Local convexity in F-spaces with the Hahn-Banach Extension Property

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Scientific Abstract

Topological vector spaces are one of the most studied structures in functional analysis. Chapter 1 will introduce this concept together with some of its properties. We will also show that every metrizable vector topology can be defined in terms of an *F-norm*. Hence we can define an *F-space* as a complete F-normed space.

Two concepts will be fundamental through the thesis: An F-space is said to have the *HBEP* if any element of the dual of a closed subspace can be extended in the dual of the whole space. A vector topology is locally convex if it can be described through a basis of neighbourhoods of the origin formed of convex sets. The main result of this paper is Theorem 3.12: **Every F-space with the HBEP is locally convex.**

For that purpose, we study these properties in different topological vector spaces in Chapter 2. We will prove the statement in Chapter 3 based on the relation of the HBEP with the weak topology and the construction of M-basic sequences in closed subspaces.

Keywords: topological vector space, basis of neighbourhoods, locally convex, Hausdorff, separable, F-norm, F-space, dual space, HBEP, point separation property, Mackey topology, weak topology, M-basic, strongly regular, polar topology.

Popular abstract

In pure mathematics, we are always working with structures. The structures describe a set of elements that have operations or relations with themselves. For example, the real numbers can be added, and we can say that one is bigger than other. Vector spaces are structures where the elements can be added and can change their size.

An important characteristic that describes some structures is the topology, which represents the "shape" of it. A topological vector space is a vector space with a topology compatible with the operations. It is a fundamental concept in this work to be locally convex, i.e. whether this topology can be described using convex sets.

In this case, we are studying topological vector spaces where we can measure, and that they are complete, that is that the sequences that "seem to converge", actually converge. That is what we call F-spaces.

The aim of this work is to prove that the F-spaces with a special property, called the HBEP, are locally convex.

Contents

1 Topological vector spaces

1.1 Definition and properties

The points of the vector space, often called *vectors*, can be added and multiplied by the elements of a field. The vector topology is defined in a way that is compatible with this structure. For the purpose of this paper, we will always assume that the vector spaces are associated with a field K, which can be $\mathbb{R}$ or $\mathbb{C}$. Now we begin to formally define this topology.

Definition 1.1. Let X be a set. A family $\tau \subset \mathcal{P}(X)$ is a *topology* on X, if $\emptyset \in \tau$, $X \in \tau$, and it is closed under finite intersections and arbitrary unions.
A set X equipped with a topology is called a *topological space*. Sometimes it will be denoted as (X, τ) to specify the topology.

Given τ, γ two topologies in X we say that τ is *weaker* than γ if $\tau \subset \gamma$. In that case, we also say that γ is *stronger* than τ.

Given (X, τ), (Y, γ) two topological spaces, we say that $f : X \to Y$ is *continuous* if for all $V \in \gamma$ we have that $f^{-1}(V) \in \tau$.

Definition 1.2. A *topological vector space* is a vector space equipped with a topology where vector addition and scalar multiplication are continuous functions in the correspondent product topologies. A topology with such properties will be called a *vector topology*.

Definition 1.3. Let (X, τ) be a topological vector space and $S \subset X$.

i) S is *open* if $S \in \tau$.

ii) S is *closed* if $X \backslash S$ is open.

iii) S is a *neighbourhood* of $x \in X$ if there is $U \in \tau$, that $x \in U \subset S$.

iv) S is *dense* if for all $U \in \tau$ we have that $U \cap S \neq \emptyset$.

v) S is *compact* if for any family of open sets whose union contains S there is a finite subfamily whose union also contains it.

vi) S is *bounded* if for any neighbourhood of the origin U there is $n \in \mathbb{N}$ such that $S \subset nU$.

vii) S is *balanced* if for all $\alpha \in K$ such that $|\alpha| \leq 1$, we have $\alpha S \subset S$

viii) S is *absorbent* if for all $x \in X$, exists $n \in \mathbb{N}$ such that $x \in nS$.

ix) S is *convex* if for $0 \leq \lambda \leq 1$ we have $\lambda S + (1-\lambda)S \subset S$.

x) S is *absolutely convex* if it is convex and balanced.

Remark 1.4. The set of neighbourhoods of a point x is usually denoted $\mathcal{F}(x)$. Note that $\mathcal{F}(x)$ has the following properties: $\emptyset \notin \mathcal{F}(x)$, $\mathcal{F}(x)$ is closed under finite intersection, and for any $A \subset X$ such that contains an $F \in \mathcal{F}(x)$, we have that $A \in \mathcal{F}(x)$.

Any family that satisfies this is called a *filter*. The last property implies that the filter of neighbourhoods of a point is uniquely defined for, what is called a basis:

Definition 1.5. Let X be a topological vector space and $x \in X$. A *basis of neighbourhoods of* x, or x*-basis*, is a family $\mathcal{B}(x) \subset \mathcal{F}(x)$ such that for all $A \in \mathcal{F}(x)$ there is $B \in \mathcal{B}(x)$ that $B \subset A$.

Remark 1.6. By definition, the translations and dilations in a topological vector space are homeomorphisms (continuous linear functions with continuous inverse). Hence, to define a vector topology, it is sufficient to describe it in terms of a basis of the neighbourhoods around one point, usually, the origin. This family can be chosen with desirable properties.

Theorem 1.7. Let X be a topological vector space. Then there is a 0-basis $\mathcal{B}$ formed by balanced and absorbent sets such that for all $U \in \mathcal{B}$ there is $V \in \mathcal{B}$ such that $V + V \subset U$. Conversely, any collection of sets on a vector space X with this property defines a unique vector topology.

Proof. We refer to [2] Section 2.2 Theorem 5. □

Observation 1.8. The use of neighbourhoods in a topological vector space gives an alternative definition of the open sets and hence for continuity. A set U is open if and only if for all $x \in U$ we have that $U \in \mathcal{F}(x)$. A function $f(X, \tau) \to (Y, \gamma)$ is continuous if, for all $y \in Y$ and $V \in \mathcal{F}(y)$, we have that $f^{-1}(V) \in \mathcal{F}(f^{-1}(y))$. It also allows us to define some other properties on a vector topology.

Definition 1.9. A topological vector space is

i) *locally convex* if it has a basis of neighbourhoods of the origin consisting of convex sets.

ii) *locally bounded* if there exists a bounded neighbourhood of the origin.

iii) *separable* if there is a countable dense set.

iv) *Hausdorff* if for all different points x, y there is U_x, U_y, neighbourhoods of x and y respectively, such that $U_x \cap U_y = \emptyset$.

Lemma 1.10. A topological vector space is Hausdorff if and only if there is a basis of neighbourhoods of the origin $\mathcal{U}$ such that $\cap_{U \in \mathcal{U}} U = \{0\}$

Definition 1.11. Let X be a topological vector space, and $S \subset X$. The *convex hull* of S (respectively the *closure* of S) is the smallest set containing S that is convex (respectively closed).

Observation 1.12. In general, we can always find a 0-basis of balanced and absorbent sets such that all of them are closed or open; see [2] Section 2.2. However, some topological vector spaces do not have a base of convex sets of the origin, moreover, being locally convex is a strong property with many consequences.

1.2 Metrizable Spaces and F-spaces

Recall that a *metric* in a set X is a function $d : X \times X \to \mathbb{R}$ such that

i) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ only if $x = y$,

ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

iii) $d(x + y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

We say that a topological space is *metrizable* if there exists a metric that generates the topology.

In any topological space X, a sequence (x_n) in X is *convergent* to $x \in X$ if for all U neighbourhood of x there is N such that $x_n \in U$ for all $n > N$. If the topology is metrizable, this notion has an equivalent definition based on the *balls*

$$B_x(\epsilon) = \{y : d(x, y) < \epsilon\},$$

forming a basis of neighbourhoods of the point. We will make an introduction to the main concepts in metrizable spaces, leaving for the next section the parallel concept for topological vector spaces.

Definition 1.13. Consider X a topological space that is metrizable by a metric d. We say that a sequence (x_n) in X is

i) *convergent* if there is $x \in X$ such that for all $\epsilon > 0$ there is N such that $d(x_n, x) < \epsilon$ for all $n > N$.

ii) *Cauchy* if for all $\epsilon > 0$ there is N such that $d(x_n, x_m) < \epsilon$ for all $n, m > N$.

Definition 1.14. A metrizable topological space X is *complete* if any Cauchy sequence with respect to a metric that generates the topology is convergent.

Remark 1.15. We say that two metrics d_1, d_2 are *equivalent* if there is $M, m > 0$ such that for all $x, y \in X$,

$$md_1(x,y) \leq d_2(x,y) \leq Md_1(x,y).$$

Note that two equivalent metrics induce the same topology but the converse is not true in general. For example, the metrics $|x-y|$ and $\frac{|x-y|}{1+|x-y|}$ induce the same topology in $\mathbb{R}$, but they are not equivalent.

The fact that the completeness of a metrizable topological space is well defined independently of the metric chosen, is a non-trivial fact proved by Klee, solving a problem from Banach [6]. Furthermore, there is always an invariant metric that generates the topology. The following is satisfied:

Theorem 1.16. Two metrics on a set determine the same topology if and only if they have the same convergent sequences.

Definition 1.17. A topological vector space is called an *F-space* if it is metrizable and complete.

Remark 1.18. The concepts of Cauchy sequence and completeness can be defined for a topological space without the necessity of being metrizable (see [10] Section 1.25). Every topological space can be embedded as a dense subspace of a complete topological space, which is called the completion. In the case of metrizable topological vector spaces, there is a unique (up to isomorphism) completion that is an F-space. For more information about this construction see [9] Section 4.8.

1.3 F-normed spaces and Δ-norms

A metric d is compatible with the structure of a vector space X if it is *translation invariant* i.e. such that

$$d(x,y) = d(x+z, y+z) \text{ for all } x, y, z \in X.$$

Recall that a *norm* is a real-valued function $\| \quad \|$ such that

i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ only if $x = 0$

ii) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in X$ and $\alpha \in K$,

iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

In general, any norm $\| \quad \|$ induces an invariant metric of the form $d(x,y) = \|x-y\|$. The converse is not true, but any invariant metric in a vector space can be induced by what is called an F-norm.

We will introduce the concept of F-normed space and show that it is equivalent to metrizable topological vector space. We will end the section by proving that any Hausdorff vector topology with a countable 0-basis is generated by an F-norm.

Observation 1.19. Every metrizable topological space is Hausdorff, but the converse is not true in general. However, let X be a Hausdorff topological vector space that has a countable basis of neighbourhoods of the origin. By Lemma 1.10, we must have that $\cap U_n = \{0\}$, in fact, we can modify this basis so $U_{n+1} + U_{n+1} \subset U_n$. Then the formula

$$\|x\| = \sup(2^{-n} : x \notin U_n).$$

defines what is called a Δ-norm:

Definition 1.20. Let X be a vector space. A Δ-norm is a map $\| \quad \| : X \to \mathbb{R}$ such that

i) $\|x\| > 0$ for all $x \in X$ such that $x \neq 0$,

ii) $\|\alpha x\| \leq \|x\|$ for all $x \in X$ and $\alpha \in K$ such that $|\alpha| \leq 1$,

iii) $\lim_{\alpha \to 0} \|\alpha x\| = 0$ for all $x \in X$,

iv) exists $C \in \mathbb{R}$, such that $\|x+y\| \leq C \cdot \max(\|x\|, \|y\|)$ for all $x, y \in X$.

We say that a Δ-norm is a *quasi-norm* if all $x \in X$ and $\alpha \in K$,

$$\|\alpha x\| = |\alpha| \|x\|.$$

Remark 1.21. Conversely to Observation 1.19, every Δ-norm $\| \quad \|$ in a vector space X, induces a Hausdorff vector topology generated by neighbourhoods of the origin of the form

$$U_n = \{x \in X : \|x\| \leq 1/n\}.$$

Note that, following the procedure of the observation, this topology induces the original Δ-norm.

Definition 1.22. An *F-norm* in a vector space X is a Δ-norm such that

$$\|x+y\| \leq \|x\| + \|y\|$$

for all $x, y \in X$.

A set X equipped with an F-norm is called an *F-normed space*. Sometimes it will be denoted as $(X, \| \quad \|)$ to specify the F-norm.

Remark 1.23. Note that any F-normed space induces a topology, metrizable by an invariant metric given by

$$d(x,y) = \|x - y\|.$$

Respectively any metrizable topological vector space, has an invariant metric d that generates the topology, hence it is an F-normed space with respect to

$$\|x\| = d(0,x).$$

Hence, from now on we will use the term F-normed space, instead of metrizable topological vector spaces. In particular, the F-spaces are simply complete F-normed spaces.

Theorem 1.24. Let X be a vector space, and let $\| \quad \|$ be a Δ-norm in X, and let $C > 1$ be such that $\|x + y\| \leq C \cdot \max(\|x\|, \|y\|)$ for all $x, y \in X$. Then for p such that $2^{1/p} = C$, the formula

$$\|x\|_F = \inf(\sum_{i=1}^{n} \|x_i\|^p : \sum_{i=1}^{n} x_i = x)$$

defines an F-norm in X that induces the same topology as $\| \quad \|$.

Proof. It can be found in [3] Theorem 1.2. $\square$

Corollary 1.25. Every Hausdorff topological vector space (X, τ) with a countable basis of neighbourhoods of the origin is an F-normed space. Hence, it is metrizable and the topology can be given by an invariant metric.

1.4 F-seminorms

Another useful characterization of a vector topology in a vector space is given by a collection of F-seminorms. This is a generalization of the concept F-norm that does not require to be non-zero for all points different from the origin.

Definition 1.26. Let X be a vector space. An *F-seminorm* is a map $p : X \to \mathbb{R}$ such that

i) $p(x) \geq 0$ for all $x \in X$,

ii) $p(\alpha x) \leq p(x)$ for all $x \in X$ and $\alpha \in K$ such that $|\alpha| \leq 1$,

iii) $\lim_{\alpha \to 0} p(\alpha x) = 0$ for all $x \in X$,

iv) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

We say that an F-seminorm p is a *seminorm* if for all $x \in X$ and $\alpha \in K$,

$$p(\alpha x) = |\alpha| p(x).$$

Note that a seminorm p is a norm if $p(x) \neq 0$ for all $x \neq 0$.

Observation 1.27. Let X be a vector space and let L be a collection of F-seminorms on X. For any finite $F \subset L$ and for any $r > 0$ we define

$$U_{F,r} = \bigcap_{p \in F} \{x \in X : p(x) < r\}.$$

The sets $U_{F,r}$ are balanced, absorbent and such that $U_{F,\frac{r}{2}} + U_{F,\frac{r}{2}} \subset U_{F,r}$. By Theorem 1.7, this family is a 0-basis that defines uniquely a vector topology in X.

Definition 1.28. Let X be a vector space and let L be a collection of F-seminorms on X. Then the *topology induced by* L, denoted τ_L is the vector topology with 0-basis the sets of the form

$$U_{F,r} = \bigcap_{p \in F} \{x \in X : p(x) < r\}$$

for any $r > 0$ and $F \subset L$ finite subset.

Lemma 1.29. Let X be a vector space and let L be a collection of F-seminorms on X. Then

i) τ_L is the weakest vector topology making all F-seminorms in L continuous.

ii) τ_L is Hausdorff if and only if for all $x \in X$ such that $x \neq 0$, there is $p \in L$ such that $p(x) > 0$.

iii) If all elements on L are seminorms, then τ_L is locally convex.

Theorem 1.30. Let X be a topological vector space and let L be the set of all continuous F-seminorms on X. Then τ_L is the original topology.

Proof. We refer to [2] Theorem 3 Section 2.7 □

Observation 1.31. If the space is locally convex, the topology is generated by the family of all continuous seminorms. In fact, it is determined by the Minkowski functionals of the open absolutely convex sets that form a 0-basis. For more details on this construction, see [9] Section 5 and Theorem 5.5.2.

1.5 Duality and linear maps

The dual space of a topological vector space gives relevant information about its structure. A linear functional establishes a relation of the space, which is usually infinite-dimensional, with the field K. The good properties of such maps define hyperplanes that can be used to separate sets. We will now introduce some of these notions and the main results related to them.

Definition 1.32. Let X, Y be topological vectors spaces over K. A *linear map* is a function $M : X \to Y$ such that for all $x, y \in X$, $k \in K$

$$M(x+y) = M(x) + M(y) \text{ and } M(kx) = kM(x).$$

In the particular case of $Y = K$, M is called a *linear functional* on X.

Theorem 1.33. (Hahn-Banach Extension Theorem) Let X be a vector space over K and let p be a real-valued subadditive function defined in X such that

i) if $K = \mathbb{R}$, for all $\alpha > 0$, $p(\alpha x) = \alpha p(x)$.

ii) if $K = \mathbb{C}$, for all $\alpha \in \mathbb{C}$, $p(\alpha x) = |\alpha| p(x)$.

Let Y be a subspace of X, and l be a linear functional in Y such that for all $y \in Y$

a) if $K = \mathbb{R}$, $l(y) \leq p(y)$.

b) if $K = \mathbb{C}$, $|l(y)| \leq p(y)$.

Then there is a linear functional l_X defined in X, such that for all $y \in Y$

$$l_X(y) = l(y),$$

and that satisfies a) if $K = \mathbb{R}$, or b) if $K = \mathbb{C}$, for all $y \in X$.

Remark 1.34. There is a known geometrical consequence of the case when $K = \mathbb{R}$, the Hyperplane separation Theorem: For any convex neighbourhood of the origin C and any $x \notin C$, there is a linear functional l such that

$$l(y) \leq l(x) \text{ for all } y \in C.$$

For more details see [8], Chapter 3.

Definition 1.35. Let X, Y be topological vectors spaces over K. The set of all continuous linear maps from X to Y is denoted by $L(X, Y)$. The set of all continuous linear functionals $L(X, K)$ is the *dual space* of X, denoted X^*.

Definition 1.36. Let X be a topological vector space. We say that a family L of linear functionals of X is *equicontinuous* if for all $\epsilon > 0$ there is a neighbourhood of the origin U such that

$$|f(x)| < \epsilon \text{ for all } x \in U \text{ and all } f \in L.$$

Note that if L is equicontinuous, then $L \subset X^*$.

Remark 1.37. The structure of the dual space determines some important properties of the original space. The following two will be studied in the following sections through different examples.

Definition 1.38. Let X be a topological vector space and M a closed proper subspace of X. M has the *Hahn-Banach Extension Property* (HBEP) if for all $l \in M^*$ there exists $\widetilde{l} \in X^*$ such that

$$l(x) = \widetilde{l}(x) \text{ for all } x \in M.$$

We say that X has the HBEP if every closed subspace of X has the HBEP.

Definition 1.39. We say that a topological vector space X has the *point separation property* if for any different points $x, y \in X$ there is $l \in X^*$ such that

$$l(x) \neq l(y).$$

Theorem 1.40. Every Hausdorff locally convex topological vector space has the HBEP.

Proof. We will assume that X is a Hausdorff topological vector space over $\mathbb{C}$. Let $\mathcal{B}$ be a 0-basis for the topology formed by absolutely convex sets. For any M closed subspace of X,

$$\{U \cap M : U \in \mathcal{B}\}$$

is a 0-basis for the topology in M also made of convex sets. We consider L the family of Minkowski functionals associated with the family $\mathcal{B}$. The topology in M is also the weakest topology making all functions in L continuous in the subspace. Hence for any $l \in M^*$ and $\epsilon > 0$ there is $\{p_1 \cdots p_n\} \subset L$ and $\delta > 0$ such that

$$|l(x)| \leq \epsilon \text{ for any } x \in \{x : \max_i p_i(x) \leq \delta\ \} \cap M.$$

Note that the basis consist of convex sets and therefore the functions p_i are seminorms. Since l is linear, for $p(x) = \max_i\{\frac{\epsilon}{\delta} p_i(x)\}$ we have that

$$|l(x)| \leq p(x) \text{ for any } x \in M.$$

The function p is subadditive and such that $p(\alpha x) = |\alpha| p(x)$. The Hahn-Banach Extension Theorem guarantees that there is a linear functional l_X defined in X extending l and dominated by p. The functional l_X is dominated by the continuous function p, so $l_X \in X^*$. $\square$

2 Examples

Recall that a topological vector space X is

i) an F-space if it is a complete F-normed space.

ii) locally convex if it has a basis of neighbourhoods of the origin consisting of convex sets.

The purpose of this paper is to show that the converse to Theorem 1.40 is satisfied in F-spaces. As we will prove in Lemma 3.6 any F-space with the HBEP has also the point separation property. However, this condition is not sufficient to imply local convexity. In this section we will study these properties in different topological vector spaces, that will show the necessity of the conditions.

2.1 Normed spaces

Note that a normed space $(X, \| \quad \|)$ is an F-normed space such that

$$\|\alpha x\| = |\alpha| \|x\|$$

for all $x \in X$ and $\alpha \in K$. It is a known fact that the HBEP is satisfied in normed spaces. See, for example, [9] Chapter 7. We will prove that it is also locally convex:

Theorem 2.1. Any normed space $(X, \| \quad \|)$ is locally convex.

Proof. There is a 0-basis of sets of the form

$$B(\epsilon) = \{x : \|x\| < \epsilon\}$$

Then for all $0 \leq \lambda \leq 1$ and $x, y \in B(\epsilon)$ we have

$$\|(\lambda x + (1 - \lambda)y)\| \leq \|\lambda x\| + \|(1 - \lambda)y\| < \lambda\epsilon + (1 - \lambda)\epsilon = \epsilon$$

Hence $\lambda x + (1 - \lambda)y \in B(\epsilon)$ and the set is convex. □

Definition 2.2. A *Banach Space* is a complete normed space.

2.2 Local p-convexity

The following generalization of convexity will be useful in certain vector spaces:

Definition 2.3. Let X be a topological vector space and $C \subset X$.

i) We say that C is *p-convex* if for all $x, y \in C$, and $a, b \in \mathbb{R}^+$ such that $a^p + b^p = 1$ we have that $ax + by \in C$.

ii) We say that C is *absolutely p-convex* if is p-convex and balanced

iii) X is *locally p-convex* if there is a basis of neighbourhoods of the origin that is p-convex.

Observation 2.4. Note that with this definition, locally convex corresponds to locally 1-convex. An alternative useful algebraic characterization for a p-convex set is given by the next lemma.

Lemma 2.5. Let X be a topological vector space, and $C \subset X$. C is absolutely p-convex if and only if for all $x, y \in C$, and $a, b \in \mathbb{R}$ such that $|a|^p + |b|^p \leq 1$ then $ax + by \in C$.

Remark 2.6. A locally p-convex topological vector space has necessarily a basis of absolutely p-convex sets, so if $0 < p < 1$ it is also locally q-convex for all $p \geq q > 0$.

2.3 Locally bounded spaces

In this section, we will study the concept of locally bounded. It is a desirable property for a topological vector space to have since it implies metrizability and local p-convexity for some p. Recall that a subset B is bounded if for any neighbourhood of zero U there is $n \in \mathbb{N}$ such that $B \subset nU$. A topological vector space is locally bounded if there exists a bounded neighbourhood of the origin.

Observation 2.7. Let B be a bounded neighbourhood of the origin in a Hausdorff locally bounded topological vector space X. For such a B we can define

$$\|x\| = \inf_{\lambda \in K} \{|\lambda| : x/\lambda \in B\}.$$

Note that since B is bounded $B + B$ is also a bounded neighbourhood of the origin, so there is $c > 0$ such that $B + B \subset cB$. That implies that

$$\|x + y\| \leq c \cdot \max(\|x\|, \|y\|),$$

and from that follows that $\| \quad \|$ is a Δ-norm. In fact, it is a quasi-norm.

The sets of the form

$$\frac{1}{n} B = \{\|x\| \leq 1/n\}.$$

define a countable basis of neighbourhoods of the origin. By Corollary 1.25, the following is satisfied:

Theorem 2.8. Every locally bounded Hausdorff topological vector space is metrizable.

Observation 2.9. If the set B is p-convex, then

$$\|x\| = \inf\{\lambda : x/\lambda \in B\}$$

is a p-subadditive Δ-norm (i.e. $\|x+y\|^p \leq \|x\|^p + \|y\|^p$). Then $(\frac{1}{n})B$ form a basis of p-convex neighbourhoods of the origin, so X is locally p-convex.

In the proof of Theorem 2.8, we have seen that there is $c > 1$ such that

$$B + B \subset cB.$$

This result combined with Theorem 1.24, proves that $\| \quad \|$ is p-subadditive for a p such that $2^{1/p} = c$. As a consequence, we have the Aoki-Rolewicz theorem:

Theorem 2.10. Every locally bounded Hausdorff topological vector space is locally p-convex for a positive p.

2.4 $C^\infty(0,1)$

We denote by $C^0[a,b]$ the space of real-valued continuous functions defined on $[a,b]$. Recursively we can define $C^k[a,b]$ for any positive integer k as the space of continuous differentiable functions such that their derivatives are in $C^{k-1}[a,b]$. In these notions, we can define the norms

$$\|f\|_n^{[a,b]} = \max\{|f^{(k)}(x)| : x \in [a,b], 1 \leq k \leq n\}.$$

It can be proven that the spaces $C^n[a,b]$ are complete in the topology induced by the norm $\|f\|_n$ (see [10] Example 1.46). The topology is generated by a norm, so by Lemma 1.29, they are locally convex F-spaces.

Now we can define the space

$$C^\infty(0,1) = \bigcap_{n\geq 0, k\geq 2} C^n[\frac{1}{k}, 1-\frac{1}{k}].$$

Each $\|f\|_n^{[a,b]}$ defines a seminorm in $C^\infty(0,1)$ for any $0 < a < b < 1$. Then the topology induced by the family

$$\{\|f\|_{n,k} := \|f\|_n^{[1/k, 1-1/k]} \text{ for } k \geq 2, n \geq 0\}$$

is locally convex.

Since they are separating, we can define the F-norm given by

$$\|f\|_\infty = \max_{k,n} 2^{-(k+n)} \frac{\|f\|_{n,k}}{\|f\|_{n,k} + 1}$$

that generates the same topology as the family of seminorms (see [10] Remarks 1.38). The space is complete with respect to this F-norm, so $(C^\infty(0,1), \| \quad \|_\infty)$ is a locally convex F-space.

2.5 L_p for $0 \leq p < 1$

In this section, we are going to study an F-space that is not locally convex and does not have the HBEP. We will start by defining the L_p spaces and showing that they are indeed F-spaces.

Given (Ω, Σ, μ) a σ-finite measure space, L_0 is the space of the classes of all Σ-measurable functions defined on Ω, identifying the ones that are equal μ-almost everywhere (see [1] Chapter 3.3). For $0 < p < \infty$, are defined the L_p spaces as

$$L_p = \{f \in L_0 : \|f\|_p = (\int |f|^p d\mu)^{1/p} < \infty\}$$

$$L_\infty = \{f \in L_0 : \|f\|_\infty = \sup_\mu(|f|) < \infty\}$$

where $\sup_\mu$ is the supremum value up to sets of μ-measure 0.

Note that for $1 \leq p \leq \infty$, the Minkowski equality is satisfied, so the previous definition of $\| \quad \|_p$ defines a norm. These spaces are known Banach spaces, in fact, they can also be defined as the completion of continuous functions such that the above is satisfied.

Another known fact is that the dual space of L_p for $1 \leq p$ can be identified with L_q with q such that $1/p + 1/q = 1$ (understanding $q = \infty$ for $p = 1$). For detailed information in these spaces, see [7] Chapter 14. Section 10.

In this section, we will focus on the spaces L_p for $0 \leq p < 1$.

Observation 2.11. Let now $0 < p < 1$. It is clear that $\|\alpha f\|_p = |\alpha|\|f\|_p$. That makes $\| \quad \|_p$ satisfy the first 3 conditions for being a Δ-norm. It is also easy to prove that for $0 \leq a, b$ and $0 < p < 1$ we have that

$$(a + b)^p \leq a^p + b^p.$$

Hence for $0 < p < 1$, we have that $f, g \in L_p$,

$$\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$$

proving that $\| \quad \|_p^p$ defines an F-norm. With a procedure similar to the one for Banach spaces, we can prove that $(L_p, \| \quad \|_p^p)$ is also complete, and hence an F-space.

Note that the balls defined by

$$B_p(r) = \{x : \|x\|_p^p) < r\},$$

form a 0-basis, and for all $r > 0$, we have that $B_p(1) \subset r^{-1/p}B_p(r)$. Hence L_p is locally bounded.

Remark 2.12. If (Ω, Σ, μ) has a finite measure, the F-norm

$$\|f\|_0 = \int \frac{|f|}{1 + |f|} d\mu$$

makes $(L_0, \| \quad \|_0)$ an F-space. However, it is not locally bounded.

Theorem 2.13. Let λ be the Lebesgue measure and consider the L_p space defined on $([0, 1], \Sigma_\lambda, \lambda)$. Then, for $0 < p < 1$, the only convex open sets in L_p, are $\emptyset$ and L_p.

Proof. Let $A \subset L_p$ be a non-empty convex open set. Taking a translation if needed, we may assume that A is a neighbourhood of 0. Since the balls form a 0-basis, A contains a ball of radius $r > 0$. Let now f be any function in L_p, we are going to prove that $f \in A$. Let n be such that $n^{p-1}\|f\|_p^p < r$. Since the integral is a continuous function in $\mathbb{R}$, there are points x_i such that for $1 \leq i \leq n$,

$$\int_{x_{i-1}}^{x_i} |f|^p d\mu = n^{-1} \int |f|^p d\mu = n^{-1}\|f\|_p^p$$

We can define the functions

$$g_i(t) = nf(t)\mathbb{1}_{[x_{i-1}, x_i]}.$$

and they are such that

$$\|g_i\|^p = n^{p-1}\|f\|_p^p < r,$$

so $g_i \in A$ for all i.

Moreover, since A is convex, we have that

$$f = \frac{g_1 + \cdots g_n}{n} \in A.$$

Since f is arbitrary we must have $A = L_p$. □

Example 2.14. The previous theorem implies that balls are not convex, i.e. there are two functions that have the same F-norm, but a convex combination of them has a bigger one. A straightforward example of this situation is with the functions $\mathbb{1}_{[0,\frac{1}{2}]}$, $\mathbb{1}_{[\frac{1}{2},1]}$ and $(\frac{1}{2})\mathbb{1}_{[0,1]}$. Note that again this is only possible since $0 < p < 1$.

Definition 2.15. We say that $A \subset \Omega$ is an *atom* if $\mu(A) > 0$ and for any measurable set $B \subset A$ such that $\mu(B) > 0$ we have $B = A$.

We say that a metric is *non-atomic* if there are no atom sets.

Remark 2.16. Making some modifications to the last proof we can also prove that for any non-atomic metric μ, the previous is satisfied for the corresponding L_p.

In fact, let Λ be a linear operator defined in such an L_p with values in a locally convex topological vector space Y. For any convex open neighbourhood of the origin in $W \subset Y$, $\Lambda^{-1}(W)$ is also an open and convex neighbourhood of the origin in L_p. Theorem 2.13 implies that $\Lambda^{-1}(W) = L_p$.

Then for all $f \in L_p$ and W a convex open neighbourhood of the origin in Y, we have that $\Lambda(f) \in W$. If Y is a Hausdorff that implies that $\Lambda \equiv 0$, so $L(L_p, Y) = \{0\}$. In particular, since the field K is locally convex, the following is satisfied:

Corollary 2.17. Let μ be a non-atomic metric in Ω, and consider L_p in the measurable space $(\Omega, \Sigma_\mu, \mu)$. Then, for $0 < p < 1$,

$$L_p^* = \{0\}$$

Remark 2.18. If μ is a metric in Ω that has an atom A, any function in $L_p(\Omega, \Sigma_\mu, \mu)$ must be measurable, and then constant in A. Then the linear functional

$$f \to f(A)$$

is non-zero and continuous, establishing a converse to the last corollary.

2.6 l_p spaces for $0 < p < 1$

As we have seen, L_p spaces induced by measures with atoms are not locally convex and have a trivial dual. One of the standard examples of atomic measures is given by letting μ be the counting measure in the natural numbers. In these spaces, denoted l_p, the functions f have a sequence associated, given by $(f(n))_{n\in\mathbb{N}}$. That allows us to define the l_p spaces as for $0 < p \leq \infty$, as:

$$l_p = \{(a_n) : \|(a_n)\|_p = (\sum |a_n|^p)^{1/p} < \infty\}$$

$$l_\infty = \{(a_n) : \|(a_n)\|_\infty = \sup|a_n| < \infty\}.$$

The spaces l_p for $p \geq 1$ are also known Banach spaces, and we will not discuss them. Similarly to the general case, we can prove that the spaces $(l_p, \| \quad \|_p^p)$ for $0 < p < 1$ are F-spaces. We will now study some of their properties.

Some of them follow from the fact that the l_p spaces have a countable basis sequence given by the functions of the form $\mathbb{1}_{\{n\}}$.

Theorem 2.19. For $0 < p < 1$, the spaces l_p are locally p-convex

Proof. We are going to prove that the unit ball B_1 is absolutely p-convex neighbourhood of the origin. Let $(x_n),(y_n) \in B_1$ and a and b such that $|a|^p + |b|^p \leq 1$. Then

$$\|a(x_n)+b(y_n)\|^p \leq \|a(x_n)\|^p + \|b(y_n)\|^p \leq |a|^p\|(x_n)\|^p + |b|^p\|(y_n)\|^p \leq |a|^p + |b|^p \leq 1.$$

So $a(x_n)+b(y_n) \in B_1$. Since B_1 is bounded, the balls $B_{\frac{1}{n}}$ define a convex basis of neighbourhoods of the origin and l_p is locally p-convex. □

Theorem 2.20. For $0 < p < 1$, l_p^* is isometric to l_∞.

Proof. Since l_p has a basis sequence (e_i), for any $\lambda \in l_p^*$, we can define $a_i = \lambda(e_i)$. In that way for $x = (x_i) \in l_p$, we have that $\lambda(x) = \sum x_i a_i$. Then for any $\lambda \in l_p^*$, we can assign $a_\lambda = (a_i)$ and it is such that

$$\|a_\lambda\|_\infty = \sup(a_n) = \|\lambda\| < \infty.$$

The converse is clear, for every $a = (a_i) \in l_\infty$, we can define a unique continuous linear map $\lambda((x_i)) = \sum x_i a_i$ with the same norm. □

Proposition 2.21. The l_p spaces are not locally convex for $0 < p < 1$.

Proof. The topology on l_p is metrizable and hence it has a 0-basis consisting of balls. If l_p is locally convex, then the convex hulls of the balls would also be a 0-basis (this will be proved in Theorem 3.2). Let (e_i) be the canonical basis sequence of l_p. We have that $\|e_i\| = 1$, meanwhile for all n

$$\|\frac{1}{n}\sum_{i=1}^{n} e_i\|_p = n^{\frac{1}{p-1}}.$$

Hence the convex hulls of the balls are unbounded. With a similar argument to the one for the spaces L_p we can prove that l_p is locally bounded, which leads to a contradiction □

Remark 2.22. Theorem 2.20 implies that l_p has the separation property. For every $s = (a_n) \neq (0)$, we have that $\|s\| > 0$. In particular, there is $i \in \mathbb{N}$ such that $0 < |a_i|$. Then the linear functional $\lambda((x_n)) = x_i$ separates s from the origin.

The space l_p has multiple convex neighbourhoods of the origin, however, there are not enough to form a basis. As we will see in Observation 3.9, this is because l_p does not have the HBEP. After these results, it is clear that the separation property does not necessarily imply locally convexity in F-spaces.

3 Main Result

The examples of the last section should be enough to wonder whether the HBEP holds in a non-locally convex topological vector space. Constructions of spaces where this happens were made in non-metrizable spaces, see [4] Section 6, Problem I. However, we will show that there is no F-space of such form. For that purpose, we will show some of the relations between the HBEP and the weak topology. But first, we will introduce a vector topology that is locally convex in every space, the Mackey topology.

3.1 The Mackey topology

Given any topological vector space, it is always possible to define the strongest locally convex topology that is weaker than the original. This is the Mackey topology, a concept that will be fundamental.

Definition 3.1. Let (X, σ) be a topological vector space. The *Mackey topology* of σ is a vector topology defined in X with the basis of neighbourhoods of the origin all convex σ-neighbourhoods of the origin.

Theorem 3.2. The Mackey topology of an F-normed space is metrizable

Proof. Let X be an F-normed space, with an induced topology σ. We will denote by μ the Mackey topology of (X, σ). Since σ is metrizable there is $\mathcal{F}$, a countable basis of σ-neighbourhoods of the origin that intersect only at the origin. For any U convex σ-neighbourhood of the origin, there is $V \in \mathcal{F}$ such that $V \subset U$. Since U is convex, the convex hull of V is also contained in U, so the collection of the convex hull of the sets in $\mathcal{F}$ is a (countable) basis for μ.

Now we are going to prove that μ is Hausdorff. Let $x \in X$ be other than the origin. Since σ is metrizable by an F-norm $\| \quad \|$, we can define the σ-balls as

$$B_\epsilon(x) = \{y : \|x - y\| < \epsilon\}.$$

For $\epsilon = \|x\|$, we have that

$$B_{\epsilon/2}(0) \cap B_{\epsilon/2}(x) = \emptyset.$$

Let $\delta < \epsilon/4$ and let consider U_0, U_x the convex hulls of the sets $B_\delta(0), B_\delta(x)$ respectively. For all $p \in U_0$ we have that there is $y, z \in B_\delta(0)$ and $0 \leq \lambda \leq 1$ such that $p = \lambda y + (1 - \lambda)z$, hence

$$\|p\| = \|\lambda y + (1 - \lambda)z\| \leq \|\lambda y\| + \|(1 - \lambda)z\| \leq \|y\| + \|z\| \leq \epsilon/2.$$

That proves that $U_0 \subset B_{\epsilon/2}(0)$ and similarly it can be proved that $U_x \subset B_{\epsilon/2}(x)$. Hence (X, μ) is Hausdorff, and by Corollary 1.25, it is also metrizable. $\square$

3.2 Weak topology and HBEP

Recall that an F-space X has the HBEP if for every M closed proper subspace of X and for all $l \in M^*$ there exists $\widetilde{l} \in X^*$ such that

$$l(x) = \widetilde{l}(x) \text{ for all } x \in M.$$

Note that in this definition the topology defined in X is only determining the continuity of the maps in X^*. That fact makes this definition related to the weak topology in X:

Definition 3.3. Let X be an F-space. The *weak topology* in X, is the weakest topology such that all maps in X^* are continuous. We say that a proper closed subspace of X is *weakly closed* if it is closed in the weak topology. Similarly, we say it is *weakly dense* if it is dense in the weak topology.

Theorem 3.4. The weak topology of an F-space X is locally convex.

Proof. There is a basis of neighbourhoods of the origin in the weak topology of sets of the form

$$B_\Omega = \{x : |l(x)| < \epsilon \text{ for all } l \in \Omega\}$$

with $\Omega \subset X^*$ a finite subset.

Then for all $0 \leq \lambda \leq 1$, $l \in \Omega$ and $x, y \in B_\Omega$ we have

$$|l(\lambda x + (1-\lambda)y)| = |\lambda l(x) + (1-\lambda)l(y)| < \lambda\epsilon + (1-\lambda)\epsilon = \epsilon$$

Hence $\lambda x + (1-\lambda)y \in B_\Omega$ and the set is convex. □

Proposition 3.5. Let X be an F-space and M a proper closed subspace of X.

i) M is weakly closed in X if and only if for all $x \notin M$ there exists $l_x \in X^*$ such that

$$l_x(M) = \{0\} \text{ and } l_x(x) \neq 0.$$

ii) M is weakly dense in X if and only if $l(M) = \{0\}$ implies that $l = 0$.

Proof. Let M be a weakly closed set in X. Then for all $x \notin M$ there is a functional $l_x \in X^*$ and $\epsilon > 0$ such that

$$\{y : |l_x(x-y)| < \epsilon\} \cap M = \emptyset.$$

Since $0 \in M$, we must have that $l_x(x) \neq 0$ and $l_x(z) = 0$ for all $z \in M$. Conversely, given a point and a linear functional satisfying these properties the set defined previously is an open set in the weak topology, so a set M with that property for every $x \notin M$, is weakly closed.

Let M be weakly dense in X. Suppose that $l(M) = \{0\}$ and let $x \in X$ such that $l(x) \neq 0$. K is Hausdorff, so there is an open set U in K that contains $l(x)$ and not 0. Since l is continuous, $l^{-1}(U)$ is an open set in the weak topology and $l^{-1}(U) \cap M \neq \emptyset$, so M would not be weakly dense. The converse is proved similarly.

□

Lemma 3.6. Let X be an F-space, then

i) X has the point separation property if and only if every finite-dimensional subspace of X has HBEP.

ii) A closed subspace M of X is weakly closed if and only if X/M has the point separation property.

iii) A closed subspace M is weakly dense in X if and only if $X/M \neq \{0\}$ and $(X/M)^* = \{0\}$.

Proof. Let X be an F-space with the point separation property. Consider a finite-dimensional subspace M. We can obtain $\{x_1 \cdots x_n\}$ linearly independent points that generate M. Note that every $l \in M^*$ is completely defined by the set $\{l(x_1) \cdots l(x_n)\}$. X has the point separation property, so for every x_i there is a functional $l_i \in X^*$ such that $l_i(x_i) \neq 0$. Since there is a finite number of generators, we can define a linear combination of l_i such that for $1 \leq i \leq n$,

$$c_1 l_1(x_i) + \cdots c_n l_n(x_i) = l(x_i).$$

Then $c_1 l_1 + \cdots c_n l_n \in X^*$ and extends l.

Conversely let $x \in X$ such that $x \neq 0$. The functional $l(kx) = k$ for $k \in K$ is defined in the subspace generated by x. Any extension of it separates x from the origin.

The other statements are straightforward consequences of Proposition 3.5. □

Remark 3.7. The result of Klee [5] show that there are (non-complete) F-normed spaces with the separation property such that the quotient with respect to a closed subspace does not have it. Even in F-spaces, there is no connection for a single closed subspace between being weakly closed and having the HBEP ([3], Chapter 4, Examples A and B). However, a global relation exist.

Theorem 3.8. An F-space has HBEP if and only if every proper closed subspace is weakly closed.

Proof. Let X be an F-space with the HBEP and let M be a proper closed subspace of X. By Lemma 3.6 ii), if X/M has the separation property then M is weakly closed. We will prove that in fact, X/M has the HBEP. Let N be a closed subset of X/M and $l \in N^*$, we define

$$\widetilde{N} = \{x + m : [x] \in N, m \in M\}.$$

For any a sequence in $\widetilde{N}$, $(x_n + m_n)$, convergent to y, $([x_n + m_n]) = ([x_n])$ is a sequence in N (which is closed) convergent to $[y]$. Hence we have that $[y] \in N$, so $y \in \widetilde{N}$. That proves that $\widetilde{N}$ is a closed subspace of X and hence it has the HBEP in X.

Since $l([0]) = 0$, we can define in $\widetilde{N}$ the continuous linear functional

$$\widetilde{l}(x) = l([x])$$

that has an extension $\check{l} \in X^*$, and it is such that $\check{l}(M) = \{0\}$. Then $\tilde{l} \in X/M$ defined by

$$\tilde{l}([x]) = \check{l}(x)$$

is well defined and extends l. Then X/M has the HBEP, so by Lemma 3.6 i), also the point separation property. Hence M is weakly closed.

Now consider X an F-space such that every closed proper subspace is weakly closed. Consider M one of these subspaces and $l \in M^*$. If $N = l^{-1}(0)$ is M, then the zero functional extends l, so we may assume that N is a proper closed subspace of M. Then there is $x \in M$ such that $l(x) \neq 0$. Note that for any y such that $l(y) \neq 0$ there is $k \in K$ such that $l(x) = kl(y)$, so $x - ky \in N$. Then we must have that

$$M = \{kx + n : k \in K, n \in N\}.$$

N is weakly closed, so by Proposition 3.5, there exists $l_x \in X^*$ such that $l_x(N) = 0$ and $l_x(x) \neq 0$. Then a scalar multiple of l_x extends l to X. □

Observation 3.9. In 1969 Shapiro and Stiles constructed a proper closed subspace of l_p that is weakly dense (see [3] Chapter 2, Section 3). Using the results of Lemma 3.6, we have that for that subspace M, $(l_p/M)^* = \{0\}$. Hence it does not have the point separation property, so M is not weakly closed. As a consequence of Theorem 3.8, we have that that l_p does not have the HBEP.

3.3 HBEP and local convexity

We have seen that the HBEP is a strong property, with relevant consequences in F-spaces. In this section, we will show that every F-space with the HBEP is locally convex, a fact firstly proven by Kalton in 1974. We will proceed following a simplification of the proof made by Drewnowski, exposed in Chapter 4 of [3]. The result relies on the concept of M-basic and Theorem 3.11, a powerful tool that enables us to construct continuous linear functionals on subspaces of an F-space. This theorem will now be stated, and it will be proved in Section 3.4.

Definition 3.10. Let X be an F-normed space. Let (x_n) be a sequence in X, and let X_0 denote the closed linear span of the elements. We say that (x_n) is *M-basic* if there is a sequence (x_n^*) in X_0^*, called the *biorthogonal functionals*, such that:

i) $x_n^*(x_m) = \mathbb{1}_{\{n\}}(m)$

ii) for any Cauchy sequence (u_n) defined in X_0 such that $x_i^*(u_n)$ converges to zero for all i, we have that (u_n) converges to the origin.

If (x_n^*) is equicontinuous, the sequence (x_n) is called *strongly regular.*

Theorem 3.11. Let X be an F-space with induced topology σ, and let α be a Hausdorff vector topology on X weaker than σ.

Let (x_n) be a sequence in X such that $x_1 \neq 0$ and such that (x_n) converges to the origin in α but not on σ. Then there is a subsequence of (x_n) starting in x_1, M-basic and strongly regular.

Theorem 3.12. Every F-space with the HBEP is locally convex.

Proof. Let X be an F-space with the HBEP. Let $\| \quad \|$ be an F-norm that generates the topology of X, denoted σ, and let μ be the Mackey topology on X. The topology μ is by definition locally convex, so we are going to prove that $\mu = \sigma$.

In Theorem 3.4 we saw that the weak topology is locally convex, and hence it is weaker than μ. Also, X has the HBEP, so by Theorem 3.8, all σ-closed subspaces are weakly closed and therefore μ-closed. We also proved in Theorem 3.2 that μ is metrizable by an F-norm $\| \quad \|_\mu$.

Claim I For any sequence (w_n) μ-convergent to the origin, the set W of the elements of the sequence is σ-bounded.

Note that if W is not σ-bounded, then there is (c_n) a sequence in K convergent to 0, such that $(c_n w_n)$ does not σ-converge to the origin.

We let u be any vector in X. For such (c_n) we define

$$x_n = c_n(u + w_n).$$

The sequence (x_n) converges to the origin with respect to μ. If it does not with respect to σ, by Theorem 3.11 there is a strongly regular M-basic subsequence for σ, $(z_n) = (x_{m_n})$.

We note that

$$c_{m_n}^{-1} z_n = u + w_{m_n}$$

so the sequence $(c_{m_n}^{-1} z_n)$ μ-converges to u. Note that for all n, u must be in the closed linear span of $\{z_i : i \geq n\}$ (which is the same in σ and μ). But by definition of the M-basic, all z_n are linearly independent, hence u must be the origin. This is contradictory since u is arbitrary, so (x_n) must σ-converge to the origin. Since

$$c_n w_n = x_n - c_n u,$$

we have that $(c_n w_n)$ σ-converge to the origin, so W is σ-bounded.

Claim II Any μ-convergent sequence is σ-convergent to the same limit. First note that since $\| \quad \|_\mu$ is an F-norm, for any positive integer k,

$$\|kx_n\|_\mu \leq k\|x_n\|_\mu.$$

We can assume (x_n) is a sequence μ-convergent to the origin, and hence there is (k_n) a non-decreasing sequence of positive integers such that

$$(k_n) \to \infty \text{ and } \|k_n x\|_\mu \leq k_n\|x\|_\mu \to 0.$$

By the Claim I, the set $K = \{k_n x_n : n \in \mathbb{N}\}$ is σ-bounded, so for all σ-neighbourhood of the origin U, there is $N \in \mathbb{N}$ such that $K \subset NU$, and therefore

$$\{\frac{k_n}{N} x_n : n \in \mathbb{N}\} \subset U.$$

We can assume U to be a balanced set, and then for m such that $k_m > N$ (note that (k_n) is non-decreasing), we have that

$$\{x_n : n \geq m\} \subset U.$$

The topology σ is Hausdorff, so we have that (x_n) must σ-converge to the origin.

Hence $\| \quad \|$ and $\| \quad \|_\mu$ have the same convergent sequences, so they generate the same topology, μ, that is locally convex. □

3.4 Polar topologies and proof of Theorem 3.11

Theorem 3.11 is fundamental in the construction of the M-basic sequence, so we proceed to prove it. For that purpose we will first prove a similar version of the theorem, relying on an additional technical useful assumption of the topology of the F-space over another vector topology.

Definition 3.13. Let V be a vector space and let σ and γ be two vector topologies on V. We say that σ is γ-polar if σ has a basis of neighbourhoods of the origin formed of γ-closed sets.

Example 3.14. Let X be a normed space and let σ be topology induced by the norm. If we denote the weak topology by γ, we have that σ is γ-polar.

Theorem 3.15. Let (X, σ) be a metrizable topological vector space and let γ be another vector topology defined in X such that σ is γ-polar. Then σ can be induced by an F-norm given by

$$\|x\| = \sup\{\lambda(x) : \lambda \in \Lambda\}$$

where Λ is a collection of γ-continuous F-seminorms. In particular, $\| \quad \|$ is lower semicontinuous with respect to γ i.e.

$$\|x\| \leq \liminf_{\substack{x_n \longrightarrow x \\ (\gamma)}} \|x_n\|$$

when (x_n) γ-converges to x.

Proof. Since (X, σ) is metrizable, it is Hausdorff with a countable basis of neighbourhoods, so we may assume that σ is induced by an F-norm $\| \quad \|_\sigma$. Let Γ be the collection of all γ-continuous F-seminorms, and $\Lambda = \{\lambda_\nu : \nu \in \Gamma\}$, where

$$\lambda_\nu(x) = \inf\{\|y\|_\sigma + \nu(x-y) : y \in X\}.$$

We claim that $\|x\| = \sup\{\lambda(x) : \lambda \in \Lambda\}$ also induces σ. Let V be a σ-neighbourhood of the origin that is γ-closed. There is $\epsilon > 0$ such that $x \in V$ for all $\|x\|_\sigma < \epsilon$. The topology induced by $\| \quad \|$ is equivalent to the original if the $\| \quad \|$-balls define a 0-basis of σ. Hence it is sufficient to prove that

$$\{\|x\|_\sigma < \epsilon\} \subset \{\|x\| < \epsilon\} \subset V.$$

By the definition of $\| \quad \|$, we already have that $\|x\| \leq \|x\|_\sigma$.

Claim For all γ-neighbourhood of the origin U, there is $\nu \in \Gamma$ such that $\nu(u) < \epsilon$ implies that $u \in U$.

By Theorem 1.30 there is a basis of γ-neighbourhoods of the origin of the form

$$U_{F,r} = \bigcap_{p \in F} \{x \in X : p(x) < r\}$$

with $F \subset \Gamma$ a finite subset and $r > 0$.

In particular, given U, a γ-neighbourhood of 0, there must be such $F = \{p_1 \cdots p_n\}$ and $\epsilon > 0$ that $U_{F,\epsilon} \subset U$. We can define $\nu \in \Gamma$ in the form

$$\nu = \sum_{j=1}^{n} 2^{-j} p_j.$$

Then ν is such that $\{\nu(u) < \epsilon\} \subset U$.

Let now x be such that $\|x\| < \epsilon$. Then, for all $\nu \in \Gamma$, we have that $\lambda_\nu(x) < \epsilon$, so there is $y \in X$ such that $\|y\|_\sigma + \nu(x - y) < \epsilon$. In particular, for all γ-neighbourhood of the origin U we have that $x = y + (x - y) \in V + U$.

Now suppose that $x \notin V$. Since V is γ-closed, there would be a γ-neighbourhood of the origin W, such that $V \cap (W + x) = \emptyset$. By Theorem 1.7 there is a basis of balanced γ-neighbourhoods of the origin, we can suppose that W is balanced. Hence we would have that

$$(V - x) \cap W = \emptyset.$$

However W is a γ-neighbourhood of x so there is $v \in V$ and $u \in W$ such that $x = v + u$, so $v - x = -u$. Since W is balanced, we have that $-u \in (V - x) \cap W$, contradictory to the hypothesis, so we must have that $x \in V$. □

Lemma 3.16. Let X be a separable F-normed space and let γ be a vector topology on X so that the original topology is γ-polar. Then

i) any Cauchy sequence in the original topology that converges to the origin in γ, also does it in the original topology.

ii) There is a metrizable vector topology γ' such that $\gamma' \subset \gamma$ and the original topology is γ'-polar.

Proof. We will denote σ the original topology, and $\| \quad \|$ the F-norm that generates σ constructed with respect to γ as in Theorem 3.15.

Let $\{x_n\}$ a sequence that converges to the origin in γ and is Cauchy in σ. Then for all $\epsilon > 0$ there is N such that for any $n, m > N$, $\|x_n - x_m\| < \epsilon$. For any fixed n, $(x_n - x_m)_m$ γ-converges to x_n. Since $\| \quad \|$ is lower semicontinuous with respect to γ, we have that $\|x_n\| < \epsilon$, so $\|x_n\|$ converges to 0.

Since X is separable, we can get $\{F_n\}$ a countable family of finite sets whose union is dense in X. By the definition of $\| \quad \|$ for every n there is a finite set Λ_n of γ-continuous F-seminorms, such that for all $x \in \bigcup_{i=1}^n F_i$

$$\max\{\lambda(x) : \lambda \in \Lambda_n\} \le \|x\| \le \max\{\lambda(x) : \lambda \in \Lambda_n\} + 1/n.$$

Now, for $\Lambda_\infty = \bigcup \Lambda_n$ and $x \in F = \bigcup F_n$ we have that

$$\|x\| = \sup\{\lambda(x) : \lambda \in \Lambda_\infty\}.$$

By continuity of the F-seminorms that is also true for all $x \in X$. Let γ' be the topology induced by the family of F-seminorms Λ_∞. Since this is a countable family γ' has a countable basis of neighbourhoods of the origin of the form

$$\bigcap_{\lambda \in F} \{x \in X : \lambda(x) \le 2^{-n}\},$$

for $n \in \mathbb{N}$ and $F \subset \Lambda_\infty$ finite. With this definition, $\lambda(x) = 0$ for all $\lambda \in \Lambda_\infty$ implies that $\|x\| = 0$, so γ' is Hausdorff. By Corollary 1.25 this topology is metrizable. It is also clear that $\gamma' \subset \gamma$ and the sets of the form $\{\|x\| \leq 1/n\}$ are γ'-closed, hence σ is γ'-polar. □

Theorem 3.17. Let X be an F-normed space, and let σ be the topology induced by the F-norm. Let $\gamma \subset \sigma$ be another vector topology in X such that σ is γ-polar.

Let (x_n) be a sequence in X such that $x_1 \neq 0$ and such that (x_n) converges to the origin in γ but not in σ. Then there is a subsequence of (x_n) starting in x_1 such that is M-basic and strongly regular in X.

Proof. We can restrict our space to the closed span of the elements in (x_n), so we will assume that X is it, and therefore it is separable. It follows from Lemma 3.16 ii), that we can assume that γ is metrizable by an F-norm $\| \quad \|_\gamma$. By Theorem 3.15, we can also assume that there is a γ-lower semicontinuous F-norm $\| \quad \|$ that induces σ.

Since $\|x_n\|_\gamma$ converges to 0, we can get a subsequence (y_n) such that $\|y_n\|_\gamma < \frac{1}{2^n}$ for all $n > 2$. But $\|y_n\|$ does not converge to 0, so there is $k > 0$ and a subsequence, we may assume (z_n), such that $\|z_n\| > k$ for all $n > 2$.

Note that we can do this process while keeping $z_1 = x_1$, and we would get

$$\textstyle\sum_{n=1}\|z_n\|_\gamma = \|x_1\|_\gamma + \sum_{n=2}\|z_n\|_\gamma < \|x_1\|_\gamma + 1 < \infty.$$

Then we will assume that our original sequence (x_n) has already that form.

Claim I There is $\delta > 0$ and a subsequence, $(y_n) = (x_{m_n})$, such that for all integer $n > 1$,

$$\|y_n - u\| \geq \delta \text{ for all } u \in A_{m_{n-1}},$$

where

$$A_n = \{\sum_{i=1}^{n} a_i x_i : |a_i| \leq 1\}.$$

Note that (x_n) cannot have a σ-Cauchy subsequence by Lemma 3.16 i). Then $Z = \{x_n : n \in \mathbf{N}\}$ is not σ-totally bounded, i.e. there is an $\epsilon > 0$ such that for any $\delta \leq \epsilon$ and any finite number of sets of the form

$$B_x(\delta) = \{y \in Y : \|x - y\| < \delta\}$$

will not cover Z.

We let δ be smaller than $\epsilon/2$. For all σ-compact set K, there is a finite amount of sets of the form $B_{p_i}(\delta)$ with $p_i \in K$ such that $K \subset \cup_{i=1}^n B_{p_i}(\delta) \subset \cup_{i=1}^n B_{p_i}(\epsilon)$. Since Z is not σ-totally bounded, there is infinitely many $x \in Z$, such that

$$x \notin \cup_{i=1}^n B_{p_i}(\epsilon).$$

Therefore, for all $N > 0$ there is $m > N$ such that for all $u \in K$,

$$\|x_m - u\| \geq \epsilon/2 \geq \delta.$$

For every $n \in \mathbb{N}$, the set A_n defines a σ-compact set. So we can get the sequence of the form of the claim.

Note that δ can be taken such that $\delta < \|x_1\|$, and we can define the sequence (y_n) as in Claim I keeping $y_1 = x_1$. By definition of the sets A_n, we have that for all $|t_i| \leq 1$ and $|t_n| = 1$,

$$\textstyle\sum_{i=1}^{n-1}(-t_n^{-1}t_iy_i) \in A_{m_{n-1}},$$

and hence

$$\|\sum_{i=1}^{n} t_iy_i\| = \|t_ny_n - \sum_{i=1}^{n-1}(-t_iy_i)\| \geq \|y_n - \sum_{i=1}^{n-1}(-t_n^{-1}t_iy_i)\| \geq \delta.$$

For such a sequence, it is also satisfied that:

Claim II For every $n \in \mathbb{N}$ there is $m_n > n$ so that for any $r \geq m_n$, $|t_i| \leq 1$ for $1 \leq i < n$ and $m_n \leq i \leq r$ and $|t_n| = 1$ we have that

$$\|\sum_{i=1}^{n} t_iy_i + \sum_{i=m_n}^{r} t_iy_i\| \geq \delta/2.$$

Otherwise, there would be n such that for all $m \geq n$, there is $r(m)$,

$$w_m = \textstyle\sum_{i=1}^{n} t_{i,m}y_i \text{ and } v_m = \sum_{i=m}^{r(m)} t_{i,m}y_i$$

with $|t_{i,m}| \leq 1$ for $1 \leq i < n$ and $m \leq i \leq r$ and $|t_{n,m}| = 1$ such that

$$\|w_m + v_m\| < \delta/2.$$

The set A_n is σ-compact, and since $w_m \in A_n$ for all m, the sequence (w_n) contains a σ-convergent subsequence to a point w. The topology σ is Hausdorff, so A_n is σ-closed. Hence $w \in A_n$ and there is $|\tilde{t_i}| \leq 1$ for $1 \leq i < n$ and $|\tilde{t_n}| = 1$ such that

$$w = \sum_{i=1}^{n} \tilde{t_i}y_i.$$

Note that (v_m) converges to the origin in γ, since

$$\|v_m\|_\gamma = \|\sum_{i=m}^{r(m)} t_{i,m}y_i\|_\gamma \leq \sum_{i=m}^{r(m)}\|y_i\|_\gamma,$$

and

$$\sum \|y_n\|_\gamma \le \sum \|x_n\|_\gamma < \infty.$$

Since $\| \quad \|$ is lower semicontinuous with respect to γ, we have that

$$\|w\| = \|\sum_{i=1}^n \tilde{t_i} y_i\| \le \inf\{\|w_m + v_m\|\} \le \delta/2,$$

contradictory to the choice of (y_n).

Hence we can take a subsequence $(z_n) = (y_{m_n})$ starting in $z_1 = y_1 = x_1$, formed recursively by $m_{m_{n-1}}$ defined as in Claim II. In particular, for all $t_1 \cdots t_n$ such that $M = \max\{t_i\} \ge 1$, we have that

$$\|\sum_{i=1}^n t_i z_i\| \ge M \|\sum_{i=1}^n (t_i/M) z_i\| \ge M\frac{\delta}{2} \ge \frac{\delta}{2}.$$

This selection of z_n implies that they are linearly independent, in fact:

Claim III The sequence (z_n) is M-basic and strongly regular.

We consider the linear span of the elements in (z_n) and the linear functionals l_n, such that $l_n(z_m) = \mathbb{1}_{\{n\}}(m)$. By linearity, we have that for all x in the linear span such that $\|x\| \le \delta/2$, we have that $|l_n(x)| \le 1$ for all n so the family l_n is equicontinuous. Then for all n, l_n can be extended to the σ-closed linear span, E, and $l_n \in E^*$.

Now, let (u_n) be any σ-Cauchy sequence in E such that $l_i(u_n)$ converges to 0 for all i. For all u_n we can get v_n in the linear span such that $\|u_n - v_n\| \le 1/n$. The sequence (v_n) is therefore also a σ-Cauchy sequence such that $l_i(v_n)$ converges to 0 for all i. The family of functions l_i is equicontinuous, so there is $K > 0$ such that for all n

$$\sup\{|l_i(v_n)| : i \in \mathbb{N}\} \le K.$$

Then,

$$\|v_n\|_\gamma = \|\sum_{i=1}^\infty l_i(v_n) z_i\|_\gamma \le \sum_{i=1}^\infty \|l_i(v_n) z_i\|_\gamma.$$

Since $l_i(v_n)$ converges to 0 for all n and $\sum_{i=1}^\infty \|z_i\|_\gamma < \infty$ we have that v_n is γ-convergent to 0. The sequence (v_n) is also σ-Cauchy, so by Lemma 3.16 i), is σ-convergent to 0, and so is (u_n). That concludes the proof that the sequence (z_n) is M-basic and strongly regular. $\square$

Observation 3.18. This Theorem seems to give the background necessary to prove Theorem 3.12, letting σ be the original topology and γ the Mackey topology. The problem is that there is not always a 0-basis of the original topology consisting of Mackey-closed sets. However, this theorem enables us to make a construction to prove Theorem 3.11, stated again and that we proceed to prove.

Theorem. Let X be an F-space with induced topology σ, and let α be a Hausdorff vector topology on X weaker than σ.

Let (x_n) be a sequence in X such that $x_1 \neq 0$ and such that (x_n) converges to the origin in α but not on σ. Then there is a subsequence of (x_n), (z_n) starting in x_1 such that (z_n) is a strongly regular M-basic.

Proof. We can again consider that X is the closed linear span of (x_n) in σ.

Let γ be the strongest vector topology weaker than σ such that (x_n) converges to the origin in γ. Therefore a σ-continuous function η would be γ-continuous if and only if $\eta(x_n)$ converges to 0. As a consequence of the definition of γ, $\alpha \subset \gamma$ so it is also a Hausdorff topology.

Now, we let ρ be the vector topology with basis all γ-closed σ-neighbourhoods of the origin. Then:

Claim I ρ is metrizable.

Let $\mathcal{F}$ be a countable basis of σ-neighbourhoods of the origin. For any ρ-neighbourhood of the origin U, there is $V \in \mathcal{F}$ such that $V \subset U$ and then, the γ-closure of V is contained in U, since U is γ-closed. Therefore the collection of the γ-closure of the sets in $\mathcal{F}$ is a (countable) basis for ρ.
Let x be a point of the intersection of all the γ-closures of the sets in $\mathcal{F}$. That implies that for all U γ-neighbourhood of x and $V \in \mathcal{F}$, $U \cap V \neq \emptyset$. The only point in the intersection of all sets in V is the origin, so it must be in U. Since γ is Hausdorff, x must be the origin. Hence ρ is also Hausdorff and by Corollary 1.25, metrizable.

For the topology γ there is a 0-basis consisting of γ-closed sets. Hence $\gamma \subset \rho \subset \sigma$ and ρ is γ-polar. In fact,

Claim II $\gamma \neq \rho$.

Suppose they are equals, and consider the identity map from (X, σ) to (X, γ), which is continuous since $\gamma \subset \sigma$. For all σ-neighbourhood of the origin V, the γ-closure of V is a neighbourhood of the origin in ρ. If $\gamma = \rho$ then the identity map is an isomorphism and $\gamma = \sigma$, contradictory since (x_n) converges to the origin in γ but does not in σ.

Then, (x_n) does not converge to the origin in ρ. By Theorem 3.17, there is a subsequence of (x_n), (z_n) M-basic and strongly regular in (X, ρ). We are going to prove that it is also in (X, σ).

Claim III (z_n) is M-basic and strongly regular in (X, σ)

The biorthogonal functionals (z_m^*) associated with the sequence (z_n) are ρ-equicontinuous. Then for all $\epsilon > 0$, there is a ρ-neighbourhood of the origin U such that

$$|z_m^*(x)| < \epsilon \text{ for all } x \in U \text{ and all } m \in \mathbb{N}.$$

Since $U \in \rho \subset \sigma$, we have that (z_m^*) is σ-equicontinuous.

Let (u_n) be a σ-Cauchy sequence in X_0 such that $(z_m^*(u_n))$ converges to zero for all m. Hence (u_n) is also ρ-Cauchy, and since (z_n) is M-basic for ρ, (u_n) ρ-converges to the origin. Since X is complete (u_n) is also σ-convergent. We have that ρ is Hausdorff, so it must σ-converge the origin. That proves that (z_n) is a strongly regular M-basic. $\square$

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