}_3^2(\chi) \rangle &= \left(\frac{1}{\sqrt{2}}_3\langle 1|_4\langle 0| - \frac{i}{\sqrt{2}}_3\langle 0|_4\langle 1| \right) (\frac{1}{2}\hat{a}_3 e^{-i\chi} + \frac{1}{2}\hat{a}_3^\dagger e^{i\chi})^2 \left(\frac{1}{\sqrt{2}}|1\rangle_3|0\rangle_4 + \frac{i}{\sqrt{2}}|0\rangle_3|1\rangle_4 \right) \\
&= \frac{1}{4} \left(\frac{1}{\sqrt{2}}_3\langle 1|_4\langle 0| - \frac{i}{\sqrt{2}}_3\langle 0|_4\langle 1| \right) (2\hat{a}_3^\dagger \hat{a}_3 + 1) \left(\frac{1}{\sqrt{2}}|1\rangle_3|0\rangle_4 + \frac{i}{\sqrt{2}}|0\rangle_3|1\rangle_4 \right) = \frac{1}{2} \quad (11)
\end{aligned}
$$

Find

$$
\langle \hat{E}_3(\chi_3)\hat{E}_4(\chi_4) \rangle \tag{12}
$$

The result of this calculation shows that the amplitudes of the output have a high degree of first order coherence.

MACH-ZEHNDER INTERFEROMETER

Now assume that two 50/50 beam splitters are in series, such that the outputs of one beam splitter are the inputs of the other beam splitter. Further, assume that the path lengths are identical. Lastly, assume that the single photon consisting of only a single plane wave mode enters only one port of the beam splitter. Then, after the first beam splitter the field operators evolve as

$$
\hat{a_1}^\dagger|0\rangle \to \frac{1}{\sqrt{2}}(\hat{a}_3^\dagger + i\hat{a}_4^\dagger)|0\rangle \tag{13}
$$

Using the assumption that the path lengths between the two beam splitters is identical, which for all intents and purposes is an obsurd assumption, we obtain

$$
\frac{1}{\sqrt{2}}(\hat{a}_3^\dagger + i\hat{a}_4^\dagger)|0\rangle \to \frac{1}{2}(\hat{a}_5^\dagger + i\hat{a}_6^\dagger + i(i\hat{a}_5^\dagger + \hat{a}_6^\dagger))|0\rangle = \hat{a}_6^\dagger|0\rangle \tag{14}
$$

where the global phase i was dropped. This shows that the photon can still have unit probability of leaving only one port of the interferometer. Dropping the assumption of equal path length, suppose now that one of the arms is longer than the other. Since the photon is an infinite plane wave mode, then this assumption means that there will be a relative phase

picked up.

$$\hat{a_1}^\dagger|0\rangle$$
$$\rightarrow \frac{1}{\sqrt{2}}(\hat{a}_3^\dagger + i\hat{a}_4^\dagger)|0\rangle$$
$$\rightarrow \frac{1}{\sqrt{2}}(\hat{a}_3^\dagger + ie^{i\phi}\hat{a}_4^\dagger)|0\rangle$$
$$\rightarrow \frac{1}{2}(\hat{a}_5^\dagger + i\hat{a}_6^\dagger + ie^{i\phi}(i\hat{a}_5^\dagger + \hat{a}_6^\dagger))|0\rangle$$
$$= \frac{1}{2}\left((1 - e^{i\phi})\hat{a}_5^\dagger + i((1 + e^{i\phi}))\hat{a}_6^\dagger\right)|0\rangle \tag{15}$$

Thus, by manipulating the path lengths between the two beam splitters, it is possible to control from which port the photon leaves. For $\phi = 0$ the photon leaves port 6 and for $\phi = \pi$, which corresponds to a halfwavelength shift, the photon leaves port 5. The visibility of the fringes obtained by scanning the phase from 0 to 2π represents the degree of first order coherence.

SINGLE PHOTON INPUT REVISITED: DENSITY MATRIX FORMALISM

As has been seen, the method outlined so far is algebraically unfriendly. However, matrix representations of all of the transformations as well as expectation values using the density matrix formalism greatly enhance the simplicity as well as the possible measurement outcomes. To start off, we define a state density matrix. Any two state system can be represented by a wavefunction of the form

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{16}$$

where α and β are complex amplitudes of the two states 0 and 1. Owing to the fact that a global phase is irrelevant, we can rewrite the system as

$$|\Psi\rangle = \alpha|0\rangle + \beta e^{i\phi(t)}|1\rangle \tag{17}$$

where α and β are now considered to be real. It is important to note that the phase factor $\phi(t)$ evolves with time. The density matrix is thus defined as

$$\rho = |\Psi\rangle\langle\Psi| \tag{18}$$

Instead of writing the superposition state in terms of the states, we rewrite the state as position and column vectors, namely

$$|\Psi\rangle = \begin{pmatrix} \alpha \\ \beta e^{i\phi(t)} \end{pmatrix} \tag{19}$$

and

$$\langle\Psi| = \begin{pmatrix} \alpha & \beta e^{-i\phi(t)} \end{pmatrix} \tag{20}$$

Hence,

$$\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} \alpha \\ \beta e^{i\phi(t)} \end{pmatrix} \begin{pmatrix} \alpha & \beta e^{-i\phi(t)} \end{pmatrix} = \begin{pmatrix} \alpha^2 & \alpha\beta e^{-i\phi(t)} \\ \alpha\beta e^{i\phi(t)} & \beta^2 \end{pmatrix} \tag{21}$$

Thus, it can be seen that if the temporal fluctuation of the phase is very fast, then the "ensemble average" of the density matrix is represented by a diagonal density matrix. This is referred to as a statistical mixture, meaning the the diagonal elements denote the probability that a given state will be measured, while the off diagonal elements denote the degree of coherence in the system. In other words, the wavefunction is not a coherent superposition on the time scale of the measurement of a given observable. A superposition state is considered to be coherent, if the relative phase fluctuations are small on the time scale of the measurement of the observable.

The next topic we will consider is the evolution of a density matrix. Since a wavefunction evolves through operator application $\hat{A}|\Psi\rangle$, then the density matrix must evolve as $\hat{A}\rho\hat{A}^\dagger$.

Beamsplitter with single photon input revisited

As an important and simple example consider again the single photon input into beamsplitter. Since there will only ever be two possible spatial modes that the photon can be in the basis states are $|1,0\rangle$ and $|0,1\rangle$, where the positions in the kets denote the number states in the two modes. In other words, the first ket states that 1 photon is in mode one and 0 photons are in mode two. If we assume that one photon is initially in mode 1, then

$$|\Psi\rangle = |1,0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{22}$$

where we have made the assumption that the amplitude of the $|1, 0\rangle$ ket occupies the first element in the column vector. This means that the density matrix is

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{23}$$

After passing through the beamsplitter the new density matrix is

$$B\rho B^\dagger = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \tag{24}$$

Now we can compute the expectation value of an observable by using the well known relation

$$\langle A \rangle = Tr[\rho A] \tag{25}$$

where A is the matrix of the observable determined by the expansion in the n dimensional basis states $|a_n\rangle$

$$A = \begin{pmatrix} \langle a_1|\hat{A}|a_1\rangle & \langle a_1|\hat{A}|a_2\rangle & \cdots \\ \langle a_2|\hat{A}|a_1\rangle & \langle a_2|\hat{A}|a_2\rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{26}$$

The basis states for the single photon state is $|0, 1\rangle$ and $|1, 0\rangle$. The number operator matrix is then

$$N_1 = \begin{pmatrix} \langle 1, 0|\hat{N}_1|1, 0\rangle & \langle 1, 0|\hat{N}_1|0, 1\rangle \\ \langle 0, 1|\hat{N}_1|1, 0\rangle & \langle 0, 1|\hat{N}_1|0, 1\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{27}$$

It should be noted here, that the operator N_1 acts on the first term in the column vector. In the previous section, after the beam splitter, the modes were labelled mode 3 and mode 4. However, using matrix notation it is simpler to consider the terms in the column vector. For example, taking the expectation value of N_1 for any number of transformations determines the expectation value of the "transmitted" mode of the beam splitter or the first term in the column vector. The expectation value of the number operator after the beam splitter is then

$$\langle N_1 \rangle = Tr[B\rho B^\dagger N_1] = \frac{1}{2} Tr\left[\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] = \frac{1}{2} \tag{28}$$

Similarly

$$\langle N_2 \rangle = Tr[B\rho B^\dagger N_2] = \frac{1}{2} Tr\left[\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] = \frac{1}{2} \tag{29}$$

The electric field vector vanishes for every element in the matrix. Thus, the mean field is zero. The electric field squared in a given output port is

$$E_1^2 = \begin{pmatrix} \langle 1,0|\hat{E}_1^2|1,0\rangle & \langle 1,0|\hat{E}_1^2|0,1\rangle \\ \langle 0,1|\hat{E}_1^2|1,0\rangle & \langle 0,1|\hat{E}_1^2|0,1\rangle \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \tag{30}$$

The expectation value of the electric field squared after the beam splitter is then

$$\langle E_1^2 \rangle = Tr[B\rho B^\dagger E_1^2] = \frac{1}{8}Tr\left[\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\right] = \frac{1}{2} \tag{31}$$

PROVE that $\langle E_2^2 \rangle = \frac{1}{2}$ after the beam splitter.

Mach-Zehnder interferometer

All of the Mach-Zehnder interferometer statistics are easily represented in density matrix formalism. Before proceeding, the phase shift matrix is introduced. The matrix which determines the relative phase shift ϕ between the two modes is given by

$$P = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} \tag{32}$$

which means that the density matrix, incorporating the relative phase shift, is given by

$$\rho'' = PB\rho B^\dagger P^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -ie^{-i\phi} \\ ie^{i\phi} & 1 \end{pmatrix} \tag{33}$$

Finally, the density matrix after the second beam splitter is given by

$$\begin{aligned} \rho''' &= B\rho''B^\dagger = \frac{1}{4}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & -ie^{-i\phi} \\ ie^{i\phi} & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} (1-e^{i\phi})^2 & i(-1+e^{2i\phi}) \\ i(-1+e^{2i\phi}) & (1+e^{i\phi})^2 \end{pmatrix}. \end{aligned} \tag{34}$$

It can be seen that the diagonal elements represent the probability that the photon will be measured in the output of each port. One can also see that the two diagonal terms are the sine and cosine squared of twice the phase angle 2ϕ. The fringe visibility is 100%.

Homework *Problems.* Find $\langle N_1 \rangle$, $\langle E_1 \rangle$ and $\langle E_1^2 \rangle$ for the Mach-Zehnder interferometer.

FRINGE VISIBILITY AND WELCHER-WEG KNOWLEDGE

The Mach-Zehnder interferometer is a very interesting apparatus. It is extensively studied, because of its strange quantum mechanical properties. It represents the 2-mode equivalent of Young's continuous-mode double slit interferometer.

It is well known that welcher-weg or which-path knowledge about the photon destroys its visibility. The more the knowledge, the less the visibility. What if there is no knowledge about which path the photon takes? One must be very careful in making such a statement. We will now consider three important examples. The first example will be to place an object in one of the arms to prevent the amplitudes from arriving at the second beam splitter. The second example will be to perform a measurement, by placing a detector in one of the arms. The third example will be to entangle the which-path knowledge of the photon with the quantum states of another photon. As guessed, all three of these methods destroy the fringe visibility of photon.

Example 1: Obstructing Object

The first example will be to determine the effects of placing an obstructing object in one of the arms of the interferometer. How does one mathematically represent this interaction? Recall that the density matrix after the beam splitter (including the arbitrary relative phase between the two paths) is

$$\rho'' = PB\rho B^{\dagger}P^{\dagger} = \frac{1}{2}\begin{pmatrix} 1 & -ie^{-i\phi} \\ ie^{i\phi} & 1 \end{pmatrix} \tag{35}$$

The new density matrix after the obstructing object is the same as the density matrix of only a single arm. In other words, after the beamsplitter, if one were to analyze only the information in a single arm, this is equivalent to placing an obstructing object in one of the two arms. The information in a single arm can be found by tracing over the states in the other arm. The definition of the trace for a given matrix A is given by

$$Tr[A] = \sum_{n}\langle a_n|A|a_n\rangle \tag{36}$$

Assume now that the obstructing object is placed in path 1. The reduced density matrix is then

$$\rho_r'' = Tr_1[\rho''] = {}_1\langle 0|\rho''|0\rangle_1 + {}_1\langle 1|\rho''|1\rangle_1 = \frac{1}{2}|1\rangle_{22}\langle 1| + \frac{1}{2}|0\rangle_{22}\langle 0| \tag{37}$$

where the subscript r denotes that this is a reduced density matrix. The mathematical result shows that there is a statistical mixture of vacuum and single photon. The well defined phase relationship of the two mode state has been lost. It should also be pointed out that the matrix representation of

$$\rho_r'' = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{38}$$

does not have the same meaning as when the basis states are represented by the two mode field. Therefore, using the beamsplitter, number, electric field and phase matrices as they have been outlined so far are not relevant. Without the coherent relationship, the interference at the second beam splitter is lost.

Example 2: Nondemolition Measurement

The next example we will study is the effect of making a "strong" nondemolition (does not destroy the photon) which-way measurement on the photon. The quantum nondemolition (QND) measurement has a back action, but that will be ignored for the moment. This measurement is done by placing a detector between the two beam splitters in one of the two paths. We will assume that the detector is in path 1. Assuming the detector has unit probability of firing when a photon strikes it, then one can which way information about the photon. If the photon is measured by the detector then the it is known that the photon is in path 1. If no photon is detected by the detector then it is known that the photon is in path 2. Therefore, a single detector is sufficient to gain which way information. Measuring the photon direction places the reduced density matrix into one of two possible states

$$\rho_r'' = |0\rangle_1|1\rangle_{21}\langle 0|_2\langle 1|\rho'' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{39}$$

or

$$\rho_r'' = |1\rangle_1|0\rangle_{21}\langle 1|_2\langle 0|\rho'' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{40}$$

after renormalization. One can the see that this measurement has "projected" the photon into one of the two modes. The photon is then only propagating in one of the two modes, except that the photon has been destroyed in one of the paths. The new density matrix is the same as the initial pure state density matrix. Thus, for the first example, in which one photon is measured in path 1, the transformation is

$$B\rho_r''B^\dagger = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \tag{41}$$

and if the photon is determined to be in path 2 the beamsplitter transformation yields

$$B\rho_r''B^\dagger = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \tag{42}$$

It can be seen that the visibility is thus

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{\frac{1}{2} - \frac{1}{2}}{\frac{1}{2} + \frac{1}{2}} = 0 \tag{43}$$

Example 3: Entanglement

Now, let us be a little clever and use entanglement to see if we can beat this predicament. Suppose we have another photon with two quantum states $|0\rangle_3$ and $|1\rangle_3$ in mode 3. These quantum states can be polarization or whatever we wish. Now suppose that we have a device which can entangle the which-path information of the photon in the Mach-Zehnder interferometer with the states of the other photon. Then, using the other photon, we can extract which-way information of the photon. Recall that the basis states are $|1,0\rangle$ and $|0,1\rangle$. The entangler (which is done after the beam splitter) performs the operation $|1,0,0\rangle$ and $|0,1,1\rangle$. Thus, after the beam splitter the density matrix has the same form as before

$$\rho_{ent}'' = PB\rho B^\dagger P^\dagger = \frac{1}{2}\begin{pmatrix} 1 & -ie^{-i\phi} \\ ie^{i\phi} & 1 \end{pmatrix} \tag{44}$$

but now it is important to realize that the additional entangled states have been incorporated in the density matrix. One can now do one of two things with the entangled state. One can choose to measure the state of the other photon in mode 3 or one can wait until after the interferometer photon has passed through the interferometer. If one chooses to measure

the state of the photon in mode 3 before the output of the interferometer is measured, the reduced density matrix of the photon in the interferometer is given by

$$k\rho_r'' = {}_3\langle 0|\rho_{ent}''|0\rangle_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{45}$$

or

$$\rho_r'' = {}_3\langle 1|\rho_{ent}''|1\rangle_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{46}$$

depending on the measurement outcome. Thus, we are left with the same result as before and there will be no interference. What if we decide not to measure the photon in mode 3 until after the photon in the interferometer has completed its trajectory. In this case, we must trace over mode 3 to determine what happens to the photon in the interferometer. The reduced density matrix is then given by

$$\rho_r'' = Tr_3[\rho''] = {}_3\langle 0|\rho''|0\rangle_3 + {}_3\langle 1|\rho''|1\rangle_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{47}$$

where the resulting reduced density matrix is described in the original basis. Application of further beamsplitter operations are to no avail, because as can be seen, the reduced density matrix is the identity matrix up to the factor of $\frac{1}{2}$, which means that $B\rho_r B^\dagger = \rho_r$ or for that matter any unitary transformation will leave the reduced density matrix unchanged. By entangling the which path information to another two-state system, we have cursed the photon to never interfere again, shame on us.

While we have explored only a few of the possible ways of gaining which-way information about a photon in an interferometer, all paths have been unsuccessful. This intriguing fact of nature forms the foundation of many important results. For example, this which way information problem is the reason why we cannot clone a quantum state perfectly.