Analysis 2 W2-2

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Goal: To show if $F : [a, b] \to \mathbb{R}$ is increasing then F is differentiable a.e.

Recall: (x_n): discontinuities of F. $\alpha_n = F(x_n^+) - F(x_n^-)$. $\exists \theta_n \in [0, 1]$ s.t. $F(x_n) = F(x_n^-) + \theta_n \alpha_n$

$$j_n(x) = \begin{cases} 0 & \text{if } x < x_n \\ \theta_n & \text{if } x = x_n \\ 1 & \text{if } x > x_n \end{cases}$$

$$J_F(x) = \sum_{n=1}^{\infty} \alpha_n j_n(x)$$

where the series converges absolutely and uniformly.

Lemma 1. Suppose that F is increasing and bounded on $[a, b]$. Then J_F is discontinuous preccisely at (c_n), and has jump at x_n equal to that of F. Moreover, $F - J_F$ is increasing and continuous.

Proof.

If $x \neq x_n$ $\forall n$, each j_n is continuous at x. By uniform convergence, J_n is continuous at x. If $x = x_k$ for some k, write

$$J_F(x) = \sum_{n=1}^{k} \alpha_n j_n(x) + \sum_{n=k+1}^{\infty} \alpha j_n(x)$$

where the first part has a jump discontinuity at x_k of size α_k and the second part is continuous. Alse, the jump size of J_F is the same as that of F. $\Rightarrow F - J_F$ is continuous. Let $x < y$.

$$J_F(y) - J_F(x) \leq \sum_{n : x < x_n \leq y} \alpha_n \leq F(y) - F(x)$$

$$\Rightarrow F(x) - J_F(x) \leq F(y) - J_F(y) \Rightarrow F - J_F \text{ increasing}$$

□

$$F = (F - J_F) + J_F$$

where $F - J_F$ is differentiable a.e.

Claim. J_F is differentiable a.e.

Proof.

Fix $\epsilon > 0$.

$$E = \{x : \limsup_{h\to 0} \frac{J_F(x+h) - J_F(x)}{h} > \epsilon\}$$

Check: E is measurable. Let $\delta = m(E)$. Want: $\delta = 0$.

$$\sum_{n=1}^{\infty} \alpha_n < \infty \Rightarrow \forall \eta > 0, \exists N \text{ s.t. } \sum_{n>N} \alpha_n < \eta$$

Consider

$$J_0(x) = \sum_{n>N} \alpha_n j_n(x)$$

$$\Rightarrow J_0(b) - J_0(a) < \eta$$

$J_F - J_0$ is just a finite sum of $\alpha_n j_n(x)$.

$$E_0 = \{x : \limsup_{h\to 0} \frac{J_0(x+h) - J_0(x)}{h} > \epsilon\}$$

E_0 differs from E by at most a finite set (which is $\{x_1, ..., x_n\}$). $\Rightarrow m(E_0) = \delta$. By inner regularitty, find compact $K \subset E_0$ s.t. $m(K) \geq \delta/2$. By definition,

$$\limsup_{h\to 0} \frac{J_0(x+h) - J_0(x)}{h} > \epsilon \ \forall x \in K$$

So for any $x \in K, \exists$interval $(a_x, b_x) \ni x$ s.t.

$$J_0(b_x) - J_0(a_x) > \epsilon(b_x - a_x)$$

K compact $\Rightarrow$ a finite collection of these intervals covers K. By Vitali's covering lemma, find disjoint intervals $I_1, ..., I_n$ s.t. $m(K) \leq 3\sum_{i=1}^n m(I_i)$. Write $I_i = (a_i, b_i)$.

$$\eta > J_0(b) - J_0(a) \overset{J_0 \text{ increasing}}{\geq} \sum_{j=1}^{n} (J_0(b_j) - J_0(a_j))$$

$$> \epsilon \sum_{j=1}^{n} (b_j - a_j) \geq \frac{\epsilon}{3} m(K) \geq \frac{\epsilon\delta}{6}$$

Let $\eta \to 0$, we have $\delta = 0$. $\square$

1 Rectifiable curves

Definition 1. Let γ be a parametrized curve in the plane by $z(t) = (x(t), y(t))$, where $a \leq t \leq b, x, y$ are continuous real-valued on $[a, b]$. (Will assume $\forall x \in \gamma, z^{-1}(\{x\})$ is a closed interval or a singleton). γ is said to be **rectifiable** if $\exists M > 0$ s.t. $\forall$ partition $a = t_0 < t_1 < \ldots < t_N = b$ of $[a, b]$,

$$\sum_{j=1}^{N} |z(t_j) - z(t_{j-1})| \leq M$$

We define the length $L(\gamma)$ of γ fo be the sup of L.H.S. (so $L(\gamma) = T_z(a, b)$). What conditions on x and y will guarantee that γ is rectifiable? If x, y are differentiable a.e., is it true that

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t?$$

Proposition 1. γ is rectifiable $\iff$ x, y are of bounded variation.

Proof. Easy. □

The answer to the second question is **no** in general. Let x, y be the Cantor function. $z(t) = (x(t), y(t))$ parametrizes γ.

Theorem 1. If x and y are absolutely continuous, then γ is rectifiable. Moreover,

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t$$

We prove something more general:

Proposition 2. If F is complex-valued and absolutely continuous on $[a, b]$, then

$$T_F(a, b) = \int_a^b |F'(t)|\mathrm{d}t$$

Proof.

By absolute continuity, for any partition $a = t_1 < \ldots < t_N = b$ of $[a, b]$,

$$\sum_{j=1}^N |F(t_j) - F(t_{j-1})| = \sum_{j=1}^N |\int_{t_{j-1}}^{t_j} F'(t)\mathrm{d}t| \le \sum_{j=1}^N \int_{t_{j-1}}^{t_j} |F'(t)\mathrm{d}t = \int_a^b |F'(t)|\mathrm{d}t$$

Take sup over all partitions

$$\Rightarrow T_F(a, b) \le \int_a^b |F'(t)|\mathrm{d}t$$

Fix $\epsilon > 0$. F absolutely continuous $\Rightarrow$ F integrable on $[a, b]$. $\exists$ a step function G on $[a, b]$ s.t.

$$\|F' - g\|_{L^1} < \epsilon$$

Let

$$h(x) = F'(x) - g(x)$$

Set

$$G(x) = \int_a^x g(t)\mathrm{d}t, H(x) = \int_a^x h(t)\mathrm{d}t$$

$$F' = g + h \Rightarrow F(x) - F(a) = G(x) + H(x)$$

$$\overset{\Delta\text{-ineq}}{\Rightarrow} T_F(a, b) \ge T_G(a, b) - T_H(a, b)$$

H is absolute continuous. From "$\le$":

$$T_H(a, b) \le \int_a^b |H'(t)|\mathrm{d}t = \int_a^b |h(t)|\mathrm{d}t < \epsilon$$

$$\Rightarrow T_F(a,b) \geq T_G(a,b) - \epsilon$$

Choose a partition $a = t_0 < \ldots < t_N = b$ s.t. g is a constant on each (t_{j-1}, t_j) (since g is a step function!).

$$T_G(a,b) \geq \sum_{j=1}^{N} |G(t_j) - G(t_{j-1})| = \sum_{j=1}^{N} \left| \int_{t_{j-1}}^{t_j} g(t)\mathrm{d}t \right|$$

since g is a constant

$$\text{R.H.S. } = \sum_{j-1}^{N} \int_{t_{j-1}}^{t_j} |g(t)|\mathrm{d}t = \int_a^b |g(t)|\mathrm{d}t \geq \int_a^b |F'(t)|\mathrm{d}t - \epsilon$$

$$T_F(a,b) \geq \int_a^b |F'(t)|\mathrm{d}t - 2\epsilon$$

□

A curve can be realized by many different parametrizations. Is there a good/natural one? Suppose that γ is parametrized by $t \to z(t)$. Write $s(t)$ for the length of the segment of γ that arises as the image of $z([a,t])$.

Check: s is a continuous increasing function from $[a,b]$ to $[0,L]$.

The arc length (re)parametrization of γ is given by $\tilde{z}(s)$ where $\tilde{z}(s) = z(t)$ for $s = s(t)$.

Check: $\tilde{z}$ is well-defined.

Note: $|\tilde{z}(s_1) - \tilde{z}(s_2)| \leq |s_1 - s_2| \ \forall s_1, s_2 \in [0,L]$ ($\tilde{z}$ is Lipschitz). So $\tilde{z}$ is absolutely continuous. Moreover, $|\tilde{z}'(s)| = 1$ a.e. By the Lipschitz inequality, $|\tilde{z}'(s)| \leq 1$ a.e. By definition,

$$L = T_{\tilde{z}}(a,b) = \int_0^L |\tilde{z}'(s)|\mathrm{d}s$$

So $|\tilde{z}'(s)| = 1$ a.e. Writing $(\tilde{z}(s)) = (\tilde{x}(s), \tilde{y}(s))$,

$$L = \int_0^L \sqrt{(\tilde{x}(s))^2 + (\tilde{y}(s))^2}\mathrm{d}s$$

2 Banach spaces

Complete normed vector spaces.

Fix $1 \leq p \leq \infty$.

$$l^p = \{(x_n) : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$

Define $\|(x_n)\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$.

$$l^\infty = \{(x_n) : \sup_{n\geq 1} |x_n| = \|(x_n)\|_\infty < \infty\}$$

Check: l^p are Banach spaces.

Remark 1. When observing a unknown space, considering the functions building on it gives some interesting information. For example, if one wants to examine the topological structure on $\mathbb{R}$, we could consider continuous functions. If we want to study the algebraic structure, we could consider linear functions, etc.

2.1 Linear functionals and dual space

Definition 2. Let X be a vector space. A **linear functional** is a linear function from X to $\mathbb{C}$.

$$L(X,\mathbb{C}) = \{\text{linear functionals from } X \text{ to } \mathbb{C}\}$$

X infinite dimensional $\Rightarrow L(X,\mathbb{C})$ infinite dimensional (Check!). A linar functional may not capture the norm structure of X if X is a normed vector space.

Definition 3. Let X be a normed space. We say that $\Lambda \in L(X,\mathbb{C})$ is **bounded** if it maps any bounded set in X to a bounded set in $\mathbb{C}$.

If Λ is bounded then for any bounded set $A \subset X, \exists C > 0$ s.t. $|\Lambda x| \leq C \ \forall x \in A$.

Proposition 3. Let $\Lambda \in L(X,\mathbb{C})$, where X is a normed space.

(a) Λ is bounded $\iff \exists C > 0$ s.t.
$$|\Lambda x| \leq C\|x\| \ \forall x \in X$$

(b) Λ is continuous $\iff$ Λ is continous at one point.

(c) Λ is continuity $\iff$ Λ is bounded.

Proof.

(a) Suppose that Λ is bounded. Take $A = \overline{B(0,1)}$. If $x \in X \setminus \{0\}$, then $x/\|x\| \in A$. $\exists C > 0$ s.t. $|\Lambda y| \leq C \ \forall y \in A$.
$$\Rightarrow \left|\Lambda\left(\frac{x}{\|x\|}\right)\right| \leq C \ \forall x \neq 0$$
$$\Rightarrow |\Lambda x| \leq C\|x\| \ \forall x \in X$$

Converse is easy.

□