MATHEMATICS

ON C^1-ISOMETRIC IMBEDDINGS. II

BY

NICOLAAS H. KUIPER

(Communicated by Prof. H. FREUDENTHAL at the meeting of September 24, 1955)

Dedicated to Prof. dr. W. van der Woude on the occasion of his 80th birthday

§ 11. *Introduction*

In our first paper ([2] § 1–10) we considered the isometric C^1-imbedding of a Riemannian n-manifold X^n in Euclidean N-space E^N, in case X^n is compact (= every infinite set of points in X^n has a point of accumulation). Now we will present some theorems concerning the case in which X^n is non-compact. Our solution of this problem is obtained in two parts. First we obtain, starting from a C^∞-imbedding of X^n in E^n, a short (in the sense of NASH) imbedding in E^{N+1} (Theorem 2, § 12). Next we prove with much reference to § 1–10, that the existence of a short imbedding of X^n in E^N, $N>n$, implies the existence of a C^1-isometric imbedding of X^n in E^N (theorem 3, § 13). Theorem 4 is the immediate consequence of theorems 2 and 3. In § 14 we apply our results and obtain a C^1-isometric imbedding of the hyperbolic plane H^2 in E^3, which is closed in E^3 [1]).

The following *definition* is needed in the sequel: If $f : X \to f(X)$ is an imbedding of the manifold X in E^N, and $x_i \in X$, $i=1, 2, \ldots$, is a divergent sequence of points in X, whereas $f(x_i)$ $i=1, 2, \ldots$ is convergent in E^N with limitpoint $u \in E^N$, then u is called a *limit point* of the imbedding. We will avoid that the image $f(X)$ meets the limit set, as is necessary for any C^1-imbedding$=$$C^1$-*homeomorphic* map with non-vanishing Jacobian.

§ 12. *Short imbeddings*

Definition: The C^∞-imbedding $f : X \to f(X)$ of a C^∞-Riemannian n-manifold X^n in E^n is called *shortable* if the ratio ds'/ds of the differential of the arclength ds' induced in $f(X)$ and X, *and* the original differential ds of the arclength in X, is uniformly bounded on the space of tangent vectors of X:

$$ds'/ds < K. \tag{12.1}$$

If a shortable imbedding exists then also a *short imbedding* exists, which

[1]) HILBERT [3] proved that a C^4-isometric imbedding of the hyperbolic plane H^2 in E^3 is impossible. BIEBERBACH [4] $n = 2$ and BLANUASA [5] $n \geqslant 2$ obtained isometric imbeddings of H^n in Hilbert space.

can be obtained as in § 2. Hence we may restrict ourselves to the consideration of shortable imbeddings.

Before we give our final result concerning short imbeddings, we present the fairly simple proof of a weaker theorem of NASH ([1] p. 395). The reader who is in a hurry may proceed with theorem 2.

Weak theorem. *Every C^∞-Riemannian n-manifold X^n has a short C^∞-imbedding in Euclidean $2n+1$-space E^{2n+1}, which does not meet its limit set.*

This theorem follows by induction from NASH's result, that X^n has a short imbedding which does not meet its limit set in a Euclidean space of some finite dimension, and the

Lemma: If X^n has a shortable C^∞-imbedding in E^N, $N > 2n+1$, which does not meet its limit set, then it has an analogous shortable imbedding in E^{N-1}.

From now on in this paragraph "Imbedding" will mean: "C^∞-imbedding which does not meet its limit-set".

Proof of the lemma. If $f : X \to f(X)$ is a shortable C^∞-imbedding of X^n into E^N, $N > 2n+1$, then the pointset T of chords and tangents of $f(X)$ has dimension $\leqslant 2n+1 < N$. The complement in E^N of the closure of T is not void and contains the interior of some hypersphere $S = S^{N-1}$, the centre of which is denoted by M. The central projection π with centre M, of $f(X)$ into S^{N-1}, is then a shortable C^∞-imbedding (!) of $f(X)$ in S^{N-1}, because π transforms any arc that does not meet the interior of the sphere S^{N-1} onto an arc that is not longer. Therefore πf is also a shortable imbedding of X into S^{N-1} [2]).

As the dimension of $\pi f(X)$ is $n < N-1$, there exists a point P in S^{N-1} such that $\pi f(X)$ has a distance $\varepsilon > 0$ to P. We next apply the analytic stereographic projection $\varkappa$ with centre P, of S^{N-1} with the exception of P, onto the hyperplane $E^{N-1} \subset E^N$ that is equatorial with respect to the sphere S^{N-1} and the north pole P. The image $\varkappa\pi f(X)$ is bounded in E^{N-1} because $\varepsilon > 0$, and so is the (variable) factor that is the quotient of the arclengths induced in $\varkappa\pi f(X)$ and $\pi f(X)$

$$ds'\,(\varkappa\pi f(X))/ds'\,(\pi f(X)) < \text{some constant}.$$

Then the C^∞-imbedding $X \to \varkappa\pi f(X)$ of X into E^{N-1} is shortable.

Theorem 2. *If a C^∞-Riemannian n-manifold X has a C^∞-imbedding in E^N which does not meet its limit set, and it has if $N = 2n$, then it has a short C^∞-imbedding in E^{N+1} which also does not meet its limit set.*

Proof: Let $f : X \to f(X)$ be a given C^∞-imbedding of the noncompact C^∞-Riemannian n-manifold X in E^N, which does not meet its limit set.

---

[2]) The symbols ψ, φ, η, τ, α used in this paragraph have no relation with the notions for which they were used in earlier paragraphs.

If $f(X)$ is not bounded then we apply the analytic transformation

$$\bar{z}^{\alpha} = \operatorname{arctg} z^{\alpha} \tag{12.2}$$

to $f(X)$. Hence we may and will assume that $f(X)$ is bounded in E^N. We also assume that f is not shortable. We want a transformation of the imbedding under which ds'/ds, defined as in the definition (12.1), is diminished considerably in those points $x \in X$, in which it assumes high values.

To obtain this new imbedding we put E^N into E^{N+2} as the linear space (with respect to preferred orthogonal coordinates $z^1, \dots z^{N+2}$):

$$z^{N+1} = 0, \quad z^{N+2} = 1.$$

Let the given C^∞-imbedding f be given by the $N+2$ real functions of $x \in X$:

$$\{z^1(x), \dots, z^N(x), 0, 1\}.$$

We replace this imbedding first by the imbedding g:

$$\{z^1(x), \dots, z^N(x), \eta(x), 1\}$$

in which $\eta(x)$ will be a positive C^∞-function of $x \in X$, which assumes very small values in points $x \in X$ in which $\max ds'/ds$ has high values. Next we project the image $g(X)$ centrally from $(0, 0, \dots 0)$ onto the unit hypersphere

$$\sum_{\alpha=1}^{N} (z^\alpha)^2 + (z^{N+1} - 1)^2 + (z^{N+2})^2 = 1 \tag{12.3}$$

which has at $(0, 0, \dots 0)$ the tangenthyperplane $z^{N+1} = 0$! The arclength of any arc that is close to this hyperplane $z^{N+1} = 0$, and in $z^{N+2} = 1$, is diminished very much under this projection, which is an analytic transformation of the pointset $z^{N+1} > 0$, $z^{N+2} = 1$ onto the part $z^{N+2} > 0$ of the hypersphere S^{N+1} given by (12.3).

The result of the projection is the C^∞-imbedding h which does not meet its limit set:

$$\{\tau(x)\, z^1(x),\ \dots,\ \tau(x)\, z^N(x),\ \tau(x)\, \eta(x),\ \tau(x)\} \tag{12.4}$$

$$\tau(x) = \frac{2\eta(x)}{1 + (\eta(x))^2 + \sum_{\alpha=1}^{N} (z^\alpha(x))^2}.$$

Now we may hope that the imbedding h is shortable for a suitable choice of the function $\eta(x)$. This choice follows now!

The limit set R of $f(X)$ in E^N is not void according to the assumption following (12.2). Let $\varrho(x)$ be the distance from the point $f(x)$ to R, measured in E^N:

$$\varrho(x) = \text{g.l.b}_{y \in R}\, [\text{distance } (f(x), y)].$$

We define a real function $\alpha(t)$ of the real variable t by

$$\alpha(t) = \max\, \{\max_{\varrho(x) \geqslant t} ds'/ds,\ 1\}.$$

The maximum between brackets is taken over the closed (!) set of all tangent unit vectors at all points x for which $\varrho(x) \geqslant t$. $\alpha(\varrho(x))$ is a continuous function of $x \in X$ which satisfies:

$$\alpha(\varrho(x)) \geqslant \max ds'/ds \ \text{(at } x).$$

We observe that $\lim_{t\to 0} \alpha(t) = \infty$; the level pointsets $\alpha(\varrho(x)) = \text{constant}$ are closed and bounded subsets of E^N.

We chose a C^∞-function $\zeta(x)$ such that

$$1 \leqslant \alpha(\varrho(x)) < \zeta(x) < \alpha(\varrho(x)) + 1 \tag{12.5}$$

$$\zeta(x) \geqslant \max ds'/ds \ \text{(at } x). \tag{12.6}$$

Then also the level pointsets of $\zeta(x)$ are closed subsets of E^N and $d\zeta/ds$ has a maximum on the set of tangent unitvectors with respect to $f(X)$ at those points $x \in X$ for which $\zeta(x) = \zeta$.

We next introduce a C^∞-function $\psi(\zeta_0)$ of one real variable such that:

$$\psi(\zeta_0) > \max \left\{ \zeta_0^2,\ \max_{\zeta(x)=\zeta_0} \left| \frac{d\zeta}{ds} \right| \right\}. \tag{12.7}$$

Finally we define

$$\chi(\zeta) = \int_\zeta^\infty \frac{d\zeta}{\psi(\zeta)},\quad \eta(x) = \chi(\zeta(x)). \tag{12.8}$$

Then the metric $(ds'')^2$ induced by the C^∞-imbedding $h: X \to h(X) \subset S^{N+1}$ so obtained, is seen to be shortable as follows:

$$\begin{aligned}
(ds'')^2 &= (d\tau z^1)^2 + \ldots + (d\tau z^N)^2 + (d\tau\eta)^2 + (d\tau)^2 \\
&= [(z^1)^2 + \ldots + (z^N)^2 + \eta^2 + 1] \left(\frac{d\tau}{ds}\right)^2 ds^2 \\
&\quad + \tau^2 \left[\left(\frac{dz^1}{ds}\right)^2 + \ldots + \left(\frac{dz^N}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2 \right] ds^2 \\
&\quad + 2\tau \frac{d\tau}{ds} \left[z^1 \frac{dz^1}{ds} + \ldots + z^N \frac{dz^N}{ds} + \eta \frac{d\eta}{ds} \right] ds^2 \\
&= O(1) \cdot ds^2,
\end{aligned}$$

because (see 12.5, 6, 7 and 8):

z^α is bounded on $f(X)$;

$$\tfrac{1}{2}\tau(x) \leqslant \eta(x) = \chi(\zeta(x)) \leqslant (\zeta(x))^{-1} \leqslant 1;$$

$$\left|\frac{d\eta}{ds}\right| = \left|\frac{d\chi}{d\zeta} \cdot \frac{d\zeta}{ds}\right| = \left|\frac{1}{\psi} \frac{d\zeta}{ds}\right| \leqslant 1;$$

$$\eta\zeta = \chi(\zeta) \cdot \zeta \leqslant \zeta \int_\zeta^\infty \frac{d\zeta}{\zeta^2} = 1;$$

$\left|\dfrac{dz^\alpha}{ds}\right| \leqslant \dfrac{ds'}{ds} \leqslant \zeta$ and:

$$\begin{aligned}
\frac{d\tau}{ds} &= O(1) \cdot \frac{d\eta}{ds} - 2\eta \, [1 + \eta^2 + \sum_{\alpha=1}^N (z^\alpha)^2]^{-2} \frac{d}{ds} \sum_{\alpha=1}^N (z^\alpha)^2 \\
&= O(1) + O(1) \cdot \eta \cdot \zeta = O(1).
\end{aligned}$$

From the shortable imbedding $h: X \to h(X) \subset S^{N+1}$ we obtain a shortable imbedding in E^{N+1} as follows: Choose a point P in S^{N+1} such that the distance from P to $h(X)$ is $\varepsilon > 0$. Then apply the stereographic projection $\varkappa$ with centre P, of S^{N+1} minus P onto the hyperplane E^{N+1} which is equatorial with respect to S^{N+1} and the northpole $P \cdot \varkappa h(X)$ is bounded in E^{N-1} and so is the (variable) factor that is the quotient of the arc-lengths induced in $\varkappa h(X)$ and $h(X)$. Then the C^∞-imbedding

$$\varkappa h : X \to \varkappa h(X)$$

of X into E^{N+1} is shortable. It does not meet its limit set either, because $f : X \to f(X)$ did not.

§ 13. *The imbedding of open Riemannian n-manifolds*

Theorem 3. *If an open n-manifold with C^∞-Riemannian metric X^n has a short C^∞-imbedding in E^N, $N > n$, which does not meet its limit set, then it has a C^1-isometric imbedding in E^N which does not meet its limit set.* (In our construction the two mentioned limit sets are identical).

Proof: In order to prove this theorem we start from a C^∞-imbedding f_1 of X in E^N, which does not meet its limit set, and for which the original and the induced metrics ds^2 and $(ds')^2$ satisfy

$$0 < (ds'/ds)^2 \leqslant \tfrac{3}{4}. \tag{13.1}$$

This is possible because a short imbedding exists (see § 2).

We will now cover X, which is assumed to be non-compact, by an increasing sequence of subsets X_T, $T = T_0, T_0 + 1, \ldots$, and we will apply the construction given in § 1–10 to these compact subsets, such that in successive stages successive subsets X_T will enter the process. This must be done carefully, however, in particular because $(ds'/ds)^2$ is not known to be uniformly bounded away from zero on X.

The choice of T_0 and X_T is as follows: T_0 is an integer which obeys:

$$T_0 > (10nP)^{32}. \tag{13.2}$$

Let $X'_T < X'_{T+1}$, $T \geqslant T_0$, be a first infinite sequence of increasing closed submanifolds of X, covering X, and neither of them meeting the boundary of the next:

$$X'_T \cap \text{closure } (X - X'_{T+1}) = \phi.$$

Let X''_T be the set of all $x \in X$ for which (in all directions at x)

$$(ds')^2/ds^2 \geqslant T^{-1/8}. \tag{13.3}$$

The sequence X''_T has the properties (which) we mentioned for the sequence X'_T except the fact that X''_T may be open.

However: $X_T = X'_T \cap X''_T$, $T \geqslant T_0$, is an infinite sequence of increasing closed submanifolds of X, covering X, neither of them meeting the boundary of the next, for which (13.3) also holds. We call X_{T_0} also Z_{T_0}, and we call the closure in X of $X_{T+1} - X_T$: Z_{T+1}. Z_{T+1} is a closed set.

In each stage or step a system of coordinate systems will be used which apart from other properties mentioned in § 1–10, will have the property that the coordinate neighborhood in X of any system meets at most two of the sets Z_T, and if so, then only two successive ones Z_T and Z_{T+1}.

In the first stages nothing is changed: $f_{T_0}=f_1$.

In stage T, in which the C^∞-imbedding f_T is replaced by f_{T+1} we first consider the neighborhoods that meet X_{T+1}. We give a C^∞-partition of unity on X_{T+1} and we construct the change of the imbedding of X_{T+1} as in paper I, however, omitting all changes with respect to coordinate neighborhoods that do not meet X_T! This has the following consequences: In stage T nothing is altered in the imbedding of $X-X_{T+1}$; the change of the imbedding of X_T is as in § 5–10; and the new imbedding of X is C^∞ again!

The construction in § 5–10 cannot be followed literally, however. We have to meet the following extra requirements: In stage T we must avoid selfintersections, not only in the image of X_T, but also in the image of X; Intersections with the (constant) limitset of the imbedding of X must be avoided; This last condition has to be fulfilled also in the limit imbedding f_∞.

These requirements are, however, easily met (see § 8–10) and we omit the details.

The construction is now completely defined.

The proof of the validity of the construction is also taken from paper I, but with the following modifications.

We study what happens with the image of Z_T under the process, and we define with respect to Z_T the numbers $c_{1,t}$, $c_{2,t}$, ε_t and ω_t by equations (5.1) (5.2′) and (7.11). In the sequel of this paragraph all considerations concern Z_T unless stated otherwise.

The following inequalities follow from the definitions and (13.1,3):

$$c_{1,t} \geqslant T^{-1/8} \tag{13.4}$$

$$c_{2,t} \leqslant \tfrac{3}{4} \text{ for } t<T, \tag{13.5}$$

and as we may assume that the change in $c_{2,t}$ is small in stage

$$t=T-1 \geqslant T_0-1$$

(5.3 only gives an *upper* bound in this stage, which is exceptional as far as Z_T is concerned, because only part of the "wanted" change in the metric is obtained):

$$c_{2,T} \leqslant \tfrac{7}{8}. \tag{13.6}$$

Substituting (13.4) in the definition of ε_t (5.2′) we get for $t \geqslant T-1$:

$$\varepsilon_t^4 = 3\,(c_{1,t}^{-1}-1)\,t^{-1/2} < 3\cdot T^{1/8-1/2}.$$

$$\varepsilon_t < 3\cdot T^{-3/32} \leqslant 3\cdot T_0^{-3/32} < (3P)^{-1} \tag{13.7}$$

uniformly in $t \geqslant T-1 \geqslant T_0-1$ (compare 13.2).

This takes care of the inequality (5.4′) for any point $x \in X$, because x is contained in at least one of the sets Z_T.

With these modifications we follow the proof of paper I, applied to Z_T, until the number ω_t appears, which is defined by (7.11). We substitute the expression (5.2′) for δ_t in (7.11) and we obtain, when applying (13.4, 5) and then (13.3). (compare the deduction of 7.14):

$$\omega_{T-1} < 0{,}1, \quad \omega_T < 0{,}1.$$

In stages following f_T, Z_T is handled according to § 5, 6, 7 and the inequality $\omega_{t+1}/\omega_t < 1$ is obtained as before. This takes care of the essential inequalities (7.14). For the rest the proof is as in paper I and theorem 3 is proved.

As a consequence of theorems 2 and 3 we have:

Theorem 4. *If an open n-manifold with C^∞-Riemannian metric X^n has a C^∞-imbedding in E^N, $N \geqslant n$, which does not meet its limit set, and it has for $N=2n$* (WHITNEY), *then it has a C^1-isometric imbedding in E^{N+1} which does not meet its limit set.*

§ 14. *The imbedding of the hyperbolic plane in euclidean three space*

Theorem 5. *The hyperbolic space H^n (e.g. the hyperbolic plane H^2) has a C^1-isometric imbedding in euclidean $n+1$-space E^{n+1}, which is closed in E^{n+1} (= limit set is void).*

Proof: If $d\sigma^2$ is the metric on a unit $n-1$ sphere in E^n with centre M, and ϱ is the distance between M and a point with polar "coordinates" (ϱ, σ) where σ is a unit vector, then the Euclidean metric in E^n is

$$(ds')^2 = d\varrho^2 + \varrho^2\, d\sigma^2. \tag{13.1}$$

A hyperbolic n-space H^n of constant negative curvature $K = -3c^2$ can be imbedded on E^n by introducing in E^n the Riemannian metric

$$ds^2 = \tfrac{4}{3}[d\varrho^2 + \varphi(\varrho)\, d\sigma^2]$$

with

$$\varphi(\varrho) = \left(\frac{\sinh 2c\varrho}{2c}\right)^2 \geqslant \varrho^2.$$

Thus we have a C^∞-imbedding of H^n in $E^n \subset E^{n+1}$ which is short because

$$(ds')^2/ds^2 \leqslant \tfrac{3}{4}.$$

Theorem 5 follows from the construction in § 13 because the limit set of the given initial imbedding is void, hence the same is true for the result of the construction: the isometric C^1-imbedding f_∞.

Wageningen, Landbouwhogeschool.

REFERENCES

1. NASH, JOHN, C^1-isometric imbeddings. Annals of Math. **60**, 383–396 (1954).
2. KUIPER, NICOLAAS H., On C^1-isometric imbeddings I, Proceedings Kon. Ned. Akad. v. Wetensch. Amsterdam **A 58** = Indagationes Math, **17**, 545–556 (1955).
3. HILBERT, D., Ueber Flächen von konstanter Gausscher Krümmung. Trans. Am. Math. Soc. 2 (1901). Also: Grundlagen der Geometrie Anhang V.
4. BIEBERBACH, L., Comm. Math. Helv. **4**, 248–255 (1932).
5. BLANUŠA, D., Monatsch. f. Math., Wien **57**, 102–108 (1953).