Analysis 2 W5-1

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Recall: $f \in L^1([-\pi,\pi])$ and extend f periodically. To each such f we can associate the Fourier series of f:

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} \mathrm{d}y$$

which is well-defined when $f \in L^1$.

Questions: Does the Fourier series converge? In what sense? Is the limit f? In fact we don't have pointwise convergence if f is justs piecewise continuous. Consider

$$f(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq \pi \\ 0 \text{ if } -\pi < x < 0 \end{cases}$$

Extend it periodically on $\mathbb{R}$.

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{\pi} e^{-inx} \mathrm{d}x = \begin{cases} \frac{1-e^{-in\pi}}{2\pi i n} & \text{if } n \neq 0 \\ \frac{1}{2} & \text{if } n = 0 \end{cases}$$

$$= \begin{cases} \frac{1}{i\pi n} & \text{if } n \text{ is odd} \\ \frac{1}{2} & \text{if } n = 0 \\ 0 & \text{if } n \text{ is even, } n \neq 0 \end{cases}$$

The Fourier series of f becomes

$$\frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi} \sin nx$$

$f(0) = 0$, but the Fourier series is $1/2$ at $x = 0$. How about continuous f? Define the partial sum

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

Consider the N^{th}-**Dirichlet kernel**

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

Check:

$$D_N(x) = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)} (D_N(0) = 2N+1)$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\mathrm{d}x = 1$$

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$$

$$= \sum_{n=-N}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\mathrm{d}y\right)e^{inx}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\left(\sum_{n=-N}^{N} e^{in(x-y)}\right)\mathrm{d}y$$

$$= f * D_N(x)$$

where

$$(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)\mathrm{d}y$$

So

$$S_N(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\left(\left(N+\frac{1}{2}\right)(x-y)\right)}{\sin\left(\frac{x-y}{2}\right)} f(y)\mathrm{d}y$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\left(\left(N+\frac{1}{2}\right)y\right)}{\sin\left(\frac{y}{2}\right)} f(x-y)\mathrm{d}y$$

Proposition 1. If f is 2π periodic and Hölder/Lipschitz continuous then $S_N(f) \to f$ pointwise.

Proof.

Let $\alpha \in (0,1]$ and $c > 0$ such that

$$|f(x+h) - f(x)| \le c|h|^\alpha \ \forall x, h \in \mathbb{R}$$

$$S_N(f)(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x-y) - f(x)\right) D_N(y)\mathrm{d}y = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)\sin\left(\left(N+\frac{1}{2}\right)y\right)\mathrm{d}y$$

where $g(y) \in L^1([-\pi,\pi])$. For $y \in [-\pi,\pi]$,

$$\left|\frac{f(x-y)-f(x)}{\sin\left(\frac{y}{2}\right)}\right| \le c'|y|^{-1+\alpha}$$

where $c'|y|^{-1+\alpha}$ is Lebesgue integrable.

Recall that if g is integrable then $\hat{g}(N) \to 0$ as $|N| \to \infty$.(Riemann-Lebesgue lemma.) Similarly, can show

$$\int_{-\pi}^{\pi} g(x)\sin\left(\left(N+\frac{1}{2}\right)x\right)\mathrm{d}y \to 0 \text{ as } N \to \infty$$

So

$$S_N(f)(x) - f(x) \to 0 \ \forall x \in \mathbb{R}, \text{ as } N \to \infty$$

□

What if f is merely continuous?

Fact: Can construct an explicit example of continuous f such that its Fourier series diverges at one point. (Stein-Shakarchi: Fourier analysis, Ch3, Section 2.2) We will give aa soft (non-constructive) proof of a stronger result. A periodic function can be viewed as a function on $\mathbb{S}^1$.

Theorem 1. $\{f \in C^0(\mathbb{S}^1) : S_N(f) \text{ diverges at } 0\}$ is dense in $C^0(\mathbb{S}^1)$.

Proof.

Consider $\Lambda_N f = S_N(f)(0) = \sum_{n=-N}^{N} \hat{f}(n)$. Check: $\Lambda_N \in (C^0(\mathbb{S}^1))^*$. Can write

$$\Lambda_N f = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(y) f(y) \mathrm{d}y$$

Claim.

$$\|\Lambda_N\| = \int_{-\pi}^{\pi} |D_N(y)| \mathrm{d}y$$

Proof.

Will prove something stronger. For $G \in C^0([a,b])$, define $\Lambda f = \int_a^b f(x) g(x) \mathrm{d}x$. Will show $\Lambda \in (C^0([a,b]))^*$, and

$$\|\Lambda\| = \int_a^b |g(x)| \mathrm{d}x$$

By Riesz representation theorem, take

$$\alpha(x) = \int_a^x g(t) \mathrm{d}t$$
$$\Rightarrow \|\Lambda\| = T_\alpha(a,b) = \|g\|_{L^1}$$

(Can prove this directly without Riesz representation theorem.) □

$$\begin{aligned}
\|\Lambda_N\| &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin\left(\left(N+\frac{1}{2}\right)y\right)}{\sin\left(\frac{y}{2}\right)} \right| \mathrm{d}y \\
&= \frac{1}{\pi} \int_{0}^{\pi} \left| \frac{\sin\left(\left(N+\frac{1}{2}\right)y\right)}{\sin\left(\frac{y}{2}\right)} \right| \mathrm{d}y \\
&\geq \frac{2}{\pi} \int_0^{\pi} \frac{\left|\sin\left(\left(N+\frac{1}{2}\right)y\right)\right|}{y} \mathrm{d}y \\
&= \frac{2}{\pi} \int_0^{\left(N+\frac{1}{2}\right)\pi} \frac{|\sin y|}{y} \mathrm{d}y \\
&\geq \frac{2}{\pi} \sum_{j=1}^{N} \int_{(j-1)\pi}^{j\pi} \frac{|\sin y|}{y} \mathrm{d}y \\
&\geq \frac{2}{\pi} \sum_{j=1}^{N} \frac{1}{j\pi} \int_{(j-1)\pi}^{j} |\sin y| \mathrm{d}y \\
& \frac{4}{\pi^2} \sum_{j=1}^{N} \frac{1}{j} \to \infty \text{ as } N \to \infty
\end{aligned}$$

That is, $\|\Lambda_N\| \to \infty$ as $N \to \infty$. So by the alternative formulation of UBP, the resonance points of (Λ_N) are dense in $C^0(\mathbb{S}^1)$. The resonance points are exactly the functions whose Fourier series diverges at 0. □

1 Open Mapping Theorem

Recall(?): A function $f : X \to Y$, where X, Y are topological spaces, is called an open map if $f(\mathcal{U})$ is open in Y whenever $\mathcal{U}$ is open in X.

Example 1. Every nonzero linear functional on a normed space X is open. Let $\Lambda \in L(X, \mathbb{C})$ be nonzero. Pick $z_0 \in X$ such that $\Lambda z_0 = 1$. Let $\mathcal{U} \subseteq X$ be open. Want: $\Lambda\mathcal{U}$ is open. Take $\Lambda x_0 \in \Lambda\mathcal{U}$. Since $\mathcal{U}$ is open, $\exists r > 0$ such that $B(x_0, r) \subseteq \mathcal{U}$. If $s \in \left(-\frac{r}{\|z_0\|}, \frac{r}{\|z_0\|}\right)$, then $x_0 + sz_0 \in B(x_0, r)$.

$$\Rightarrow \Lambda(x_0 + sz_0) = \Lambda x_0 + s \in \Lambda\mathcal{U}$$

$$\Rightarrow \left(\Lambda x_0 - \frac{r}{\|z_0\|}, \Lambda x_0 + \frac{r}{\|z_0\|}\right) \subseteq \Lambda\mathcal{U}$$

So $\Lambda\mathcal{U}$ is open.

Theorem 2. Every surjective bounded linear operator from a Banach space to another Banach space is an open map.

Proof.

Let $T \in \mathcal{B}(X, Y)$ be surjective, where X, Y are Banach spaces.

Step 1: $\exists r > 0$ such that $B_Y(0, r) \subseteq \overline{TB_X(0, 1)}$. Why? Since T is onto,

$$Y = \bigcup_{j=1}^{\infty} TB_X(0, j) = \bigcup_{j=1}^{\infty} \overline{TB_X(0, j)}$$

Y is complete. By Baire category theorem, $\exists j_0$ such that $\overline{TB_X(0, j_0)}$ contains a ball $B_Y(y_0, \rho)$. $TB_X(0, j_0)$ is dense in $\overline{TB_X(0, j_0)} \Rightarrow B_Y(y_0, \rho) \cap TB_X(0, j_0) \neq \phi$. By replacing $B_Y(y_0, \rho)$ by a smaller ball if necessary, we may assume $y_0 = Tx_0$ for some $x_0 \in B(0, j_0)$. Then

$$B_Y(y_0, \rho) \subseteq \overline{TB_X(0, j_0)} \overset{\Delta-\text{ineq}}{\subseteq} \overline{TB_X(x_0, j_0 + \|x_0\|)}$$

$$\overset{\text{Translation}, y_0 = Tx_0}{\Rightarrow} B_Y(0, \rho) \subseteq \overline{TB_X(0, j_0 + \|x_0\|)}$$

$$\Rightarrow B_Y(0, r) \subseteq \overline{TB_X(0, 1)}, \text{ where } r = \frac{\rho}{j_0 + \|x_0\|}$$

Step 2: $B_Y(0, r) \subseteq TB_X(0, 3)$. Exercise: This implies the theorem. It remains to show Step 2. Let $y \in B_Y(0, r)$. Want to find $x^* \in B_X(0, 3)$ such that $Tx^* = y$. By Step 1, for any $n \geq 0$, we have

$$B_Y\left(0, \frac{r}{2^n}\right) \subseteq \overline{TB_X\left(0, \frac{1}{2^n}\right)}$$

$n = 0$: $\exists x_1 \in B_X(0, 1)$ such that $\|y - Tx_1\| < \frac{r}{2}$. Now, $y - Tx_1 \in B_Y\left(0, \frac{r}{2}\right)$.
$n = 1$: $\exists x_2 \in B_X\left(0, \frac{1}{2}\right)$ such that $\|y - Tx_1 - Tx_2\| < \frac{r}{2^2}$. Inductively, we obtain (x_n) with $x_n \in B_X\left(0, \frac{1}{2^{n-1}}\right)$ and $\|y - Tx_1 - Tx_2 - \cdots - Tx_n\| < \frac{r}{2^n}$. Set $z_n = \sum_{j=1}^{n} x_n$. Then (z_n) is Cauchy: for $r_n < n$,

$$\|z_m - z_n\| \leq \|x_{m+1}\| + \cdots + \|x_n\| \leq \frac{1}{2^{m-1}}$$

X is complete $\Rightarrow x^* = \lim_{n\to\infty} z_n$ exists. Check: $x^* \in B_X(0,3)$ and $Tx^* = y$.

$$\|z_n\| \leq \sum_{j=1}^{n} \|x_j\| \leq 2$$

$$\Rightarrow x^* \in \overline{B_X(0,2)} \subseteq B_X(0,3)$$

$$\|y - Tx^*\| \leq \|y - Tz_n\| + \|Tz_n - Tx^*\| < \frac{r}{2^n} + 0 \to 0$$

$y = Tx^*$. □

Corollary 1 (Banach inverse mapping theorem)**.** Let $T \in \mathcal{B}(X,Y)$ be a bijection, where X, Y are Banach spaces. Then T is invertible.

Proof.

Want: T^{-1} is bounded. From the proof of the open mapping theorem, $\exists r > 0$ such that $B_Y(0,r) \subseteq TB_X(0,3)$.

$$\Rightarrow T^{-1}B_Y(0,r) \subseteq B_X(0,3)$$

$\Rightarrow T^{-1}$ maps a ball (and hence any ball) in Y to a bounded set in X. $\Rightarrow T^{-1}$ is bounded. □

2 Closed Graph Theorem

Theorem 3. Let $T : X \to Y$ be linear, where X, Y are vector spaces. The graph of T is

$$G(T) = \{(x, Tx) : x \in X\} \subseteq X \times Y\}$$

Further suppose that X, Y are normed. We say that T is closed if $G(T)$ is a cclosed subset of $X \times Y$ under the product topology. Equivalently, T is closed if whenever $x_n \to x$ in X and $Tx_n \to y$ then $y = Tx$.

T bounded $\Rightarrow T$ closed. T closed $\not\Rightarrow T$ bounded. Differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ on $C^1([a,b]) \subseteq C^0([a,b])$ with sup norm is closed by unbounded.

Theorem 4 (Closed Graph Theorem)**.** Any closed map from a Banach space to another Banach space is bounded.

Proof.

Let $T \in L(X,Y)$ be a closed map. Since $X \times Y$ is complete and $G(T)$ is closed in $X \times Y$, $G(T)$ is also a Banach space. Define the projection operator $P : G(T) \to X$ by

$$P(x, Tx) = x \ \forall x \in X$$

Clearly, P is bijective. Also,

$$\|P(x,Tx)\| = \|x\| \leq \|x\| + \|Tx\|$$

$\Rightarrow P \in \mathcal{B}(G(T), X)$. By Banach's inverse mapping theorem, P^{-1} is bounded. So $\exists, > 0$ such that

$$\|x\| + \|Tx\| = \|(x,Tx)\| = \|P^{-1}x\| \leq c\|x\| \ \forall x \in X$$

$\Rightarrow T$ is bounded. □

We showed open mapping theorem $\Rightarrow$ closed graph theorem. Can also show closed graph theorem $\Rightarrow$ open mapping theorem. These two are equivalent.