MATHEMATICAL FOUNDATIONS OF QUANTUM THEORY

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1978

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PREFACE

*This quantum question is so incredibly important and difficult
that everyone should busy himself with it.*
Albert Einstein writing to Jakob Laub, 1908

Nearly 70 years after Einstein's letter to Laub the most inclusive and ambitious physical theory ever invented still retains the qualities of importance and difficulty seen by Einstein. Responding to something akin to the urgency of Einstein's counsel (suggested as the conference motto by J. A. Wheeler), a group of physicists and mathematicians met together in New Orleans near the beginning of summer 1977 to discuss and study intensively the mathematical and logical foundations of the theory. The participants came from a wide variety of backgrounds and specializations, but all shared a common interest in answering the "incredibly important and difficult" quantum question.

The results are contained in this volume. They are obviously not the results of three days but of accumulated years of work. The papers range from the general to the very specialized, and require for full understanding correspondingly varied backgrounds. However, we believe that the reader with the necessary background in each case will find, if not always full answers, at least excellent questions and some answers.

This first-time conference coordinator and editor has, after extensive search, been able to discern two rather pleasurable aspects to his position: he has the privilege of placing the burden for any errors of substance on the shoulders of the individual contributors (pleading lack of broad competence), while accepting for himself (reluctantly, and pleading lack of narrow competence) the onus of minor typographical errors; and he has the pleasant duty of thanking all those whose time, efforts, and support contributed to the conference and this volume. We hereby claim the privilege fully and shall now attempt to discharge the duty. Our sincerest gratitude goes to the Research Corporation (matching funds grant), James C. Carter, S. J., President of Loyola University

During the conference at least one principle of general importance emerged. It was formulated by P. A. M. Dirac in his opening presentation as the result of a long and incredibly productive career in theoretical physics. It might be paraphrased as, "Good mathematics makes good physics." It is in this spirit that we present this volume.

THE MATHEMATICAL FOUNDATIONS OF QUANTUM THEORY

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I am glad to have this opportunity to talk to you about the mathematical foundations of quantum theory. It allows me to express my own views on the subject. They differ somewhat from those of the majority of physicists.

I want to emphasize the necessity for a sound mathematical basis for any fundamental physical theory. Any philosophical ideas that one may have play only a subordinate role. Unless such ideas have a mathematical basis they will be ineffective.

As an example of a philosophical idea without a precise mathematical basis I would like to mention Mach's principle. Einstein has stated that he was indebted to this principle in his line of thought which led him to general relativity. But I do not see how this could be. I do not see how the principle can be formulated in a sufficiently definite way to be of any value in the search for a precise physical theory.

One should keep the need for a sound mathematical basis dominating one's search for a new theory. Any physical or philosophical ideas that one has must be adjusted to fit the mathematics. Not the other way around.

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Too many physicists are inclined to start from some preconceived physical ideas and then to try to develop them and find a mathematical scheme that incorporates them. Such a line of attack is unlikely to lead to success. One runs into difficulties and finds no reasonable way out of them. One ought then to realize that one's whole line of approach is wrong and to seek a new starting point with a sound mathematical basis.

The need for putting the mathematics first comes from its more rigid nature. One can tinker with one's physical or philosophical ideas to adapt them to fit the mathematics. But the mathematics cannot be tinkered with. It is subject to completely rigid rules and is harshly restricted by strict logic.

The reason I feel so strongly about the views expressed above is because of the success I have had with them in the past. My early research work, in the early 1920's, was based on Bohr orbits, and was completely unsuccessful. I was taking the Bohr orbits as physically real and trying to build up a mathematics for them. The Bohr orbits apply to individual electrons and for an atom containing several electrons one required a theory of Bohr orbits in interaction. I worked hard on this problem, along with other people.

One sees now how futile such work was. Heisenberg showed that one needed a completely new mathematics, involving non-commutative algebra. The Bohr orbits were an unsound physical concept and should not be used as the basis for a theory.

I learnt my lesson then. I learnt to distrust all physical concepts as the basis for a theory. Instead one should put one's trust in a mathematical scheme, even if the scheme does not appear at first sight to be connected with physics. One should concentrate on getting an interesting mathematics.

The non-commutative algebra was developed and one soon found how to

connect it with dynamics, by means of the analogy between the commutator and the Poisson bracket of the Hamiltonian form of mechanics. A general quantum mechanics was set up, a beautiful and powerful theory.

The basic equations of the theory were worked out before their physical meaning was obtained. The physical meaning had to follow behind the mathematics. The complete physical interpretation was obtained only a few years after the mathematical basis. It involved probabilities, subject to uncertainty relations.

The new theory was very satisfactory except in one respect: It was non-relativistic. It did not even fit with Einstein's special theory of relativity, so was not applicable to particles with high velocities. The difficulty was a fundamental one. An important feature of the theory was the wave equation, which had to be linear in the operator $\partial/\partial t$, and thus treated time differently from the space coordinates.

This difficulty did not bother many physicists at the time. They worked with a wave equation based on the relativistic operator

$$\frac{\partial^2}{c^2 \partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \, ,$$

the so-called Klein-Gordon equation. But with this equation the probabilities that one was led to were not positive definite, which was ridiculous. Only equations linear in $\partial/\partial t$ give probabilities that are positive definite.

But the physicists at that time were not dismayed. They just said, let us replace probability density by charge density. A charge density can be negative or positive.

I found this development quite unacceptable. It meant departing from the fundamental ideas of the non-relativistic quantum mechanics, ideas which demanded a wave equation linear in $\partial/\partial t$. It meant abandoning the whole beautiful mathematical scheme for the sake of introducing

certain physical ideas.

My dissatisfaction with the views that most physicists accepted led me to worry over this problem and eventually to find a new wave equation, satisfying both the requirements of relativity and the condition of being linear in $\partial/\partial t$. It turned out to be a satisfactory wave equation to describe an electron, with a spin 1/2.

Surprisingly, the particle with spin 1/2 appeared as simpler to treat than the particle with no spin. One would have expected that one would first have to solve the problem of the particle with no spin and then subsequently bring in the spin. But the mathematics showed otherwise. The mathematics led the way.

The appearance of this equation did not solve the general problem of making quantum mechanics relativistic. It applied only to the problem of a single electron, not several particles in interaction.

People tried to build up a general theory by putting several particles together and introducing interaction between them by inserting interaction terms into the equations. One took the interaction terms to be those suggested by classical electrodynamics, adapted to agree with special relativity.

The resulting theory was not satisfactory. One had a definite Schrodinger wave equation, but when one tried to solve it, using standard perturbation methods, one always obtained divergent integrals. It appeared that the wave equation had no solutions.

Theoretical physicists were not stopped by this situation. They proceeded to develop rules for discarding the infinities and continuing to develop their equations in terms of finite quantities. The discard could be expressed as a renormalization of fundamental physical constants. The results of the calculations were well-defined. They turned out to be in agreement with observation.

Most physicists are very satisfied with this situation. They argue that if one has rules for doing calculations and the results agree with observation, that is all that one requires.

But it is not all that one requires. One requires a single comprehensive theory applying to all physical phenomena. Not one theory for dealing with non-relativistic effects and a separate disjoint theory for dealing with certain relativistic effects.

Furthermore, the theory has to be based on sound mathematics, in which one neglects only quantities that are small. One is not allowed to neglect infinitely large quantities. The renormalization idea would be sensible only if it was applied with finite renormalization factors, not infinite ones.

For these reasons I find the present quantum electrodynamics quite unsatisfactory. One ought not to be complacent about its faults. The agreement with observation is presumably a coincidence, just like the original calculation of the hydrogen spectrum with Bohr orbits. Such coincidences are no reason for turning a blind eye to the faults of a theory.

Quantum electrodynamics is rather like the Klein-Gordon equation. It was built up from physical ideas that were not correctly incorporated into the theory and it has no sound mathematical foundation. One must seek a new relativistic quantum mechanics and one's prime concern must be to base it on sound mathematics. How can one proceed to find it?

Let us disregard the gravitational force. Consider only the other forces of physics, the electromagnetic, strong and weak nuclear forces. These seem to be the only forces of importance in atomic physics.

One is then working in the space of special relativity, Minkowski space. It is subject to a certain group, the Poincare group.

Let us accept the fundamental principle of quantum mechanics, the

principle of the superposition of states. With this principle, the states
of any quantum system provide a representation of the Poincare group. We
are led to the mathematics of representations of the Poincare group.

To study the group we may work with the infinitesimal operators.
There are ten independent ones, four translation operators $P_\mu (\mu = 0,1,2,3)$
and six rotation operators about a point $M_{\mu\nu} = -M_{\nu\mu}$. They satisfy definite
commutation relations,

$$[P_\mu, P_\nu] = 0 \quad [P_\mu, M_{\nu\rho}] = g_{\mu\nu}P_\rho - g_{\mu\rho}P_\nu$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -g_{\mu\rho}M_{\nu\sigma} + g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\nu\sigma}M_{\mu\rho}.$$

Any representation of the Poincare group provides ten operators
satisfying these commutation relations. Conversely, any set of ten opera-
tors satisfying these relations gives a representation of the Poincare
group (provided a certain global condition is also satisfied, that conti-
nual application of an infinitesimal rotation such as M_{12} so as to build
up a complete revolution gives 1 or -1). The ten operators then provide
the mathematical basis for a dynamical system in relativistic quantum
mechanics.

The whole problem of relativistic quantum mechanics, which has been
holding up the development of theoretical physics for decades, reduces to
finding suitable sets of ten operators satisfying the commutation rela-
tions.

One can look at the problem purely mathematically. Mathematicians
have worked out all the irreducible representations of the group. But the
irreducible representations just correspond to isolated particles and do
not take us very far.

For dealing with interesting physical systems involving particles
in interaction we need representations which are far from irreducible. The

problem of finding the physically important representations is the big unsolved problem.

One way of attacking the problem is to take a subgroup of the Poincaré group and first set up a representation of the subgroup in a trivial way, corresponding to some particles with no interaction between them. One then completes the group, bringing in suitable interaction terms.

This method leads to different forms of theory with different choices of the subgroup. Each subgroup corresponds to leaving invariant some geometrical object in the Minkowsky space.

One might take the subgroup which leaves invariant one instant of time, say $x_o = 0$. I call the resulting theory the <u>instant form</u>. One might take the subgroup which leaves invariant some point in the Minkowski space. I call the resulting theory the <u>point form</u>. Finally, one might take the subgroup which leaves invariant a wave front travelling with the speed of light, such as the 3-dimensional surface $x_o-x_3 = 0$. I call the resulting theory the <u>front form</u>.

The instant form is the one that is usually used by physicists. It involves setting up for P_r, M_{rs} (r,s = 1,2,3) their expressions in terms of the dynamical variables of particles without interaction terms,

$$P_r = \Sigma\ p_r, \qquad M_{rs} = \Sigma\ (x_r p_s - x_s p_r)$$

where the variables p_r, x_r refer to individual particles.

It remains to determine P_o and M_{ro}. We put

$$P_o = \Sigma\ p_o + U \qquad M_{ro} = \Sigma x_r p_o + V_r$$

where U and V_r are the interaction terms, involving the dynamical variables of all the particles. The problem is to find the four quantities U, V_r. They must be chosen to make the P_μ, $M_{\mu\nu}$ satisfy the required commutation relations.

Some of the conditions on these four quantities are linear. They are easily satisfied. They will just say that U is a three-dimensional scalar and V_r is a three-dimensional vector. The other conditions are quadratic. They are the difficult ones to satisfy and provide the real problem.

People have been making guesses at U and V_r from physical intuition. It is such guesses that lead to quantum electrodynamics. But the mathematics is not sound and the commutation relations are not really satisfied.

If we work with the point form we get essentially the same situation. There are again four unknown quantities to express the interaction. Again they have to satisfy some linear conditions, which are easy, and some quadratic ones, which are difficult.

The situation is different with the front form. Here the operators of the subgroup which leave the surface $x_o - x_3 = 0$ invariant are

$$P_1, P_2, P_0 - P_3, M_{12}, M_{03}, M_{01} - M_{31}, M_{02} - M_{32}.$$

There are seven of them, instead of six, as we had with the instant and point forms.

The only remaining operators of the Poincare group are now

$$P_0 + P_3, \qquad M_{01} + M_{31}, \qquad M_{02} + M_{32}.$$

There are three of them, and they will contain three unknown interaction terms, instead of the four that we had with the instant and point forms.

Thus mathematically the problem is substantially simpler when we work with the front form.

Working with a front is a process that is unfamiliar to physicists. But still I feel that the mathematical simplification that it introduces is all-important. I consider the method to be promising and have recently been making an extensive study of it. It offers new opportunities, while the familiar instant form seems to be played out.

The "Past" and the
"Delayed-Choice" Double-Slit Experiment

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Partway down the optic axis of the traditional double-slit experiment stands the central element, the doubly-slit screen. Can one choose whether the photon (or electron) *shall have* come through both of the slits, or only one of them, after it has *already* transversed this screen? That is the new question raised and analyzed here.

Known since the days of Young is the possibility to use the receptor at the end of the apparatus to record well defined interference fringes. How can they be formed unless the electromagnetic energy has come through both slits? In later times Einstein noted that in principle one can determine the lateral kick given to the receptor by each arriving quantum. How can this kick be understood unless the energy came through only a single slit?

Einstein's further reasoning as reported by Bohr (1) is familiar. Record both the kicks and the fringes. Conclude from the kicks that each quantum of energy comes through a single slit alone; from the fringes, that it nevertheless also comes through both slits. But this conclusion is self-contradictory. Therefore quantum theory destroys itself by internal inconsistency.

Bohr's reply (1) has become by now a central lesson of quantum physics. One can record the fringes or the kicks but not both. The arrangement for the recording of the one automatically rules out the recording of the other. The quantum has momentum p, de Broglie wave length $\lambda = h/p$, and reduced wave length $\lambdabar = \hbar/p$. To record for it well defined interference fringes one must fix the location of the

receptor within a latitude

$$\Delta y < (fringe\ spacing)/2\pi = (L/2S)\lambdabar. \qquad (1)$$

To tell from which slit the quantum of energy arrives one must register the transverse kick it gives to the receptor within a latitude small enough to distinguish clearly between a momentum $p = \hbar/\lambdabar$ coming from below, at the inclination S/L, and a momentum coming from above at a like inclination; thus,

$$\Delta p_y < (S/L)(\hbar/\lambdabar). \qquad (2)$$

However, for the receptor simultaneously to serve both functions would be incompatible with what the principle of indeterminancy has to say about receptor dynamics in the Y-direction,

$$\Delta y \Delta p_y > \hbar/2. \qquad (3)$$

Not being able to observe simultaneously the two complementary features of the radiation, it is natural to focus on the one and forego examination of the other. Either one will insert the pin through the hole shown in Fig. 1. It will couple the receptor to the rest of the device. It will give the receptor a well defined location. Then one will be able to check on the predicted pattern of inter-ference fringes. Or one will remove the pin. Then one can measure the through-the-slot component of momentum of the receptor before and after the impact of the quantum. Then one will say that one knows through which slit the energy came.

Pin in or pin out: when may the choice be made? Must it be made before the quantum of energy passes through the doubly slit screen? Or may it be made after? That is the central question in this paper as that question first seems to impose itself. However, a closer look shows that the measurement of transverse momentum kick, in principle conceivable, is practically almost out of the question. Therefore it is appropriate to alter the idealized experi-ment before taking up the question of "before" *versus* "after." What is the difficulty and what is the change?

PORTAL   LENS AND SCREEN   R E C E P T O R
                                         PLATE AND   PHOTO-
                                         PLATE HOLDER   DE-
                                                      TECTORS

BOHR-EINSTEIN
DIALOG

PLATE
CAN SLIDE
IN SLOT FOR
P_y MEASURE-
MENT WHEN   SILVER BRO-
PIN IS RE-   MIDE GRAINS   PLATE
MOVED   RECORD INTER-   CAN
                   FERENCE   HINGE,
                   FRINGES WHEN   EXPOSE
                   PLATE IS   PHOTO-
                   FIXED IN   DETEC-
                   SLOT   TORS
                   BY
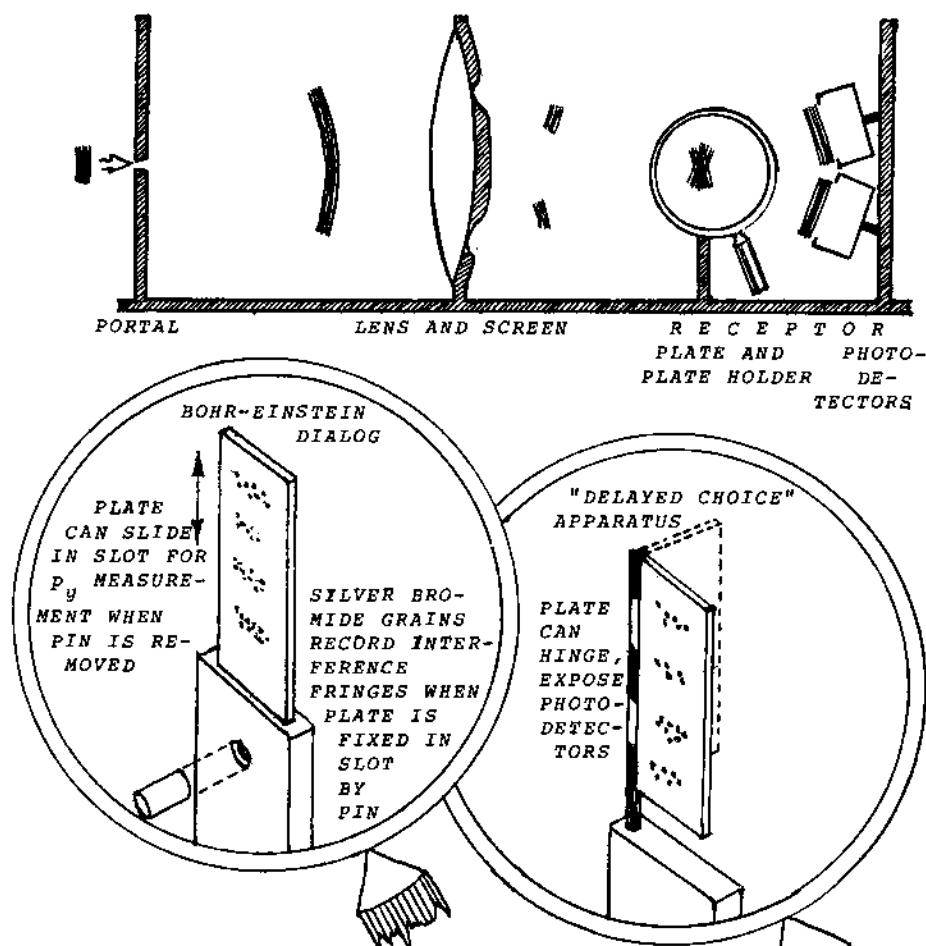   PIN

"DELAYED CHOICE"
APPARATUS

Fig. 1. Top: Idealized double-slit experi-
ment. Distance of each slit from optic axis, S; from
photographic plate, L. For simplicity, details of the plate
and plate holder are omitted from the circle encompassed by
the magnifying glass and are presented below, magnified and
in perspective. Lower left: The version of the Bohr-Einstein
dialog. The plate catches every photon. It registers pre-
cisely the y-coordinate of impact or the y-component of im-
pulse delivered--but does not and cannot do both. Omit the
photodetectors. Lower right: The present "delayed choice"
version. Include the photodetectors. One or other of them
is sure to catch the quantum of energy when the plate is
swung aside. Whether to expose the plate or expose the photo-
detectors, whether thus to infer that a single quantum of
energy _shall_ _have_ gone through both slits in the screen or
only one, is subject to the free choice of the observer after
the energy has _already_ traversed the screen.

An irreversible act of amplification (2, 3); a phenomenon brought to a close by observation (4); "indelibility" (5): all these features a photographic plate has; and all these features the receptor of Fig. 1--if   it is to record fringes--must have. A receptor acceptable in this sense that is electronic and heavier than a photographic plate is easy to conceive, but not one much lighter. Moreover, the lightest plate is heavy, as seen by a look at the following factors and their product:

Minimum number of fringes with which one would be satisfied,	~ 5
Minimum number of y-values or "domains" required adequately to spell out one fringe,	$\sim 2\pi$
Minimum cross sectional area required to intercept photons of wave length λ incident on one domain,	$\sim \lambda^2$
Minimum thickness required for absorption and irreversible amplification in silver halide grain,	$\sim 10\lambda$
Product of factors so far, evaluated for $\lambda = 5000 A =$ 5×10^{-5}cm,	$\sim 4 \times 10^{-11}$cm^3
Minimum density,	~ 1g/cm^3
Minimum mass,	$\sim 4 \times 10^{-11}$g
Uncertainty, $(MkT)^{1/2}$, in momentum of so great a mass by reason of thermal agitation at $T = 300$deg$K$,	$\sim 10^{-12}$g cm/s
Lateral momentum of incident photon, say $10^{-2}\hbar\omega/c$ or less,	$\sim 10^{-24}$g cm/s
Factor by which this lateral momentum fails of being measurable,	$\sim 10^{12}$

The origin of the difficulty is not far to seek. The measuring equipment--the photographic plate--has itself been made the subject of measurement. One might as well put a Geiger counter or Schrödinger's cat or Wigner's friend into a box and try to measure the lateral momentum communicated to that enormity! An irreversible act of amplification, yes; but make it part of the system under study? No. Principle does not call for it and practicality forbids it.

The difficulty is overcome by a simple change:

(1) Give up measuring the y-component of the momentum of the photographic plate.

(2) Hold its y-coordinate fixed.

(3) By means of a hinge parallel to the y-axis arrange that this high narrow plate can be swung out of the way of the incident light--at the last minute option of the observer-- quicker than the flight of light from screen to plate. (Switch from "operative" to "open" position.)

(4) Sufficiently far beyond the region of the plate, the beams from upper and lower slits cease to overlap and become well separated. There place photodetectors. Let each have an opening such that it records with essentially 100 percent probability a quantum of energy arriving in its own beam, and with essentially zero probability a quantum arriving in the other beam.

Now the choice is clear; and the objective, too. We today cannot argue, and Einstein in his later years would not even have wanted to argue, his erstwhile case of logical inconsistency against quantum theory: the photon goes through both slits, as evidenced in interference fringes, and yet simultaneously through only one, as evidenced in lateral momentum kick. Choose we know we must between the two complementary features open to study; and choose we do, by putting the plate athwart the light or turning it out of the line of fire. In the one case the quantum will trans- form a grain of silver bromide and contribute to the record of a two-slit interference fringe. In the other case one of the two counters will go off and signal in which beam-- and therefore from which slit--the photon has arrived.

In our arrangement the photographic plate registers
only the point of impact of a photon. In the earlier
idealized experiment it could additionally (Einstein) or
alternatively (Bohr) record the transverse momentum delivered
by the impact. We have assigned the two distinct kinds of
measurement to two distinct kinds of register. We have de-
moted the plate from a privileged status. That demotion
is irrelevant to any question now at issue. Equally irrel-
evant is the different distance--and time of flight--from
entry portal to plate, or photodetector, according as the
one or other register is exposed. But the essential new
point is the timing of the *choice*--between observing a two-
slit effect and a one-slit one--until after the single
quantum of energy in question has *already* passed through the
screen.

Let the reasoning be passed in review that leads to
this at first sight strange inversion of the normal order
of time. Then let the general lesson of this apparent
time inversion be drawn: "No phenomenon is a phenomenon
until it is an observed phenomenon." In other words, it is
not a paradox that we choose what *shall* have happened after
"it has *already* happened." It has not really happened, it
is not a phenomenon, until it is an observed phenomenon.

ANALYSIS OF THE "DELAYED CHOICE" EXPERIMENT

Whatever we now do to spell out the otherwise idealized
experiment, we will leave idealized its most unusual features:
the "swinging door photographic plate." That term includes
the arrangement, whatever it may be,

(1) for a last minute choice, to swing the plate
aside or leave it athwart the beam, after the arriving energy
has already traversed the doubly slit screen, and

(2) for completion of that movement before the energy
arrives at the plate. In practice it will be more reasonable
to swing the beam than swing the plate. Fix the plate.
Halfway from screen to it, position a Kerr cell. Apply to
it a positive or a negative voltage (6) according as one
wishes to record fringes on the plate, or register "which
beam" on a counter. Or, still better, Manfred Fink suggests,

replace the experiment with the photon by an experiment with
an electron. Then the last minute deflection of the elec-
tron beam can be accomplished by a localized magnetic field
centered between screen and plate. One or another of these
arrangements to swing the beam will be understood hereafter
to apply in practice when in principle we speak of swinging
the plate.

Now we turn to other requirements and presuppositions
of the idealized experiment. First among them is the "window
in time" that admits the incoming radiation to the entry
portal. Well known windows today tailor the pulse of
radiation that implode a pellet of D-T (7). The window in
the present device we take to be "Gaussian in time." The
incident amplitude is given by

$$E(x,t) = Re \ const \ exp \ [ik_o(x-t) - (x-t)^2/2\tau^2] \qquad (4)$$

Here the origin in x and origin in time are so chosen as to
bring the center of the wave train to the doubly slit screen,
$x = 0$, at $t = 0$; and the symbol Re stands for "real part of."
In this expression and hereafter three simplifications are
made. First, time is expressed in cm of "light travel time;"
that is, in the same geometrical units that are employed for
distance. Second, all reference is eliminated to polarization.
Only that scalar function is considered which multiplies the
familiar polarization factor in a full-dress treatment of a
propagating electromagnetic wave. This simplification will
be legitimate provided that we reduce sufficiently the
angles of diffraction of the beams, and the angle of their
crossing, and the consequent departures from the forward
direction.

The third simplification requires more discussion.
One part of the factor that multiplies the scalar amplitude
(4) and subsequent expressions for diffracted waves is a
quantum-field-theory creation operator, a^+. It will be
understood so unmistakably to be a part of all such ex-
pressions that it will never even be written down. The
rationale for this omission was given originally by Heisenberg
and Pauli (8) and was subsequently spelled out by Breit (9).
In the present context that rationale goes as follows: The

wave in question is to be Fourier-analyzed into elementary
waves. Each elementary wave is to bear its own creation
operator. The probability for a particular silver halide
grain to be transformed, or for a particular point in the
photodetector to initiate an electron cascade, is evaluated
by the following sequence of mathematical operations. Take
the square of the amplitude of the elementary wave at the
point in question. Multiply it by the number operator
$N = a^+a$ for the elementary wave in question. Sum over all
elementary waves to get the desired probability for bringing
about a local excitation. But this calculation of the
probability is closely equivalent in its result to a cal-
culation made along semiclassical lines, in which no mention
is made of creation and annihilation operators: Take the
electric field as calculated classically as a function of
position and time. Square. Multiply by $2c/8\pi$ to get energy
flow. Integrate over time to get energy per unit area.
Divide by the average energy per photon, $\hbar c k_o$. The quotient
is very nearly the probability that a photon will strike a
unit once normal to the direction of flow. Close equiva-
lence can be converted to exact equivalence by making the
calculation more elaborate. For this purpose the electric
field $E(t)$ at the place in question should be Fourier
analyzed before squaring; and the squared Fourier amplitude,
$|E(k)|^2$, should be divided by the quantum energy, $\hbar c k$,
appropriate to the frequency in question before integrating--
this time over frequency, rather than time--to determine the
expectation value of the number of photons that arrive.
However, the length τ of the wave train is envisaged here to
be long in comparison with one vibration period of the
radiation; or more precisely, we take the dimensionless
quantity $k_o\tau$ to be large compared to unity:

$$N \equiv k_o\tau \gg 1. \tag{5}$$

Therefore the spread of frequencies, $\Delta k \sim 1/\tau$, in the wave
packet is small compared to the frequency k_o itself. For
this reason we dispense with the complication of Fourier
analysis, to make manageable the mathematics of which would
force us to a more sophisticated profile for the wave packet.

In summary, our simplification gives for the probability per unit area for impact of a photon the expression

$$w = (1/4\pi\hbar k_o) \int E^2 (position, t) dt. \qquad (6)$$

More than any other design feature, the delayed-choice double-slit experiment calls for beams directed from the two slits towards the plane of interference and--when the photographic plate normally located there is swung aside-- continuing on to make clean entry into well separated photodetectors. It is appropriate to begin with the optics of a single beam. For simplicity let it be conceived to come from a slit centered on the optic axis, $y = 0$, and let it run in the x-direction. Later on the slit location can be moved and the beam direction changed to describe the beams actually desired in the double-slit experiment.

It is easy to complicate the double-slit interference pattern. For that purpose it is enough to have a complicated single-slit diffraction pattern, and let the waves from two such slits interfere. In contrast, we want fringes, when fringes are observed, to originate in, and serve as evidence for, interference. Therefore we rule out any single-slit design that gives a diffraction pattern with a zero in it. We propose to meet this requirement by operating with a Gaussian diffraction pattern, obtained from a slit with a Gaussian distribution of transmissivity,

$$(transmission) = exp(-y^2/2a^2). \qquad (7)$$

Lenses have been used in recent times with a distribution of transmissivity designed to eliminate any zero in the diffrac- tion pattern (10), and it takes no feat of the imagination to think of applying the same technique of "apodisation" to one slit, and by extension to both slits.

The pattern of diffraction from the "Gaussian slit" defined by (7) we evaluate by decomposing the incoming wave train (4) into monochromatic constituents of the form $exp\ ik(x-t)$ with a Fourier amplitude

$$[\tau/(2\pi)^{\frac{1}{2}}] exp[-\tau^2 (k-k_o)^2/2] dk, \qquad (8)$$

determining in the familiar Fresnel approximation the diffraction pattern produced by each, and recomposing these separate contributions. From the portion $d\bar{y}$ of the wave front of a monochromatic wave of wave number k and unit amplitude at location $x = 0$ the Fresnel prescription gives us at any distance $x \gg \lambdabar = k^{-1}$ and at any elevation y above the optic axis the contribution

$$(2\pi i \lambdabar r)^{-\frac{1}{2}} e^{ikr} d\bar{y};\qquad\qquad(9)$$

or, at the level of approximation at which we work,

$$(2\pi i \lambdabar x)^{-\frac{1}{2}} exp[ikx + ik(y-\bar{y})^2/2x]d\bar{y}.\qquad(10)$$

For a detailed analysis we cannot overlook the presence of a third dimension, z, perpendicular to the (x,y)-plane of Fig. 1. We tailor the transmissivity of each slit in its dependence upon z according to a Guassian law, $exp(-z^2/2b^2)$, in order to avoid any zeros in the diffraction pattern. However, the Gaussian widths, b and a, in the z- and y- directions are very different in order of magnitude ($b \gg a$). Thus the angle of diffraction in the z-direction, $\sim \lambdabar/b$, may be considered to be negligible in comparison to the angle of diffraction in the y-direction $\sim \lambdabar/a$.

More important than any diffraction-induced spreading of the beam in the z-direction is the lens-induced convergence of the beam in that direction. This effect allows one to use a truly narrow protographic plate, as indicated schematically in the inset diagram at the lower right in Fig. 1. However, it is appropriate not to make the plate location, $x = L$, identical with the focal point, $x = F$, of the lens. The reason is simple. The convergence in the z-direction, coming to a head in the immediate neighborhood of the focus, produces there a sudden change in phase of $\pi/2$, analogous to the well known change in phase of π that occurs at a "two-way" focus (11, 12, 13). To operate too close to this point is to run the risk that some parts of the photographic plate will cross the place of change of phase, giving rise to anomalies in the interference pattern. Granted the elementary precaution needed to insure against

this effect, $|L-F|>>\lambda F^2/b^2$, we neither want to give nor will give any more attention to the development of the beam in the z-direction. We shall idealize it as if it developed entirely in the (x,y)-plane.

The diffraction-induced spreading of the beam in the (x, y)-plane we compensate insofar as feasible with a lens-induced convergence. That means that the wave can be regarded as starting at the plane of the slit, $x = 0$, with not only its amplitude dependent upon y, as described by the Gaussian of (7), but also its phase dependent upon y, as appropriate for a wave that would come to a focus at $x = F$; thus,

$$(amplitude) = exp[-(ik/2F)y^2-(1/2a^2)y^2].\qquad(11)$$

Composing this elementary amplitude with the Fresnel propagator (10) and the Fourier coefficient (8) we have for the beam amplitude at the point (x,y) at the time t the expression

$$(beam\ amplitude) = (\tau/2\pi)\iint(k/ix)^{\frac{1}{2}}exp\{-ikt-(\tau^2/2)(k-k_o)^2$$

$$+ikx+(ik/2x)(y-\bar{y})^2-(ik/2F)\bar{y}^2$$

$$-(1/2a^2)\bar{y}^2\}d\bar{y}dk.\qquad(12)$$

In the exponent, in the coefficients of $(y-\bar{y})^2$ and $\bar{y}^2$, and also in the multiplying coefficient before the exponential, we replace k by $k_o = \lambda^{-1}$ to make the integral manageable. The resultant error is readily estimated. In the case of interest two conditions will be fulfilled:

(1)   The focal length will be of the order of the distance x to the plate.

(2)   The contributions to the beam width from diffraction, $x\lambda/a$, and from the original (focusing neglected) width of the beam, a, will be comparable; that is, we will operate at a distance x of the order of a^2/λ. Under these conditions we find that the error in the exponent is of the order of $\lambda/\tau \sim 1/N$. This correction can be neglected when the number of waves in the incident wave packet is sufficiently great. In this approximation the integration (12) gives the result,

$$(beam\ amplitude)\ =\ Re\{1-x/F+i\lambda x/e^2\}^{-\frac{1}{2}}exp\{ik_o(x-t)-(x-t)^2/2\tau^2$$

$$-\frac{(y^2/2)}{a^2(1-x/F)^2+(x\lambda/a)^2}$$

$$+\ (iky^2/2)\frac{x(F\lambda/a^2)^2-(F-x)}{(F-x)^2+(xF\lambda/a^2)^2}\}\ .\qquad(13)$$

Every term in (13) has its simple interpretation. The first two terms in the exponent come over unaltered from the phase and the profile of the primary radiation. The next term in the exponent shows that the parameter describing the width of the beam in the y-direction is the square root of the sum of the squares of two separate width parameters. Of these one describes a diffraction-induced spreading of the beam by the amount $(distance)\cdot(angle) = x\lambda/a$. The other describes a focus-induced convergence of the original beam width a to a figure $(1-x/F)a$. Both effects also show up in the multiplicative factor before the exponential. It describes the change of amplitude caused by the focussing and diffraction of the beam. In agreement with these considerations, the square of its absolute value, the intensity, is correctly given by dividing the original width parameter of the beam, a, at $x = 0$ by the value of the width parameter at $x = x$.

The phase of the amplitude factor evidently changes by $\pi/2$, as anticipated, when x passes through $x = F$. However, the distance over which this change of phase comes about is not of the order of the reduced wave length, λ, itself, as one might at first have expected. It has the much larger value $(F/a)^2\lambda$. What this distance means is best seen by shifting attention from the slit, $x = 0$, to the place, $x = x_{min}$, where the beam has its minimum width:

$$(width)^2\ =\ a^2(1-x/F)^2+(x\lambda/a)^2,$$

$$(F^2/2)d(width)^2/dx\ =\ -a^2(F-x)+x(F\lambda/a)^2,$$

$$(width)^2_{min}\ =\ F^2\lambda^2/[a^2+(F\lambda/a)^2],$$

at

$$x_{min}\ =\ a^2F/[a^2+(F\lambda/a)^2].\qquad(14)$$

Viewed from x_{min} as origin, the beam, tracked either forward
or backward, is propagating and diffracting as if it were a
nearly plane wave emerging from an aperture of opening
$(width)_{min}$. Therefore at a distance d from this virtual
aperture there are two contributions to the width. One is
this aperture opening itself. The other is diffraction,
with a diffraction angle $\sim(reduced\ wave\ length)/(aperture) =$
$\lambdabar/(width)_{min}$. Thus the beam width is given by the formula

$$(width)^2 = (width)^2_{min}+d^2\lambdabar^2/(width)^2_{min}. \tag{15}$$

As a check, we can verify that this widening is just enough
at the distance $d = x_{min}$ to send the beam back through the
original slit $(width = a)$.

The transition from the "inner region" of propagation
of the beam as a nearly plane wave of nearly constant width
to the "outer region" of propagation as a cylindrically
converging or diverging wave occurs at that distance from
the point of minimum width where the two contributions of
(15) to the width are of the same order of magnitude,

$$d = \frac{(F/a)^2\lambdabar}{1+(F\lambdabar/a^2)^2}. \tag{16}$$

The "correction factor" in the denominator of (16) is of
the order of unity under conditions where the diffraction-
induced broadening of the beam does not substantially exceed
the slit width, a, itself. Then in the present Huygens
approximation, the calculated length of the inner region of
propagation is of the same order as the calculated length,
$\sim(F/a)^2\lambdabar$, of the region in which the phase of the wave rises
by the amount $\pi/2$. It will be convenient for the subsequent
consideration of the idealized experiment to regard the
photographic plate as located in this inner region,
$d<<(F/a)^2\lambdabar$, and yet staying away from the focus in the
z-direction by the already specified margin, $d>>(F/b)^2\lambdabar$.

These two foci, for the y-convergence and the z-con-
vergence, are not the same. This difference comes about,
not because of any imperfection in the lens, but because the
y-diffraction caused by the narrow slit of width a fights
against the normal geometrical-optics convergence. Thus the

beam reaches its minimum extension in the y-direction, not
at the distance F from the lens, but at the shorter distance
x_{min} of (14). Adopting for the slit width here and here-
after a figure

$$a = (F \hbar)^{\frac{1}{2}}, \tag{17}$$

we have

$$x_{min} = F/2. \tag{18}$$

The final term in the exponent of (13) tells us that the
wave crests and troughs in the beam at the distance x in
their progression towards the right are warped into circular
arcs centered on a point, $x = \bar{x}+D$, located to the distance
D to the right of $\bar{x}$, where

$$\frac{1}{D} = \frac{(F-\bar{x})-\bar{x}(F\hbar/a^2)^2}{(F-\bar{x})^2+\bar{x}^2(F\hbar/a^2)^2} \tag{19}$$

In the limit of geometrical optics, where diffraction is
negligible ($\hbar \to 0$), (19) gives $D = (F-\bar{x})$, so in this limit
the point of convergence is located at $x = \bar{x}+D = F$, as
expected. However, to operate as we propose, where the
wave fronts are plane, that is to say, centered at infinity,
we look for the $\bar{x}$ value that annuls the numerator of (19),
thus

$$\bar{x} = F/[1+(F\hbar/a^2)^2]. \tag{20}$$

It is not surprising that this point is identical with the
point, $x = x_{min}$, where the wave front achieves its minimum
width. This is one more instance of a general theorem [see
for example reference (14) for a survey and citations of the
literature] relating width of the wave with radius of
curvature,

$$d(width)/(width) = -(dx)/D, \tag{21}$$

a theorem that one verifies explicitly by comparing (21) with
(14).

There is no better epitome of these considerations on
beam width, beam phase, diffraction, and curvature of the

wave front than the factor

$$[(1-x/F)a+ix(\bar{\lambda}/a)]^{-\frac{1}{2}} \tag{22}$$

which (along with the constant $a^{\frac{1}{2}}$) multiplies the
exponential in expression (13) for the beam amplitude.
Out of (22) are to be read out immediately the $\pi/2$ change
of phase going through the region of convergence and the two
separate contributions to the square of the beam width; and
from this beam width the radius of convergence or divergence
of the waves follows directly out of (21).

   In summary, we specify the reduced wave length
$\bar{\lambda} = \lambda/2\pi = 1/k_O$ of the incident wave train and the parameter
$\tau = N\bar{\lambda}$ that measures the length of the incident wave train.
We specify the focal length F of the lens. We fix the
slit width a to be $(F\bar{\lambda})^{\frac{1}{2}}$. We locate the photographic plate
at the point $x = F/2$ where the beam has reached its minimum
width

$$w = a/2^{\frac{1}{2}} = (F\bar{\lambda}/2)^{\frac{1}{2}} \tag{23}$$

and where the wave fronts are planar. Well past this
region of convergence the wave spreads out with an angular
width parameter

$$\theta_{diffraction} = \bar{\lambda}/w = 2^{\frac{1}{2}}\bar{\lambda}/a = (2\bar{\lambda}/F)^{\frac{1}{2}}. \tag{24}$$

   We now replace the one slit beaming energy along the
optic axis by two slits, one sending a wave from an elevation
$s*$ above the optic axis on a downward slant, the other
beaming a wave on an upward slant from a point $s*$ units of
distance below the optic axis. Were each wave converging
as predicted by geometrical optics, the two would come to
a common focus at $x = F$. In actuality each comes to its
narrowest width and to planarity at $x = F/2$. There the
center of the one wave lies above the center of the other
wave by half the original $2s*$ separation. We want to make
the wave centers coincide. Only so will we secure optimum
interference between the two waves. For this purpose we
saw through the lens on the plane $y = s*/2$, also on the
plane $y = -s*/2$. We throw away the intervening slab of

glass. We glue back together the upper and lower portion
of the lens. Now the slit centers have the elevations
$\pm S \equiv S*/2$ with respect to the optic axis, and optimum inter-
ference is achieved.

The difference in angle of fire of the two beams,

$$\theta_{fire} = \binom{difference}{in\ elevation} \Big/ \binom{distance\ to}{overlap\ of\ wave\ centers}$$

$$= (2S)/(F/2) = 4S/F,\tag{25}$$

is not all pure gain in cleanly separating the beams as
they pass the plane $x = F/2$ of interference (and potential
location of the photographic plate) on their way to the two
photodetectors. Only when θ_{fire} exceeds the diffraction
angle θ_{diff} of (24) by a very substantial factor, say 20-
fold, will we get from the strong Gaussian fall off of the
diffraction pattern the assurance we want against a contri-
bution from either beam to a count in the wrong photo-
detector. With

$$\theta_{fire} = 20\ \theta_{diffraction}\tag{26}$$

we have for the required off-axis slit distance the value

$$S = (F/4)\theta_{fire} = 5\theta_{diffraction}F = 5(2\lambdabar F)^{\frac{1}{2}} = 5 \cdot 2^{\frac{1}{2}}a.\tag{27}$$

In other words, a suitable separation between the centers of
the two slits is $10 \cdot 2^{\frac{1}{2}}a$ or about 14 times the Gaussian
width parameter of either slit individually.

Separation of the beams well past $x = F/2$ being thus
assured, we turn back to the details of the interference
pattern at $x = F/2$ itself. Were one of the beams directed
along the optic axis, it would admit in this region of its
development the simple mathematical expression

$$(electric\ field) = E_o \cos k_o(x-t)\ exp[-(y^2/2w^2)-(x-t)^2/2\tau^2],\tag{28}$$

with $w^2 = a^2/2 = F\lambdabar/2$. We get this result by substituting
into our original formula (13) the expressions $x = (F/2) + \bar{x}$
and $t = (F/2) + \bar{t}$. These substitutions put the origin of

the coordinate x at the plane of interference and the origin
of the time variable $\bar{t}$ at the instant when the peak of the
wave train arrives at this plane. For simplicity we drop
the bars over the new coordinates in (28) and hereafter. We
have only to rotate the x and y axes by the small angle
$\phi = S/(F/2)$ or the small angle $-\phi$ to obtain the expression
for the one interfering wave or the other. We add the two
expressions and evaluate the sum at the plane of the
photographic plate, $x = 0$. In this way we find

$$\begin{pmatrix} electric\ field \\ at\ x\ =\ 0 \end{pmatrix} = E_o \cos k_o(t+\phi y) exp[-y^2/2w^2-(t+\phi y)^2/2\tau^2]$$

$$+(corresponding\ term\ with\ \phi\rightarrow-\phi). \qquad (29)$$

We square this sum and integrate over time as in Eq. (6) to
arrive at a measure of the probability that a photon will
strike a unit area at elevation y. Dropping the factor E_o
of primary field strength and associated normalization
factors, and dropping terms of the type $exp(-k_o^2\tau^2) = exp(-N^2)$
in comparison with unity because we take $N\gg1$, and concerning
ourselves solely with relative probabilities, we have

$$\begin{pmatrix} relative\ probability \\ for\ photon\ to\ hit \\ at\ the\ elevation\ y \end{pmatrix} = e^{-y^2/w^2}[1+1+2\cos(2k_o\phi y)e^{-\phi^2 y^2/\tau^2}]. (30)$$

Here the first "1" can be considered to measure the con-
tribution of the lower slit; the second "1," the upper slit;
and the final term, positive or negative, the effect of
interference between the waves from the two slits. This
interference is governed by the retardation, $2\phi y$, of the one
beam relative to the other at the elevation y, one beam
being tilted up at the angle $\phi = S/(distance\ from\ slit) =
2S/F$, the other being tilted down at the same angle. The
same tilt puts the peak of the one wave train behind the peak
of the other wave train by the same amount. When that
retardation is comparable to the length parameter of the wave
train, τ, the interference drops off or practically disappears,
as evidenced by the final exponential in (30). The first
exponential in (30) affects all terms alike, and is a
consequence of the wave's having the natural width, w.

Now measure elevation above the optic axis in units of the parameter a of slit width, which we have taken to have the value $(\lambda F)^{\frac{1}{2}}$; thus,

$$y = aY = (\lambda F)^{\frac{1}{2}}Y. \tag{31}$$

Also insert for all other parameters the values that we have assigned them for convenience and for illustration:

$$w^2 = \alpha^2 2 = \lambda F/2,$$

$$\phi = 2S/F = 5 \cdot 2^{\frac{1}{2}}(\lambda/F)^{\frac{1}{2}},$$

$$\tau = N\lambda. \tag{32}$$

Then the expression for the relative probability for a photon to strike at the elevation y is

$$(relative\ probability) = e^{-2Y^2}[1+1+2\cos(10\sqrt{2}Y)e^{-50Y^2/N^2}]. \tag{33}$$

When the wave train is indefinitely long $(N\to\infty)$, expression (33) (divided by 4) becomes $exp(-2Y^2)\cos^2 5\sqrt{2}Y$. It is natural to write $n\pi$ for the argument of the cosine. Where the fringes would peak in the absence of modulation, $n = 0$, $n = \pm 1$, $n = \pm 2$, there the modulation makes the relative intensities 1, 0.674, 0.206. Thus there is a well developed interference pattern. Moreover, one has only to increase the separation $2S$ of the two slits or the relative inclination 2ϕ of the two plane wave fronts to have more fringes and a still more fully developed interference pattern showing up within the effective width of the beam. This modification, far from causing difficulty, only helps the photodetectors to discriminate between the two beams when the photographic plate is swung aside.

Having thus dealt with all other considerations, we conclude that the length of the wave train, $\tau = N\lambda$, is the central factor in deciding whether one can have good interference fringes and still have "delayed choice." When N is finite, the intensity no longer goes to zero at fringe minima; the fringes are more or less "washed out." Returning to expression (30) for the distribution of intensity in the interference pattern, we can use the words "purity of the

interference pattern" and "background" to speak of the
quantities

$$(\text{"local purity of pattern"}) \equiv e^{-\phi^2 y^2/\tau^2}, \tag{34}$$

$$(\text{"local background"}) \equiv 1-e^{-\phi^2 y^2/\tau^2}. \tag{35}$$

A fuller analysis would go into the information carried by
the interference pattern in the technical sense in which in-
formation is defined (15, 16). However, it is enough for the
present purpose to multiply (34) and (35) by the overall
modulation factor e^{-y^2/w^2} and integrate each over y and
compare the thus weighted integral of the "local pattern
purity" (34) with the sum of the two integrals to arrive at
one useful index of overall pattern purity,

$$\left(\begin{array}{c}\text{"index of overall}\\ \text{pattern purity"}\end{array}\right) \equiv \frac{[\text{weighted integral of (34)}]}{[\text{weighted integral of (34) plus (35)}]}$$

$$= \frac{1}{(1+w^2\phi^2/\tau^2)^{\frac{1}{2}}}, \tag{36}$$

an expression which reduces in the illustrative example to

$$\left(\begin{array}{c}\text{"index of overall}\\ \text{pattern purity"}\end{array}\right) = \frac{1}{(1+25/N^2)^{\frac{1}{2}}} \tag{37}$$

Then and only then can one speak with assurance of both slits
being involved in what happens to a quantum of energy when
this index is close to unity.

When the incident wave train is very long ($N\to\infty$), Eq.
(37) gives a purity index of unity. In other words, fringe
intensity approaches zero at the canonical locations;
background in this sense is negligible; and any limitation
on the number of fringes that can be seen arises solely
from the finite width parameter of the beam. Thus with
$N=\infty$ we found five fringes ($n=0,\pm1,\pm2$) with intensities
within an order of magnitude of the intensity of the central
fringe. Those five fringes still stand out with nearly
that much absence of background so long as the parameter
N for the length of the incident wave train is of the order
of several times $N_{crit}=5$ or greater, we conclude from (37);

but when the length parameter is little by little decreased
below N_{crit}=5, first the outer two of these five fringes
are washed out and then the next two. How this effect
comes about is seen from an examination of the two wave
trains depicted in Fig. 1 where they cross in the zone of
interference. We conclude that there is a natural limit
of the order τ_{crit}=$N_{crit}\lambda$=5λ below which it is not appro-
priate to reduce the length of the wave train. Only by
arranging for the incident wave train to be that long or
longer do we obtain well defined fringes. Only so will the
interference pattern convey the message, "the energy came
through both slits."

The requirement

$$\tau = N\lambda > w\phi = N_{crit}\lambda \qquad (N_{crit}=5 \text{ in the example}) \qquad (38)$$

makes no difficulty for the new feature of the "delayed
choice" in the double slit experiment. The distance $F/2$
from the doubly slit screen to the photographic plate,
like λ, is a primary parameter in the design of the idealized
apparatus. It can be made as big as one pleases. For
example, it can well be 10cm or $(2\times10^5)\cdot(5\times10^{-5}\text{cm})$ or
$1.256\times10^6\lambda>>>>N\lambda$. Thus the wave train travels many times
its own length on the way from the doubly slit screen to
the plate. In conclusion, there is ample time, after the
energy of the single quantum has already passed through the
screen, to decide electronically, by a random-number
generator or by a plan of choice programmed in advance in
any way one pleases, what kind of indelible evidence shall
be produced: "which-slit" evidence, or "double-slit"
evidence.

Light travels from screen to plate on a null cone.
Therefore we have good reason to ask, does the passage
through the screen really take place in the past light
cone of the "decision," the swinging of the photographic
plate into, or out of, the zone of interference? Unless
the answer is clearly "Yes," we cannot speak of an "apparent
inversion of the normal order of time" in the delayed-choice
double-slit experiment.

As one way to put the "traversal of the screen" clearly in the past light cone of "the decision" it is enough to switch from a light beam to an electron beam travelling at a speed, v, equal for example to $c/2$. However, it avoids the introduction of the new parameter v into all the foregoing analysis to stay with light, and make a simple alteration, not shown, in the arrangement of Fig. 1. Let a mirror be introduced half way from the doubly slit screen to the zone of interference. It folds the path of travel of the light almost back on itself. It puts the zone of interference almost on top of the screen. It makes the separation in space between the event of passage of the screen and the event of arrival in the zone of interference small compared to the separation in time of those two events. In this way the decision about "what shall have happened" in the passage of the screen is placed quite clearly in the future light cone of that passage itself.

"DELAYED CHOICE" AS AN ADDITIVE OPTION IN OTHER IDEALIZED EXPERIMENTS

To be forced to choose between complementary modes of observation is familiar, but it is unfamiliar to make this choice after the relevant interaction has already come to an end. Moreover, one can assert this "voice in what shall have happened, after it appears already to have happened" in illustrations of complementarity other than the double slit, by suitable modification of the idealized apparatus.

In the gamma ray microscope as described by Bohr (17) and Heisenberg (18), a lens of angular opening ε, receiving and bringing to a focus light of the reduced wave length $\hbar\lambda$, tells the position of the electron that scatters the light into the lens within a latitude of the order of

$$\Delta x \sim \hbar\lambda/\varepsilon. \tag{39}$$

When the lens is thus used to fix position, the quantum of energy scattered into the lens gives the electron a lateral kick, the amount of which is subject to an uncertainty of the order

$$\Delta p \sim \left(\begin{array}{c} photon \\ momentum \end{array} \right) \left(\begin{array}{c} angular\ opening \\ of\ the\ lens \end{array} \right) \sim (\hbar/\lambda)\,\varepsilon. \qquad (40)$$

This magnitude is coordinated with the Δx of (39) by the usual indeterminacy relation. However, the uncertainty in the lateral kick can be reduced to a very small fraction of (40) by placing a sufficiently great collection of sufficiently small photodetectors at a little distance above the lens. Whichever one of them goes off signals the direction of the scattered photon and thus the momentum imparted to the electron. When the lens is operated in this mode it ceases to serve as a lens in the true sense of the word. Lost then is the possibility to know the position of the electron within anything like the narrowness of limits implied by the Δx of (39). All this is the standard and well known lesson of complementarity.

We now add the feature of delayed choice. Only after the quantum of electromagnetic energy has already passed through the lens do we decide which lattice work of photodetectors to swing into action. One lattice is located in the focal plane of the lens. Let it be the lattice that is swung into action. Then one of the counters in this lattice goes off. This irreversible act of amplification tells us where the electron was when it scattered the radiation, within the lattitude Δx of (39). In drawing this conclusion about resolving power we accept the fact that the radiation made use of the entire aperture of the lens.

Or let the other lattice of photodetectors be swung into action. They are located in a plane some small fraction of the way up from lens to focal plane. One of these devices thereupon registers an event. This indelible record tells us within a certain range, much smaller than the Δp of (40), what lateral kick was given to the electron in the Compton process. The reasoning is simple. The kick to the electron is deduced from the direction of the scattered photon. The direction of this photon is revealed by the coordinates of the photocounter that registered, because it responds to a quantum of energy only when that

energy goes through a highly restricted portion of the aperture of the lens.

Shall we say that "the whole aperture transmitted the energy" or that "only a very small fraction of the aperture transmitted the energy?" We can freely decide one way or the other, according as we activate one set of photocounters or the other, after the lens has *already* finished transmitting the energy. That is the unfamiliar feature that "delayed choice" brings to the idealized gamma-ray microscope.

The split-beam experiment provides another example, Bohr tells us (19), "to which Einstein very early called attention and often has reverted. If a semi-reflecting mirror is placed in the way of a photon, leaving two possibilities for its direction of propagation, the photon may either be recorded on one, and only one, of two photographic plates situated at great distances in the two directions in question, or else we may, by replacing the plates by mirrors, observe effects exhibiting an interference between the two reflected wave-trains. In any attempt of a pictorial representation of the behaviour of the photon we would, thus, meet with the difficulty: to be obliged to say, on the one hand, that the photon always chooses *one* of the two ways and, on the other hand, that it behaves as if it had passed *both* ways."

Stapp (20) has proposed a more specific experimental arrangement. In it the two parts of the split beam are brought back together with a phase difference, δ, at a second half-silvered mirror. The two alternative ways for a quantum of radiant energy to leave that mirror have relative probabilities $\sin^2\delta$ and $\cos^2\delta$, as can be measured by suitably positioned counters. Alternatively the counters can be moved to intercept the arriving energy before it hits the half-silvered mirror. According as we dispose the counters in the one way or the other we observe the consequences of interference or we find out in which of the two beams the quantum arrives; but, in conformity with the principle of complementarity, we cannot do both kinds of observation on the same photon.

When we add to this experiment the feature of delayed
choice, we arrive at the arrangement of Fig. 2. We swing
the photodetectors into the positions c_1, c_2 or the
positions c_1', c_2' at our free choice and at the very last
instant. In other words, after the quantum of electromag-
netic energy has *already* finished interacting with the doubly
silvered mirror, A, we choose whether this quantum shall
manifest a two-beam interference phenomenon or shall arrive
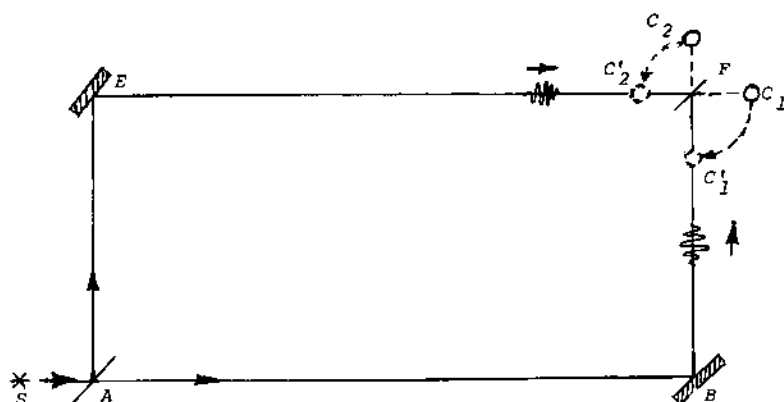via a single beam.

*Fig. 2.   The "delayed choice" split-beam experiment. S,
pulsed source, giving a wave train long compared to one
wave length but short compared to the total travel distance.
It is operated at an intensity level so low that at most
one quantum of energy is transported per pulse. A and F,
semi-reflecting mirrors. B and E, fully reflecting mirrors.
C_1 and C_2, photodetectors set to determine, by their
relative counting rates, the phase difference between the
two alternative routes to the point of observation. C_1'
and C_2', the same photodetectors swung into position to
determine on which route the quantum of energy arrives.*

Fig. 3 shows the feature of delayed choice added to
another familiar idealized experiment. Here we have to
choose between measuring the time of emission of a photon
and measuring its energy. When the faces of the tiltable
teeth of the grating are aligned, it becomes a spherical
mirror. Then the time-of-arrival counter alone is operative.
The information it gives, projected back in time, tells
us we had to do with a sharp pulse and tells us the time
at which it left the atom. Excluded from measurement in
this option, and therefore without meaning, is the energy
with which the photon was emitted. In the other option
we determine this energy, but forego any possibility of a
precise measurement of the time of emission. For this
purpose we turn the teeth of the grating--and in an improved
dispensation, also adjust their shape--so that, in the lan-
guage of spectroscopy, they "blaze" practically all of the
energy into the first order spectrum. In this case one
of the spectrum-analysis counters registers an event. From
it we project back

(1) to conclude that the quantum of energy had a
narrowly defined wave length and

(2) to deduce the energy it carried away from the
atom.

The atom is very far away in the idealized arrange-
ment we contemplate. There is ample time between the emission
of the radiation and its arrival at the grating to choose
which way to tilt the teeth. Thus after the radiation has
already been emitted, we choose whether it is to manifest
itself as pulselike or as nearly monochromatic.

In example No. 5 we make a delayed choice between measur-
ing the direction (θ, ϕ) of an emitted photon and measuring
the (l, m, π)-mode to which it belongs. For this purpose we
can use a natural but rather complicated extension of the
tiltable tooth arrangement to encompass the whole 4π solid
angle around, and some distance from, the emitting atom.
Then, according as we make the one or the other last minute
choice of alignment of the tiltable teeth, we get one or
other very different story--directivity or multipolarity--
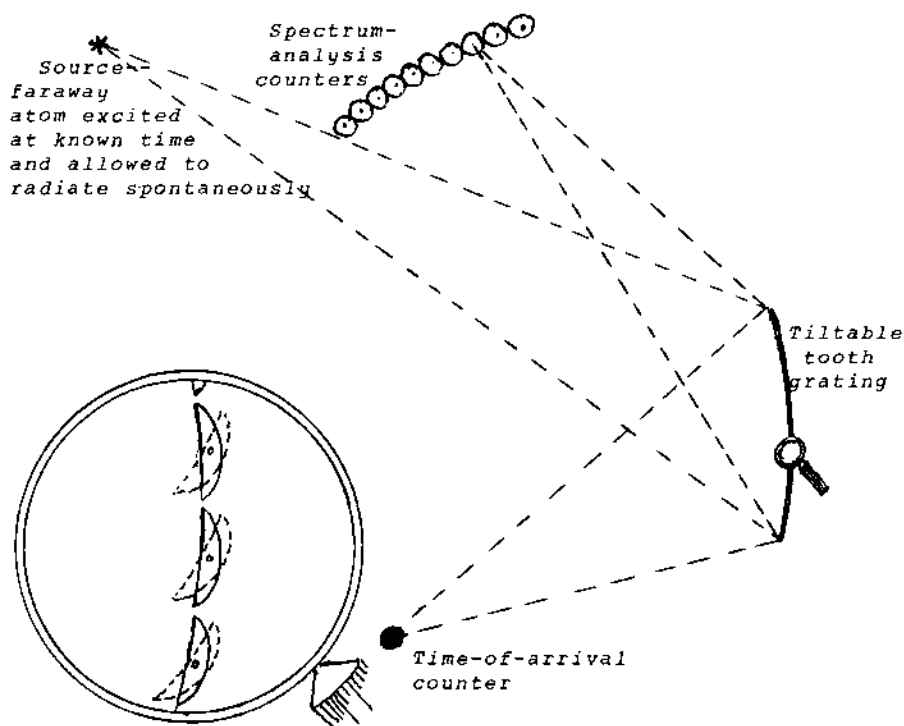for the character of the radiation that had *already* left

*Fig. 3. Use of a tiltable tooth grating to make a
delayed choice between alternative pictures (wave train
versus pulse) of the radiation that the atom has already
emitted. The magnified view shows a few typical teeth
of the grating depicted (1) by dashed lines when the teeth
are so oriented as to reflect or "blaze" the light to the
spectrum-analyzing counters and (2) by full lines when
the teeth are aligned. Then the grating becomes a spherical
mirror, and the time-of-arrival counter alone is operative.*

the atom at the time the choice was made.

Example No. 6 of delayed choice between complementary
alternatives makes polarization the focus of attention.
We arrange for a distant atom to be excited at time $t = 0$.
We allow it to radiate spontaneously. The arriving quantum
of energy we analyze with a Nichol prism. It is turned to
let through radiation with an azimuth of polarization $\phi = \phi_0$.
A transmitted quantum triggers a photodetector A ("accepted
quantum"). A photon of the orthogonal polarization
$\phi = \phi_0 + (\pi/2)$ would normally run into the layer of black
paint that is part of the traditional design of a Nichol

prism and be absorbed. However, to provide extra assurance
of a reliable analysis of polarization, we strip away that
paint and arrange for such a photon to enter into, and
trigger, a second photodetector R ("rejected quantum").
We make the choice of the acceptable azimuth, ϕ_o, and
in effect[1] rotate the Nichol prism and associated A and R
counters to that azimuth, after the quantum of energy has
already left the distant atom, and before it reaches the
analyzer.

As seventh and last example of adding delayed choice
to a standard illustration of complementarity, consider
the spontaneous annihilation of the light "atom" (e^+, e^-) in
its singlet ground state. Let two analyzers be used to
investigate the polarizations of the two 0.511MeV quanta
given off in opposite directions in this annihilation
process.[2]

Let the two analyzers first be set to "corresponding"
polarizations. That means "orthogonal azimuths" in the case
when both detectors are adjusted to analyze for linear
polarization, as in the experiments done so far. There is
a prospect that new experiments will look for circular

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[1] *In actuality, under ordinary experimental circum-
stances, the equipment will be too large and fragile and
the available time too short to permit any significant
rotation of the Nichol prism. Therefore the equivalent of
the last minute rotation is understood to be accomplished by
an appropriate electronic rotator-of-polarization located
just before the Nichol prism.*

[2] *Theory,(21-23); observations with two analyzers of
linear polarization, (24-34); theory for the case when the
analyzers can be set to accept photons of arbitrary Stokes
parameters (36) observations on the polarization of the
photons given out in a related process, the two-step
transition of an atom from an excited state to its ground
state, (37, 38); a result in such an experiment in contra-
diction with quantum predictions, (39); failure to confirm
this experiment, (40); confirmation of quantum predictions
(40-42).*

polarization. In that case "corresponding" means opposite
senses of rotation of the electric vector as judged by a
faraway observer of the equipment looking along the common
line of travel of the two photons, one approaching him,
the other receding. For the more general case of elliptic
polarization, "corresponding" means that the Stokes vector (35)
of the polarization accepted by the one analyzer appears
to the same faraway assessor of the apparatus to be
antipodal to the Stokes vector of the polarization
accepted by the other analyzer. Let the information in each
Stokes vector be summarized in the location of a point on
the surface of the unit sphere. Then the two polarizations
"correspond," Kagali shows (36), when their separation,
θ, on the unit sphere is $\theta = 180°$.

When the two analyzers are set to polarizations that
correspond in the foregoing sense, then the predictions of
quantum mechanics are simple; and the observations, so
far as they go (linear polarization) support those pre-
dictions: Every time an annihilation quantum triggers
the "accept counter," A, of the one analyzer, its partner
annihilation quantum sets off the accept register, A,
of the other analyzer; and every time the first photon
is recorded by the "reject counter," R, of the first analyzer,
its mate triggers the "reject photodetector," R, of the
other analyzer.

When the Stokes vectors of the two analyzers do not
correspond; when their separation, θ, on the unit sphere
is not $180°$, but some lesser angle, then the predictions
are still simple (36), and the observations for the case
of linear polarization support those predictions:

When the accept counter of the first analyzer goes off,
the accept counter of the second analyzer goes off
with a probability $\sin^2(\theta/2)$; and
the reject counter of the second analyzer goes
off with a probability $\cos^2(\theta/2)$.

When the reject counter of the first analyzer goes off,
the accept counter of the second analyzer goes off
with a probability $\cos^2(\theta/2)$; and
the reject counter of the second analyzer goes

off with a probability $\sin^2(\theta/2)$.   (41)

Very different are these predictions from what one would have expected had the two annihilation photons started out from their point of origin with "already defined polarizations:" linear polarization of the one quantum at an azimuth $\phi_{emission} = \phi_e$, where ϕ_e is random; and then, of necessity, linear polarization of its mate at the perpendicular azimuth $\phi_e+(\pi/2)$. Were this a proper description of what goes on, were the polarizations thus defined independently of the act of observation, then the chance of an "accept count" in the first analyzer (set at an azimuth ϕ_1) along with an accept count in the second analyzer [set at an azimuth $\phi_1+(\theta/2)$] would have been

(chance of an "accept-accept coincidence")

$= \cos^2(\phi_e-\phi_1) \cos^2[(\phi_e+\pi/2)-(\phi_1+\theta/2)]$,

averaged over azimuths,$\phi_{emission}$

$= (1/8) + (1/4) \sin^2(\theta/2)$,   (42)

instead of the quantum-mechanically predicted--and observed-- and twice as strong (*zero* to full strength) dependence of coincidence rate on relative azimuth, $\theta/2$, of the two analyzers,

(chance of an A-A coincidence) $= (1/2)\sin^2(\theta/2)$ (43)

as illustrated in Fig. 4.

Set the analyzers at one relative azimuth and measure the probability of an accept-accept coincidence; repeat for a second relative azimuth and for a third. Between these three probabilities Bell established the existence of an inequality, by now well known, and used by him and others-- in conjunction with the observations--to exclude various theories of "hidden variables" and other proposed alternatives to quantum mechanics (43, 44). We have no need to examine that watch-dog inequality here. We are not interested in "alternatives." We seek, not to escape standard battle-tested quantum theory, but to learn what lesson it has to teach in "delayed choice experiments."
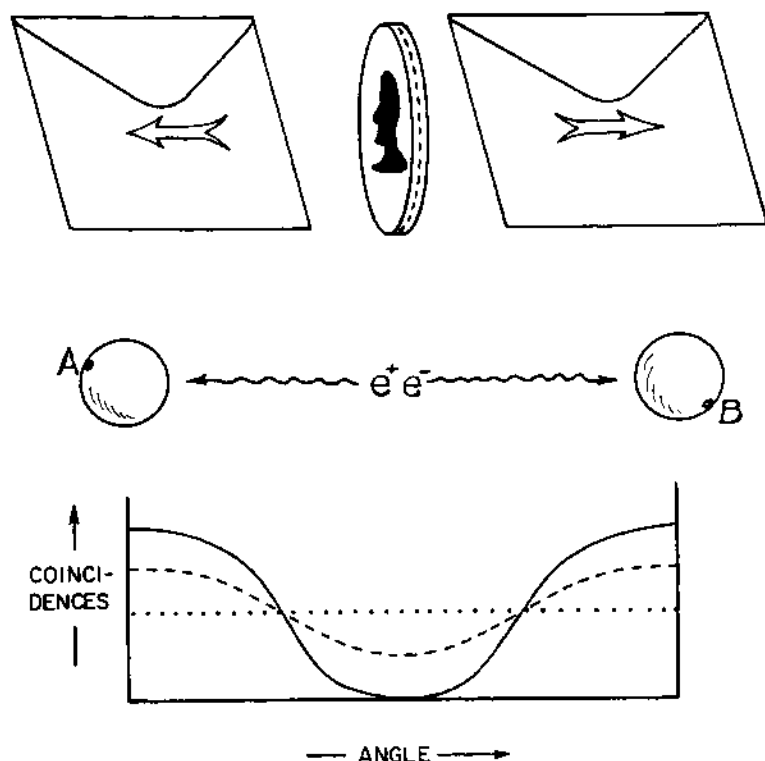
Fig. 4.   In contrast to splitting a coin, putting the
two slices into envelopes, shuffling them, and sending them
to two remote observers (upper part of diagram) the experi-
ment on the annihilation photons (middle part of diagram)
permits a double infinity of choices for the polarization to
be looked for in the right-hand (e^+e^-) annihilation photon,
with corresponding consequences for the polarization that
will be found for the left-hand photon.   If the polariza-
tions were determined in the act of emission ("hidden
variables") the coincidences between the two photons,
when both analyzers are set to accept linear polarization,
would show only half the dependence on relative orientation
of the two polarizations (dashed curve in lower diagram) that
is predicted by quantum mechanics--and observed (full curve).

In the "delayed-choice and arbitrary-polarization"
version of the experiment on annihilation quanta, we wait
until the two photons are on their way from source to
receptors to fix the experimental conditions. At the last
instant before the one quantum arrives at the one receptor
we set the one analyzer to one set of Stokes parameters
(i.e., select one point on the right-hand unit sphere of
Fig. 4); similarly set the other analyzer to another set
of Stokes parameters (i.e., select a point on the left-hand
unit sphere in Fig. 4--equivalent to a reflected point on
the right-hand unit sphere). It does not matter how differ-
ent the ellipticities are to which the two analyzers are
set nor how late in the travel of the two photons these
settings are made; the probabilities of AA, AR, RA and RR
coincidences are still given by Eqs. (41). To specialize,
pick at the very last minute whatever Stokes vector one
pleases for the one analyzer, and the complementary polari-
zation (antipodal Stokes vector) for the other analyzer.
Then one is assured of a perfect concord between the causally
disconnected records made at the two receptors. When the
accept counter goes off at the one receptor, the accept
counter goes off at the other receptor. When the reject
register triggers at the one, the reject register goes off
at the other. It is unpredictable whether the photon
travelling to the left will set off the accept counter or
the reject one. Whichever it is, the other photon will do
the *same*. It does not matter that the two photons are
out of causal contact with each other. It does not matter
that only at the last instant, while they were *already* on
their way, did we make that choice of complementary
polarizations to which the two photons, in perfect unison,
align or antialign.

LESSON OF DELAYED-CHOICE EXPERIMENTS

The double slit experiment, like the other six
idealized experiments (microscope, split beam, tilt-teeth,
radiation pattern, one-photon polarization, and polarization
of paired photons), imposes a choice between complementary

modes of observation. In each experiment we have found a
way to delay that choice of type of phenomenon to be
looked for up to the very final stage of development of
the phenomenon, *whichever* type we then fix upon. That
delay makes no difference in the experimental predictions.
On this score everything we find was foreshadowed in that
solitary and pregnant sentence of Bohr (45), ". . .it. . .
can make no difference, as regards observable effects ob-
tainable by a definite experimental arrangement, whether
our plans for constructing or handling the instruments are
fixed beforehand or whether we prefer to postpone the
completion of our planning until a later moment when the
particle is already on its way from one instrument to
another."

Not one of the seven delayed choice experiments has
yet been done. There can hardly be one that the student
of physics would not like to see done. In none is any
justification whatsoever evident for doubting the obvious
predictions.

We search here, not for new experiments or new pre-
dictions, but for new insight. Experiments dramatize and
predictions spell out the quantum's consequences; but what
is its central idea? A pedant of Copernican times could have
calculated planetary positions from the equations of
Copernicus as well as Copernicus himself; but what would we
think of him if his eyes were closed to the main point, that
the "Earth goes around the Sun"?

In the absence of an equally simple statement of *its*
central idea, quantum theory appears to many as strange,
unwelcome, and forced on physics as it were from outside
against its will. In contrast, if the essential point could
be grasped in a single phrase, we can well believe that the
quantum would seem so natural that we would recognize at
once that the universe could not even have come into being
without it.

Special relativity's findings shower out like fire-
works from a single compact package, "The laws of physics are
the same in every inertial reference system." No leap of the

imagination to a comparably compact and explodable formulation
of quantum theory being forthcoming, experience recommends
step-by-step progress. Of such steps none in recent times
moved our understanding forward more than the Einstein-Bohr
dialog (1). Out of that dialog no concept emerged of
greater fruitfulness than "phenomenon" (46): ". . .[In my
discussions with Einstein] I advocated the application of
the word *phenomenon* exclusively to refer to the observations
obtained under specified circumstances, including an account
of the whole experimental arrangement" (47). No other
point does the present analysis of idealized delayed-choice
experiments have but to investigate what "phenomenon"
means as applied to the "past."

 After the quantum of energy has *already* gone through
the doubly slit screen, a last-instant free choice on
our part--we have found--gives at will a double-slit-
interference record or a one-slit-beam count. Does this
result mean that present choice influences past dynamics,
in contravention of every formulation of causality? Or
does it mean, calculate pedantically and don't ask questions?
Neither; the lesson presents itself rather as this, that
the past has no existence except as it is recorded in the
present. It has no sense to speak of what the quantum
of electromagnetic energy was doing except as it is
observed or calculable from what is observed. More
generally, we would seem forced to say that no phenomenon
is a phenomenon until--by observation, or some proper
combination of theory and observation--it is an observed
phenomenon. The universe does not "exist, out there,"
independent of all acts of observation. Instead,it is
in some strange sense a participatory universe.

 That present choice of mode of observation in the
double-slit experiment should influence what we say about
the "past" of the photon; that the "past" is undefined and
undefinable without the observation, may be illustrated
by a little story.

 The "game of twenty questions" will be recalled. One
of the party is sent out of the room. The others agree

on a word. The one fated to be questioner returns and
starts his questions. "Is it a living object?" "No."
"Does it belong to the mineral kingdom?" "Yes." So the
questions go from respondent to respondent around the room
until at length the word emerges: victory if in twenty
tries or less; otherwise, defeat.

 Well does one participant recall the evening when
he, fourth to be sent out, returned to find a smile on
everyone's face, sign of a joke or a plot. He innocently
started his questions. But each question he put took longer
in the answering--strange, when the answer itself was only
a simple "yes" or "no." At length, feeling hot on the trail,
he asked, "Is the word 'cloud'?" "Yes," came the reply and
everyone burst out laughing. They explained that when he
had gone out, they had agreed not to agree in advance on
any word at all. Everyone could respond "yes" or "no"
as he pleased to whatever question was put to him. But
however he answered, he had to have a word in mind compatible
with his own reply--and with all the replies that went before.
No wonder it took time to answer!

 It is natural to compare the game in its two versions
with physics in its two formulations, classical and quantum.
First, the puzzled participant thought the word already
existed "out there" as physics once thought that the position
and momentum of the electron existed "out there," indepen-
dent of any act of observation. Second, the information
about the word was brought into being step by step through
the questions that the interrogator raised, as the infor-
mation about the electron is brought into being step by
step by the experiments that the observer chooses to make.
Third, if the participant had chosen to ask different
questions he would have ended up with a different word, as
the experimenter would have ended up with a different story
for the doings of the electron if he had done different
measurements or the same measurements in a different
sequence. Fourth, whatever power the interrogator had in
influencing the outcome for the word was partial only. A
major part of the decision lay in the hands of the other
participants. Similarly, the experimenter has some

substantial influence on what will happen to the electron
by the choice of experiments he will do on it; but he is
well aware that there is much impredictability about what
any given one of his measurements will disclose. Fifth,
there was a "rule of the game" that required of every
participator that his choice of yes or no should be
compatible with *some* word. Similarly, there is a consisten-
cy about the observations made in physics. One person can
tell another in plain language what he finds and the second
person can verify the observation. Interesting though
this comparison is between the world of physics and the
world of the game, there is an important point of difference.
The game has a finite number of participants and terminates
after a finite number of steps. In contrast, the making
of observations is a continuing process. Moreover, it is
extraordinarily difficult to state sharply and clearly

>   (1)   where the community of observer-participators
>   begins and where it ends, and
>
>   (2)   what the degree of amplification must be to
>   define an observation: "The amplification
>   of atomic effects, which makes it possible
>   to base the account on measurable quantities
>   and which gives the phenomenon a peculiar
>   closed character, only emphasizes the irreversi-
>   bility characteristic of the very concept of
>   observation" (48).

It is not necessary to understand every point about
the quantum principle in order to understand something about
it. Of all the points that stand forth from comparing the
world of quantum observations with the game of twenty
questions, none is more central than this: As in the game
no word was the word until that word had been promoted to
reality by the choice of questions asked and answers given,
so no phenomenon is a phenomenon until it is an observed
phenomenon.

REFERENCES

1. Bohr, N., "Discussion with Einstein on epistemological
    problems in atomic physics,"in "Albert Einstein:
    Philosopher-Scientist," pp. 199-241. Library of
    Living Philosophers, Evanston, Illinois, 1949.
2. Bohr, N., "Atomic Physics and Human Knowledge,"
    p. 73. Wiley, New York, 1958.
3. George, C., I. Prigogine and L. Rosenfeld, "The
    macroscopic level of quantum mechanics," Kgl. Danske
    Videnskab. Selskab, Mat.-fys. Meddelelser 38, No. 12
    (1972).
4. Bohr, N., "Atomic Physics and Human Knowledge," pp. 88-
    89. Wiley, New York, 1958; see also extensive
    references to the literature on p. 174 of Petersen,
    A., "Quantum Physics and the Philosophical Tradition,"
    M. I. T. Press, Cambridge, Massachusetts, 1968.
5. Belinfante, F. J., "Measurement and Time Reversal
    in Objective Quantum Theory," p. 39. Pergamon, Oxford
    and New York, 1975.
6. Duguay, M. A., "The ultrafast optical Kerr shutter,"
    pp. 161-193 in "Progress in Optics, vol. 14 (E.
    Wolf, Ed.), North Holland, Amsterdam, 1976, compares
    the new ultrafast optical Kerr shutter to the older
    and slower conventional and electrically operated
    Kerr cell.

7. Lubkin, G. B., "Tests show CO_2 laser is suitable for fusion," Physics Today 30, 19 (September 1977).

8. Heisenberg, W., and W. Pauli, "Zur Quantendynamik der Wellenfelder," Zeits.f. Physik 56, 1-61 (1929); and "Zur Quantentheorie der Wellenfelder. II," Zeits. f. Physik 59, 168-190 (1930).

9. Breit, G. "Quantum theory of dispersion," Rev. Mod. Phys. 4, 504-576 (1932); see especially pp. 521-532 and 553-554; and "Quantum theory of dispersion (continued). Parts VI and VII," Rev. Mod. Phys. 5, 91-140 (1933); see especially pp. 104-105 and 127-128.

10. Lapedes, E. N., Ed., "McGraw-Hill Dictionary of Scientific and Technical Terms," article on apodisation, McGraw-Hill, New York, 1974; Jacquinot, P., and B. Roizen-Dossier, "Apodisation," in "Progress in Optics," Vol. 3 (E. Wolf, Ed.), North Holland, Amsterdam, 1964.

11. Gouy, L. G., "Sur une proprieté nouvelle des ondes lumineux," C. R. Acad. Sci. Paris 110, 1251 (1890).

12. Bruhat, G., "Optique," p. 59, theory; p. 95, experiment, Masson, Paris, 1942.

13. Simms, D. J. and N. M. J. Woodhouse, "Lectures on Geometric Quantization," list of references on pp. 159-163, especially citations of the work of Maslov, Springer, Berlin and New York, 1976.

14. Misner, C. W., K. S. Thorne and J. A. Wheeler, "Gravitation," section of focusing, pp. 570-583, Freeman, San Francisco, 1973.

15. Everett, H., III, "The theory of the universal wave function," Princeton University doctoral thesis, 1956; published in abbreviated form as "'Relative State' Formulation of Quantum Mechanics," Rev. Mod. Phys. 29, 454-462 (1957); and in full in "The Many-Worlds Interpretation of Quantum Mechanics," (B. De Witt and N. Graham, Eds.) Princeton University Press, Princeton, New Jersey, 1973, pp. 3-140; see there pp. 43-53 for the considerations on information theory cited in the text.

16. Wootters, W. K., unpublished October 1977 lecture on the information content of an interference pattern.

17. Bohr, N., "The quantum postulate and the recent development of atomic theory," Nature 121, 580-590 (1928); see especially pp. 582,583.

18. Heisenberg, W., translated by C. Eckart and F. C. Hoyt, "The Physical Principles of the Quantum Theory," pp. 21-22, University of Chicago Press, Chicago, Illinois, 1930.

19. Bohr, N., "Discussion with Einstein on epistemological problems in atomic physics," pp. 201-241 and specifically p. 222 in Schilpp, P. A., "Albert Einstein: Philosopher-Scientist," Library of Living Philosophers, Evanston, Illinois, 1949.

20. Stapp, H., oral report, The University of Texas at Austin, spring, 1977.

21. Wheeler, J. A., "Polyelectrons," Ann. New York Acad. Sci. 48, 219-238 (1946). On p. 235 the figures, 1.08 at 90° and 1.10 at 74°30', should be corrected to 2.6 at 90° and 2.85 at 82°, according to Snyder et al (23).

22. Pryce, M. H. L. and J. C. Ward, "Angular correlation effects with annihilation radiation," Nature 160, 435 (1947).

23. Snyder, H. S., S. Pasternack and J. Hornbostel, "Angular correlation of scattered annihilation radiation," Phys. Rev. 73, 440-448 (1948).

24. Bleuler, E., and H. L. Bradt, "Correlation between the states of polarization ef the two quanta of annihilation radiation," Phys. Rev. 73, 1398 (1948).

25. Hanna, R. C., "Polarization of annihilation radiation," Nature 162, 332 (1948).

26. Vlasov, N. A. and B. S. Dzeljepov, "Poljarizatzija annigiljatzionnikh gamma-kvantov," Dokl. Akad. Nauk. SSSR 69, 777-779 (1949).

27. Wu, C. S. and I. Shakhov, "The angular correlation of scattered annihilation radiation," Phys. Rev. 77, 136 (1950).

28. Hereford, F., "The angular correlation of photoelectrons ejected by annihilation quanta," Phys. Rev. 81, 482 (see also pp. 627-628)(1951).

29. Bertolini, G., M. Bettoni and E. Lazzarini, "Angular correlation of scattered annihilation radiation," Nuovo Cim. 2, 661-662 (1955).

30. Langhoff, H., "Die Linearpolarisation der Vernich-tungsstrahlung von Positronen," Zeits. f. Physik 160, 186-193 (1960).

31. Kasday, L., J. Ullman and C. S. Wu, "The Einstein-Podolsky-Rosen argument: positron annihilation experiment," Bull. Amer. Phys. Soc. 15, 586 (1970).

32. Kasday, L., "Experimental test of quantum predictions for widely separated photons," pp. 195-210 in "Foundations of Quantum Mechanics, Proceedings of the International School of Physics 'Enrico Fermi,' Course 49" (B. d'Espagnat, Ed.), Academic, New York, 1971.

33. Faraci, G., D. Gutkowski, S. Notarrigo and A. R. Pennisi, "An experimental test of the EPR paradox," Lettere al Nuovo Cimento, 9, 607-611 (1974).

34. Wilson, A. R., J. Lowe and D. K. Butt, "Measurement of the relative planes of polarization of annihilation quanta as a function of separation distance," J. Phys. G: Nucl. Phys. 2, 613-624 (1976).

35. Born, M., and E. Wolf, "Principles of Optics," 4th Ed., Pergamon, New York, 1970.

36. Kagali, Basavaraj A., "Correlation of polarizations in e^+-e^- annihilation," preprint, Center for Theoretical Physics, University of Texas (1977); submitted for publication.

37. Kocher, C. A. and E. D. Commins, "Polarization correlation of photons emitted in an atomic cascade," Phys. Rev. Lett. 18, 575-577 (1967).

38. Freedman, S. J. and J. F. Clauser, "Experimental test of local hidden-variable theories," Phys. Rev. Lett. 28, 938-941 (1972).

39. Holt, R. A., "Atomic cascade experiments," Doctoral dissertation, Harvard University, Cambridge, Massachusetts, unpublished, available from University Microfilms, Inc., Ann Arbor, Michigan 48106. (1973).

40. Clauser, J. F., "Experimental investigation of a polarization correlation anomaly," Phys. Rev. Lett.36, 1223-

1226 (1976).

41. Freedman, S. J. and R. A. Holt, "Tests of local hidden-variable theories in atomic physics," Comments on Atomic and Molecular Physics 5, 55-62 (1975).

42. Fry, E. S. and R. C. Thompson, "Experimental test of local hidden variable theories," Phys. Rev. Let. 37, 465-468 (1976).

43. Bell, J. S., "On the Einstein Podolsky Rosen paradox," Physics 1, 195-200 (1964).

44. Wigner, E. P., "On hidden variables and quantum mechanical probabilities," American J. of Physics 38, 1005-1009 (1970).

45. Bohr, N., see reference 1, p. 230.

46. Petersen, A., "Quantum Mechanics and the Philosophical Tradition," M. I. T. Press, Cambridge, Massachusetts, 1968, gives a preliminary account of the stages in the development of the concept of "phenomenon."

47. Reference (2), p. 64.

48. Reference (2), p. 89.