Analysis 2 W2-1

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Remark 1. If $p = \infty$

$$\|f\|_\infty = \text{esssup}(|f|)$$

where

$$\text{esssup}(f) \equiv \inf\{y > 0 : m(|f| > y) = 0\}$$

Last time: Try to show if F is increasing and continuous then F is differentiable a.e. It remains to show

$$D^+(F)(x) \leq D_-(F)(x) \text{ for a.e. x}$$

Fix $r < R$ rationals. Define

$$E = \{x \in [a, b] : D^+(F)(x) < r < R < D_-(F)(x)\}$$

If we can show $m(E) = 0$ then we are done (Take union over $r < R$). Assume that $m(E) > 0$. Recall: $\forall \epsilon > 0, \exists$ open $\mathcal{U} \supset E$ s.t. $m(\mathcal{U} < m(E) + \epsilon$. $R/r > 1 \Rightarrow$ take $\epsilon = (R/r - 1)m(E) > 0$. Find an open set $E \subset \mathcal{U} \subset (a, b)$ s.t.

$$m(\mathcal{U}) < \frac{R}{r} m(E)$$

Write $\mathcal{U} = \cup_n I_n$ where I_n are disjoint open intervals (may assume $I_n \cup E \neq \phi \ \forall n$).

Recall: $G : [a, b] \to \mathbb{R}$ continuous.

$$E = \{x \in (a, b) : G(x + h) > G(x) \text{ for some } h > 0\}$$

If $E \neq \phi$, write $E = \cup_k (a_k, b_k)$. We have $G(a_k) \leq G(b_k)$. We apply the above to $G(x) = F(-x) + rx$ on $-I_n$.

$$E_n = \{x \in -I_n : G(x + h) > G(x) \text{ for some } h > 0\}$$

$E_n \neq \phi$? Now

$$G(x + h) > G(x) \Leftrightarrow F(-x - h) + r(x + h) > F(-x) + rx$$

$$\Leftrightarrow F(-x - h) - F(-x) > -rh$$

$$\Leftrightarrow \frac{F(-x - h) - F(-x)}{-h} < r$$

If $x \in (-E) \cap (-I_n)$ then $x \in E_n$. So $E_n \neq \phi$. Write $E_n = \cap_k (-b_k, -a_k)$. Then we have $G(-b_k) \leq G(-a_k)$.

$$\Leftrightarrow F(b_k) - F(a_k) \leq r(b_k - a_k)$$

On each (a_k, b_k), we can apply the recall to

$$G(x) = F(x) - Rx$$

By a similar argument, we obtain an open set $E'_n = \cup_{k,j}(a_{k,j}, b_{k,j})$ s.t. $(a_{k,j}, b_{k,j}) \subset (a_k, b_k)$ $\forall j$ and $F(b_{k,j}) - F(a_{k,j}) \geq R(b_{k,j} - a_{k,j})$.

$$\begin{aligned} m(E'_n) &= \sum_{k,j}(b_{k,j} - a_{k,j}) \\ &\leq \frac{1}{R}\sum_{k,j}(F(b_{k,j}) - F(a_{k,j})) \\ &\overset{\text{F increasing}}{\leq} \frac{1}{R}\sum_{k}(F(b_k) - F(a_k)) \\ &\leq \frac{r}{R}\sum_{k}(b_k - a_k) \\ &\leq \frac{r}{R}m(I_n) \end{aligned}$$

We can show similarly that $I_n \cap E \subset E'_n$.

$$\begin{aligned} m(E) = \sum_n m(E \cap I_n) &\leq \sum_n m(E'_n) \\ \leq \frac{r}{R}\sum_n m(I_n) &= \frac{r}{R}m(\mathcal{U}) < m(E) \end{aligned}$$

A contradiction!!!!! So $m(E) = 0$.

Corollary 1. If F is increasing and continuous, the F' exists a.e. Moreover, F' is measurable, nonnegative, and

$$\int_a^b F'(x)\mathrm{d}x \leq F(b) - F(a)$$

In particular, if F is bounded on $\mathbb{R}$, then F' is integrable on $\mathbb{R}$.

Proof.

We only prove the statement in the middle.

$$F'(x) = \lim_{n\to\infty} n(F(x + \frac{1}{n}) - F(x))$$

exists a.e. So F' is measurable.

$$\int_a^b F'(x)\mathrm{d}x \leq \liminf_{n\to\infty} n\int_a^b (F(x + \frac{1}{n}) - F(x))\mathrm{d}x$$

$$\Rightarrow \int_a^b F'(x)\mathrm{d}x \leq n\int_b^{b+\frac{1}{n}} F(x)\mathrm{d}x - n\int_a^{a+\frac{1}{n}} F(x)\mathrm{d}x = F(b) - F(a)$$

where the last line comes from continuity. We do not expect equality holds for all increasing and continuous F. Counterexapmle: Cantor function.

$\square$

1 Absolutely continuous functions

Definition 1. A function $F : [a, b] \to \mathbb{R}$ is said to be **absolutely continuous** if $\forall \epsilon > 0, \exists \delta > 0$ s.t.

$$\sum_{k=1}^{N} |F(b_k) - F(a_k)| < \epsilon \text{ whenever } \sum_{k=1}^{N} (b_k - a_k) < \delta$$

where $(a_k, b_k), k = 1, ..., N$ are disjoint intervals.

Observations:

- Absolute continuity $\Rightarrow$ Uniform continuity
- Absolute continuity $\Rightarrow$ bounded variation. In this case, the total variation is also absolutely continuous.

So every absolutely continuous function can be written as a difference of two increasing functions.

Example 1.

- Lipschitz functions.
- $F(x) = \int_a^x f(x)\mathrm{d}x$, where f integrable. F is absolutely continuous. Recall: $\forall \epsilon > 0, \exists \delta > 0$ s.t. $m(E) < \delta \Rightarrow \int_E |f| < \epsilon$. ($f$ is integrable)

From the second example, if we want $F(b) - F(a) = \int_a^b F'(x)\mathrm{d}x$, then F has to be absolutely continuous. It turns out that absolute continuity is sufficient!

Definition 2. A collection $\mathcal{B}$ is said to be a **Vitali covering** of a set E if $\forall x \in E$ and $\forall \eta > 0$, exists a ball $B \in \mathcal{B}$ s.t. $x \in B$ and $m(B) < \eta$.

Lemma 1. Suppose that $m(E) < \infty$, and $\mathcal{B}$ is a Vitali covering of E. For any $\delta > 0$, we can find finitely many balls $B_1, ..., B_n \in \mathcal{B}$ that are disjoint and s.t.

$$m(E) < \sum_{i=1}^{N} m(B_i) + \delta$$

Proof.

May assume $m(E) > \delta$. Take any compact $E' \subset E$ s.t. $m(E') \geq \delta$. We can cover E' by finitely many balls in $\mathcal{B}$. Recall: if $\{B_1, ..., B_k\}$ are open balls, we can find a disjoint collection $B_{i_1}, ..., B_{i_j}$ of balls s.t.

$$m\left(\bigcup_{i=1}^{k} B_i\right) \leq 3^d \sum_{l=1}^{j} m(B_{i_l})$$

So we obtain disjoint balls from $\mathcal{B}$, say $B_1, ..., B_{N_1}$, s.t.

$$\delta \leq 3^d \sum_{i=1}^{N_1} m(B_i)$$

Two cases:

1. $m(E) \leq \sum_{i=1}^{N_1} m(B_i) + \delta$. Then done.

2. $m(E) > \sum_{i=1}^{N_1} m(B_i) + \delta$. Consider $E_2 = E \setminus \cup_{i=1}^{N_1} \overline{B_i}$. Then $m(E_2) > \delta$. Moreover, the balls in $\mathcal{B}$ that are disjoint from $\cup_{i=1}^{N_1} \overline{B_i}$ is a Vitali covering for E_2. Therefore from this remaining collection of balls we can find disjoint balls, say

$$B_{N_1+1}, ..., B_{N_2}$$

s.t.

$$\delta \leq 3^d \sum_{i=N_1+1}^{N_2} m(B_i)$$

Again two cases:

(a) $m(E) < \delta + \sum_{i=1}^{N_2} m(B_i)$ Then done.

(b) $m(E) > \delta + \sum_{i=1}^{N_2} m(B_i) \geq 2\delta 3^{-d}$.

At each step we increase by a factor of $\delta \cdot 3^{-d}$. $m(E) < \infty$, this procedure must stop at some point.

□

Theorem 1. If F is absolutely continuous on $[a, b]$, then F' exists almost everywhere. Moreover, if $F'(x) = 0$ for a.e. x, then F is a constant.

Proof.

To prove the second statement, we just need to show that $F(b) = F(a)$ (can apply the argument to subintervals). Let

$$E = \{s \in (a, b) : F'(x) \text{ exists and } F'(x) = 0\}$$

Then $m(E) = b - a$. Fix $\epsilon > 0$. For each $x \in E$, we have

$$\lim_{h\to 0} \left| \frac{F(x + h) - F(x)}{h} \right| = 0$$

So for each $\eta > 0$, $\exists$ open interval $I = (a_x, b_x) \subset [a, b]$ containing x s.t.

$$|F(b_x) - F(a_x)| \leq \epsilon(b_x - a_x) \text{ and } (b_x - a_x) < \eta$$

These intervals form a Vitali covering of E. By the previous lemma, $\forall \delta > 0$, we can find finitely many intervals $I_1, ..., I_N$ s.t. they are disjoint and

$$b - a = m(E) \leq \sum_{i=1}^{N} m(I_i) + \delta$$

$$m\left(\bigcup_{i=1}^{N} I_i\right) = \sum_{i=1}^{N} m(I_i) \geq (b - a) - \delta$$

Write $I_i = (a_i, b_i)$. The complement of $\cup_{i=1}^{N} I_i$ in $[a, b]$ has measure $\leq \delta$. Also, it is a union of finitely many closed interals $\cup_{k=1}^{M} [\alpha_k \beta_k]$. By absolute continuity, we can choose $\delta > 0$ small s.t.

$$\sum_{k=1}^{M} |F(\beta_k) - F(\alpha_k)| < \epsilon$$

On the other hand,

$$|F(b_i) - F(a_i)| \leq \epsilon(b_i - a_i)$$
$$\Rightarrow \sum_{i=1}^{N} |F(b_i) - F(a_i)| \leq \epsilon(b - a)$$

Combine everything,

$$|F(b) - F(a)| \leq \sum_{k=1}^{M} |F(\beta_k) - F(\alpha_k)| + \sum_{i=1}^{N} |F(b_i) - F(a_i)| \leq \epsilon(b - a + 1)$$

$\epsilon > 0$ is arbitrary, so $F(b) = F(a)$.

□

Corollary 2. Suppose F is absolutely continuous on $[a, b]$. Then F' exists a.e. and is integrable. Moreover

$$F(x) - F(a) = \int_a^x F'(y)\mathrm{d}y \ \forall x \in [a, b]$$

Conversely, if f is integrable on $[a, b]$, then $\exists$ absolutely continuous function F s.t. $F' = f$ a.e.

Proof.

F' integrable because

$$\int_a^b G'(x)\mathrm{d}x \leq G(b) - G(a)$$

if G is increasing and continuous. To prove the equality, let $G(x) = \int_a^x F'(y)\mathrm{d}y$. So G is absolutely continuous, and so is $G - F$. By the Lebesgue differentiation theorem, $G'(x) = F'(x)$ for a.e. x. So $(G - F)' = 0$ a.e. $\Rightarrow G - F$ is a constant.

$$\Rightarrow \int_a^x F'(y)\mathrm{d}y = F(x) - F(a)$$

For the converse part, take $F(x) = \int_a^x f(y)\mathrm{d}y$. □

2 Differentiability of jump functions

Recall: We haven't finished if F is of bounded variation on $[a, b]$ then it is differentiable a.e. Let $F : [a, b] \to \mathbb{R}$ be increasing and bounded. Write (x_n) for the points at which F is discontinuous. Let

$$\alpha_n = F(x_n^+) - F(x_n^-)$$

which is the jump of F at x_n. $\exists \theta_n \in [0, 1]$ s.t. $F(x_n) = F(x_n^-) + \theta_n \alpha_n$. Define

$$j_n(x) = \begin{cases} 0 & \text{if } x < x_n \\ \theta_n & \text{if } x = x_n \\ 1 & \text{if } x > x_n \end{cases}$$

Finally, define the jump function associated to F by

$$J_F(x) = \sum_{n-1}^{\infty} \alpha_n j_n(x)$$

Observe:

$$\sum_{n=1}^{\infty} \alpha_n \leq F(b) - F(a) < \infty$$

So the series defining J_F converges absolutely and uniformly.

Lemma 2. Suppose that F is increasing and bounded on $[a, b]$. Then J_F is dscontinuous precisely at (x_n), and has jump at x_n equal to that of F. Moreover, $F - J_F$ is increasing and continuous.