# Analysis 2 W3-2

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Last time:

**Theorem 1** (Hahn-Banach theorem)**.** Let $X$ be a vector space over $\mathbb{C}$ and let $p$ a gauge on $X$. Let $Y$ be a proper subspace of $X$. Let $\Lambda \in L(Y, \mathbb{C})$ satisfy

$$Re(\Lambda x) \leq p(x) \ \forall x \in Y$$

Then $\exists$ an extension $\tilde{\Lambda} \in L(X, \mathbb{C})$ s.t.

$$Re(\tilde{\Lambda} x) \leq p(x) \ \forall x \in X$$

*Proof.*

We will first prove the case that $X$ is over $\mathbb{R}$.

$$\mathcal{D} = \{(Z, T) : Y \subseteq Z \subseteq X, T \in L(Z, \mathbb{R}) \text{ extends } \Lambda \text{ and } T_x \leq p(x) \ \forall x \in Z\}$$

$(Y, \Lambda) \in \mathcal{D}$, so $\mathcal{D} \neq \phi$. Define $\leq$ on $\mathcal{D}$ as follows: $(Z_1, T_1) \leq (Z_2, T_2)$ if $Z_1 \subseteq Z_2$ and $T_2$ extends $T_1$. Let $\mathcal{C}$ be a chain (totally ordered subset) in $(\mathcal{D}, \leq)$.

**Claim.** $\mathcal{C}$ has an upper bound in $(\mathcal{D}, \leq)$.

*Proof.*

Define

$$Z = \bigcup_{\alpha \in \mathcal{C}} Z_\alpha, T_z = T_\alpha z \text{ if } z \in Z_\alpha$$

$Z$ is a vector sapce. Let $z_1, z_2 \in Z$. Then $z_1 \in Z_\alpha$ and $z_1 \in Z_\beta$ for some $\alpha, \beta$. Either $Z_\alpha \subseteq Z_\beta$ or the converse. Assume the former. Then $z_1, z_2 \in Z_\beta$. So $\lambda_1 z_1 + \lambda_2 z_2 \in Z_\beta \subseteq Z \ \forall \lambda_1, \lambda_2 \in \mathbb{R}$. $\Rightarrow Z$ is a vector space. Similarly, you can show that $T_\alpha z = T_\beta z$ if $z \in Z_\alpha \cap Z_\beta$. $\Rightarrow T$ is well-defined. If $z \in Z$, then $z \in Z_\alpha$ for some $\alpha$.

$$T_z = T_\alpha z \leq p(z)$$

$\Rightarrow (Z, T) \in \mathcal{D}$ is an upper bound for $\mathcal{C}$. By Zorn's lemma, $\mathcal{D}$ has a maximal element $(\bar{Z}, \bar{T})$.

□

**Claim.** $\bar{Z} = X$

*Proof.*

Suppose not. Pick $x_0 \in X \setminus \bar{Z}$. By one-step extension, we can find $T_1$ on $\bar{Z}' = \bar{Z} \bigoplus span\{x_0\}$ s.t. $T_1$ extends $\bar{T}$ and $T_1(x) \leq p(x)\ \forall x \in \bar{Z}$. Then $(\bar{Z}, \bar{T}) \leq (\bar{Z}', T_1) \in \mathcal{D}$. Impossible but maximal. So $\bar{Z} = X$. $\square$

$\tilde{\Lambda} = \bar{T}$ is our desired extension.

What can we do if $X$ is over $\mathbb{C}$? A complex linear functional is uniquely determined by its real or imaginary part.

**Lemma 1.** (a) Let $\Lambda \in L(X, \mathbb{C})$. Then its real part and imaginary part are in $L(X, \mathbb{R})$ when $X$ is regarded as a vector space over $\mathbb{R}$. Moreover,

$$\Lambda x = Re(\Lambda x) - iRe(\Lambda(ix))\ \forall x \in X$$

(b) Conversely, if $\Lambda_1 \in L(X, \mathbb{R}), \exists! \Lambda \in L(X, \mathbb{C})$ s.t. $Re(\Lambda) = \Lambda_1$.

*Proof.*

(a) $\Lambda x = Re(\Lambda x) + iIm(\Lambda x) = \Lambda_r x + i\Lambda_i x$. We claim that $\Lambda_r(ix) = -\Lambda_i x, \Lambda_i(ix) = \Lambda_r x$. Clearly, this implies (a). To prove this,

$$\Lambda(ix) = i\Lambda x$$

$$\Rightarrow \Lambda_r(ix) + i\Lambda_i(ix) = -\Lambda_i x + i\Lambda_r x$$

(b) Define $\Lambda x = \Lambda_1 x - i\Lambda_1(ix)$. Then $Re(\Lambda x = \Lambda_1 x$. Need to check $\Lambda \in L(X, \mathbb{C})$. Let $x, y \in X$.

$$\Lambda(x+y) = \lambda_1(x+y) - i\Lambda_1(i(x+y))$$

$$= \Lambda_1 x - i\Lambda_1(ix) + \Lambda_1 y - i\Lambda_1(iy) = \Lambda x + \Lambda y$$

Also, let $a, b \in \mathbb{R}$.

$$\Lambda((a+bi)x) = \Lambda_1((a+bi)x) - i\Lambda_1((-b+ai)x)$$

$$= \Lambda_1(ax) - i\Lambda_1(iax) + \Lambda_1(ibx) - i\Lambda_1(-bx)$$

$$= a\Lambda_1 x - ia\lambda_1(ix) + b\Lambda_1(ix) + ib\Lambda_1 x$$

$$= a\Lambda x + ib(\Lambda_1 x - i\Lambda_1(ix))$$

$$= a\Lambda x + ib\Lambda x = (a+bi)\Lambda x$$

So $\Lambda \in L(X, \mathbb{C})$. Uniqueness is easy to check.

$\square$

Let $\Lambda \in L(Y, \mathbb{C})$ s.t. $Re(\Lambda x) \leq p(x)\ \forall x \in Y$. $\exists$ a real extension $\Lambda_1$ of $Re(\Lambda)$ s.t. $\Lambda_1 x \leq p(x)\ \forall x \in X$. By lemma, $\exists \tilde{\Lambda} \in L(X, \mathbb{C})$ s.t. $Re(\tilde{\Lambda}) = \Lambda_1$. $\square$

# 1 Hahn-Banach on normed spaces

**Theorem 2.** Let $(X, \|\cdot\|)$ be a normed sapce and let $Y \subseteq X$ be a proper subspace. Then any $\Lambda \in Y^*$ admits an extension to some $\tilde{\Lambda} \in X^*$ with $\|\tilde{\Lambda}\| = \|\Lambda\|$.

*Proof.*

If such an extension exists, then

$$\|\tilde{\Lambda}\| = \sup_{x \in X, x \neq 0} \frac{|\tilde{\Lambda}x|}{\|x\|} \geq \sup_{x \in Y, x \neq 0} \frac{|\tilde{\Lambda}x|}{\|x\|} = \|\Lambda\|$$

It suffices to show $\leq$. Consider the real case. Take $p(x) = \|\Lambda\|\|x\|$. $|\Lambda x| \leq \|\Lambda\|\|x\| \; \forall x \in Y = p(x)$. By Hahn-Banach, $\exists \tilde{\Lambda} \in L(X, \mathbb{R})$ s.t.

$$\tilde{\Lambda}x \leq p(x) = \|\Lambda\|\|x\| \; \forall x \in X$$

Replace $x$ by $-x$,

$$-\tilde{\Lambda}x \leq \|\Lambda\|\|x\|$$

$$\Rightarrow |\tilde{\Lambda}x| \leq \|\Lambda\|\|x\|$$

$$\Rightarrow \|\tilde{\Lambda}\| \leq \|\Lambda\|$$

So $\tilde{\Lambda} \in X^*$. For the complex case, let $\tilde{\Lambda}$ be an extension of $\Lambda$ s.t.

$$Re(\tilde{\Lambda}x) \leq \|\Lambda\|\|x\| \; \forall x \in X$$

For any $x \in X, \exists e^{i\theta}$ s.t.

$$\tilde{\Lambda}x = |\tilde{\Lambda}x|e^{i\theta}$$

Then

$$|\tilde{\Lambda}x| = e^{-i\theta}\tilde{\Lambda}x = \tilde{\Lambda}(e^{i\theta}x)$$

$$= Re(\tilde{\Lambda}(e^{i\theta}x)) \leq \|\Lambda\|\|e^{i\theta}x\| = \|\Lambda\|\|x\|$$

$$\Rightarrow \|\tilde{\Lambda}\| \leq \|\Lambda\|$$

□

Consequences:

**Proposition 1.** Let $(X, \|\cdot\|)$ be normed space and let $Y$ be a closed proper subspace of $X$. For any $x_0 \in X \setminus Y, \exists \Lambda \in X^*$ s.t. $\|\Lambda\| = 1$.

$$\Lambda x_0 = dist(x_0, Y), \Lambda y = 0 \; \forall y \in Y$$

*Proof.*

Write $d = dist(x_0, Y)$. $d > 0$ because $Y$ is closed and $x_0 \in X \setminus Y$. Consider $y' = Y \bigoplus span\{x_0\}$. Define $\Lambda_0$ on $Y'$ by setting

$$\Lambda_0(y + cx_0) = cd$$

$\Lambda_0$ is linear and vanishes on $Y$.

$$0 < d = \inf_{z \in Y} \|x_0 + z\| \le \frac{1}{|c|}\|cx_0 + y\| \ \forall y \in Y \ \forall c \neq 0$$

$$\Rightarrow |c| \le \frac{1}{d}\|cx_0 + y\| \ \forall y \in Y, \ \forall c \in \mathbb{C}$$

$$|\Lambda_0(y + cx_0)| \le \|cx_0 + y\|$$

$$\Rightarrow \|\Lambda_0\| \le 1$$

$\Lambda_0 \in (Y')^*$.

**Claim.** $\|\Lambda_0\| = 1$.

*Proof.*

Pick $y_n \in Y$ s.t. $\|y_n + x_0\| \to d$. Then

$$d = \Lambda_0(y_n + x_0) \le \|\Lambda_0\|\|y_n + x_0\| \to d\|\Lambda_0\|$$

$$\|\Lambda_0\| \ge 1$$

□

By Hahn-Banach, we obtain $\Lambda \in X^*$ s.t. $\|\Lambda\| = 1$, $\Lambda x_0 = d$, $\Lambda y = 0 \ \forall y \in Y$. □

**Corollary 1.** $\forall x_0 \in X \setminus \{0\}, \exists \Lambda \in X^*$ s.t. $\Lambda x_0 = \|x_0\|$ and $\|\Lambda\| = 1$.

*Proof.*

Take $Y = \{0\}$. □

**Remark 1.** A bounded linear functional with these properties is called a "dual point" of $x_0$. Dual points may not be unique.

**Example 1.** $X = \mathbb{R}^2, \|(x,y)\| = |x| + |y|, x_0 = (1,0)$. Check: $\Lambda_1(x,y) = x, \Lambda_2(x,y) = x + y$ are dual points of $x_0$.

**Corollary 2.** For any $x \in X$,

$$\|x\| = \sup_{\Lambda \in X^*, \Lambda \neq 0} \frac{|\Lambda x|}{\|\Lambda\|}$$

*Proof.*

Assume $x \neq 0$.

$$|\Lambda x| \le \|\Lambda\|\|x\|$$

$$\Rightarrow \|x\| \ge \sup_{\Lambda \in X^*, \Lambda \neq 0} \frac{|\Lambda x|}{\|\Lambda\|}$$

From the previous cor., $\exists \Lambda_0 \in X^*$ s.t. $\Lambda_0 x = \|x\|$ and $\|\Lambda_0\| = 1$.

$$\|x\| = \frac{|\Lambda_0 x|}{\|\Lambda_0\|} \le \sup_{\Lambda \in X^*, \Lambda \neq 0} \frac{|\Lambda x|}{\|\Lambda\|}$$

□

**Remark 2.** Not only we can recover the norm of a vector from the bounded linear functionals, but the sup can be strengthened to max as it is attained by $\Lambda_0$.

## 2 The dual space of $C^0([a,b])$

Examples: For $x \in [a,b], \Lambda_x f = f(x)$. Fix $g \in C^0([a,b]), \Lambda_g f = \int_a^b fg$. Are there any others? These can be described by the Stieltjes integrals.

**Definition 1.** Suppose that $f$ and $\alpha$ are complex-valued function defined on $[a,b]$. Suppose that $P, T$ a partition pair of $[a,b]$. We define the **Riemann-Stieltjes sum**

$$R(f,\alpha,P,T) = \sum_{j=1}^{n} f(t_j)(\alpha(x_j) - \alpha(x_{j-1}))$$

We say that $f$ is **Riemann-Stieltjes integrable** w.r.t. $\alpha$ if the sum converges to a number when $mesh(P) \to 0$. Write the integral as $\int_a^b f(x)\mathrm{d}\alpha(x)$ or $\int f\mathrm{d}\alpha$.

$$\mathcal{R}_\alpha([a,b]) = \{\text{R-S integrable functions w.r.t. } \alpha \text{ on } [a,b]\}$$

**Remark 3.** You can define Lebesgue-Stieltjes integral, by using the measure $\mu([a,b]) = \alpha(b) - \alpha(a)$. Since we are working on $C^0([a,b])$, doesn't matter if it is Lebesgue or Riemann.

Easy to verify:

- R-S integral is linear in both $f$ and $\alpha$.
- Every $f \in C^0([a,b])$ is R-S integrable w.r.t a bounded variation function $\alpha$ on $[a,b]$. Also

$$\left| \int_a^b f\mathrm{d}\alpha \right| \leq T_\alpha(a,b) \cdot \|f\|_\infty$$

Evaluation function is a R-S integral:
Fix $c \in (a,b)$. Take $\alpha = \chi_{[c,b]}$. When $c \in P, \exists! i$ s.t. $c \in (x_{i-1}, x_i)$.

$$\sum_{j=1}^{n} f(t_j)(\alpha(x_j) - \alpha(x_{j-1})) = f(t_i)$$

When $mesh(P) \to 0$, this converges to $f(c)$

$$\int f\mathrm{d}\chi_{[c,b]} = f(c)$$

Similarly,

$$\int f\mathrm{d}\chi_{(c,b]} = f(c)$$

There should be some correspondence between $(C^0([a,b]))^*$ and R-S integrals. But the example above shows that it is not 1-1. Restrict to a smaller class:

**Definition 2.** Let $BV_0([a,b])$ be set of all functions $\alpha$ of bounded variation s.t. $\alpha(a)=0$. Define

$$V([a,b]) = \{\alpha \in BV_0([a,b]) : \alpha \text{ is right-continuous on } [a,b)\}$$

Norm on $V([a,b])$: Total variation.

Here are some facts:

- Every $\alpha \in BV_0([a,b])$ is equal to a unique $\tilde{\alpha} \in V([a,b])$ except at possibly countably many points.
- $\int f \mathrm{d}\alpha = \int f \mathrm{d}\tilde{\alpha}$ $\forall f \in C^0([a,b])$.
- If $\int f \mathrm{d}\alpha_1 = \int f \mathrm{d}\alpha_2$ $\forall f \in C^0([a,b])$ and $\alpha_1, \alpha_2 \in V([a,b])$, then $\alpha_1 = \alpha_2$.

Using these facts, we see the map $\alpha \to \Lambda_\alpha$, where

$$\Lambda_\alpha f = \int f \mathrm{d}\alpha$$

defines a linear injective map $\Phi : V([a,b]) \to (C^0([a,b]))^*$. Moreover,

$$|\Phi(\alpha) f| = |\Lambda_\alpha f| \le T_\alpha(a,b)\|f\|$$

$$\Rightarrow \|\Phi(\alpha)\| \le T_\alpha(a,b) \ \forall \alpha \in V([a,b]).$$

**Theorem 3** (Riesz representation theorem, baby version)**.** $\exists$ an isometric isomorphism from $(C^0([a,b]))^*$ to $V([a,b])$.