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# $C^1$ ISOMETRIC IMBEDDINGS

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## Introduction

The question of whether or not in general a Riemannian manifold can be isometrically imbedded in Euclidean space has been open for some time. The local problem was discussed by Schlaefli [1] in 1873 and treated by Janet [2] and Cartan [3] in 1926 and 1927.

This question comes up in connection with the alternative extrinsic and intrinsic approaches to differential geometry. The historically older extrinsic attitude sees a manifold as imbedded in Euclidean space and its metric as derived from the metric of the surrounding space. The metric is considered to be given abstractly from the intrinsic viewpoint.

això es pot dir també al tfg

This intrinsic approach has seemed the more general, so long as there was no contravening evidence. Now it develops that the two attitudes are equally general, and any (positive) metric on a manifold can be realized by an appropriate imbedding in Euclidean space.

This paper is limited to the construction of $C^1$ isometric imbeddings. It turns out that the $C^1$ case is easier to treat and that surprisingly low dimensional Euclidean spaces can be used. A closed $n$-manifold always has $C^1$ isometric imbeddings in $E^{2n}$. But to get a $C^3$ imbedding of an $n$-manifold with $C^3$ metric I have (as of this writing) needed $1\frac{1}{2}n^2 + 5\frac{1}{2}n$ dimensions. One expects this number to be reduced, but it is clear that there will always be a sharp transition between the $C^1$ case and more differentiable imbeddings. At least $(n^2 + n)/2$ dimensions will be required beyond the $C^1$ case. This many dimensions were used in the analytic local theory.

important

The $C^1$ proof uses a sequence of successive corrections applied to an initial imbedding of the manifold. In this process the first derivatives of the imbedding functions are kept carefully under control but the others grow without bound. So the limit imbedding is $C^1$ but not $C^2$.

The corrections applied are "spiralling" perturbations of the imbedding. Each perturbation is localized to a small area of the imbedding and is supposed to increase the metric in a certain direction, the direction of the "spiralling". By piecing together an infinite sequence of these one ultimately obtains the desired imbedding. This process always acts to increase distances, so we must begin with what we call a "short" imbedding, where all distances measured along paths in the manifold are less than they should be. Beginning with a short imbedding we can hope to "stretch it out" until it is isometric.

We can also obtain a result on isometric immersions from the proof of the imbedding theorem, so this is done. An immersion is defined, as by Whitney [4, 5], to be a non-singular imbedding except that self-intersections are allowed.

## The proofs

Let us begin with a differentiable Riemannian $n$-manifold $M$ with a positive definite continuous metric tensor. We may regard $M$ as a $C^\infty$ manifold with $C^0$ metric, using results of Whitney [4, 5] to justify the upgrading of the differentiability to $C^\infty$. We must start with a "short" $C^\infty$ immersion of $M$ in some $E^k$. In using the term immersion we mean to include imbeddings too. The criterion for the immersion to be short can be given symbolically. Let $g_{ij}$ be the intrinsic metric; let $\{x^i\}$ be internal coordinates in $M$, $1 \leqq i \leqq n$; and let $\{z^\alpha\}$ be the coordinates in the Euclidean space $E^k$. The immersion makes the $z^\alpha$ functions of the $x^i$ and the metric $h_{ij}$ induced by it is

què vol dir amb això :(

$$h_{ij} = \sum_\alpha \frac{\partial z^\alpha}{\partial x^i} \frac{\partial z^\alpha}{\partial x^j} . \tag{1}$$

podem parlar de mètriques induides

The immersion is short if the difference $g_{ij} - h_{ij}$ is always a positive definite matrix.

és aquesta caracterització diferent a la de Sung Jin OH?

Given any immersion of a closed manifold in $E^k$ one obtains a short immersion by simply changing the scale in $E^k$, if necessary. For open manifolds it is not that simple and we have to make a construction, which is described later. For the present we may regard the short $C^\infty$ immersion as given.

The correction process has an infinite sequence of "stages". Each stage is like any other stage. The first stage of correction begins with the initial short immersion, the second begins with the immersion constructed by the first, etc. Each stage has the purpose of modifying the immersion so that the induced metric is closer to the intrinsic metric, but in such a way that the immersion is still short, although less so. Actually each stage should make approximately half the correction still needed at its beginning. For example the metric $\overline{h_{ij}}$ after the first stage should be

$$\overline{h_{ij}} \approx h_{ij} + \tfrac{1}{2}(g_{ij} - h_{ij}) = g_{ij} - \tfrac{1}{2}(g_{ij} - h_{ij}) \tag{2}$$

and the remaining error would be

$$g_{ij} - \overline{h_{ij}} \approx \tfrac{1}{2}(g_{ij} - h_{ij}). \tag{3}$$

The errors involved in a stage are the reason for the infinite number of stages. Each succeeding stage must compensate the errors left from the one it follows. And because the stage process we use happens to be limited to positive changes in the metric we must leave over for each stage a portion of the originally needed positive metric correction. In this way we can insure that each stage has a positive metric change to accomplish.

Each stage is in turn divided into a large number of "steps". A step affects only a local part of the immersion and produces an increase in the metric in only one direction (approximately). In order to divide a stage into steps we must have a certain type of covering of the manifold by neighborhoods. Each (closed cell) neighborhood should have $n$ $C^\infty$ non-singular parameters. No neighborhood should meet more than a finite number of the others. Every point of $M$ should be interior to at least one neighborhood. We need not detail the rather standard construction of such a neighborhood system.

Associated with each neighborhood, say $N_p$, we need a $C^\infty$ weighting function $\varphi_p$ which is positive interior to $N_p$ and zero on the boundary and outside $N_p$. These functions should be so balanced that at any point the sum of all the $\varphi_p$ is unity. This is also standard. These are used to distribute the correction "load" of a stage among the neighborhoods.

Within a neighborhood $N_p$ the "load" is further subdivided into parts to be handled by the "steps" associated with the neighborhood. Let

$$\delta_{ij} = g_{ij} - h_{ij} \tag{4}$$

be the amount by which the metric still needs to be increased after a number of stages have been performed, yielding the metric $h_{ij}$. At the next stage we try to increase the metric by approximately $\frac{1}{2}\delta_{ij}$. In neighborhood $N_p$ we try to approximate the increase $\frac{1}{2}\varphi_p \delta_{ij}$. To do this we must first approximate $\delta_{ij}$ by a $C^\infty$ positive definite tensor $\beta_{ij}$ within $N_p$. The accuracy needed in this approximation will be discussed later.

The division of the correction assigned to $N_p$ into steps depends on expressing $\frac{1}{2}\varphi_p\beta_{ij}$ as a special sort of sum:

$$\tfrac{1}{2}\varphi_p \beta_{ij} = \sum_\nu a_\nu \frac{\partial\psi^\nu}{\partial x^i} \frac{\partial\psi^\nu}{\partial x^j}. \tag{5}$$

Here the $x^i$ are the local parameters of $N_p$, the $a_\nu$ should be non-negative $C^\infty$ functions, and the $\psi^\nu$ are to be a finite number of linear functions of the $x^i$. Each step will correspond to one of the $\psi^\nu$. Observe that (5) could not be satisfied if $\beta_{ij}$ were not positive definite within $N_p$.

LEMMA 1. *Equation* (5) *can be satisfied as required above.*

PROOF. The positive definite symmetric matrices of rank $n$ form an $\frac{1}{2}n(n+1)$ dimensional cone. We need a covering of this cone by open simplicial neighborhoods where the simplices lie entirely within the cone and are geometrical simplices, not merely topological. Each point of the cone must be interior to one of these simplices. And this covering must be "uniformly star-finite", which means that no point should be in more than some fixed number $W$ of the simplices.

It is a well known topologico-geometrical fact that such a covering can be obtained and that how large $W$ needs to be depends only on the dimensionality of the linear space involved. This dimensionality is here $\frac{1}{2}n(n+1)$ so we may regard $W$ as a function of $n$.

A matrix represented by a point interior to a simplex can be regarded as a sum with positive coefficients of the matrices represented by the vertices of the simplex. We want to set up a system by which each matrix of the cone is assigned a representation as such a weighted sum of matrices corresponding to vertices of the simplices of the covering. Let the sets of coefficients

$$\begin{array}{l} C_{1,1}\,;\quad C_{1,2}\,;\quad C_{1,3}\ \cdots \\ C_{2,1}\,;\quad C_{2,2}\,;\ \cdots \\ \cdot \\ \cdot \\ \cdot \\ C_{q,1}\,;\quad C_{q,2}\,;\ \cdots \end{array} \tag{6}$$

be the coefficients for $q$ different representations of a matrix which is interior to $q$ of the covering simplices as a weighted sum of vertex matrices. We define new coefficients as

$$C^*_{\mu,\nu} = \frac{C_{\mu,\nu} \exp\{-\sum_\sigma 1/C_{\mu,\sigma}\}}{\sum_\rho \exp\{-\sum_\rho 1/C_{\rho,\sigma}\}} \tag{7}$$

where an exponential is to be regarded as zero if any of the coefficients appearing in the exponent are either zero or undefined. This is just a simple $C^\infty$ weighting device analogous to that used (the $\varphi_p$) for the neighborhoods. Now the matrix is a sum with positive coefficients of all the matrices which correspond to vertices of simplices containing it. Also the coefficients are $C^\infty$ functions of the matrix as it moves throughout the cone. For any particular matrix at most $[\frac{1}{2}n(n+1)+1]$ $W$ of the $C^*_{\mu,\nu}$ can be non-zero.

Now consider $\beta_{ij}$ as defining as $C^\infty$ mapping of $N_p$ into this cone. Then the coefficients $C_{\mu,\nu}$ will become $C^\infty$ functions on $N_p$ and we can write

$$\beta_{ij} = \sum\nolimits_{\mu,\nu} C^*_{\mu,\nu}\, M_{(\mu,\nu)ij} \tag{8}$$

where each $M_{(\mu,\nu)ij}$ is a matrix at a vertex of one of the simplices. For each vertex matrix $M_{(\mu,\nu)ij}$ we can obtain a set of $n$ unit orthogonal eigenvectors $\{V_r\}$ and $n$ associated positive eigenvalues $v_r$. Then if $\psi_r$ is for each $r$ the linear function of the local parameters for which $\sqrt{v_r}\, V_r$ is the gradient vector we shall have

$$M_{(\mu,\nu)ij} = \sum_r \frac{\partial \psi_r}{\partial x^i} \frac{\partial \psi_r}{\partial x^j}. \tag{9}$$

Now all we need do is to put (8) and (9) together and multiply the coefficients by $\frac{1}{2}\varphi_p$ and we have satisfied (5) as desired. Also this has been done in such a way that there are never more than $n[\frac{1}{2}n(n+1)+1]$ $W$ of the coefficients $a_\nu$ of (5) that are simultaneously non-zero. Furthermore only a finite number of coefficients $a_\nu$ will be involved with the neighborhood, because $N_p$ maps into a compact set in the (open) cone of matrices.

### The normal fields

In performing a step for a neighborhood $N_p$ we need two $C^\infty$ unit orthogonal vector fields normal to the immersion of $N_p$. If $\zeta^\alpha$ and $\eta^\alpha$ denote these fields

entenc que això és per construcció, i que és el que canvia kuiper? Sí we should have

$$\sum_\alpha (\zeta^\alpha)^2 = 1 = \sum_\alpha (\eta^\alpha)^2 \qquad \text{(Unit.)} \tag{10}$$

$$\sum_\alpha \zeta^\alpha \eta^\alpha = 0 \qquad \text{(Orthogonal)} \tag{11}$$

$$\sum_\alpha \zeta^\alpha \frac{\partial z^\alpha}{\partial x^i} = 0 = \sum_\alpha \eta^\alpha \frac{\partial z^\alpha}{\partial x^i} \qquad \text{(Normal)} \tag{12}$$

and they should be represented by $C^\infty$ functions in $N_p$.

If the immersion of $N_p$ is $C^\infty$ and non-singular before this step is performed one can prove that such fields exist in several ways. We need to assume that the dimensionality of $E^k$ satisfies $k \geqq n + 2$. And the neighborhood should be a closed cell, of course. The existence of the fields follows from a general theorem on fibre bundles [see Steenrod [6] Th. 6. 7 page 25, Cor. 11.6 page 5.3]. Also they could be obtained by orthogonal propagation.

### The step device

Each step associated with a stage and a neighborhood $N_p$ is intended to give a metric change approximately equal to one term of the sum in equation (5). For the step associated with $a_\nu$ and $\psi^\nu$ we define perturbed immersion functions $\bar{z}^\alpha$ by the equation

$$\boxed{\bar{z}^\alpha = z^\alpha + \zeta^\alpha \frac{\sqrt{a_\nu}}{\lambda} \cos \lambda\, \psi^\nu + \eta^\alpha \frac{\sqrt{a_\nu}}{\lambda} \sin \lambda \psi^\nu,} \tag{13}$$

where $\lambda$ is a large positive constant that we shall use to control the accuracy of the effect. The metric change will be

$$\sum_\alpha \frac{\partial \bar{z}^\alpha}{\partial x^i} \frac{\partial \bar{z}^\alpha}{\partial x^j} - \sum_\alpha \frac{\partial z^\alpha}{\partial x^i} \frac{\partial z^\alpha}{\partial x^j}. \tag{14}$$

We can avoid computing this in detail because many of the terms would contain $1/\lambda$ or $1/\lambda^2$. These will be made small by using $\lambda$ sufficiently large. Such a term would be of the form

$$\begin{bmatrix} 1/\lambda \\ \text{or} \\ 1/\lambda^2 \end{bmatrix} \cdot \begin{bmatrix} \sin \lambda\psi^\nu \text{ or } \cos \lambda\psi^\nu \\ \text{or a product of these} \\ \text{or a square of either} \end{bmatrix} \cdot f \tag{15}$$

where $f$ would not involve $\lambda$ and would be a $C^\infty$ function zero on the boundary of $N_p$ and outside $N_p$. Such a term approaches zero uniformly as $\lambda \to \infty$.

Another type of term contains only first derivatives of $z^\alpha$. One such term comes from the first part of (14) and cancels with the subtracted second part.

Any other term must come from the differentiation of the trigonometric functions appearing in (13). When $\cos \lambda\psi^\nu$ or $\sin \lambda\psi^\nu$ is differentiated we get a $\lambda$ in the numerator to cancel the one in the denominator. Some of these involve only one trigonometric function and a derivative of $z^\alpha$. The sum of these terms is

$$(16) \qquad \sum_\alpha \frac{\partial z^\alpha}{\partial x^i} \zeta^\alpha \sqrt{a_\nu}(-\sin \lambda\psi^\nu) \frac{\partial x^\nu}{\partial x^j} + \sum_\alpha \frac{\partial z^\alpha}{\partial x^i} \eta^\alpha \sqrt{a_\nu}(\cos \lambda\psi^\nu) \frac{\partial \psi^\nu}{\partial x^j}$$

plus two other summations with $i$ and $j$ interchanged. Equation (12), expressing the normality of $\zeta^\alpha$ and $\eta^\alpha$ to the immersion, makes each of the four summations vanish.

The remaining terms will contain two trigonometric functions or a square of one. Those that contain $(\cos \lambda\psi^\nu)(\sin \lambda\psi^\nu)$ will also contain $\zeta^\alpha \eta^\alpha$ and the summation over $\alpha$ will make them cancel by virtue of the orthogonality of $\zeta^\alpha$ and $\eta^\alpha$, equation (11). The terms with trigonometric squares collect to

$$(17) \qquad \sum_\alpha (\zeta^\alpha)^2 a_\nu (\sin^2 \lambda\psi^\nu) \frac{\partial \psi^\nu}{\partial x^i} \frac{\partial \psi^\nu}{\partial x^j} + \sum_\alpha (\eta^\alpha)^2 a_\nu (\cos^2 \lambda\psi^\nu) \frac{\partial \psi^\nu}{\partial x^i} \frac{\partial \psi^\nu}{\partial x^j}.$$

Now since $\zeta^\alpha$ and $\eta^\alpha$ are unit vectors, see equation (10), and since $\sin^2 + \cos^2 = 1$, expression (17) may be rewritten as

$$(18) \qquad a_\nu \frac{\partial \psi^\nu}{\partial x^i} \frac{\partial \psi^\nu}{\partial x^j}.$$

This is the metric change we were trying to approximate. So we see that the error is controllable by the size of $\lambda$; it is uniformly $0(1/\lambda)$.

La clau aquí seria comparar amb el que fa SJO al baby nash theorem, que crec que és simplement aquesta part.

## Size of immersion change from a step

We need to know how much the immersion functions and their first derivatives change when the immersion is perturbed by a step. One observes immediately that the perturbation itself, $\bar{z}^\alpha - z^\alpha$, is $0(1/\lambda)$. And for the change in first derivatives we have easily

$$(19) \qquad \left| \frac{\partial \bar{z}^\alpha}{\partial x^i} - \frac{\partial z^\alpha}{\partial x^i} \right| \leqq 2\sqrt{a_\nu} \left| \frac{\partial \psi^\nu}{\partial x^i} \right| + 0(1/\lambda).$$

This estimate is not so useful directly, but we can use it to derive a useful estimate for the total change in the first derivatives of the immersion functions due to all the steps associated with a stage and a neighborhood.

To obtain this estimate we consider a diagonal entry of the matrices equated in (5):

$$(20) \qquad \tfrac{1}{2}\varphi_p \beta_{ii} = \sum_\nu a_\nu \left( \frac{\partial \psi^\nu}{\partial x^i} \right)^2 = \sum_\nu \left( \sqrt{a_\nu} \left| \frac{\partial \psi^\nu}{\partial x^i} \right| \right)^2.$$

Now remember that in proving Lemma 1 we arranged that at most

$$n[\tfrac{1}{2}n(n+1) + 1]\, W$$

of the $a_\nu$ could be simultaneously non-zero. Using this fact and the Schwarz inequality we obtain

$$\begin{aligned} \sum_\nu \sqrt{a_\nu} \left| \frac{\partial \psi^\nu}{\partial x^i} \right| &\leqq \{n[\tfrac{1}{2}n(n+1)+1]W\tfrac{1}{2}\phi_p\beta_{ii}\}^{\frac{1}{2}} \\ &\leqq \{K\tfrac{1}{2}\phi_p\beta_{ii}\}^{\frac{1}{2}} \\ &\leqq \{K\beta_{ii}\}^{\frac{1}{2}} \end{aligned} \tag{21}$$

where $K$ is a constant depending only on $n$.

Now we combine (21) and (19) to bound the total change in the derivatives from the steps of $N_p$ and this stage:

$$\begin{aligned} &\left| \left(\frac{\partial z^\alpha}{\partial x^i}\right)_{\text{after}} - \left(\frac{\partial z^\alpha}{\partial x^i}\right)_{\text{before}} \right| \\ &\qquad \leqq 2\sqrt{K\beta_{ii}} + 0(1/\lambda_1) + 0(1/\lambda_2) + \cdots + 0(1/\lambda_\nu) + \cdots \end{aligned} \tag{22}$$

This assumes of course that all the steps could be performed properly. We must later consider the general question of a breakdown of the process.

### Global considerations and convergence

We have now already treated the actual mechanism of the process but we must verify its operation in the large, consider the accuracies needed at various points, consider convergence questions, put in various points of rigor, etc. The only difficulty in all this is in forming a clear picture.

Some rule must exist to determine how large $\lambda$ need be taken at each step. Let $B_1$ be the maximum permissible error in the approximation to the metric change $a_\nu(\partial\psi^\nu/\partial x^i)(\partial\psi^\nu/\partial x^j)$. This is to apply to all $i, j$ pairs and all points in the relevant neighborhood. Let $B_2$ similarly be the maximum permissible value of the $0(1/\lambda)$ part of the change $(\partial\bar{z}^\alpha/\partial x^i) - (\partial z^\alpha/\partial x^i)$ of first derivatives. And let $B_3$ be the bound on the change $\bar{z}^\alpha - z^\alpha$.

Any triad $B_1$, $B_2$, $B_3$ of bounds for a step can be satisfied by making $\lambda$ sufficiently large. So we need hereafter only consider what values to choose (systematically) for the $B_1$, $B_2$, $B_3$ of the steps to insure convergence, etc.

First consider the $B_1$'s. To decide on the metric error permissible in a step we must know how much is acceptable for the neighborhood and stage with which the step is associated. Equation (4)

$$\delta_{ij} = g_{ij} - h_{ij} \tag{4}$$

represents the correction $\delta_{ij}$ still needed at the beginning of the stage in terms of the intrinsic metric $g_{ij}$ and the metric $h_{ij}$ induced by the immersion as it stands at this point. $\delta_{ij}$ should be positive definite and continuous, as is $g_{ij}$, and $h_{ij}$ should be $C^\infty$ and positive definite. Notice that this requirement that the metric induced be positive definite implies that the immersion inducing it (which is supposed to be $C^\infty$) is non-singular.

Now in the stage being considered, which follows the attainment of the induced metric $h_{ij}$, we are to approximate a metric increase $\frac{1}{2}\delta_{ij}$. This will give us a new induced metric $h'_{ij}$ with

$$h'_{ij} \approx h_{ij} + \tfrac{1}{2}\delta_{ij}\,, \text{ or } h'_{ij} = g_{ij} - \tfrac{1}{2}\delta_{ij} + e_{ij} \tag{23}$$

where $e_{ij}$ stands for the error. We can also write

$$\delta'_{ij} = \tfrac{1}{2}\delta_{ij} - e_{ij}\,. \tag{24}$$

Two objectives are important. First, $\delta'_{ij}$ must be positive definite, and second we must make $e_{ij}$ small enough to assure convergence. We can use the maximum of the absolute values of its components as a measure of the size of a tensor. If in $N_p$ we require

$$\begin{bmatrix}\text{max size}\\ \text{in } N_p \text{ of } e_{ij}\end{bmatrix} \leqq \frac{1}{6}\begin{bmatrix}\text{min size}\\ \text{in } N_p \text{ of } \delta_{ij}\end{bmatrix} \tag{25}$$

then we obtain

$$\begin{bmatrix}\text{max size}\\ \text{in } N_p \text{ of } \delta'_{ij}\end{bmatrix} \leqq \frac{2}{3}\begin{bmatrix}\text{max size}\\ \text{in } N_p \text{ of } \delta_{ij}\end{bmatrix}. \tag{26}$$

This will take care of the metric convergence problem. To make $\delta'_{ij}$ positive definite we employ an existential argument. In each compact $N_p$ there will be a sufficiently small $\varepsilon_p$ such that requiring in $N_p$

$$[\text{size } e_{ij}] \leqq \varepsilon_p \tag{27}$$

assures that $\delta'_{ij} = \frac{1}{2}\delta_{ij} - e_{ij}$ is positive definite in $N_p$.

Now requirements (25) and (27) must be related to requirements on the steps associated with the neighborhoods and the stage, remembering that the neighborhoods overlap so that errors from a step formally associated with one neighborhood can have effects in any overlapping neighborhoods also.

For any size limitation imposed in an $N_p$ and for any overlapping $N_q$ there will be a bound in $N_q$ which is sufficient to ensure satisfying the $N_p$ limitation in the overlap. So if $N_p$ meets $\sigma$ neighborhoods, including itself, we can divide the $N_p$ limits of (25) and (27) by $\sigma$ and impose on $N_p$ and the overlapping neighborhoods the derived limits corresponding to these reduced limits of $N_p$. Now each neighborhood will have an enlarged but still finite set of size limitations on $e_{ij}$. These new bounds, however, refer to the total error due to steps directly associated with the neighborhood whereas the old ones referred to the total error accumulated from all steps affecting the metric in $N_p$. Notice the use in this argument of the star-finite property of the covering.

For brevity we can use $d_{ij}$ for the error due to steps associated with $N_p$ and use $\varepsilon_p^*$ for the minimum of the various size limitations on $d_{ij}$. Then we want

$$[\text{size } d_{ij}] \leqq \varepsilon_p^*\,. \tag{28}$$

In the steps there are two sources of error. One is the preliminary approximation of $\delta_{ij}$ by a $C^\infty$ tensor $\beta_{ij}$. And the other is the individual errors of the steps. So we require size $(\delta_{ij} - \beta_{ij}) \leqq \varepsilon_p^*$. This makes size $(\frac{1}{2}\varphi_p\delta_{ij} - \frac{1}{2}\varphi_p\beta_{ij}) \leqq \frac{1}{2}\varepsilon_p^*$. Then for the $B_1$'s of the sequence of steps associated with $N_p$ we stipulate

$$B_{11} \leqq \tfrac{1}{4}\varepsilon_p^*, \quad B_{12} \leqq \tfrac{1}{8}\varepsilon_p^*, \quad B_{13} \leqq \tfrac{1}{16}\varepsilon_p^*, \quad \text{etc.} \tag{29}$$

The added suffix on the $B_1$'s indexes the steps in order. This arrangement makes the total error due to the step errors not more than $\frac{1}{2}\varepsilon_p^*$. So the two sources give together no more than $\varepsilon_p^*$ as we wanted.

### Performability of the steps

We have above assumed that the steps were always performable. The essential requirement for this is that we have a $C^\infty$ non-singular immersion. But if the induced metric is positive definite and the immersion functions are $C^\infty$ the immersion cannot be singular.

All processes we have used employ and yield $C^\infty$ functions, so the question reduces to the positiveness of the metric. We can consider this inductively. Each step begins with an immersion producing a positive metric (inductive hypothesis) and has the purpose of effecting approximately the increase $a_\nu(\partial\psi^\nu/\partial x^i)(\partial\psi^\nu/\partial x^j)$ in the metric. This increase is itself a non-negative tensor. Thus the only question stems from the error of the step. And this can be made arbitrarily small by control of the $\lambda$ involved. So let us assume that the $\lambda$ of a step is always taken large enough to give a positive metric after the step. One should note that until the end of the process any point of the manifold will have been affected by only finitely many steps.

### General organization

This is put in to clarify the general picture of the process.

The process consists of a sequence of stages. Each is like any other, but the first requires a $C^\infty$ short immersion to begin with. After each stage we have a $C^\infty$ immersion which is still short but for which the metric error is not more than $\frac{2}{3}$ of what it was before the stage (equation 26).

It is best to use the same covering by neighborhoods for all the stages. This helps in measuring convergence.

Each stage is divided into a sequence of steps. First the correction is apportioned among the $N_p$ by the weighting functions $\varphi_p$. Then the correction assigned to each neighborhood is divided into portions to be accomplished by the individual steps (equation 5).

The important thing to keep clear about the steps of a stage is that they must be performed serially rather than simultaneously. We may assume that at each stage the component steps are arbitrarily enumerated and performed in that order. For an open manifold the sequence of steps of a stage would be infinite, but any compact portion would be effected by only a finite number of them. So this particular infiniteness does not raise any convergence problems.

### Convergence of the immersion

The $\frac{2}{3}$ result of (26) assures the convergence of the metric but does not show that the sequence of immersions produced converges to a $C^1$ limit. For the convergence of the immersion functions we require that the $B_3$ of step $r$ (in performance order) of stage $s$ is less than $2^{-r-s}$. This majorizes the changes in the immersion functions by a convergent series.

We can use this same series, $2^{-r-s}$, on the $B_2$'s to take care of the $0(1/\lambda)$ part of the change in the first derivatives (equations 19, 22). The irreducible component of this change is estimated by (22) for the cumulative effect of all the steps of a neighborhood. Remember that the $K$ of (22) depends only on the dimension $n$. So we must show that the series formed by the values of $2\sqrt{K\beta_{ii}}$ for the different stages converges. It should converge for each $i$ and uniformly in each $N_p$.

The tensor $\beta_{ij}$ approximates $\delta_{ij}$ and $\delta_{ij}$ decreases by $\frac{1}{3}$ or more each step. We must merely show that $\beta_{ij}$ decreases similarly. If we require that $\beta_{ij}$ approximates $\delta_{ij}$ so closely that

$$\frac{9}{10} \leqq \frac{\text{max size in } N_p \text{ of } \beta_{ij}}{\text{max size in } N_p \text{ of } \delta_{ij}} \leqq \frac{9}{8} \tag{30}$$

as a general rule to be applied at every stage for each $N_p$, then we can conclude from (26) and (30) that $\beta'_{ij}$ (the $C^\infty$ approximation to the metric deficiency $\delta'_{ij}$ at the next stage) will satisfy the inequality

$$\begin{bmatrix} \text{max size of } \beta'_{ij} \\ \text{in } N_p \end{bmatrix} \leqq \frac{5}{6} \begin{bmatrix} \text{max size of } \beta_{ij} \\ \text{in } N_p \end{bmatrix} \tag{31}$$

which means that $\beta_{ij}$ will decrease geometrically as the stages pass.

Since the derivative change depends on $2\sqrt{K\beta_{ii}}$ it will be dominated by a geometrical series with term ratio $\sqrt{5/6}$. So the first derivatives of the immersion functions converge uniformly in each neighborhood.

Summing up, we have a sequence (with the stages) of $C^\infty$ functions which converge uniformly and the first derivatives converge uniformly. Hence they converge to $C^1$ functions and the expression (1) giving the induced metric converges to its value for the limit immersion. In view of (26) the limit is an isometric immersion.

### Isometric imbeddings

The case of imbeddings can be handled by modifying the immersion proof above. We apply the same process but we begin with an imbedding and if the manifold is open the limit set of the imbedding (if any) must not meet the imbedding. It happens that all we need do is to carefully control the size of the perturbation (i.e., control $|\bar{z}^\alpha - z^\alpha|$ ) at each step and self-intersections can be prevented.

The problem divides into two parts: First, to prevent any step of the process from producing a self-intersection; and second, to prevent self-intersections in the limit.

Concerning the limit set, if any, the restriction (involving $2^{-r-s}$) we imposed above on the $B_3$'s of the steps has the effect of holding the limit set fixed throughout the process.

Now consider a single step. The essential fact is that the perturbation is normal to the imbedding, i.e.

$$\sum_\alpha (\bar{z}^\alpha - z^\alpha) \frac{\partial z^\alpha}{\partial x^i} = 0. \tag{32}$$

This follows from the normality of $\zeta^\alpha$ and $\eta^\alpha$. As we increase $\lambda$ we decrease the size of the perturbation $\bar{z}^\alpha - z^\alpha$ indefinitely. We must show that if it is made small enough it cannot produce a self-intersection. We also do not want the imbedding to touch the limit set. Suppose the step is associated with neighborhood $N_p$.

Let $M$ be a closed cell on the imbedding that is slightly larger than $N_p$ and contains $N_p$ in its interior. $M$ will have an open surrounding $H$ which is sufficiently close to $M$ that a point $P$ in $H$ cannot have two different line segments within $H$ normal to $M$ and ending at $P$. That is, $P$ cannot have two perpendiculars to $M$ lying within $H$. This is a general property of $C^2$ imbeddings. For reference see Cairns [7] or Lemma 1 of Nash [8].

Now let $Q$ be the union of the limit set with the part of the imbedding which is not in the interior of $M$. Then $Q$ will be a closed set. Clearly a sufficiently small perturbation affecting $N_p^*$ will not meet $Q$. Also, if small enough, it will lie within $H$ so that two points of $M$ cannot be brought together (normality of the perturbation). This takes care of all the possibilities.

The argument above shows that we can prevent a step from introducing a self-intersection or a contact with the limit set assuming that neither of these existed before the step. This inductively takes care of the self-intersection problem until the final limit of an infinite number of stages is reached. Remember that until then each neighborhood is affected by but a finite number of steps and that the limit set stays fixed throughout the process.

To consider the problem of self-intersection in the limit let the neighborhoods be enumerated in some way. Then let $S_r$ be the set of all pairs of points from the first $r$ neighborhoods for which the separation distance in the original short imbedding was at least $2^{-r}$. Obviously any point pair will be ultimately included in some $S_r$. At stage $r$ we put all pairs of $S_r$ under protection from self-intersection by requiring that the $B_3$ of step $\mu$(in performance order) of stage $\nu$ be not more than $\varepsilon_r 2^{-\mu-\nu}$, where $\nu \geqq r$, of course. Clearly if $\varepsilon_r$ is taken small enough the points of any pair in $S_r$ cannot move enough to come together in the limit.

So at each stage $r$ we choose $\varepsilon_r$ small enough to prevent point pairs of $S_r$ from coincidence in the limit. Ultimately all point pairs are protected.

### Constructing the initial immersion or imbedding

According to Whitney's results [4, 5] any differentiable $n$-manifold has a non-singular imbedding (which can be analytic if desired) in $E^{2n}$. So for a closed manifold we can take an analytic imbedding in $E^{2n}$ and adjust the scale to make it short.

Obtaining the initial $C^\infty$ short immersion or imbedding for an open manifold requires more work. One sets up a system of neighborhoods $N_p$ like that used before with an associated system of $C^\infty$ weighting functions $\varphi_p$ and with parameters $x_{pi}$. For later convenience we assume the parameters are bounded so that $| x_{pi} | \leqq 1$. Here we need a "uniformly star-finite" system where there is a fixed number $s$ such that no neighborhood touches more than $s$ different neighborhoods including itself. The realizability of this requirement is a standard topological fact.

We first divide the neighborhoods among $s$ classes such that no two neighborhoods which overlap are in the same class. This is done by taking them in order and classifying each in a different class from any of its already classified overlapping neighborhoods, which can occupy at most s $-$ 1 classes.

Next we construct an imbedding in $s(n + 2)$ dimensional space by defining imbedding functions as follows:

$$u_\sigma = \varepsilon_p \varphi_p \tag{33}$$

if the point is in a neighborhood $N_p$ of class $\sigma$, otherwise $u_\sigma = 0$.

$$v_\sigma = \varepsilon_p^2 \varphi_p \tag{34}$$

etc. like $u_\sigma$.

$$w_{\sigma i} = \varepsilon_p \varphi_p x_{pi} \tag{35}$$

if the point is in $N_p$ and $N_p$ is in class $\sigma$, and otherwise zero, where $x_{pi}$ stands for the $i$th parameter of the coordinate system of $N_p$.

The numbers $\varepsilon_p$ are to be a strictly monotone decreasing sequence of constants converging to zero. Observe first that the $s(n + 2)$ functions defined are $C^\infty$. Notice that any two points not interior to the same neighborhoods are distinguished by having different sets of ratios $v_\sigma/u_\sigma$. And two points interior to a common neighborhood are distinguished by the $w_{\sigma i}$. Finally the limit set has to be the origin but no point maps into the origin.

Now we have only to assure that the imbedding is short by proper control of the $\varepsilon_p$. This is done inductively. If the epsilons up to $\varepsilon_p$ have been selected in such a way that the metric already induced on the first $p$ neighborhoods is everywhere short by the factor $\frac{1}{3}(2 - 1/p)$ or better then we can obviously select $\varepsilon_{p+1}$ so that the metric induced on the first $p + 1$ neighborhoods is short by the factor $\frac{1}{3}(2 - (1/p + 1))$ or better. And we can do this in such a way that the epsilons decrease monotonically to zero.

This gives us a short $C^\infty$ imbedding in $E^{s(n+2)}$ with the origin as the limit set. The imbedding plus the origin is an $n$-dimensional set. By the classical process of generic linear projection we can reduce the dimensionality of the surrounding Euclidean space to $2n + 1$ without introducing any singularities or self-intersections, without lengthening the imbedding, and without letting the limit set (the origin) meet the imbedding. For an immersion we can go down to $E^{2n}$.

## General summary

We now state as theorems the results obtained above.

THEOREM 1. *Any closed Riemannian n-manifold has a $C^1$ isometric imbedding in $E^{2n}$.*

THEOREM 2. *Any Riemannian n-manifold has a $C^1$ isometric immersion in $E^{2n}$ and an isometric imbedding in $E^{2n+1}$*

THEOREM 3. *If a closed Riemannian n-manifold has $C^\infty$ immersion or imbedding in $E^k$ with $k \geqq n + 2$ it also has respectively an isometric immersion or imbedding in $E^k$.*

THEOREM 4. *If an open Riemannian n-manifold has a short $C^\infty$ immersion (or imbedding) in $E^k$ with $k \geqq n + 2$ which does not meet its limit set (if any) then it also has an isometric immersion (or respectively imbedding) in $E^k$ of the same type.*

Actually the condition $k \geqq n + 2$ might be replaced by $k \geqq n + 1$. This would come from use of a less easily controlled perturbation process needing only one direction normal to the imbedding.

An interesting illustration (due to Milnor) of these results is given by the $n$-torus when regarded as the metrical product of $n$ circles. Tompkins [9] has shown that this torus has no $C^4$ isometric imbedding in less than $2n$ dimensions. However it is easy to construct a differentiable imbedding in $E^{n+1}$. So there is a $C^1$ isometric imbedding in $E^{n+2}$! Additional negative results on $C^4$ isometric imbeddability have been given by Chern and Kuiper [10].

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