MATHEMATICS

ON C^1-ISOMETRIC IMBEDDINGS. I

BY

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§ 1. *Introduction*

Some time ago JOHN NASH [1] made a very interesting and surprising contribution to the problem of the isometric imbedding of Riemannian n-manifolds X^n in euclidean N-space E^N. His main result is: If X^n compact with boundary has a C^1-imbedding in E^N with $N \geqslant n+k$, $k=2$ (and it always has for $N=2n$ as WHITNEY [2] proved; compare also VAN KAMPEN [3]), then there exists a C^1-isometric imbedding of X^n in E^N. (C^r = of differentiability class r = derivatives of order r exist and are continuous). A local consequence of this theorem is: Some neighborhood of any point of a X^n can be imbedded C^1-isometrically in E^{n+k}, $k=2$.

At the end of his paper NASH presents a conjecture to the effect that in the above result $k=2$ might be replaced by $k=1$.

In this paper we prove Nash' conjecture and hence obtain the Theorem. *If a compact C^1-Riemannian C^1-manifold with boundary (possibly void) of dimension n has a C^1-imbedding in euclidean N-space E^N* (and it always has if $N=2n$) $N \geqslant n+1$, *then it has a C^1-isometric imbedding in E^N.* [1])

Example: a locally Euclidean torus of dimension n, that is the metric product of n circles, can be imbedded C^1-isometrically in E^{n+1}.

Local consequence: *Some neighborhood of any point of a C^1-Riemannian n-manifold has a C^1-isometric imbedding in E^{n+1}.*

Our proof follows Nash' proof with the exception of a different kind of one step device: a strain. This strain however requires considerations concerning the convergence of the process which are even more delicate then those required with Nash' one-step device. We therefore give a complete proof independent of Nash' paper.

In the sequel we will give an explicit proof of the theorem: If a compact C^∞-Riemannian n-manifold has a C^1-imbedding in E^N, $N>n$, then it has a C^1-isometric imbedding in E^N. With some slight alterations one obtains a proof for the case of a compact manifold with boundary and validity for a C^1-Riemannian C^∞-manifold. But any C^1-manifold has a compatible C^∞-structure, so then the theorem will also be seen to be true for a C^1-Riemannian C^1-manifold.

[1]) C^1-Riemannian = the metric is defined by a tensor g_{ij} which is continuous on X^n.

It is our aim now to construct an isometric imbedding of a compact C^∞-Riemannian n-manifold X^n in some higher dimensional Euclidean space. Let f_0 be a C^1-imbedding of X^n in E^N. Then also a C^∞-imbedding f_1 of X^n in E^N exists (WHITNEY). Starting from f_1 we shall try to define a sequence of C^∞-imbeddings $f_t\,(=1, 2, 3 \ldots)$ which converges to a C^1-*isometric* imbedding f_∞ of X in E^N. The "improvement" $(f_1; f_\infty)$ is obtained in stages $(f_t; f_{t+1})$, divided in successive substages, again divided in successive steps.

After a preliminary step (§ 2) and some preliminary work (§ 3, 4) we indicate in § 5, and 6 how we want to obtain the improvement (f_1, f_∞) in successive steps. We study the required convergence of our process in the final §'s. In a second paper we study the isometric imbedding of non-compact manifolds.

§ 2. *The preliminary step* (f_1, f_2)

If x^i, $i=1, \ldots n$, are local (C^∞-) coordinates in X, and z^β, $\beta=1, \ldots N$, are the coordinates of E^N, then $f_1(X)$ and X have the induced metric (represented by the line element):

$$(ds')^2 = h_{ij}\, dx^i\, dx^j = \sum_\beta \frac{\partial z^\beta}{\partial x^i} \frac{\partial z^\beta}{\partial x^j}\, dx^i\, dx^j. \tag{2.1}$$

If ds^2 is the given metric of X, then $(ds')^2/(ds)^2$, a continuous function on the closed space of tangent unit vectors of X, attains a maximum k and a minimum $4ck/3 > 0$. The second imbedding f_2 is now defined as the product of f_1 and the geometrical multiplication in E^N:

$$\bar{z}^\beta = z^\beta/\sqrt{4k/3}.$$

f_2 has the induced metric $(ds'')^2$ for which $(ds'')^2/(ds)^2$ has maximum $^3/_4$ and minimum $c \leqslant {}^3/_4$. NASH calls an imbedding for which this ratio is smaller then one a *short* imbedding. All imbeddings to be obtained, except f_1 and f_∞, will be short.

To simplify formulas later on we leave the imbedding fixed in the next t_0-1 stages, where t_0 is a large constant to be chosen later

$$f_{t_0} = f_2. \tag{2.2}$$

§ 3. *A C^∞-partition of one on an open C^∞-manifold*

Let X be an open C^∞-manifold of dimension n. Let $\varkappa_p$ be a C^∞-coordinate system which maps $N_p \subset X$ homeomorphically onto a solid closed sphere $\varkappa_p(N_p)$ in Euclidean coordinate-space. For $x \in N_p$, we denote by $\tau_p(x)$ the square of the distance from $\varkappa_p(x)$ to the centre of $\varkappa_p(N_p)$ divided by the square of the radius of $\varkappa_p(N_p)$. A C^∞-function on X is now defined as follows:

$$\varphi_p^*(x) = \begin{cases} 0 & \text{for } x \notin \text{interior } N_p, \\ \exp\left((\tau_p(x)-1)^{-1}\right) & \text{for } x \in \text{interior } N_p. \end{cases}$$

It is known that there exists an integer $W(n)$ such that it is possible to chose a denumerable subset of the coordinate systems $\varkappa_p$, with the property that any $x \in X$, together with some neighborhood of x, is interior to at least one and at most $W(n)$ of the coordinate neighborhoods of the subset. From now on the index p will run over such a subset. Then the C^∞-functions on X:

$$\varphi_p(x) = \frac{\varphi_p^*(x)}{\sum_{p'} \varphi_{p'}^*(x)}. \tag{3.1}$$

$p = 1, 2, \dots$ form a so called C^∞-partition of one with respect to the neighborhoods N_p:

$$\sum_i \varphi_p(x) = 1, \quad \varphi_p(x) = 0 \text{ for } x \notin N_p,$$

and for any x at most $W(n)$ terms do not vanish. In case X is the compact manifold which we want to imbedd, p will take only a finite number of values.

§ 4. *A C^∞-vector-decomposition of M, the space of symmetric positive definite $n \times n$ matrices $\mathfrak{m}$*

The space M is a convex open subset of the vector space of all $n \times n$-symmetric matrices. With every point $\mathfrak{m} \in M$ we choose a closed *linear* simplex S_q and a coordinate system of the kind described in § 3:

$$\varkappa_q : N_q \to \varkappa_q(N_q)$$

for which

$$\mathfrak{m} \in \text{interior } N_q \subset N_q \subset S_q \subset M.$$

This set can be chosen and a denumerable subset found with the property that any point $\mathfrak{m} \in M$ together with some neighborhood of $\mathfrak{m}$ is interior to at least one and at most $W(n')$ ($n' = n(n+1)/2 =$ dimension M) of the coordinate systems of the subset. From now on the index q will run over such a subset. The functions of the corresponding C^∞-partition of *one* (see § 3) will be called $\psi_q(\mathfrak{m})$.

We then have for any $\mathfrak{m} \in M$:

$$1 = \sum_q \psi_q(\mathfrak{m}); \quad \mathfrak{m} = \sum_q \psi_q(\mathfrak{m}) \cdot \mathfrak{m}. \tag{4.1}$$

$\mathfrak{m} \in S_q$ is a unique linear combination with positive coefficients $c_{q\mu}$ of the $n'+1$ positive definite symmetric (constant) matrices $\mathfrak{m}_{q\mu}$ that are the vertices of S_q:

$$\mathfrak{m} = \sum_\mu c_{q\mu}(\mathfrak{m}) \cdot \mathfrak{m}_{q\mu}. \tag{4.2}$$

We next observe that any positive definite quadratic form in n variables is the sum of n squares of linear forms, hence any positive definit symmetric matrix is the sum of n non-negative symmetric matrices of rank one. We fix such a decomposition for each matrix $\mathfrak{m}_{q\mu}$:

$$\mathfrak{m}_{q\mu} = \sum_\nu \mathfrak{m}_{q\mu\nu}. \tag{4.3}$$

Substitution in (4. 1,2) yields the required decomposition

$$\mathfrak{m} = \sum_q \sum_\mu \sum_\nu \psi_q(\mathfrak{m}) \cdot c_{q\mu}(\mathfrak{m}) \cdot \mathfrak{m}_{q\mu\nu}. \tag{4.4}$$

Replacing the triples of indices $q\mu\nu$ by one index τ we have:

$$\mathfrak{m} = \sum_{\tau} \chi_\tau(\mathfrak{m}) \cdot \mathfrak{m}_\tau . \tag{4.4}$$

$\chi_\tau(\mathfrak{m})$ is non-negative and a C^∞-function of $\mathfrak{m}$. The matrices $\mathfrak{m}_\tau$ are constant symmetric matrices of rank one. For some neighborhood of any $\mathfrak{m} \in M$ the sum (4.4) has at most

$$V = n \cdot (n'+1) \cdot W(n') \qquad (n' = n(n+1)/2) \tag{4.5}$$

terms not identically zero.

Remark: In the application $\mathfrak{m} = \mathfrak{m}(x^1, \ldots, x^n)$ is a C^∞-function of local coordinates $x^1, \ldots, x^n$ of X; hence also $\chi_\tau(\mathfrak{m})$ is a C^∞-function of $x^1, \ldots, x^n$.

If M' is a closed subset of M then only a finite number of the matrices $\mathfrak{m}_\tau$ will occur in the decomposition of all $\mathfrak{m} \in M'$.

§ 5. *On the distribution of the improvement* (f_{t_0}, f_∞) *over the stages and steps*

We denote the metric of the initial short imbedding f_t of the stage (f_t, f_{t+1}), with $t \geqslant t_0$, by $(ds^b)^2$ and we define c_1 and c_2 by:

$$0 < c_1 = \text{minimum } \frac{(ds^b)^2}{ds^2} < \text{maximum } \frac{(ds^b)^2}{ds^2} = c_2 < 1 . \tag{5.1}$$

In case confusion is possible we also indicate the stage number by an index t: for example $c_{1,t}$ etc.

We *want* to obtain at the end of the stage an imbedding with induced metric $(ds^w)^2$ for which

$$\frac{(ds)^2 - (ds^w)^2}{(ds)^2 - (ds^b)^2} = 1 - t^{-\frac{1}{2}} \tag{5.2}$$

or

$$\frac{(ds^w)^2 - (ds^b)^2}{(ds)^2 - (ds^b)^2} = t^{-\frac{1}{2}} = \varepsilon_t^4 \frac{c_1}{3(1-c_1)}, \tag{5.2'}$$

and we will *succeed* (§ 7) in obtaining a short imbedding f_{t+1} with induced metric $(ds^e)^2$ for which

$$0 < 1 - 2t^{-\frac{1}{2}} < \frac{(ds)^2 - (ds^e)^2}{(ds)^2 - (ds^b)^2} < 1 - \tfrac{1}{2} t^{-\frac{1}{2}} < 1 \tag{5.3}$$

(ε_t is defined by the last equality in (5.2')).

Because $\Pi_{t=t_0}^{\infty} (1 - \frac{1}{2} t^{-\frac{1}{2}}) = 0$, (5.3) applied to successive stages, implies that if the metrics induced by f_t converge with $t \to \infty$ to the induced metric $(ds^\infty)^2$ of a C^1-imbedding f_∞, then $ds^2 - (ds^\infty)^2 = 0$ and this limit-imbedding is as required.

We will make the inductive assumption that (5.3) is true for all stages after f_{t_0} that have been performed before the particular stage under consideration. In the sequel we will occasionally make analogous inductive assumptions concerning substages and steps, without explicitly saying so. For example f at the beginning of a stage or step under consideration is always assumed to be a C^∞-imbedding and not just a mapping.

Because $c_{1,t}$ and $c_{2,t}$ are increasing functions of $t \geqslant t_0$ (see 5.1, 3) we conclude from § 2:

(5.4) $$0 < c = c_{1,t_0} \leqslant c_1 = c_{1,t} < 1;\ \tfrac{3}{4} = c_{2,t_0} \leqslant c_2 = c_{2,t} < 1.$$

We also observe that ε_t is a decreasing function of $t \geqslant t_0$:

(5.4) $$0 < \varepsilon_t \leqslant \varepsilon_{t_0} < (3 \cdot P)^{-1}$$

t_0 is chosen so large that $\varepsilon_{t_0} < (3 \cdot P)^{-1}$; $P = W(n) \cdot V$.

The following equalities and inequalities are used later:

(5.5) $$\varepsilon_t^4 = 3\, t^{-\frac{1}{2}} (c_1^{-1} - 1) \leqslant 3\, t_0^{-\frac{1}{2}} (c^{-1} - 1)$$

$$c_1\, ds^2 \leqslant (ds^b)^2$$

$$(ds^w)^2 - (ds^b)^2 = - \frac{\varepsilon_t^4 c_1}{3(1-c_1)} (ds^b)^2 + \frac{\varepsilon_t^4}{3(1-c_1)} c_1\, ds^2 \text{ from } (5{,}2'),$$

(5.6) $$0 < (ds^w)^2 - (ds^b)^2 \leqslant \tfrac{1}{3} \varepsilon_t^4 (ds^b)^2.$$

The improvement wanted in one stage:

(5.7) $$(ds^w)^2 - (ds^b)^2,$$

is divided in improvements wanted in successive *sub-stages*:

(5.8) $$\varphi_p(x)\ \{(ds^w)^2 - (ds^b)^2\}$$

with the help of the C^∞-partitioning functions $\varphi_p(x)$, defined for X in § 3. Without restriction we may assume that some neighborhood of any $x \in X$ is effectively involved in at most $W(n)$ substages in a stage (§ 3).

As (5.8) vanishes outside the coordinate neighborhood $N_p \subset X$ of the p-th coordinate system $\varkappa_p$, this can be expressed in terms of *one* coordinate system $\varkappa_p$ with

$$ds^2 = g_{ij}\, dx^i\, dx^j;\ (ds^b)^2 = h^b_{ij}\, dx^i\, dx^j;\ (ds^w)^2 = h^w_{ij}\, dx^i\, dx^j;$$

as:

(5.9) $$\varphi_p(x) \cdot (h^w_{ij} - h^b_{ij})\, dx^i\, dx^j \overset{\text{def}}{=} \varphi_p(x) \cdot m_{ij}\, dx^i\, dx^j.$$

The matrix $\mathfrak{m}$ with components m_{ij} is positive definite, C^∞-function of $x \in N_p$.

In the choice in § 3 of the coordinate systems $\varkappa_p$ we had much freedom which we shall restrict now. We want h^b_{ij} to be close to the unit matrix δ_{ij}. Therefore we choose new coordinate systems of the kind given in § 3, at the beginning of each stage such that in any of them the expression for $(ds^b)^2 = h^b_{ij} dx^i dx^j$ obeys

(5.10) $$-\varepsilon_t^3 < \frac{(h^b_{ij} - \delta_{ij})\, dx^i\, dx^j}{\delta_{ij}\, dx^i\, dx^j} < \varepsilon_t^3,$$

and such that also any point $x \in X$ together with some neighborhood is interior to at least one and at most $W(n)$ of the coordinates systems $\varkappa_p$. Putting two or one among $dx^1, \ldots, dx^n$ equal to one and the others zero, we obtain from (5.10):

(5.10′) $$i \neq j \quad |h^b_{ii} + 2\, h^b_{ij} + h^b_{jj} - 2| < 2\, \varepsilon_t^3;\ |h^b_{ii} - 1| < \varepsilon_t^3;\ |h^b_{ij}| < 2\, \varepsilon_t^3;$$

Observe that (5.10) is invariant under orthogonal coordinate transformations.

Next we divide the wanted improvement (5.9), with the help of (4.4) in the following improvements wanted in successive *steps* ($m_{\tau ij}$ are the elements of the symmetric matrix $\mathfrak{m}_\tau$ of rank one):

$$\varphi_p(x)\,\chi_\tau(\mathfrak{m})\cdot m_{\tau ij}\,dx^i\,dx^j, \tag{5.11}$$

$m_{\tau ij}dx^i dx^j$ is the square of a linear form with constant coefficients. $\varphi_p(x)\,\chi_\tau(\mathfrak{m})$ is a C^∞-function of $x \in N_p$ vanishing on the boundary. For some neighborhood of any $x \in N_p \subset X$ at most V steps in one substage have non-vanishing terms (5.11) and *at most* $P = W(n)\cdot V$ *steps in one stage* (see 5.4). We stress that the wanted improvement in any step of a stage is fixed at the beginning of the stage.

§ 6. *The strain*

Before we present our strain leading to a desired one-step result, we illustrate the idea of it for the simple case of a line imbedded as the line $y=0$ in the euclidean plane with orthogonal coordinates (x, y).

In this case the change of imbedding consists in replacing $y=0$ by a sinusoide. This might be done by the mapping:

$$(x, 0) \to (x, \alpha\ \sin x),\ \alpha\ \text{constant}. \tag{6.1}$$

However in this way the change of the induced metric vanishes almost near the tops and the dells of the sinusoide and it is relatively large in the flexes. A more uniform distribution of the change might be obtained from

$$(x, 0) \to \{x - k\sin 2x,\ \alpha\sin(x - k\sin 2x)\}.$$

It is sufficiently uniform if we chose moreover $8k = \alpha^2$:

$$(x, 0) \to \{x - \tfrac{1}{8}\alpha^2\sin 2x,\ \alpha\sin(x - \tfrac{1}{8}\alpha^2\sin 2x)\}. \tag{6.2}$$

If ds is the differential of the arclength of the sinusoide, image of dx of the line $y=0$ under (6.2), then

$$\begin{aligned}\left(\frac{ds}{dx}\right)^2 &= (1 - \tfrac{1}{4}\alpha^2\cos 2x)^2 + \alpha^2\cos^2(x - \tfrac{1}{8}\alpha^2\sin 2x)\cdot(1 - \tfrac{1}{4}\alpha^2\cos 2x)^2\\ &= (1 - \tfrac{1}{2}\alpha^2\cos 2x + \tfrac{1}{16}\alpha^4\cos^2 2x)\,[1 + \alpha^2\cos^2(x - \tfrac{1}{8}\alpha^2\sin 2x)]\\ &= (1 + \tfrac{1}{2}\alpha^2 - \alpha^2\cos^2 x + \tfrac{1}{16}\alpha^4\cos^2 2x)\,(1 + \alpha^2\cos^2 x + \theta\,\tfrac{1}{8}\alpha^4),\ \ |\theta| < 1.\end{aligned}$$

Assuming $\alpha < \varepsilon^2 < \frac{1}{3}$ we find

$$\left(\frac{ds}{dx}\right)^2 = 1 + \tfrac{1}{2}\alpha^2 + R, \quad |R| < \tfrac{5}{4}\alpha^4 < \tfrac{5}{2}\varepsilon^4\cdot\tfrac{1}{2}\alpha^2. \tag{6.3}$$

Under a geometrical multiplication of the x-y-plane with factor λ^{-1} we get from (6.2):

$$(x, 0) \to \left\{x - \frac{\alpha^2}{8\lambda}\sin 2\lambda\,x,\ \ \frac{\alpha}{\lambda}\sin\left(\lambda\,x - \frac{\alpha^2}{8}\sin 2\lambda\,x\right)\right\} \tag{6.4}$$

with (6.3) as before. Intrinsically the resulting imbedding is therefore in first approximation independent of λ!

In the general case we aim at the improvement (5.11) which we write

$$(\alpha^2/2)\,(dn_{\tau i}\,x^i)^2 \tag{6.5}$$

with $$\sum_i (n_{\tau i})^2 = 1; \quad n_{\tau i} \text{ constant};$$

$$\alpha^2/2 = \varphi_p(x)\cdot\chi_\tau(\mathfrak{m})\cdot K; \;\; K^{-1}\cdot m_{\tau ij}\,dx^i\,dx^j = (dn_{\tau i}\,x^i)^2.$$

After a suitable Euclidean coordinate transformation on $x^1, \ldots, x^n$, $n_{\tau i}x^i$ is replaced by x^1. We apply this in order to simplify formulas. It does not hurt the argument. In the step under consideration we therefore *want* to replace the C^∞-imbedding $f : z^\beta = z^\beta(x^1, \ldots, x^n)$ with induced metric $h_{ij}dx^i dx^j$, by $\bar{f}$: $\bar{z}^\beta = \bar{z}^\beta(x^1, \ldots, x^n)$ with induced metric $\bar{h}_{ij}dx^i dx^j$, such that $\bar{h}_{ij}dx^i dx^j - h_{ij}dx^i dx^j$ is approximately

$$(\alpha^2/2)\,dx^1\,dx^1 \overset{\text{def}}{=} (\bar{\alpha}^2/2)\,h_{11}\,dx^1\,dx^1, \tag{6.6}$$

In order to obtain (6.6) approximately we introduce two perpendicular unit vector fields ξ^β and η^β on the image $f(N_p)$ of $N_p \subset X$: The unit vector ξ^β is defined by: $\partial z^\beta/\partial x^1 = h\cdot\xi^\beta$, $h > 0$, $h^2 = h_{11}$; The C^∞ unit vector field η^β is chosen normal to $f(N_p)$.

We define $\bar{f}$ by:

$$\bar{z}^\beta = z^\beta - \frac{\bar{\alpha}^2 \sin 2\lambda x^1}{8\lambda}\,h\,\xi^\beta + \frac{\bar{\alpha}}{\lambda}\sin\left[\lambda x^1 - \frac{\bar{\alpha}^2 \sin 2\lambda x^1}{8}\right]\cdot h\,\eta^\beta. \tag{6.7}$$

λ is a constant to be chosen for each step. In the sequel we shall impose several restrictions on λ's. All these will be positive lower bounds, and the conditions for λ in a step will be independent of the following steps and of other conditions concerning λ in the same step. Therefore all these restrictions can be obeyed at the same time.

§ 7. *How to keep the results in the successive stages close to the wanted results*

At the beginning of a stage (5.10) holds at any point $x \in X$ with respect to any of the used coordinate-systems. The image of any point $x \in X$, is effectively involved in at most P steps in a stage. We will make the inductive assumption (proved in 7.8), true for the first step in the stage (5.10), that at a point x the image of which has been effected in p_x steps in the stage before the step under consideration, the induced metric h_{ij} at the beginning of this step obeys

$$|(h_{ij} - \delta_{ij})\,dx^i\,dx^j| < (\varepsilon_t^3 + p_x\,\varepsilon_t^4)\,\delta_{ij}\,dx^i\,dx^j. \tag{7.1}$$

Each step in the stage takes care of a non-negative part like (6.6) of the wanted improvement (5.7) which is according to (5.6) majorised by

$$\tfrac{1}{3}\,\varepsilon_t^4\,(ds^b)^2 = \tfrac{1}{3}\,\varepsilon_t^4\,h_{ij}\,dx^i\,dx^j.$$

Hence

$$(\bar{\alpha}^2/2)\,h_{11}dx^1dx^1 \leqslant (\varepsilon_t^4/3)\,h_{11}dx^1dx^1. \tag{7.2}$$

For ε_t sufficiently small (see 5.10, 7.1 and 5.4), h_{11} and h^b_{11} are very close to *one* and (7.2) gives:

$$\bar{\alpha} < \varepsilon_t^2 < \varepsilon_{t_0}^2 < 1/3, \quad \alpha < \varepsilon_t^2. \tag{7.3}$$

To simplify computations we introduce $O(\lambda^{-1})$, the O-symbol of LANDAU. In our applications it will represent any real function $g(x, \lambda)$ of $x \in X$ and λ, for which $|g(x, \lambda)| < K\lambda^{-1}$, for some constant K, independent of x. The symbol $O(\lambda^{-1})$ will be allowed to represent different functions at two places in one or more equations.

From (6.7) we obtain (observe that $x \in N_p$ compact)

$$\text{(7.4)}\quad \begin{cases} \dfrac{\partial \bar{z}^\beta}{\partial x^1} = \dfrac{\partial z^\beta}{\partial x^1} - \dfrac{\bar{\alpha}^2 \cos 2\lambda x^1}{4} \cdot h\, \xi^\beta + \bar{\alpha}\left(1 - \dfrac{\bar{\alpha}^2 \cos 2\lambda x^1}{4}\right) \cos\left[\lambda x^1 - \dfrac{\bar{\alpha}^2 \sin 2\lambda x^1}{8}\right] \cdot \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot\, h\eta^\beta + O(\lambda^{-1}) \\ \dfrac{\partial \bar{z}^\beta}{\partial x^i} = \dfrac{\partial z^\beta}{\partial x^i} + O(\lambda^{-1}) \qquad \text{for } i \neq 1, \end{cases}$$

and (compare 7.3 and 6.3, and observe that the vector η^β is orthogonal to $\partial z^\beta/\partial x^1 = h\xi^\beta$ and to $\partial z^\beta/\partial x^i$)

$$\begin{aligned} \bar{h}_{ij} - h_{ij} &= \sum_i \left(\frac{\partial \bar{z}^\beta}{\partial x^i}\frac{\partial \bar{z}^\beta}{\partial x^j} - \frac{\partial z^\beta}{\partial x^i}\frac{\partial z^\beta}{\partial x^j}\right) \\ \bar{h}_{11} - h_{11} &= \tfrac{1}{2}\bar{\alpha}^2 (1 + \tfrac{5}{2}\theta\, \varepsilon_t^4)\, h_{11} + O(\lambda^{-1}) \qquad |\theta| < 1 \\ \bar{h}_{i1} - h_{i1} &= -\frac{\bar{\alpha}^2 \cos 2\lambda x^1}{4}\, h_{i1} + O(\lambda^{-1}) \qquad \text{for } i \neq 1 \\ \bar{h}_{ij} - h_{ij} &= O(\lambda^{-1}) \qquad \text{for } i \text{ and } j \neq 1. \end{aligned}$$

Summarised:

$$\text{(7.5)}\quad \begin{cases} (\bar{h}_{ij} - h_{ij})\, dx^i\, dx^j = (\alpha^2/2)\, dx^1\, dx^1 + \dfrac{\bar{\alpha}^2}{2}\left\{\dfrac{5}{2}\,\theta\, \varepsilon_t^4\, h_{11}\, dx^1\, dx^1\right. \\ \left. - \dfrac{\cos 2\lambda x^1}{2} \displaystyle\sum_{i \neq 1} h_{i1}\, dx^i\, dx^1\right\} + O(\lambda^{-1})\, \delta_{ij}\, dx^i\, dx^j \\ = \text{wanted improvement} + \text{error } I + \text{error } II. \end{cases}$$

Our problem in this paragraph is to keep error I small in successive steps. This is the reason why we put condition (7.1) on h_{ij}.

The improvement wanted in the step is majorised by the improvement wanted in the stage (see 5.4, 5.6 and 7.1):

$$\text{(7.6)}\quad \begin{cases} (\alpha^2/2)\, dx^1 dx^1 \leqslant (ds^w)^2 - (ds^b)^2 \leqslant \tfrac{1}{3}\varepsilon_t^4\, (ds^b)^2 \\ \qquad\qquad\qquad\quad \leqslant \tfrac{1}{3}\varepsilon_t^4\, (1 + \varepsilon_t^3)\, \delta_{ij} dx^i dx^j. \end{cases}$$

With $|h_{11}| < 1 + \varepsilon_t^3 + P\varepsilon_t^4$, $|h_{i1}| < 2(\varepsilon_t^3 + P\varepsilon_t^4)$, (compare (5.10′), (7.1, 2, 3)) we get

$$\text{error } I = \tfrac{1}{2}\bar{\alpha}^2 2\, \varepsilon_t^3 dx^1 \left\{\theta_1 dx^1 + (1 + P\varepsilon_\tau) \sum_{i \neq 1} \theta_i dx^i\right\} \qquad |\theta_i| < 1.$$

Because the absolute values of the quadratic forms $dx^1 dx^1$ and $2dx^1 dx^i$ $i \neq 1$ are majorised by $\delta_{ij} dx^i dx^j$, the absolute value of error I is majorised by

$$\tfrac{1}{2}\bar{\alpha}^2 \cdot n\varepsilon_t^3\, \delta_{ij} dx^i dx^j$$

or also by, what is about the same: (see 7.3): (7.7).

$$\tfrac{1}{2}\alpha^2 n\, \varepsilon_t^3\, (ds^b)^2 \leqslant \tfrac{1}{2} n\, \varepsilon_t^7\, (ds^b)^2. \tag{7.7}$$

From (7.5, 6, 7) we get with (7.2, 3) for λ sufficiently large:

$$(\bar{h}_{ij} - h_{ij}) dx^i dx^j < \varepsilon_t^4 \delta_{ij} dx^i dx^j\ . \tag{7.8}$$

This proves *the inductive assumption* (7.1).

In the steps of a stage any point is involved effectively in at most P steps. Therefore with the above inequalities we obtain for the metric $(ds^e)^2$ induced at the end of the stage:

$$(ds^e)^2 - (ds^b)^2 = (ds^w)^2 - (ds^b)^2 + R_I + R_{II}.$$

The improvement wanted in the stage is

$$(ds^w)^2 - (ds^b)^2 = t^{-\frac{1}{2}} \{ds^2 - (ds^b)^2\} \geqslant t^{-\frac{1}{2}} (c_2^{-1} - 1)\,(ds^b)^2. \tag{7.9}$$

The error R_I is for any $x \in X$ the sum of at most P terms of the kind "error I" in (7.5) each of which is majorised as indicated in (7.7).

Hence (see 5.2′)

$$|R_I| \leqslant P \cdot \frac{n}{2}\, \varepsilon_t^7\, (ds^b)^2 < 3\, nPt^{-\frac{1}{2}} (c_1^{-1} - 1)\, \varepsilon_t^3\, (ds^b)^2. \tag{7.10}$$

The error R_{II} consists of at most P terms $O(\lambda^{-1})\,(ds^b)^2$. In each term λ is chosen so large that this term is majorised by P^{-1} times the right hand side of (7.10). Then, compare 7.9,:

$$\left\{\begin{aligned} &\left|\frac{(ds^e)^2 - (ds^w)^2}{(ds^w)^2 - (ds^b)^2}\right| = \left|\frac{R_I + R_{II}}{(ds^w)^2 - (ds^b)^2}\right| < \\ &< 6\, n P\, \varepsilon_t^3\, \frac{1/c_1 - 1}{1/c_2 - 1} = \omega_t \text{ (by definition).}\end{aligned}\right. \tag{7.11}$$

With (5.2) we deduce from (7.11):

$$1 - (1 + \omega_t)\, t^{-\frac{1}{2}} < \frac{ds^2 - (ds^e)^2}{ds^2 - (ds^b)^2} < 1 - (1 - \omega_t)\, t^{-\frac{1}{2}}. \tag{7.12}$$

Now we assume t_0 so large (ε_{t_0} so small) that $\omega_{t_0} < 0{,}1$.

From (5.1) and (7.12) we obtain inequalities for the constants c_1 and c_2 at the beginning and at the end of stage (f_t, f_{t+1}):

$$\left\{\begin{aligned} &(1 - c_{1,\,t+1})/(1 - c_{1,\,t}) < 1 - (1 - \omega_t)/\sqrt{t} \\ &(1 - c_{2,\,t+1})/(1 - c_{2,\,t}) > 1 - (1 + \omega_t)/\sqrt{t}.\end{aligned}\right. \tag{7.13}$$

Eliminating ε_t^3 from (7.11) with the help of (5.2′) we then get

$$\frac{\omega_{t+1}}{\omega_t} < \left(\frac{t}{t+1}\right)^{3/8} \left(\frac{1 - c_{1,\,t+1}}{1 - c_{1,\,t}}\right)^{7/4} \left(\frac{1 - c_{2,\,t+1}}{1 - c_{2,\,t}}\right)^{-1} \left(\frac{c_{1,\,t+1}}{c_{1,\,t}}\right)^{-7/4} \left(\frac{c_{2,\,t+1}}{c_{2,\,t}}\right)$$

$$< 1 \cdot \left(1 - \frac{1 - \omega_t}{\sqrt{t}}\right)^{7/4} \left(1 - \frac{1 + \omega_t}{\sqrt{t}}\right)^{-1} \cdot 1 \cdot \left\{1 + \frac{1 + \omega_t}{\sqrt{t}} \left(\frac{1}{c_2} - 1\right)\right\}.$$

With $\frac{1}{c_2} - 1 < \frac{1}{\frac{3}{4}} - 1 = 1/3$ (§ 2) we find that ω_{t+1}/ω_t is less then *one*, for $t > 10$ and $\omega_t < 0{,}1$.

By induction it follows with $\omega_{t_0} < 0{,}1$ that $\omega_t < 0{,}1$ for all $t \geqslant t_0$, and (7.12) implies for all $t \geqslant t_0$:

$$(7.14) \qquad 1 - 2t^{-\frac{1}{4}} < \frac{ds^2 - (ds^e)^2}{ds^2 - (ds^b)^2} < 1 - \tfrac{1}{2}t^{-\frac{1}{4}},$$

the relation we promised in (5.3).

§ 8. *How to avoid self-intersections in a step*

Assuming inductively that at the beginning of a step f is a C^∞-imbedding and observing that the displacement (6.7) is also C^∞, $\bar{f}$ too is seen to be a C^∞-mapping of X into E^N. The metric induced by $\bar{f}$ is allready known to be positive definite (7.1). This however implies that $\bar{f}$ is a C^∞-*immersion* (= the restriction of $\bar{f}$ to some neighborhood of any point $x \in X$ is an imbedding).

We have to prove that this immersion is an imbedding (= no self-intersection) for λ sufficiently large. For that reason we first consider a part of the product space $X \times X$ close to the diagonal. A point of this space is a pair of points x_0, x in X. All entities that refer to x_0 will be given a lower index 0, and we obtain from (6.7):

$$\begin{aligned}
&\bar{z}_0^\beta - \bar{z}^\beta = z_0^\beta - z^\beta - (\bar{\alpha}_0^2 h_0 \xi_0^\beta - \bar{\alpha}^2 h \xi^\beta)(8\lambda)^{-1} \sin 2\lambda x_0^1 \\
&- (8\lambda)^{-1}(\sin 2\lambda x_0^1 - \sin 2\lambda x^1)\, \bar{\alpha}^2 h \xi^\beta + (\bar{\alpha}_0 h_0 \eta_0^\beta - \bar{\alpha} h \eta^\beta)\, \lambda^{-1} \sin \lambda \\
&[x_0^1 - (8\lambda)^{-1}(\alpha_0^2 \sin 2\lambda x_0^1)] + \{\sin \lambda\, [x_0^1 - (8\lambda)^{-1}\, \bar{\alpha}_0^2 \sin 2\lambda x_0^1] - \\
&- \sin \lambda\, [x^1 - (8\lambda)^{-1}\, \bar{\alpha}^2 \sin 2\lambda x^1]\}\, \lambda^{-1}\, \bar{\alpha} h \eta^\beta.
\end{aligned}$$

Denoting the distance of the images of x_0 and x under f and $\bar{f}$ by $d(x_0, x)$ $\bar{d}(x_0, x)$ respectively, we find after some computation

$$(8.1) \qquad \sum_\beta (\bar{z}_0^\beta - \bar{z}^\beta)^2 = \bar{d}^2(x_0, x) \geqslant d^2(x_0, x)\,[1 + O\{d(x_0, x)\} + O(\lambda^{-1})].$$

In view of (7.3) this implies the existence of an open neighborhood N of the diagonal in $X \times X$, and a constant λ_2, such that for $\lambda > \lambda_2$ and the pair $(x_0, x) \in N$:

$$(8.2) \qquad \bar{d}(x_0, x)/d(x_0, x) > \sqrt[2P]{(2/3)}.$$

The choice of this constant < 1 will be motivated in § 10.

For any point $x \in X$ the formula for the displacement (6.7) gives

$$\lim_{\lambda \to \infty} \bar{f}(x) = f(x),$$

hence for any $x_0 \neq x$, both fixed,

$$\lim_{\lambda \to \infty} \bar{d}(x_0, x)/d(x_0, x) = 1.$$

The complement of N in the space $X \times X$ is a closed set of pairs $x_0 \neq x$. Then $\lambda_3 \geqslant \lambda_2$ exists such that for $\lambda > \lambda_3$ and any pair (x_0, x) in this complement, also (8.2) holds. (8.2) is then obeyed for any pair $x_0 \neq x$. It implies in particular $\bar{d}(x_0, x) \neq 0$ for $x_0 \neq x$, hence $\bar{f}$ is a C^∞-imbedding for $\lambda > \lambda_3$.

§ 9. *How to get convergence to a C^1-immersion f_∞*

The distance over which the image of a point $x \in X$ is displaced under a step is less then (compare 6.7 and 7.3)

$$\lambda^{-1}. \tag{9.1}$$

This image is effectively displaced under at most P steps in a stage. We chose λ for all steps in stage t greater then 2^t. The total distance over which the image of $x \in X$ is displaced in stage t is then less then

$$P \cdot 2^{-t} \tag{9.2}$$

and also the change in the coordinate $z^\beta (\beta = 1, \ldots, \text{or } N)$ in stage t is less then $P \cdot 2^{-t}$, a term of a geometrical series. Therefore the images of (any) $x \in X$ under f_t converge for $t \to \infty$ and as a matter of fact uniformly in x.

Next we have to prove the convergence of the derivatives of the imbedding with respect to fixed local coordinates in X. From (7.4) we observe that the derivatives in a step with respect to the coordinates used in the stage obey

$$\left\{\begin{aligned} &\sum_\beta \delta^{ij} \left(\frac{\partial \bar{z}^\beta}{\partial x^i} - \frac{\partial z^\beta}{\partial x^i}\right)\left(\frac{\partial \bar{z}^\beta}{\partial x^j} - \frac{\partial z^\beta}{\partial x^j}\right) = \\ &\qquad = \sum_i \sum_\beta \left(\frac{\partial \bar{z}^\beta}{\partial x^i} - \frac{\partial z^\beta}{\partial x^i}\right)^2 < 3\,\alpha^2 + O(\lambda^{-1}) < 3\,\varepsilon_t^4 + O(\lambda^{-1}) < 4\,\varepsilon_t^4 \end{aligned}\right. \tag{9.3}$$

for λ sufficiently large.

In later stages ds^2 and $(ds^b)^2$ are not much different any more and the expression for both of them will be not much different from $\delta_{ij} dx^i dx^j$. δ^{ij} is then not much different from g^{ij}, and (9.3) implies:

$$\sum_\beta g^{ij} \left(\frac{\partial \bar{z}^\beta}{\partial x^i} - \frac{\partial z^\beta}{\partial x^i}\right)\left(\frac{\partial \bar{z}^\beta}{\partial x^j} - \frac{\partial z^\beta}{\partial x^j}\right) < 9\,\varepsilon_t^4 = (3\,\varepsilon_t^2)^2, \tag{9.4}$$

which is invariant undercoordinate-transformations in $x^1, \ldots x^n$.

We know already that the length of the image of a unit vector tangent to X, under f_t tends to one for $t \to \infty$, because the induced metric tends to the required metric. (9.4) then implies for t sufficiently large, that the angle between the images of a unitvector under the imbeddings at the beginning and the end of a step is smaller then $6\varepsilon_t^2$. The images themselves converge, because the sum of terms $6\varepsilon_t^2$ at most P in each stage, (9.4), is convergent:

$$\sum_t P \cdot 6 \cdot \varepsilon_t^2 < \infty.$$

Proof:

$$\varepsilon_t^4 = 3\,t^{-\frac{1}{2}}\,(c_1^{-1} - 1). \tag{5.2'}$$

From (7.13) and $\omega_t < 0{,}1$ and t sufficiently large: $t > t_1 > t_0$ we deduce

$$\varepsilon_{t+1}^4 / \varepsilon_t^4 < \sqrt{\frac{t}{t+1}}\,(1 - \tfrac{1}{2} t^{-\frac{1}{2}}) < 1 - \tfrac{1}{2} t^{-1}$$

hence $\varepsilon^2_{t+1}/\varepsilon^2_t < 1 - \frac{1}{8}\, t^{-\frac{1}{2}}$

$$\varepsilon^2_t < (\varepsilon_{t_1})^2 \cdot \prod_{k=t_1}^{t} (1 - \tfrac{1}{8} k^{-\frac{1}{2}}) = O\,[\exp. - \int_{t_1}^{t} \tfrac{1}{8}\, k^{-\frac{1}{2}}\, dk]$$

$$< O\,[\exp\,(-\tfrac{1}{4}\sqrt{t})] \qquad (\text{Landau's } O\text{-symbol}).$$

Therefore ε^2_t is a term of a convergent series.

As the convergence is uniform in $x \in X$, the derivatives of the limit-mapping f_∞ not only exist, but they are even continuous with respect to x. f_∞ is therefore a C^1-mapping, which is an isometric immersion because the induced metric is $(ds^\infty)^2 = (ds)^2$ (§ 5).

§ 10. f_∞ *is a* C^1-*imbedding*

After the imbedding f_t, the image of any point $x \in X$ is according to (9.2) displaced over a distance less then

$$\sum_{k=t}^{\infty} P \cdot 2^{-k} = P \cdot 2^{1-t}$$

in all stages to come.

The distance between the images of two points is changed after f_t with an amount less then twice this amount:

$$P \cdot 2^{2-t}. \tag{10.1}$$

If the distance between the images under f_{t_0} is d, then the distance of the images under f_t is at least (compare 8.2 and observe that two points are effectively displaced in at most $2P$ steps per stage).

$$d \cdot \left\{ \sqrt[2P]{(2/3)} \right\}^{2P\cdot(t-t_0)} = d \cdot (2/3)^{t-t_0}. \tag{10.2}$$

For these two points, and for any two points in X, a number $t_2 > t_1$ exists such that (10.2) exceeds (10.1) for $t > t_2$. After that stage the two points in X can never get the same image in E^N, not even under f_∞. Hence f_∞ is an isometric C^1-imbedding.

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