Analysis 2 W1-2

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Theorem 1. If (K_δ) is an approximation to the identity and $f \in L^1(\mathbb{R}^d)$ then

$$(f * K_\delta)(x) \to f(x) \text{ as } \delta \to 0 \forall x \in Leb(f)$$

Proof.

$$(f * K_\delta)(x) - f(x) = \int f(x-y)K_\delta(y)dy - \int f(x)K_\delta(y)dy$$

$$= \int (f(x-y) - f(x))K_\delta(y)dy$$

$$\Rightarrow |(f * K_\delta)(x) - f(x)| \leq \int |f(x-y) - f(x)||K_\delta(y)|\mathrm{d}y$$

$$\int_{|y|\leq\delta} |f(x-y) - f(x)||K_\delta(y)|dy \overset{(b)}{\leq} \frac{C}{\delta^d} \int_{|y|\leq\delta} |f(x-y) - f(x)|dy = CA(\delta)$$

$$\int_{2^k\delta<|y|\leq 2^{k+1}\delta} |f(x-y) - f(x)||K_\delta(y)|dy \leq C\delta \int_{2^k\delta<|y|\leq 2^{k+1}\delta} |f(x-y) - f(x)| \frac{1}{|y|^{d+1}} dy$$

$$\leq \frac{D\delta}{(2^k\delta)^{d+1}} \int_{|y|\leq 2^{k+1}\delta} |f(x-y) - f(x)|dy$$

$$\leq C' 2^{-k} A(2^{k+1}\delta)$$

$$\Rightarrow |(f * K_\delta)(x) - f(x)| \leq CA(\delta) + C' \sum_{k=0}^{\infty} 2^{-k} A(2^{k+1}\delta)$$

Given $\epsilon > 0$, choose N large s.t. $\sum_{k\geq N} 2^{-k} < \epsilon$. By the Lemma, we can choose $\delta > 0$ small s.t.

$$A(2^{k+1}\delta) < \frac{\epsilon}{N} \text{ for } k = 0, 1, \ldots, N-1$$

Also, A is bounded.

$$\Rightarrow |(f * K_\delta)(x) - f(x)| \leq C''\epsilon$$

This proves the theorem.

So $f * K_\delta \to f$ a.e. as $\delta \to 0$. We also have L^1-convergence.

□

Theorem 2. $f \in L^1(\mathbb{R}^d)$, $(K_\delta)_{\delta>0}$ approximation to the identity. Then $f(K_\delta) \in L^1(\mathbb{R}^d)$ and

$$||f * K_\delta - f||_{L^1} \to 0 \text{ as } \delta \to 0$$

Proof. $f * K_\delta \in L^1(\mathbb{R}^d)$ follows from Fubini (check!)

$$|f * K_\delta(x) - f(x)| \leq \int |f(x-y) - f(x)||K_\delta(y)|dy$$

$$||f * K_\delta - f||_{L^1} \leq \int\int |f(x-y) - f(x)||K_\delta(y)|dydx$$

Write $f_y(x) = f(x-y)$.

$$\Rightarrow ||f * K_\delta - f||_{L^1} \leq \int ||f_y - f||_{L^1}|K_\delta(y)|dy$$

Recall: Given $\epsilon > 0$, $\exists \eta > 0$ s.t. if $|y| < \eta$, then $||f_y - f||_{L^1} < \epsilon$.

$$\Rightarrow ||f * K_\delta - f||_{L^1} \leq C\epsilon + \int_{|y|\geq\eta} ||f_y - f||_{L^1}|K_\delta(y)|dy$$

$$\leq C\epsilon + 2||f||_{L^1}\int_{|y|\geq\eta} |K_\delta(y)|dy$$

Where the second term converges to 0 as $\delta \to 0$.

$\Rightarrow f * K_\delta \to f$ in L^1.

□

For which class of F we have

$$F(b) - F(a) = \int_a^b F'(x)dy?$$

If F is an indefinite integral then this must be true. When can a function be written as an indefinite integral? Will study a wider class of functions.

Definition 1. Let $F : [a, b] \to \mathbb{C}$ be a function, and let $a = t_0 < t_1 < \ldots < t_N = b$ be a partition of $[a, b]$. The variation of F on this partition is defined by

$$\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})|$$

F is said to be of **bounded variation** if

$$\sup_{\text{all partitions of } [a,b]} \sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| < \infty$$

Example 1.
- If $F : [a, b] \to \mathbb{R}$ is monotone, then F is of bounded variation.
- If F is Lipschitz, then F is of bounded variation.

Remark 1. If $\mathcal{P}$ is a partition of $[a, b]$ and if $\mathcal{P}'$ is a refinement of $\mathcal{P}(\mathcal{P} \subset \mathcal{P}')$ then the variation of F on $\mathcal{P}'$ is at least the variation of F on $\mathcal{P}$.

Definition 2. Let $F : [a, b] \to \mathbb{C}$ be a function of bounded variation. Let $x \in [a, b]$. The **total variation** of F on $[a, x]$ is

$$T_F(a, x) = \sup_{\text{all partitions of } [a,x]} \sum_{j=1}^{N} |F(t_j) - F(t_{j-1})|$$

If F is real-valued, the positive variation of F on $[a, x]$ is

$$P_F(a, x) = \sup \sum_{(+)} F(t_j) - F(t_{j-1})$$

Where the $(+)$ means summing over all j s.t. $F(t_j) \leq F(t_{j-1})$. The negative variation of F on $[a, x]$ is

$$N_F(a, x) = \sup \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

Where $(-)$ means summing over all j s.t. $F(t_j) \leq F(t_{j-1})$.

Lemma 1. Suppose $F : [a, b] \to \mathbb{R}$ is of bounded variation. Then $\forall x \in [a, b]$ one has

$$F(x) - F(a) = P_F(a, x) - N_F(a, x)$$

$$T_F(a, x) = P_F(a, x) + N_F(a, x)$$

Proof.

Let $\epsilon > 0$. $\exists$ a partition $a = t_0 < \ldots < t_N = b$ s.t.

$$|P_F(a, x) - \sum_{(+)} (F(t_j) - F(t_{j-1}))| < \frac{\epsilon}{2}$$

$$|N_F(a, x) - \sum_{(-)} -(F(t_j) - F(t_{j-1}))| < \frac{\epsilon}{2}$$

Observe

$$F(x) - F(a) = \sum_{j=1}^{N} (F(t_j) - F(t_{j-1}))$$

$$= \sum_{(+)} F(t_j) - F(t_{j-1}) - \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

$$\Rightarrow |F(x) - F(a) - (P_F(a,x) - N_F(a,x))| < \epsilon$$

For any partition $a = t_0 < \ldots < t_N = x$,

$$\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| = \sum_{(+)} F(t_j) - F(t_{j-1}) + \sum_{(-)} -(F(t_j) - F(t_{j-1}))$$

$$\Rightarrow T_F(a,x) = P_F(a,x) + N_F(a,x)$$

where one should verify this line carefully.

□

Theorem 3. $F : [a,b] \to \mathbb{R}$ is of bounded variation $\Leftrightarrow$ F is a difference of two increasing bounded functions.

Proof.

If $F = F_1 - F_2$, where F_1, F_2 are increasing and bounded, then $T_F(a,x) \le T_{F_1}(a,x) + T_{F_2}(a,x) < \infty$. On the other hand, if F is of bounded variation, take

$$F_1(x) = P_F(a,x) + F(a),\ F_2(x) = N_F(a,x)$$

where both the functions are increasing and bounded. Clearly $F(x) = F_1(x) - F_2(x)$. □

Consequence: Any complex-valued function of bounded variation is a linear combination of 4 increasing function.

Theorem 4. If F is of bounded variation on $[a,b]$, then F is differentiable a.e.

It suffices to show that increasing functions are differentiable a.e. We will first consider the case that the function is continuous.

Lemma 2. Suppose that G is real-valued and continuous on $\mathbb{R}$. Let $E = \{x \in \mathbb{R} : G(x+h) > G(x)$ for some $h = h_x > 0\}$. If $E \neq \phi$, then E is open, and we can write $E = \bigcup_k (a_k, b_k)$ where the intervals are disjoint. If (a_k, b_k) is a finite interval in the union, then $G(b_k) - G(a_k) = 0$.

Proof.

Suppose that (a_k, b_k) is a finite interval in the union. $a_k \notin E \Rightarrow$ we cannot have $G(b_k) > G(a_k)$. Suppose that $G(b_k) < G(a_k)$. By continuity, $\exists c \in (a_k, b_k)$ s.t.

$$G(c) = \frac{G(a_k) + G(b_k)}{2}$$

We may choose c to be the rightmost such point in (a_k, b_k). Then $c \in E$. So $\exists d > c$ s.t. $G(d) > G(c)$. Also, $b_k \notin E$. So $G(x) \le G(b_k) \forall x \le b_k$.

$$G(d) > G(c) > G(b_k) \Rightarrow d < b_k$$

By continuity again, $\exists c' \in (d, b_k)$ s.t. $G(c') = G(c)$, contradicting that c is the rightmost point. So $G(a_k) = G(b_k)$. $\square$

Remark 2. If $G : [a, b] \to \mathbb{R}$ is continuous, and if we define $E = \{x \in (a, b) : G(x + h) > G(x)$ for some $h = h_x > 0\}$, then the above result still holds, except possibly when $a_k = a$. In this case we only have $G(a_k) \leq G(b_k)$.

Define

$$\Delta_h(F)(x) = \frac{F(x+h) - F(x)}{h}$$

The Dini derivatives of F at x are

$$D^+(F)(x) = \limsup_{h \to 0^+} \Delta_h(F)(x)$$

$$D_+(F)(x) = \liminf_{h \to 0^+} \Delta_h(F)(x)$$

$$D^-(F)(x) = \limsup_{h \to 0^-} \Delta_h(F)(x)$$

$$D_-(F)(x) = \liminf_{h \to 0^-} \Delta_h(F)(x)$$

Want: $D^+ = D_+ = D^- = D_- < \infty$ a.e. clearly, $D_+ \leq D^+$ and $D_- \leq D_-$. It suffices to verify that for all increasing and continuous F,

(a) $D^+(F)(x) < \infty$ for a.e. x

(b) $D^+(F)(x) \leq D_-(F)(x)$ for a.e. x.

If (a) and (b) hold for all increasing and continuous F, we apply (b) to $-F(-x)$ (increasing and continuous) on the interval $[-b, -a]$.

$$\Rightarrow D^+(-F)(-x) \leq D_-(-F)(-x)(\text{write } \tilde{F}(x) = -F(-x))$$

while

$$\text{L.H.S.} = \limsup_{h \to 0^+}(-\frac{\tilde{F}(x+h) - \tilde{F}(x)}{h}) = \limsup_{h \to 0^+}(-\frac{F(-x-h) - F(-x)}{-h})$$

$$= \limsup_{h \to 0^+}(\frac{F(-x-h) - F(-x)}{-h}) = \limsup_{h \to 0^-}(\frac{F(-x+h) - F(-x)}{h})$$

$$= D^-(F)(-x) = D^-(F)(y)$$

$$D_-(-F)(y) = D_+(F)(y)$$

$$\Rightarrow D_- \leq D_+$$

$$\Rightarrow D^+ \overset{(b)}{\leq} D_- \leq D^- \leq D_+ \leq D_+ \overset{(a)}{<} \infty$$

So all are equal and finite. $\Rightarrow F'$ exists a.e. Remains to verify (a) and (b). For $\gamma > 0$, define

$$E_\gamma = \{x : D^+(F)(x) > \gamma\}$$

E_γ is measurable. Apply the remark to $G(x) = F(x) - \gamma_x$

$$E = \{x \in (a,b) : G(x+h) > G(x) \text{ for some } h > 0\}$$

$$\Rightarrow E_\gamma a \subset E = \bigcup_k (a_k, b_k)$$

By Remark, $G(a_k) \leq G(b_k)$

$$\Rightarrow F(a_k) - \gamma a_k \leq F(b_k) - \gamma b_k$$

$$\Rightarrow F(b_k) - F(a_k) \leq \gamma(b_k - a_k)$$

$$m(E_\gamma) \leq m(E) \leq \sum_k m((a_k, b_k)) \leq \frac{1}{\gamma} \sum_k (F(b_k) - F(a_k))$$

$$\overset{\text{F increasing}}{\leq} \frac{1}{\gamma}(F(b) - F(a))$$

$$\Rightarrow \lim_{\gamma \to \infty} m(E_\gamma) = 0$$

$$\Rightarrow \{x : D^+ = \infty\} \text{ has measure zero}$$