Classification

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Content

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Binary Classification

Suppose we want to build an email spam classification software:

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> Hello
> I want to know about different topics that relate to qualitative reinforcement learning and make abstraction&aggregation... to solve problem compactly . I have read some survey to know exactly but sometimes I doubt about some topics are related to or not. for example Qualitative Spatial Representation and Reasoning. can anyone tell me different categorized topics? I need to know the general classification of them.
>
> ...

Binary Classification

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- The x may be some features of some piece of email.

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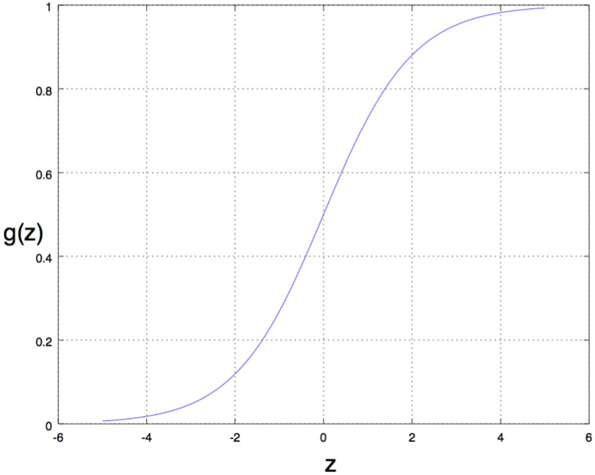
- where $g(z) = \frac{1}{1+e^{-z}}$

Binary Classification

$$g(z) = \frac{1}{1+e^{-z}} \qquad g : \mathbb{R} \to (0,1)$$



Derivative of the Sigmoid Function

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&= g(z)\left(1-g(z)\right).
\end{aligned}
$$

A Probabilistic Approach

Lets assume that:

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P(y = 1|x;\theta) &= h_\theta(x) \\
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Note that this can be written more compactly as:

$$p(y|x;\theta) = (h_\theta(x))^y(1-h_\theta(x))^{1-y}$$

Likelihood of the Training Data's Labels

Assuming that the m training examples were generated independently, we can then write:

$$
\begin{aligned}
L(\theta) &= p(\mathrm{y}|\mathrm{X};\theta) \\
&= \prod_{i=1}^{m} p(y_i|\mathrm{x}_i;\theta) \\
&= \prod_{i=1}^{m} (h_\theta(x_i))^{y_i}(1-h_\theta(x_i))^{1-y_i}
\end{aligned}
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$$L(\theta) = \prod_{i=1}^{m} (h_\theta(x_i))^{y_i} (1 - h_\theta(x_i))^{1-y_i}$$

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Using gradient ascent we get an update rule like this:

$$\theta := \theta + \alpha \nabla_\theta \ell(\theta)$$

Maximizing the Likelihood

Working with one example, the derivatives are as follows:

$$\frac{\partial}{\partial \theta_j} \ell(\theta) = \frac{\partial}{\partial \theta_j} \Big[y \log h(x) + (1-y) \log(1-h(x)) \Big]$$

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&= \left(y\frac{1}{g(\theta^\top \mathrm{x})} - (1-y)\frac{1}{1-g(\theta^\top \mathrm{x})}\right)\frac{\partial}{\partial\theta_j}g(\theta^\top \mathrm{x})
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&= \Big(y(1-g(\theta^\top \mathrm{x})) - (1-y)g(\theta^\top \mathrm{x})\Big)x_j
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&= \Big(y(1-g(\theta^\top \mathrm{x})) - (1-y)g(\theta^\top \mathrm{x})\Big)x_j \\
&= (y - h_\theta(\mathrm{x}))x_j
\end{aligned}
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The LMS Update Rule for Classification

Given that:

$\frac{\partial}{\partial\theta_j}\ell(\theta) = (y - h_\theta(\mathrm{x}))x_j,$

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where $h_\theta(\mathrm{x}_i) = g(\theta^\top \mathrm{x}_i) = \frac{1}{1+e^{(-\theta^\top \mathrm{x}_i)}}$ is now defined as a non-linear function of $\theta^\top \mathrm{x}_i$.

So, we end up with the same update rule for a different algorithm and learning problem.

LMS Algorithms for Classification

Batch Gradient Descent

Repeat until convergence {

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{m} \left[y_i - g(\theta^\top x_i) \right] (x_i)_j \qquad \text{(for every } j\text{)}.$$

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Stochastic Gradient Descent

Loop {

for $i = 1$ to m {

$$\theta_j := \theta_j + \alpha \left[y_i - g(\theta^\top x_i) \right] (x_i)_j \qquad \text{(for every } j\text{)}.$$

}

}

LMS Algorithms for Classification

Mini-Batch Gradient Descent

Repeat until convergence $\{$

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{k} \left[y_i - h_\theta(x_i) \right] (x_i)_j \qquad \text{(for every } j\text{).}$$

$\}$

Here we use mini-batches containing 10 to 1000 examples. This is $k \in [10, 1000]$.

Error of a Binary Classifier (1/2)

True Positive $=$ A positive example correctly identified as positive.

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It answers the question: How many of the examples identified as positives are indeed positives?

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Recall (Sensitivity)

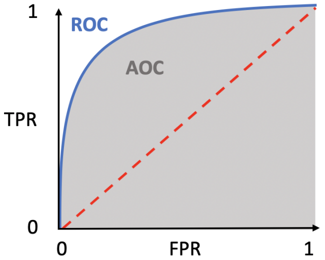
It answers the question: How many of the positive examples were correctly identified as positive?

$$\text{Recall} = \frac{TP}{P} = \frac{TP}{TP+FN}.$$

Receiver Operating Characteristic (ROC) curve

The Receiver Operating Characteristic curve is a graph showing the performance of a binary classifier with all the classification thresholds.

It shows two parameters: the true positive rate (TPR) and the false positive rate (FPR).



$$TPR = Recall = Sensitivity = \frac{TP}{TP+FN}$$

$$Specificity = \frac{TN}{FP+TN}$$

$$FPR = \frac{FP}{FP+TN} = 1 \text{ - specificity}$$

AOC = Area under the ROC curve.

The Perceptron Learning Algorithm

Consider modifying the logistic regression to output either 1 or 0:

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By making $h_\theta(x) = g(\theta^\top x)$, then we have the update rule:

$$\theta_j := \theta_j + \alpha(y_i - h_\theta(x_i))(x_i)_j.$$

Newton's Method for Finding a Zero of a function

Suppose we have a function $f : \mathbb{R} \to \mathbb{R}$ and we want to find a value of θ such that $f(\theta) = 0$, with $\theta \in \mathbb{R}$.

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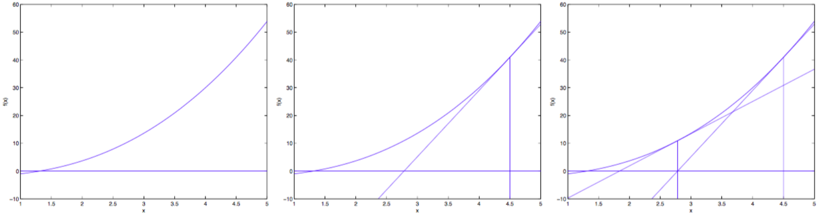

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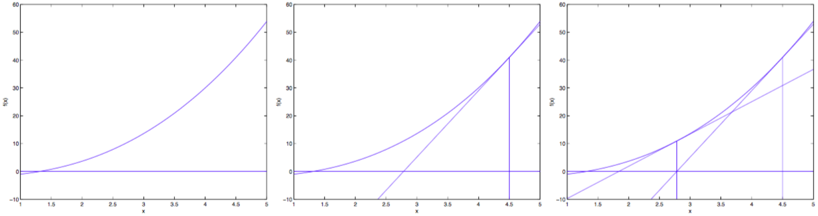

Newton's method performs the following update rule:

$$\theta := \theta - \frac{f(\theta)}{f'(\theta)}.$$

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Now, suppose we want to maximize a function ℓ. The maxima of ℓ correspond to points where its first derivative $\ell'(\theta)$ is zero.

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Now, suppose we want to maximize a function ℓ. The maxima of ℓ correspond to points where its first derivative $\ell'(\theta)$ is zero.

So, by letting $f(\theta) = \ell'(\theta)$, we can use the same algorithm to maximize ℓ:

$$\theta := \theta - \frac{\ell'(\theta)}{\ell''(\theta)}.$$

Newton-Raphson Method

In our regression setting θ is vector-valued. The generalization of Newton's method to this multidimensional setting is given by

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where H is the Hessian matrix

$$H_{ij} = \frac{\partial^2 \ell(\theta)}{\partial \theta_i \partial \theta_j}.$$

Thank you!

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