# Analysis 2 W4-2

fat

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**Definition 1.** Let $T \in \mathcal{B}(X, Y)$. We define the **transpose** $T^t : Y^* \to X^*$ by

$$T^t y^*(x) = y^*(Tx) \ \forall y^* \in Y^*, \ \forall x \in X$$

Some people write

$$\langle x^*, x \rangle := x^*(x), x^* \in X^*, x \in X$$

$$\langle T^t y^*, x \rangle = \langle y^*, Tx \rangle$$

Exercises:

(a) $T^t \in \mathcal{B}(Y^*, X^*)$. Furthermore, $\|T^t\| = \|T\|$

(b) $T \mapsto T^t$ is linear from $\mathcal{B}(X, Y)$ to $\mathcal{B}(Y^*, X^*)$

(c) If $S \in \mathcal{B}(Y, Z)$, then $(ST)^t = T^t S^t$

This transpose extends the one in finite dimensions. Let $T : \mathbb{C}^n \to \mathbb{C}^m$ be linear, and let $\{e_j\}, \{f_j\}$ be the canonical basis for $\mathbb{C}^n, \mathbb{C}^m$, respectively. If $x \in \sum_j \alpha_j e_j$, then $\exists a_{kj}$ s.t. $Tx = \sum_{k,j} a_{kj} \alpha_j f_k$. $T$ is represented by the matrix $(a_{kj})$. Can also represent $T^t$ as a matrix by using the dual canonical bases $\{f_j^*\}$ and $\{e_j^*\}$. For $y^* = \sum_j \beta_j f_j^*, T^t y^* = \sum b_{lj} \beta_j e_l^*$.

$$T^t f_k^*(e_j) = \sum_l b_{lk} e_l^*(e_j) = b_{jk}$$

$$Te_j = \sum_l a_{lj} f_l$$

$$f_k^*(Te_j) = a_{kj}$$

$$\Rightarrow a_{kj} = b_{jk}$$

**Definition 2.** $T \in \mathcal{B}(X, Y)$, the **range** $R(T) = T(X)$, the **null space** $N(T) = \{x \in X : Tx = 0\}$.

**Definition 3.** For $Y \subseteq X$, we define its **annihilator** to be

$$Y^\perp = \{x^* \in X^* : x^*(y) = 0 \ \forall y \in Y\}$$

For a subspace $G$ of $X^*$, its annihilator is

$$^\perp G = \{x \in X : x^*(x) = 0 \ \forall x^* \in G\}$$

Check:

- The annihilators in both cases are closed subspaces.
- $Y \subseteq^\perp (Y^\perp), G \subseteq (^\perp G)^\perp$.

**Lemma 1.** Let $X$ be a normed space, and let $Y$ be a closed subspace of $X$. Then $Y =^\perp (Y^\perp)$. If $X$ is reflexive and $G$ is a closed subspace of $X^*$, then $G = (^\perp G)^\perp$.

*Proof.*

It suffices to show $^\perp(Y^\perp) \subseteq Y$. Let $x \in^\perp (Y^\perp)$. Then $\Lambda x = 0\ \forall \Lambda \in Y^\perp$. Such a $\Lambda$ has to vanish on $Y$. If $x \notin Y$, then $\exists \Lambda \in Y^\perp$ s.t. $\Lambda x = \operatorname{dist}(x, Y) \neq 0$. So $x \in Y$. For the other equality, let $\Lambda \in (^\perp G)^\perp$. Then $\Lambda x = 0\ \forall x \in^\perp G$. If $\Lambda \notin G$, then $\exists x \in X \simeq X^{**}$ s.t.

$$\Lambda x = \operatorname{dist}(\Lambda, G) \neq 0 \text{ and } x \in^\perp G$$

Impossible. So $\Lambda \in G$. □

**Proposition 1.** Let $X, Y$ be normed spaces and $T \in \mathcal{B}(X, Y)$. Then

$$N(T^t) = \overline{R(T)}^\perp$$

$$N(T) =^\perp \overline{R(T^t)}$$

$$^\perp N(T^t) = \overline{R(T)}$$

$$N(T)^\perp = (^\perp \overline{R(T^t)})^\perp$$

*Proof.*

The $3^{rd}$ and $4^{th}$ follows from the first 2 by Lemma. We will only show $N(T^t) = \overline{R(T)}^\perp$. Let $y^* \in N(T^t)$. By definition, $T^t y^* = 0 \Rightarrow y^*(Tx) = T^t y^*(x) = 0\ \forall x \in X$. $y^*$ vanishes on $R(T)$. By continuity, $y^*$ vanishes on $\overline{R(T)}$. So $y^* \in \overline{R(T)}^\perp$. Reverse the reasoning gives $\supseteq$. □

**Corollary 1.** Let $X, Y$ be normed and $T \in \mathcal{B}(X, Y)$. Then $R(T)$ is dense in $Y \Leftrightarrow T^t$ is injective.

*Proof.*

$T^t$ injective $\Leftrightarrow N(T^t) = \{0\} \Leftrightarrow \overline{R(T)}^\perp = \{0\} \Leftrightarrow \overline{R(T)} = Y$. □

# Examples of linear operators

Let $x = (x_n)$ be a sequence. If $T$ is a linear operator that maps sequences to sequences, we can write $Tx = y = (y_n)$. Each $y_n$ depends linearly on $x$. Formally, we can write

$$y_n = \sum_{j=1}^{\infty} c_{jn} x_j$$

Depending on which sequence space and the growth of $c_{jn}, T$ defines a bounded linear operator or an unbounded one.

**Example 1.** Two examples.

(a) Let $(a_n)$ be a sequence s.t. $\lim_{n\to\infty} a_n = 0, a_n \neq 0 \ \forall n$. Define $T : \ell^p \to \ell^p$ by

$$Tx = (a_1x_1, a_2x_2, ...).$$

$$\|Tx\|_p \leq \|a\|_\infty \|x\|_p$$

so $T$ is bounded. How about $T^{-1}$?

**Definition 4.** We say that $T \in \mathcal{B}(X, Y)$ is **invertible** if it is bijective and $T^{-1} \in \mathcal{B}(Y, X)$.

The $T$ above is not invertible. Why? If $T^{-1}$ exists, then $T^{-1}e_j = a_j^{-1}e_j \ \forall j$. If $T^{-1} \in \mathcal{B}(\ell^p)$, then

$$\|a_j^{-1}e_j\|_p = |a_j|^{-1} \leq \|T^{-1}\|$$

$\Rightarrow (|a_j|^{-1})$ is bounded, impossible.

**Example 1.** (b) Right shift operator $S_R : \ell^p \to \ell^p$

$$S_R(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$$

Check $S_R \in \mathcal{B}(\ell^p), \|S_R\| = 1$. $S_R$ is not onto, so $S_R$ is not invertible.

## Integral operators

Consider 1-dimensional case for simplicity. Fix $K \in C^0([a, b] \times [a, b])$ (can be relaxed) $K$ is usually called the integral kernel. Define

$$Tf(x) = \int_a^b K(x, y)f(y)\mathrm{d}y \ \forall f \in C^0([a, b])$$

$T$ is linear and bounded on $C^0([a, b])$.

$$|Tf(x)| \leq \int_a^b |K(x, y)||f(y)\mathrm{d}y \leq \|f\|_\infty \int_a^b |K(x, y)|\mathrm{d}y$$

$$\leq (\sup_x \int_a^b |K(x, y)|\mathrm{d}y)\|f\|_\infty$$

$$\Rightarrow \|Tf\|_\infty \leq M\|f\|_\infty \Rightarrow \|T\| \leq M$$

(with more effort, can show $\|T\| = M$.) Can define integral operators on other spaces, e.g. $L^p$-spaces. For $1 \leq p < \infty$,

$$\|Tf\|_{L^p}^p = \int_a^b \left| \int_a^b K(x, y)f(y)\mathrm{d}y \right|^p \mathrm{d}x$$

$$\leq (b - a)\|K\|_\infty^p \left( \int_a^b |f(y)|\mathrm{d}y \right)^p$$

by Holder's inequality.

$$\leq (b - a)\|K\|_\infty^p \left( \|f\|_{L^p}(b - a)^{\frac{1}{q}} \right)^p$$

$$= (b - a)^{1+\frac{p}{q}}\|K\|_\infty^p\|f\|_{L^p}^p = (b - a)^p\|K\|_\infty^p\|f\|_{L^p}^p$$

$\Rightarrow T \in \mathcal{B}(L^p([a, b]))$.

## Differential operator

$X = C^1([0,1])$ with sup norm $X$ is a subspace of $C^0([0,1])$, but $X$ is not a Banach space. $\frac{\mathrm{d}}{\mathrm{d}x} : X \to C^0([0,1])$ is linear but unbounded. (e.g. $f_k(x) = \sin(kx)$)

## Uniform boundedness principle/Banach-Steinhaus theorem

**Theorem 1.** Let $\mathcal{F}$ be a family of bounded linear operators from a Banach space $X$ to a normed space $Y$. Suppose that $\mathcal{F}$ is bounded pointwise:

$$\forall x \in X, \exists \text{ a constant } C_x \text{ s.t. } \|Tx\| \leq C_x \ \forall T \in \mathcal{F}$$

Then $\exists M > 0$ s.t. $\|T\| \leq M \ \forall T \in \mathcal{F}$.

**Lemma 2.** Let $T : X \to Y$ be linear, where $X, Y$ are normed. Fix $x_0 \in X$. Suppose that $\exists c, \rho > 0$ s.t. $\forall x \in B(x_0, \rho), \|Tx\| \leq c$. Then $\|T\| \leq \frac{2c}{\rho}$

*Proof.*

By linearity, $B(0, \rho) = B(x_0, \rho) - x_0$. If $v \in B(0, \rho)$, then $v + x_0 \in B(x_0, \rho)$.

$$\|Tv\| \leq \|T(v + x_0)\| + \|Tx_0\| \leq 2c$$

$$\Rightarrow \|T\| \leq \frac{2c}{\rho}$$

□

*Proof of UBP.*

Recall: $M$ complete metric space. If $M$ is a countable union of closed sets, then at least one of them has nonempty interior. Let $E_k = \{x \in X : \|Tx\| \leq k \ \forall T \in \mathcal{F}\}$. Then $X = \bigcup_{k=1}^{\infty} E_k$. Each $E_k$ is closed. $X$ is complete. At least one of the $E_k$'s has nonempty interior. By the Lemma, $\exists M > 0$ s.t. $\|T\| \leq M \ \forall T \in \mathcal{F}$. □

The uniform boundedness principle may not hold if the completeness assumption of $X$ is removed.

Alternative formulation:

**Definition 5.** A vector $x_0$ is called a **resonance point** for a family $\mathcal{F}$ of bounded linear operators if $\sup_{T \in \mathcal{F}} \|Tx_0\| = \infty$.

**Theorem 2.** Let $\mathcal{F}$ be a family of bounded linear operators from $X$ to $Y$, where $X$ is Banach and $Y$ is normed. Suppose that $\sup_{T \in \mathcal{F}} \|T\| = \infty$. Then the resonance points for $\mathcal{F}$ are dense in $X$.

*Proof.*

Suppose not. $\exists$ a ball $\overline{B(x_0, \rho)}$ on which $\mathcal{F}$ is bounded pointwise. $\forall x \in \overline{B(x_0, \rho)}, \|Tx\| \leq Cx \ \forall T \in \mathcal{F}$. Pick $0 \neq x \in X$. Then

$$z := \frac{\rho x}{\|x\|} + x_0 \in \overline{B(x_0, \rho)}$$

$$\Rightarrow \|Tz\| = \left\| T\left(\frac{\rho x}{\|x\|} + x_0\right) \right\|$$

$$\Rightarrow \|Tx\| \leq \frac{Cz + \|Tx_0\|}{\rho} \|x\| \ \forall T \in \mathcal{F}$$

$\Rightarrow \mathcal{F}$ is bounded pointwise on $X$. By uniform boundedness principle, $\exists M > 0$ s.t. $\|T\| \leq M \ \forall T \in \mathcal{F}$. Impossible! $\square$

## Interlude: Fourier series

Let $f \in L^1_{\text{loc}}(\mathbb{R})$ of period $2\pi$. We would like to express $f$ as a series of sines and cosines.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

In the complex form,

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Question: What should $c_n$ be if we want

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}?$$

Forget about the convergence issue at this point. Fact:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} \mathrm{d}x = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ if } k \neq 0 \end{cases}$$

If $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \mathrm{d}x\text{''} = \text{''} \sum_{n=-\infty}^{\infty} c_n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} \cdot e^{-ikx} \mathrm{d}x = c_k.$$

We (expectedly) call the left hand side $\hat{f}(k)$, the $k$-th Fourier coefficient of $f$.

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(x) e^{-inx}$$

When does the Fourier series converge? If yes, is it equal to $f$?