Chapter 8

Hydrodynamics

Landau considered hydrodynamics as the fascinating branch of theoretical physics. Not everyone among his disciples would agree with him probably, but howbeit, everyone was obliged to pass hydrodynamics as one of the exams of the Landau "Theoretical Minimum".

8.1 The Bernar's convection

Paper [8a] was the first paper that I have finished after being admitted to the Landau group in 1956. I took an interest in the formal analogy between the Bernar convection and the Landau theory of the second order phase transitions. In fact, in Landau theory the free energy is expanded in powers of the order parameter near the critical temperature. Similarly, it seemed possible to expand the hydrodynamic equations near the threshold of stability of the homogenous heat flow between two parallel plates. From the physical point of view, the two problems, of course, have nothing in common because heat transfer means the dissipative process.

I have had no full access to the hydrodynamic literature of those days, but from several references to my paper, I gather that I probably was among the first ones who applied such an approach to studying weekly non-linear hydrodynamic regimes.

To simplify calculations, I modeled the onset of a Bernar convection in the heat flow between two *free* surfaces, the lower one being at a temperature higher than the upper; for such geometry I arrived to the hexagonal convection cells. The stability of the solution was not studied, although it was known that depending on the experimental conditions, the cells may have other symmetries (cells of the tetragonal symmetry, or the parallel rolls).

I had no time to think of more, for soon, Landau, Abrikosov and I were bogged down in analysis of singularities at the second order phase transitions. We have even elaborated the diagrammatic technique reinvented later by many who used it in the *scaling* approach to the problem, but, instead of calculating the critical indexes, we were trying to single out the "leading" contributions into the specific heat from fluctuations and determine the *exact form* of the singularity at the transition; understandably, our attempts led us nowhere.

8.2 Acoustic coagulation

At the end of the 50's, one group at the Acoustic Institute of the Soviet Academy of Sciences in Moscow was studying the behavior of small particles suspended in a field of standing ultrasound waves. The group probably worked on the problem of air purification for some industries plagued by the formation of dangerous combustive suspensions, such as the sugar powder of sugar-mills, or the precipitation of harmful aerosols. What they have been observing is the clustering of suspended particles into fanciful complexes of various symmetries. The head of the group, N. N. Andreev, an academician, and one from the old school, asked Landau for advice and help. As I have had the interests in hydrodynamics, Landau told me to go and see what could be done.

I derived for them the general expression for the average forces acting on a particle suspended in the acoustic field of the *arbitrary* profile [8b]. The approach could be directly extended to the forces between the particles forming their clusters. I do not know whether my paper was of consequence in the Soviet Union back in the day, for, most probably, the project was classified; recently I have discovered that the results of [8b] became popular in different biological and medical applications in which acoustic forces can be employed for manipulations by small objects, e.g. as non-invasive "acoustic micro-tweezers", or for coagulation of gas bubbles in a liquid (say, air bubbles in blood).

Few omissions and typos made in the English version in translation must be corrected as follows:

$$\text{Eq. (4): } \varphi_{sc} = \frac{a(t-r/c)}{r} - \frac{(\vec{A}(t-r/c)\cdot n)}{r^2} + \cdots \quad \text{Eq. (6): } a(t) = -\frac{R^3}{3\rho}\dot{\rho}_{in}(t);$$

$$\text{Eq. (7): } \varphi_{sc} = -\frac{R^3}{3\rho}\dot{\rho}_{in} f_1 - \frac{R^3}{2} f_2 div\left(\vec{v}_{in}\frac{1}{r}\right);$$

In Eqs. (8, 9) (under the integral):

$$v_{in}v_{in} \to v_{in}v_{sc};$$

Unnumbered, below Eq. (11):

$$\Delta\varphi_{sc} - \frac{1}{c^2}\frac{\partial^2\varphi_{sc}}{\partial t^2} = \frac{4\pi}{3\rho} f_1 R^3 \dot{\rho}_{in}\delta(\vec{r}) + 2\pi f_2 R^3 div(\vec{v}_{in}\delta(\vec{r})).$$

8.3 Running shock wave at acoustic resonance

To the two papers above I would be glad to add one more. Unfortunately, it was published only in Russian in a rather specialized engineering journal.

The work was commissioned by Kapiza. Apart from the discovery of superfluidity of *liquid He II* Petr Kapiza made many other important contributions in the field of the low temperature physics. Recall also that he was a brilliant engineer famous for the legendary Kapiza's helium liquefiers; he conducted experiments and has published several papers on hydrodynamic problems.

One day Kapiza came forward with an idea of the cooling making use of non-linear standing acoustic waves in a gas. That is, in a closed cylinder in the regime close to the acoustic resonance when the amplitude of the ultrasound wave becomes large; inside the cylinder forms a shock wave running back and forth between the piston and closed end of the cylinder. With negligible viscosity, the shock wave becomes the only mechanism of dissipation. I don't think that Kapiza had thought it over in all the technical details, but somehow his idea was to take away the heated air behind the passing shock wave with a few synchronized valves (at that, the interior of the cylinder should, of course, be thermally isolated).

I solved the hydrodynamic equations and obtained the time and the spatial profile of the non-linear oscillations. From the formulas the engineers from the Kapitza group could calculate the amplitude of the shock wave and, hence, the losses at the given amplitude of the piston oscillations. To my thinking, the solution was quite elegant. I regret not knowing what happened to Kapiza's idea, for I left the institute too soon. Howbeit, it was worth seeing that huge steel tube of ten meters long and hearing the rumble!

List of original papers

[8a] L. P. Gor'kov, *Stationary convection in a plane liquid layer near the critical heat transfer point*, Sov. Phys. JETP **6**, 31 (1958) {96}

[8b] L. P. Gor' kov, *On the force acting on a small particle in an acoustic field in an ideal liquid*, Sov. Phys. Doklady **6**, 773 (1962) {431}

SOVIET PHYSICS JETP VOLUME 6(33) NUMBER 2 FEBRUARY, 1958

STATIONARY CONVECTION IN A PLANE LIQUID LAYER NEAR THE CRITICAL HEAT TRANSFER POINT

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Submitted to JETP editor February 5, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 402-407 (August, 1957)

Thermal convection which arises above the limit of stability is investigated for a plane liquid layer. A dependence of the amplitude of motion on a parameter which characterizes the departure from critical conditions has been obtained. Possible symmetry of the flow is discussed.

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*An analogous situation can in principle arise for the scattering of electrons by heavy atoms. Here an $\sim E^{2/3}$ dependence for the current is also obtained, but with different numerical coefficients; this is understandable, since A_2 is different for ions. In this case, however, the electrons have sufficient energy for ionization, which must be taken into account.

HEAT transfer in a plane liquid layer with constant temperature gradient was studied experimentally by Benard,[1] who showed that for certain values of a downward-directed temperature gradient pure thermal conduction is replaced by convection with certain characteristics in a stationary mode; the flow is divided among regular vertical hexagonal prisms and thus possesses hexagonal symmetry. The critical gradient at which convection occurs has been calculated in several papers (see Ref. 2), but the utilization of linearized hydrodynamic equations permitted the determination of only the magnitude of the periods. The determination of the symmetry and amplitude of the motion for supercritical gradients requires study of the nonlinear equations. The present paper presents the results of such an investigation for modes close to the critical value.

A plane horizontal liquid layer of thickness h is bounded above and below by two planes between which a constant temperature difference is maintained. The boundary conditions can be taken in three forms: (a) two rigid planes, (b) one rigid plane and one free surface, and (c) two free surfaces. The last case is distinguished by the simplicity of its equations. Benard's experiments were performed under conditions (b). We shall write the basic equations of stationary convection as follows (see Ref. 3):

$$(\mathbf{v}\nabla)\,\mathbf{v} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \beta \mathbf{g}\theta + \nu\Delta\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0, \quad (\mathbf{v}\nabla\theta) - Av_z = \chi\Delta\theta. \tag{1}$$

Here p and θ are small deviations, due to convection, from the equilibrium pressure P and temperature T:

$$P = p_0 - \rho g z + p; \quad T = -Az + T_0 + \theta,$$

$\mathbf{v}$ is the velocity of the motion; $\beta = -\rho^{-1}(\partial\rho/\partial T)_p$. The boundary conditions at the rigid surface are:

$$\theta = 0, \quad v_z = 0; \quad \partial v_z/\partial z = 0 \quad (v_x = v_y = 0), \tag{1a}$$

and at the free surface:

$$\theta = 0; \quad v_z = 0; \quad \partial^2 v_z/\partial z^2 = 0 \quad (\sigma_{xz} = \sigma_{yz} = 0). \tag{1b}$$

For convenience we shall retain the vector notation only for variables in the horizontal plane $\mathbf{x} = (x, y)$, separating the z coordinate; for all quantities we shall employ a Fourier expansion in these variables:

$$\mathbf{v}(x, y, z) = \sum_k \mathbf{v}_k(z)\, e^{i\mathbf{kx}} \tag{2}$$

and similarly for v_z, p and θ. By virtue of the boundary conditions $v_{0z} = 0$. The velocity $\mathbf{v}_k(z)$ is resolved into components which are parallel and perpendicular to the wave vector $\mathbf{k}$:

$$\mathbf{v}_k = \tilde{\mathbf{v}}_k + \mathbf{k}(\mathbf{k}\mathbf{v}_k)/k^2.$$

The second and third equations of (1) become

$$i(\mathbf{k}\mathbf{v}_k) + dv_{kz}/dz = 0, \tag{3}$$

$$\chi\left(\frac{d^2}{dz^2} - k^2\right)\theta_k + Av_{kz} = \sum_{k'+k''=k}\left\{i(\mathbf{k}\mathbf{v}_{k'})\,\theta_{k''} + \frac{d}{dz}(v_{k'z}\theta_{k''})\right\}. \tag{4}$$

Eliminating the pressure in the first equation,

$$\frac{1}{\rho}p_k = \frac{\nu}{k^2}\left(\frac{d^2}{dz^2} - k^2\right)\frac{dv_{kz}}{dz} + \frac{i}{k^2}\sum_{k'+k''=k}\left\{i(\mathbf{k}\mathbf{v}_{k'})(\mathbf{k}\mathbf{v}_{k''}) + \frac{d}{dz}(v_{k'z}(\mathbf{k}\mathbf{v}_{k''}))\right\},$$

we also obtain

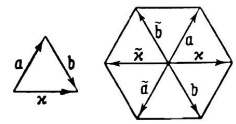
$$-\nu\left(\frac{d^2}{dz^2} - k^2\right)^2 v_{kz} + k^2\beta g\theta_k = \sum_{k'+k''=k}\left\{k^2\left[i(\mathbf{k}\mathbf{v}_{k'})\,v_{k''z} + \frac{d}{dz}(v_{k'z}v_{k''z})\right] + i\frac{d}{dz}\left[i(\mathbf{k}\mathbf{v}_{k'})(\mathbf{k}\mathbf{v}_{k''}) + \frac{d}{dz}(v_{k'z}(\mathbf{k}\mathbf{v}_{k''}))\right]\right\}, \tag{5}$$

$$\nu\left(\frac{d^2}{dz^2} - k^2\right)\tilde{\mathbf{v}}_k = \sum_{k'+k''=k}\left\{i(\mathbf{k}\mathbf{v}_{k'})\left[\mathbf{v}_{k''} - \frac{\mathbf{k}(\mathbf{k}\mathbf{v}_{k''})}{k^2}\right] + \frac{d}{dz}\left(v_{k'z}\left[\mathbf{v}_{k''} - \frac{\mathbf{k}(\mathbf{k}\mathbf{v}_{k''})}{k^2}\right]\right)\right\}. \tag{6}$$

Finally, we give the equations for the Fourier components with $k = 0$:

$$\chi \frac{d^2\theta_0}{dz^2} = \frac{d}{dz}\sum_{k'}(v_{k'z}\theta_{-k'});\quad \frac{d}{dz}\left\{\nu\frac{dv_0}{dz} - \sum_{k'} v_{k'z}\mathbf{v}_{-k'}\right\} = 0;\quad \frac{1}{\rho}\frac{dp_0}{dz} = \beta g\theta_0 - \frac{d}{dz}\sum_{k'}(v_{k'z}v_{-k'z}).$$

Before proceeding to study the equations which have been derived we shall determine the vectors **k** of the Fourier series (2). As was shown by the investigation of stability in Ref. 2, when the critical temperature gradient is attained there arises convective motion of the form $f(z)e^{i\mathbf{kx}}$ with a certain value $k = \kappa_0$ which depends on the experimental conditions (a, b or c). In (2), and similarly in the expansions for v_z and θ, we shall call v_{kz} with $k = \kappa_0$ the principal term. It is easy to determine all such vectors of the flow. Indeed, as can be seen from (3)—(6), corrections to the principal terms with $k = \kappa_0$ will consist of terms with wave vectors which are again equal in magnitude to κ_0. Therefore, if (2) contains a term with the wave vector $\kappa(|\kappa| = \kappa_0)$ there must also be terms with the vectors a and b ($a = b = \kappa_0$), thus forming the equilateral triangle of Fig. a. It is clear that there are altogether six such vectors $\kappa, \mathbf{a}, \mathbf{b}, \tilde{\kappa}, \tilde{\mathbf{a}}, \tilde{\mathbf{b}}$, which are represented together in Fig. b. The Fourier components which are thus designated, for all quantities, comprise the set of principal terms of the solution; the remaining terms correspond to wave vectors representing various sums of the six principal vectors and will be small for temperature gradients near the critical value.



a b

We shall write the principal terms in the form

$$v_{iz} = X_i f(z);\quad \theta_i = X_i\varphi(z),$$

where[2]

$$\{(d^2/dz^2 - \kappa_0^2)^3 + (A\beta g\kappa_0^2/\chi\nu)\} f(z) = 0;\ \varphi(z) = (\nu/\kappa_0^2\beta g)(d^2/dz^2 - \kappa_0^2)^2 f(z). \tag{7}$$

It will be sufficient hereinafter to confine ourselves to terms of the third order in X, i.e., to take into account all triple combinations of the principal vectors. This means that all quantities v_{kz} and θ_k can be obtained up to terms of the second order in X; in other words, in addition to the corrections $\delta v_{\kappa z}$ and $\delta\theta_\kappa$ (and similarly for **a**, **b** etc.) to the expressions in (7), it is necessary to obtain the Fourier components for the velocities and temperature corresponding to wave vectors each of which is a sum of two principal vectors; for example $\mathbf{f} = \boldsymbol{\kappa} + \mathbf{b}$, $2\boldsymbol{\kappa}$, $\mathbf{o}$ etc. We note that by virtue of the equation of continuity (3) expressions of the form $(\mathbf{qv}_l)$ in (4) and (5) can be rewritten as

$$(\mathbf{qv}_l) = (\mathbf{q}\tilde{\mathbf{v}}_l) + i(\mathbf{ql})\, l^{-2} dv_{lz}/dz.$$

From (6) it follows, to terms of the second order, that

$$\nu\left(\frac{d^2}{dz^2} - l^2\right)(\mathbf{q}\tilde{\mathbf{v}}_l) = i\sum_{k'+k''=l}\left\{\left[\frac{(\mathbf{qk}'')}{k''^2} - \frac{(\mathbf{ql})(\mathbf{lk}'')}{l^2k''^2}\right]\left[\frac{d}{dz}\left(v_{k'z}\frac{dv_{k''z}}{dz}\right) - \frac{(\mathbf{lk}')}{k'^2}\frac{dv_{k'z}}{dz}\frac{dv_{k''z}}{dz}\right]\right\}.$$

Substituting the expressions in (7) we see that for all required $\boldsymbol{l}$ and $\mathbf{q}$ the right-hand side vanishes. Second-order terms in $(\mathbf{q}\tilde{\mathbf{v}}_l)$ vanish so that in (4) and (5) we can write

$$(\mathbf{qv}_l) \to i(\mathbf{ql})\, l^{-2} dv_{lz}/dz.$$

For brevity we introduce the notation

$$D_\kappa = (d^2/dz^2 - \kappa_0^2).$$

From (4) and (5), including third-order terms in X, we obtain

$$D_k^3\theta_k + \frac{k^2A\beta g}{\chi\nu}\theta_k = \frac{1}{\chi}\sum_{k'+k''=k} D_k^2\left\{\frac{d}{dz}(v_{k'z}\theta_{k''}) - \frac{(\mathbf{kk}')}{k'^2}\frac{dv_{k'z}}{dz}\theta_{k''}\right\} + \frac{A}{\chi\nu}\sum_{k'+k''=k}\left\{k^2\left[\frac{d}{dz}(v_{k'z}v_{k''z}) - \frac{(\mathbf{kk}')}{k'^2}\frac{dv_{k'z}}{dz}v_{k''z}\right]\right.$$
$$\left. + \frac{(\mathbf{kk}'')}{k''^2}\frac{d}{dz}\left[\frac{(\mathbf{kk}')}{k'^2}\frac{dv_{k'z}}{dz}\frac{dv_{k''z}}{dz} - \frac{d}{dz}\left(v_{k'z}\frac{dv_{k''z}}{dz}\right)\right]\right\}. \tag{8}$$

The magnitude of the dimensionless parameter $\gamma = A\beta gh^4/\chi\nu$ determines the possibility of the existence of stationary convective motion. The critical value γ_0 at which instability arises[2] is for two rigid surfaces ≈ 1708, for one free surface ≈ 1100 and for two free surfaces ≈ 657. We shall write (8) for the κ component as follows:

$$D_\kappa^3\theta_\kappa + \gamma_0\theta_\kappa = -\Delta\gamma\theta_\kappa + R_\kappa(\theta^3)$$

where $R_\kappa(\theta^3)$ is the right-hand side of (8) and $\Delta\gamma = \gamma - \gamma_0$. The homogeneous equation has the solution $\varphi(z)$, with γ_0 possessing the same meaning. The condition

$$\int f\{R_\varkappa(\theta^3) - \Delta\gamma\theta_\varkappa\}\,dz = 0, \tag{9}$$

which is required for the solution of (8), leads to the determination of the amplitude of motion X. This condition has the form

$$\frac{\varkappa_0^2\Delta\gamma}{h^4}X_\varkappa\int f\varphi dz = \frac{1}{\chi}\int(D_\varkappa^2 f)\sum_{k'+k''=\varkappa}\left\{\frac{d}{dz}(v_{k'z}\theta_{k''}) - \frac{(\varkappa\mathbf{k}')}{k'^2}\frac{dv_{k'z}}{dz}\theta_{k''}\right\}dz + \frac{A\varkappa_0^2}{\chi\nu}\int f\sum_{k'+k''=\varkappa}\left\{\frac{d}{dz}(v_{k'z}v_{k''z}) - \frac{(\varkappa\mathbf{k}')}{k'^2}\frac{dv_{k'z}}{dz}v_{k''z}\right\}dz$$

$$+\frac{A}{\chi\nu}\int\frac{df}{dz}\sum_{k'+k''=\varkappa}\left\{\frac{(\mathbf{k}\mathbf{k}'')}{k''^2}\left[\frac{d}{dz}\left(v_{k'z}\frac{dv_{k''z}}{dz}\right) - \frac{(\mathbf{k}\mathbf{k}')}{k'^2}\frac{dv_{k'z}}{dz}\frac{dv_{k''z}}{dz}\right]\right\}dz.$$

Second-order terms in X drop out, as is shown by substitution of (7) into the right-hand side;* for the purpose of calculating third-order terms it is necessary to obtain the corrections to v_z and θ. Such corrections appear for the following quantities (see Fig. b):

$$\delta\theta_a = X_{\tilde b}X_\varkappa\xi_1(z);\ \delta\theta_b = X_{\tilde a}X_\varkappa\xi_1(z);\ \delta\theta_f = X_\varkappa X_b\xi_2(z);\quad \delta\theta_g = X_aX_\varkappa\xi_2(z);\ \delta\theta_{2\varkappa} = X_\varkappa^2\xi_3(z);$$

$$\delta\theta_0 = (|X_a|^2 + |X_a|^2 + |X_\varkappa|^2)\xi_4(z).\ \delta v_{az} = X_{\tilde b}X_\varkappa\eta_1(z);\ \delta v_{bz} = X_{\tilde a}X_\varkappa\eta_1(z);\ \delta v_{fz} = X_\varkappa X_b\eta_2(z);$$

$$\delta v_{gz} = X_\varkappa X_a\eta_2(z);\ \delta v_{2\varkappa} = X_\varkappa^2\eta_3(z).$$

Using these designations (9) becomes

$$\frac{\chi\nu\Delta\gamma}{\beta g\hbar^4}\int(fD^2f)\,dz = (|X_a|^2 + |X_b|^2)\int\left\{\frac{A}{2\nu}[Df'f(\eta_1+\eta_2) + 2f'Df\eta_1] - fD^2f'(\xi_1+\xi_2+\xi_4) - \frac{D^2f}{2}f'(\xi_1-\xi_2+2\xi_4)\right\}dz$$

$$+|X_\varkappa|^2\int\left\{\frac{A}{2\nu}[f'''f - f''f']\eta_3 + \left[\xi_3(D^2ff' - D^2f'f) - \xi_4\frac{d}{dz}(fD^2f)\right]\right\}dz$$

where primes denote differentiation with respect to z. Two other equations are then obtained directly by the substitutions: $\kappa\to a$, $a\to b$, $b\to\kappa$, or $\kappa\to b$, $a\to\kappa$, $b\to a$.

The solution of these equations is obviously $|X_\kappa| = |X_a| = |X_b|$; the amplitude of convective motion is then proportional to the square root of the parameter Δ_γ which characterizes the supercritical heat transfer mode. But our equations contain only the absolute value of X so that the phase relation cannot be determined. It is clear how this has come about: when we write (9) for θ_κ, say, we must select in R all combinations corresponding to any three principal vectors whose vector sum is x. It is easily seen from the figure that any such combination must be of the form $X_\kappa|X_i|^2$ so that X_κ is cancelled and (10) contains only the square of the modulus of X. Fourth-order terms in X in (9) obviously drop out. In the fifth order it is possible to obtain a combination which is not proportional to X_κ: the sum $2\mathbf{a} + 2\mathbf{b} + \tilde{\boldsymbol\kappa}$ in the right-hand side gives a term which is proportional to $X_\kappa \sim X_a^2X_b^2$. Writing $X = |X|e^{i\delta}$, we obtain because of the reality of the coefficients the following phase relation:

$$\delta_a + \delta_b - \delta_\varkappa = 0,\ \pi/2.$$

In the first case it is possible to make all phases equal to zero by shifting the coordinate origin. The principal term in the solution becomes

$$v_z = B\left(\frac{\chi}{\nu}\right)f(z)\sqrt{\frac{\Delta\gamma}{\gamma}}\left\{\cos\varkappa_0 y + \cos\frac{\varkappa_0}{2}(y + \sqrt{3}x) + \cos\frac{\varkappa_0}{2}(y - \sqrt{3}x)\right\} \tag{11a}$$

where the y axis is parallel to κ. This solution possesses hexagonal symmetry; in the (x, y) plane it represents a periodic structure of regular hexagonal prisms of the length $4\pi/\kappa_0$ along each side of the base; the liquid flows upward at the center of each prism and downward along the walls.[2]

In the second case, with a suitable choice of the coordinate origin, the solution can be written as follows:

$$v_z = B\left(\frac{\chi}{\nu}\right)f(z)\sqrt{\frac{\Delta\gamma}{\gamma}}\left\{\sin\frac{\varkappa_0}{2}(y + \sqrt{3}x) + \sin\frac{\varkappa_0}{2}(y - \sqrt{3}x) - \sin\varkappa_0 y\right\}. \tag{11b}$$

This flow permits three-fold rotations around the vertical axis (the z axis) and is symmetric with respect to the x axis, but changes sign with the substitution $y\to -y$. In the horizontal plane it breaks up

*For symmetrical boundary conditions this is clear from simple considerations, since the solution of f(z) is symmetrical with respect to z = 0.

into alternate equilateral triangles of side length $4\pi/\sqrt{3}\kappa_0$, in each of which the liquid flows upward or downward, respectively.

A decision as to which of the two types of flow will occur in reality could be reached theoretically only by investigating fifth-order terms in X in the basic equations of convection, but in actuality it would be difficult to do this. For a liquid layer on a rigid plate experiment[1] favors the first type of flow.

The equations for the ξ functions are

$$\left[D^3+\frac{\varkappa_0^2\gamma}{h^4}\right]\xi_1=\frac{2\nu}{\varkappa_0^2\beta g\chi}\left\{D^2\left(fD^2f'+\frac{f'}{2}D^2f\right)-\frac{A\beta g\varkappa_0^2}{\nu^2}\left(fDf'+\frac{f'}{2}Df\right)\right\};$$

$$\left[(D-2\varkappa_0^2)^3+\frac{3\varkappa_0^2\gamma}{h^4}\right]\xi_2=\frac{2\nu}{\varkappa_0^2\beta g\chi}\left\{(D-2\varkappa_0^2)^2\left(fD^2f'-\frac{f'}{2}D^2f\right)-\frac{3A\beta g\varkappa_0^2}{2\nu^2}fDf'\right\}; \tag{12}$$

$$\left[(D-3\varkappa_0^2)^3+\frac{4\varkappa_0^2\gamma}{h^4}\right]\xi_3=\frac{\nu}{\varkappa_0^2\beta g\chi}\left\{(D-3\varkappa_0^2)^2(fD^2f'-f'D^2f)+\frac{2A\beta g\varkappa_0^2}{\nu}(f'f''-ff''')\right\};\quad d^2\xi_4/dz^2=(2\nu/\varkappa_0^2\beta g\chi)\,d(fD^2f)/dz,$$

and the expressions for the η functions are

$$A\eta_1=-\chi D\xi_1+2(f\varphi'+{}^1/_2\,\varphi f');\quad A\eta_2=-\chi(D-2\varkappa_0^2)\xi_2+2(f\varphi'-{}^1/_2\,\varphi f');\quad A\eta_3=-\chi(D-3\varkappa_0^2)\xi_3+(f\varphi'-f'\varphi). \tag{13}$$

The calculation of these quantities subject to the boundary conditions 1a and 1b would be very laborious for the general case. The problem becomes simpler only in the somewhat hypothetical case of a liquid with two free surfaces. We present the results in so far as they are of a general character.

In this case the critical value is $\gamma_0 = 27\pi^4/4$, $\kappa_0^2 = \pi^2/2h^2$ and $f(z)$, which determines the dependence of the vertical velocity component on height, can be taken in the form

$$f(z)=\cos\frac{\pi z}{h}\left(\text{ or }\quad \varphi(z)=\frac{9\nu\pi^2}{2\beta gh^2}\cos\frac{\pi z}{h}\right).$$

Equations (12) and (13) can be solved simply and give for ξ and η

$$\left.\begin{aligned}\xi_1&=\frac{9\nu\pi[9+\chi/\nu]}{52\chi\beta gh}\\ \xi_2&=\frac{9\nu\pi[121+27\chi/\nu]}{2500\chi\beta gh}\\ \xi_4&=\frac{9\nu\pi}{4\chi\beta gh}\end{aligned}\right\}\times\sin\frac{2\pi z}{h},\quad \left.\begin{aligned}\eta_1&=\frac{h[1+3\chi/\nu]}{26\pi\chi}\\ \eta_2&=\frac{9h[3+11\chi/\nu]}{1250\pi\chi}\end{aligned}\right\}\times\sin\frac{2\pi z}{h},$$

$$\xi_3=0.$$

By substituting these expressions in (10) we can determine X. For example, the temperature distribution for the first type of flow is given by

$$\theta(xyz)=\frac{4Ah}{3\pi}\sqrt{\frac{\Delta\gamma}{\gamma}}\frac{\cos(\pi z/h)}{\sqrt{1.35+0.08\chi/\nu+0.14\chi^2/\nu^2}}\left\{\cos\frac{\pi y}{2h}+\cos\frac{\pi}{4h}(y+\sqrt{3}x)+\cos\frac{\pi}{4h}(y-\sqrt{3}x)\right\} \tag{14}$$

The development of convection is conveniently characterized by the ratio of the maximum temperature change in the (x, y) plane, $T_{max} - T_{min}$, to the vertical temperature difference $T_1 - T_2 = Ah$:

$$\frac{T_{max}-T_{min}}{T_1-T_2}=\frac{4}{\pi}\sqrt{\frac{\Delta\gamma}{\gamma}}\left(1.35+0.08\,\frac{\chi}{\nu}+0.14\,\frac{\chi^2}{\nu^2}\right)^{-1/2}.$$

We note in conclusion that the form of the dependence of the coefficient in (14) on the Prandtl number is the same for any boundary conditions. This results from the fact that the solution of the linear equation $f(z) \equiv f(z/h)$ depends only on the critical value γ_0, and also on how (12), (13) and (10) contain the different dimensional quantities. Thus the strength of stationary thermal convection in a plane liquid layer for modes near the critical point is proportional to the square root of the parameter which characterizes supercriticality, whereas dependence on the Prandtl number appears in a denominator as the square root of a polynomial of not higher than the second degree. The flow belongs to one of the two types of symmetry represented by (11a) and (11b).

The author is indebted to Academician L. D. Landau for valuable suggestions and assistance.

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Translated by I. Emin

SOVIET PHYSICS – DOKLADY VOLUME 6, NUMBER 9 MARCH, 1962

PHYSICS

ON THE FORCES ACTING ON A SMALL PARTICLE IN AN ACOUSTICAL FIELD IN AN IDEAL FLUID

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(Presented by Academician L. D. Landau, April 28, 1961)
Translated from Doklady Akademii Nauk SSSR, Vol. 140, No. 1, pp. 88-91, September, 1961
Original article submitted April 22, 1961

When a particle is suspended in the field of a sound wave, the fluid exerts hydrodynamical forces on it. In the linear approximation, these forces are proportional to the velocity of the fluid [1] and, on the average, do not lead to a displacement of the particle. In problems of acoustical coagulation, an important part is played by average forces acting on the particle which arise as the result of second order effects. These forces are of a nature related to that of the radiation-pressure forces in a sound wave [2, 3]. The magnitude of the forces has been found only for particles which are in an ideal fluid, in a paper by King [2]. The method of [2] was to solve exactly the problem of the flow around a small sphere in the field of a sound wave. So far as is known to us, no account was taken of effects associated with viscosity and thermal conductivity.

We present here a simple method which allows us to determine the magnitude of the average forces that act on the particle in an arbitrary acoustical field when the size of the particle is much smaller than the wavelength of the sound. In this paper we shall treat the case of an ideal fluid.

The magnitude of the force is equal to the average flux of momentum through any closed surface in the fluid which encloses the particle:

$$F_i = -\oint \overline{\Pi_{ik}} df_k.$$

Here Π_{ik} is the momentum flux density tensor in the ideal fluid: $\Pi_{ik} = p\,\delta_{ik} + \rho v_i v_k$. We shall take as the surface a sphere with a radius much larger than λ, the wavelength of the sound. In the linear approximation the velocity potential φ is the sum of waves incident on the particle and scattered by it:

$$\varphi(\mathbf{r}, t) = \varphi_{in}(\mathbf{r}, t) + \varphi_{sc}(\mathbf{r}, t). \tag{1}$$

At the same time, we have from Euler's equations ($p = p_0 + p'$)

$$p' = -\rho\frac{\partial\varphi}{\partial t} - \rho\frac{v^2}{2} + \frac{\rho}{2c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2$$

Since $\overline{(\partial\varphi/\partial t)} \equiv 0$, we get for the average force

$$F_i = -\oint\left\{\left[-\rho\frac{\overline{v^2}}{2} + \frac{\rho}{2c^2}\overline{\left(\frac{\partial\varphi}{\partial t}\right)^2}\right]\delta_{ik} + \rho\overline{v_i v_k}\right\} df_k.$$

Thus, for the calculation of the average force correct up to terms of the second order in the velocity, it is sufficient to find the solution of the linear scattering problem.

In the wave zone the potential of the scattered wave can be expanded in terms of multipoles:

$$\varphi_p = -\frac{a(t - r/c)}{r} - \frac{(\dot{\mathbf{A}}(t - r/c), \mathbf{n})}{rc} + \dots \tag{4}$$

For particles of size $R \ll \lambda$, the coefficients in Eq. (4) are in turn determined from the solution of the problems of the flow of an incompressible fluid around the body (cf. [1]). In fact, for $r \ll \lambda$ the wave equation goes over into the equation of an incompressible liquid, $\Delta\varphi_{sc} = 0$, and the decreasing solutions of this equation are:

$$\varphi_{sc} = -\frac{a(t)}{r} - \frac{(\mathbf{A}(t), \mathbf{n})}{r^2} + \dots \tag{4'}$$

For the potential flow of an ideal fluid around a stationary sphere the second term is of the form:

$$-(\mathbf{v}_{in}(t), \mathbf{n})\frac{R^3}{2r^2}, \tag{5}$$

where $\mathbf{v}_{in}(t)$ is the velocity in the incident wave at the place where the sphere is located. The first term corresponds to the "ejection" of mass owing to the compression of the gas in the incident wave, and is given by

$$a(t) = -\frac{R^3}{3\rho}\dot{\rho}_{in}(t). \tag{6}$$

The expressions (5) and (6) relate only to a stationary rigid sphere. In the general case, the sphere is partially entrained by the fluid and acquires a velocity $\mathbf{u}(t)$:

$$\mathbf{u}(t) = \frac{3\rho}{\rho + 2\rho_0}\mathbf{v}_{in}(t),$$

where ρ_0 is the density of the sphere. Therefore, if ρ_0 is comparable with the density of the fluid we must insert in Eq. (5) the relative velocity $\mathbf{v}_{in} - \mathbf{u}$.

If one takes into account compressibility of the sphere itself, one gets instead of Eq. (6),

$$a(t) = -\frac{R^3}{3\rho}\dot{\rho}_{in}(t)\left(1 - \frac{c^2\rho}{c_0^2\rho_0}\right)$$

Collecting the results, we find for the potential (4) of the waves scattered by the sphere the expression

$$\varphi_p = -\frac{R^3}{3\rho r}\dot{\rho}_{in} f_1 - \frac{R^3}{2} f_2 \operatorname{div}\left(\mathbf{v}_{in}\frac{1}{r}\right) \qquad (7)$$

where

$$f_1 = 1 - c^2\rho/c_0^2\rho_0, \quad f_2 = 2(\rho_0 - \rho)/(2\rho_0 + \rho).$$

As was shown in [2], the magnitude of the average force in a standing wave is larger than in a plane running wave. In the former case, in the quadratic expression (3) for the force there are also important contributions from the interference terms between the incident and scattered waves, whereas for a running wave the magnitude of the momentum imparted to the particle by the waves is determined only by the momentum carried away by the scattered wave. In fact, by using the relations between the quantities in a plane running wave, we can rewrite Eq. (3) in the following form:

$$F_x = -\rho\oint\left[\overline{v_{in}v_{in}}(1 + \cos\theta) + \overline{v_{sc}^2}\cos\theta\right]df \qquad (8)$$

(the x axis is taken in the direction of propagation of the incident wave, and v_{sc} is the projection of the velocity of the scattered waves in the direction of the radius of the sphere). On the other hand, in the absence of dissipation, the average flux of energy through the closed surface is zero. Expanding the expression for the energy flux $\mathbf{Q} = \rho\mathbf{v}(v^2/2 + w)$ [1] to terms of second order, we find

$$\oint \mathbf{Q}\, d\mathbf{f} = \rho c\oint\left[\overline{v_{in}v_{in}}(1 + \cos\theta) + \overline{v_{sc}^2}\right]df \equiv 0. \qquad (9)$$

From this we have

$$F_x = \rho\oint \overline{v_{sc}^2}(1 - \cos\theta)\, df. \qquad (8')$$

From a monochromatic wave we get

$$F = \frac{4\pi I}{9c} R^2 (kR)^4\left[f_1^2 + f_1 f_2 + \frac{3}{4} f_2^2\right], \qquad (10)$$

where $I = \rho c u_0^2/2$ is the average energy flux density in the incident waves.

The formula (8') is valid only in a plane running wave. In an arbitrary acoustical field the main contribution to the expression (3) for the force comes from interference terms between the incident and scattered radiation, which lead to forces much larger than that given by Eq. (10). Confining ourselves to these terms in Eq. (3), we find

$$F_i = -\oint\left\{\left[-\rho\left(\overline{v_{sc}v_{in}}\right) + \frac{c^2}{\rho}\overline{\rho'_{in}\rho'_{sc}}\right]\delta_{ik} + \rho\left(\overline{v_{isc}v_{kin}} + \overline{v_{in}v_{ksc}}\right)\right\} df_k.$$

Changing to a volume integral, and using the Euler equations and the fact that the flow is a potential flow, we get

$$\mathbf{F} = -\int_V \overline{\mathbf{v}_{in}\left(\rho \operatorname{div}\mathbf{v}_{sc} + \frac{\partial\rho_{sc}}{\partial t}\right)}\, dV$$

$$= -\rho\int_V \overline{\mathbf{v}_{in}\left(\Delta\varphi_{sc} - \frac{1}{c^2}\frac{\partial^2\varphi_{sc}}{\partial t^2}\right)}\, dV; \qquad (11)$$

but, according to Eq. (7):

$$\Delta\varphi_{sc} - \frac{1}{c^2}\frac{\partial^2\varphi_p}{\partial t^2} = \frac{4\pi}{3\rho} f_1 R^3 \dot{\rho}_{in}\,\delta(\mathbf{r}) + 2\pi f_2 R^3 \operatorname{div}(\mathbf{v}_{in}\delta(\mathbf{r})).$$

Substituting this in Eq. (11) and introducing the quantity $U(\mathbf{r})$, the potential of the forces $\mathbf{F} = -\nabla U$ we get

$$U = 2\pi R^3\rho\left\{\frac{\overline{p'^2_{in}}}{3\rho^2c^2} f_1 - \frac{\overline{v^2_{in}}}{2} f_2\right\} \qquad (12)$$

($\overline{p'^2_{in}}$ and $\overline{v^2_{in}}$ are the mean square fluctuations of the pressure and velocity in the wave at the point where the particle is located). The force in a standing wave $\varphi_{in} = -(u_0/k)\cos\omega t\cos kx$:

$$F = 4\pi\bar{E}R^2(kR)\sin 2kx\left\{\frac{\rho_0 + \frac{2}{3}(\rho_0 - \rho)}{2\rho_0 + \rho} - \frac{1}{3}\frac{c^2\rho}{c_0^2\rho_0}\right\} \qquad (13)$$

($\bar{E}$ is the average density of the acoustical energy). The formulas (10) and (11) agree with the results of [2, 3].

In the field $U(\mathbf{r})$, particles collect near the minimum of the potential energy (12). The equilibrium density distribution of the suspended particles $n(\mathbf{r})$ follows the Boltzmann formula

$$n(\mathbf{r}) \backsim \exp\{-U(\mathbf{r})/kT\}. \qquad (14)$$

At the places where the density is greatest the further coagulation of the particles is facilitated. At ordinary sound intensities $I \sim 0.1$ w/cm^2 and $\omega \sim 10^4$ sec^{-1} the accumulation of particles at the nodes or antinodes of the wave plays an important role. For example, the width Δ of the region in which the particles are concentrated in a standing wave, Eq. (13), is given in order of magnitude by*

*For very small distances between the particles, or near a wall, forces of interaction between the particles and between the particles and the wall become important.

$$(k\Delta)^2 = \frac{3\kappa T}{10\pi R^3 \overline{E}} .$$

(For $E \approx 4 \cdot 10^{10}$ erg /cm^3, $\nu \approx 1$ kc, $\Delta \approx 1.5 \cdot 10^{-6} R^{-3/2}$) The time after which the distribution (4) is established is $\tau \sim \eta \ (kR)^{-2} \ (E)^{-1}$, where η is the viscosity of the medium.

We emphasize that Eq. (12) holds for an arbitrary field, with the exception of fields closely similar to that of a plane running wave. In particular, in a spherically converging or diverging wave

$$U(r) = \frac{\overline{Q}R^3}{2c}\left\{\frac{f_1}{3r^2} - \frac{f_2}{2}\left(\frac{1}{r^2} + \frac{1}{k^2r^4}\right)\right\} \qquad (15)$$

(Q is the power of the source).

When $f_2 > 0$, $f_1 > {}^3/_2 f_2$, the particles either collect at the center or move out to infinity, depending on the distance from the center; for $f_2 > 0$, $f_1 < {}^3/_2 f_2$ there is a collapse to the center; if $f_1 < {}^3/_2 f_2$, $f_2 < 0$ Eq. (15) has a minimum at distances of the order of a wavelength; finally, if $f_1 > {}^3/_2 f_2 > 0$, the particles are pushed out to infinity. Of course, at very large distances from the center the quantity (15) becomes small, the spherical wave goes over into a plane wave, and the main part is played by the forces (10):

$$U(r) = \pm \frac{\overline{Q}R^2}{9cr}(kR)^4\left\{f_1^2 + f_1 f_2 + \frac{3}{4} f_2^2\right\} \qquad (16)$$

(the plus sign is for a diverging wave and the minus sign for a converging wave).

The values of (16) and (15) are comparable for $r \sim R\,(kR)^{-4}$. For the results given here to be valid it is necessary for the condition $\sqrt{\eta/\rho\omega} \ll R$. to hold.

In conclusion the author expresses his gratitude to Academician L. D. Landau for a discussion of this work and for valuable advice.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.