Analysis 2 W6-1

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March 26, 2024

Last time: K closed, convex, proper, nonempty subset of X where X is a Hilbert space. Let $x_0 \in X \setminus K$. Then $\exists! x^* \in K$ such that

$$\|x_0 - x^*\| = \inf_{x \in K} \|x_0 - x\|$$

1 Orthogonal Decomposition

Theorem 1. Let Y be a closed proper subspace of a Hilbert space X. Let $x_0 \in X \setminus Y$. Then the point $y_0 \in Y$ that minimizes the distance between x_0 and Y satisfies

$$\langle x_0 - y_0, y \rangle = 0 \quad \forall y \in Y$$

Conversely, if $z \in Y$ such that $\langle x_0 - z, y \rangle = 0 \quad y \in Y$, then $z = y_0$. In this case, we have $\|x_0 - y_0\|^2 + \|y_0\|^2 = \|x_0\|^2$.

Proof.

For $y \in Y$ and $h \in \mathbb{R}$,

$$\varphi(h) = \|x_0 - y_0 - hy\|^2$$

attains a minimum at $h = 0$.

$$\varphi(h) = \|x_0 - y_0\|^2 - h\langle x_0 - y_0, y \rangle - h\langle y, x_0 - y_0 \rangle + h^2\|y\|^2$$

$$\varphi'(0) = 0 \Rightarrow \mathrm{Re}\langle x_0 - y_0, y \rangle = 0$$

Replacing y by iy, we obtain $\mathrm{Im}\langle x_0 - y_0, y \rangle = 0$. Conversely, if $\langle x_0 - z, y \rangle = 0 \quad \forall y \in Y$, then

$$\|x_0 - y\|^2 = \|x_0 - z + z - y\|^2 = \|x_0 - z\|^2 + \|z - y\|^2 \geq \|x_0 - z\|^2$$

Since y is arbitrary, z minimizes the distance between x_0 and $Y \Rightarrow z = y_0$. □

Definition 1. Let Y be a closed subspace of a Hilbert space X. The **projection operator** P of X onto Y is given by

$$Px_0 = \begin{cases} y_0 & \text{if } x_0 \in X \setminus Y \\ x_0 & \text{if } x_0 \in Y \end{cases}$$

We may call Px the **best approximation** of x in Y/**orthogonal projection** of x on Y.

Check:

- $P \in \mathcal{B}(X, Y)$
- $P^2 = P$.
- $\|P\| = 1$.

Example 1. To show that P is linear, we just need to show that

$$\langle \alpha x_1 + \beta x_2 - (\alpha P x_1 + \beta P x_2), y \rangle = 0 \quad \forall y \in Y$$

By uniquenesss we have $P(\alpha x_1 + \beta x_2) = \alpha P x_1 + \beta P x_2$.

Check: If $x, z \in X$, then $z = Px \Leftrightarrow z$ satisfies $\langle x - z, y \rangle = 0 \quad \forall y \in Y$.

Two consequences:

1. Hilbert space is self-dual. To each $z \in X$, we associate a bounded linear function

$$\Lambda_z x = \langle x, z \rangle \quad \forall x \in X$$

Easy to check: $\Lambda_z \in X^*$ and $\|\Lambda_z\| = \|z\|$. The map $\Phi : z \mapsto \Lambda_z$ defines a **sesquilinear** map from X to X^*, i.e.

$$\Phi(\alpha x_1 + \beta x_2) = \bar{\alpha}\Phi(x_1) + \bar{\beta}\Phi(x_2)$$

Will show: Φ is isometric sesquilinear isomorphism. $\Rightarrow$ Hilbert space is self-dual.

Theorem 2 (Riesz Representation Theorem)**.** Let X be a Hilbert space. For every $\Lambda \in X^*$, $\exists! z \in X$ such that $\Lambda_z = \Lambda$ and $\|z\| = \|\Lambda\|$.

Theorem $\Rightarrow \Phi$ is surjective and isometric.

Proof.

Let $\Lambda \in X^*$. Consider

$$Y = \{x \in X : \Lambda x = 0\}. \quad (\text{ So } Y = N(\Lambda).)$$

Y is closed. If $Y = X$, then $\Lambda = 0$. Take $z = 0$. Otherwise, take any $x_0 \in X \setminus Y$. Then $\langle x_0 - Px_0, y \rangle = 0$ $\forall y \in Y$. Set $z_0 = x_0 - Px_0$ and $z = \frac{\overline{\Lambda z_0}}{\|z_0\|^2} z_0$. Fix $x \in X$. Let $u = (\Lambda x) z_0 - (\Lambda z_0) x$. Then $\Lambda u = 0 \Rightarrow u \in Y$.

$$\Rightarrow 0 = \langle z_0, u \rangle = \langle u, z_0 \rangle = \langle (\Lambda x) z_0 - (\Lambda z_0) x, z_0 \rangle$$

$$= (\Lambda x)\|z_0\|^2 - \langle (\Lambda z_0) x, z_0 \rangle = (\Lambda x)\|z_0\|^2 - \langle x, (\overline{\Lambda z_0}) z_0 \rangle$$

$$\Rightarrow \Lambda x = \frac{1}{\|z_0\|^2} \langle x, (\overline{\Lambda z_0}) z_0 \rangle = \langle x, z \rangle$$

Check: $\|\Lambda\| = \|z\|$ and z is unique. $\square$

2. Direct sum decomposition in a Hilbert space. $X = X_1 \oplus X_2$. If X is normed, we hope that the projections $X \to X_1$ and $X \to X_2$ are bounded to preserve the toopology. Also hope that X_1, X_2 are closed (so X Banach $\Rightarrow X_1, X_2$ Banach). If X_1, X_2 are closed, then by the closed graph theorem the projections are bounded.

Another question: Gieven any closed subspace X_1 of a Banach space X, can wew find a closed subspace X_2 such that $X = X_1 \oplus X_2$? Not always possible in the general situation. Fact: If a Banach space possesses the property that any closed subspace satisfies this complementary property, then its norm must be equivalent to a norm induced by an inner product.

Let X be a Hilbert space. $X_1 \subseteq X$ be a closed subspace. The orthogonal complement $X_1^\perp$ is defined as

$$X_1^\perp = \{x \in X : \langle x, x_1 \rangle = 0 \quad \forall x_1 \in X_1\}$$

By the Riesz Representation Theorem, $X_1^\perp$ is also the annihilation of X_1. The notation is consitent. $X_1^\perp$ is a closed subspace of X.

$$x = Px + (x - Px) \in X_1 + X_1^\perp$$

Claim. This is a direct sum.

Proof.

If $x_0 \in X_1 \cap X_1^\perp$, then $\langle x_0, x_1 \rangle = 0 \quad \forall x_1 \in X_1$. $\Rightarrow \langle x_0, x_0 \rangle = 0 \Rightarrow x_0 = 0$. □

Moreover, $P, I - P$ are precisely the projection maps of $X_1 \oplus X_1^\perp$. To summarize:

Theorem 3. $\forall$ closed subspace X_1 of a Hilbert space X, $X = X_1 \oplus X_1^\perp$. Moreover, if $P : X \to X_1$ is the projection operator, then $\forall x \in X$, Px is the unique point in X_1 that satisfies $\|x - Px\| = \mathrm{dist}(x, X_1)$ and the projection $Q : X \to X_1^\perp$ is given by $Qx = x - Px$.

2 Complete Orthonormal Set

Question: How can we determine Px_0 when x_0, Y are given? In finite dimensions, if $\{x_1, ..., x_m\}$ is a basis for Y, then any projection y_0 of x_0 can be written as $y_0 = \sum_{k=1}^m \alpha_k x_k$. Also $\langle y_0 - x_0, x_j \rangle = 0 \quad j = 1, ..., m$. To determine α_k's, we just need to solve

$$\sum_{k=1}^m \langle x_k, x_j \rangle \alpha_k = \langle x_0, x_j \rangle \quad \forall j = 1, ..., m$$

If $\{x_1, ..., x_m\}$ is orthonormal, we see $\alpha_j = \langle x_0, x_j \rangle$, so $y_0 = \sum_{k=1}^m \langle x_0, x_k \rangle x_k$. In general, if we have an orthonormal spanning set in Y, then we should be able to find Px_0 easily.

Lemma 1 (Bessel's Inequality)**.** Let S be an orthonormal set in a Hilbert space X. Then for each $x \in X$, $\langle x, x_\alpha \rangle = 0$ for all but at most countable many α. If we write B for this exceptional indices, then for any sequence (α_k) in B,

$$\sum_k |\langle x, x_{\alpha_k} \rangle|^2 \leq \|x\|^2$$

Proof.

Step 1: We show the Bessel inequality for a finite orthonormal set $\{x_1, ..., x_N\}$.

Claim. $\sum_{k=1}^N |\langle x, x_k\rangle|^2 \leq \|x\|^2$.

Proof.

Let $y = \sum_{k=1}^N \langle x, x_k\rangle x_k$. Then

$$\langle x - y, x_k\rangle = \left\langle x - \sum_{l=1}^N \langle x, x_l\rangle x_l, x_k \right\rangle$$
$$= \langle x, x_k\rangle - \langle x, x_k\rangle = 0$$

So y is the orthogonal projection of x onto span$(\{x_1, ..., x_N\})$. Also

$$\sum_{k=1}^N |\langle x, x_k\rangle|^2 = \|y\|^2 = \|x\|^2 - \|x - y\|^2 \leq \|x\|^2$$

This proves the claim. □

Step 2: Let $x \in X$ and $l \in \mathbb{N}$. Define

$$S_l = \{x_\alpha \in S : |\langle x, x_\alpha\rangle| \geq \frac{1}{l}\}$$

We show that S_l is a finite set. Pick $x_{\alpha_1}, ..., x_{\alpha_N}$ from S_l. Then by Step 1,

$$\frac{N}{l^2} \leq \sum_{k=1}^N |\langle x, x_{\alpha_k}\rangle|^2 \leq \|x\|^2$$

$\Rightarrow N \leq l^2\|x\|^2$. So S_l has at most $l^2\|x\|^2$ many elements.

Step 3: Fix $x \in X$. Then there are at most countable manny $\langle x, x_\alpha\rangle$ are nonzero. Ler $S_x = \{x_\alpha \in S : \langle x, x_\alpha\rangle \neq 0\}$. Then

$$S_x = \bigcup_{l=1}^\infty S_l$$

So S_x is at most countable.

Finally the Bessel inequality follows from the finite case by taking $N \to \infty$. □

Theorem 4. Let Y be a closed subspace of a Hilbert space X. Suppose that S is an orthonormal subset of Y such that span(S) is dense in Y. Then $\forall x \in X$, its orthogonal projection on Y is given by $\sum_k \langle x, x_k\rangle x_k$, where (x_k) is any ordering of those x_α in S with $\langle x, x_\alpha\rangle \neq 0$. (So we can write $\sum_\alpha \langle x, x_\alpha\rangle x_\alpha$ without any ambiguity.)

Proof.

We first verigy that $\sum_k \langle x, x_k\rangle x_k$ is convergent. Let $z_n = \sum_{k=1}^n \langle x, x_k\rangle x_k$. We just need to show that (z_n) is Cauchy. For $m < n$,

$$\|z_m - z_n\|^2 = \left\|\sum_{k=m+1}^n \langle x, x_k\rangle x_k\right\|^2 = \sum_{k=m+1}^n |\langle x, x_k\rangle|^2$$

where the last term is small as m, n are large, by Bessel's inequality. So (z_n) is Cauchy. Also, it is easy to check that

$$\left\langle x - \sum_k \langle x, x_k \rangle x_k, y \right\rangle = 0 \quad \forall y \in Y$$

So $\sum_k \langle x, x_k \rangle x_k$ is the orthogonal projection of x on Y. □

Definition 2. A subset B of a Hilbert space X is called a **complete orthonormal set** if

(a) It is an orthonormal set.

(b) $\overline{\operatorname{span}(B)} = X$.

Recall: A Hamel basis B satisfies

(a') All vectors in B are linearly independent.

(b') $\operatorname{span}(B) = X$.

(a)⇒(a'), but (b') is stronger than (b). A complete orthonormal set is a good substitution for a Hamel basis. (Some books called a complete orthonormal set as an orthonormal basis.)

Theorem 5. Every nonzero Hilbert space admits a complete orthonormal set.

Proof.

Let $\mathcal{O}$ be the collection of all orthonormal sets in X. $\mathcal{O} \neq \phi$, partially ordered by $\subseteq$. If $\mathcal{C}$ is a chain in $\mathcal{O}$, then

$$O^* = \bigcup_{O \in \mathcal{C}} O$$

is an upper boundd of $\mathcal{C}$ in $\mathcal{O}$. By Zorn's lemma, $\mathcal{O}$ has a maximal element B.

Claim. B is a complete orthonormal set.

Proof.

B is clearly orthonormal. Suppose $\exists z \in X \setminus \overline{\operatorname{span}(B)}$. By orthogonal decomposition, if P is the projection operator onto $\overline{\operatorname{span}(B)}$, then

$$z' = \frac{z - Pz}{\|z - Pz\|}$$

is a unit vector $\perp \overline{\operatorname{span}(B)}$. Then $B \cup \{z'\} \in \mathcal{O}$, a contradiction. □

□

When is an orthonormal set complete?

Theorem 6. Let B be an orthonormal set in a Hilbert space X. The following are equivalent.

(a) B is a complete orthonormal set.

(b) $x = \sum_\alpha \langle x, x_\alpha \rangle x_\alpha$ holds for all $x \in X$.

(c) $\|x\|^2 = \sum_\alpha |\langle x, x_\alpha\rangle|^2$ holds for all $x \in X$.

(d) $\langle x, x_\alpha\rangle = 0 \quad \forall x_\alpha \in B$ then $x = 0$.

(c) is called the **Parseval Identity**.

Proof.

"(a)⇒(b)" Suppose that B is a complete orthonormal set. The orthogonal projection on $\overline{\mathrm{span}(B)}$ is just I. So (b) holds.

"(b)⇒(c)"

$$\|x\|^2 - \sum_{k=1}^{n} |\langle x, x_k\rangle|^2 = \|x - \sum_{k=1}^{n} \langle x, x_k\rangle x_k\|^2 \to 0$$

"(c)⇒(d)" is obvious.

"(d)⇒(a)" Suppose that $\overline{\mathrm{span}(B)} \subset X$ is a proper subset of X. Then $\exists$ nonzero $x_0 \in X \setminus \overline{\mathrm{span}(B)}$ such that $\langle x_0, x_\alpha\rangle = 0 \quad \forall x_\alpha \in B$. (similar to the proof of existence of a complete orthonormal set). $\Rightarrow x_0 = 0$ by (a), a contradiction. $\square$