Analysis 2 W4-1

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Recall: $BV_0([a,b])$: set of all functions α of bounded variation on $[a,b]$ s.t. $\alpha(a)=0$. $V([a,b])=\{\alpha \in BV_0([a,b]) : \alpha \text{ is right-continuous on } [a,b)\}$. Norm on $V([a,b])$: Total variation.

Theorem 1 (Riesz representation theorem)**.** $\exists$ an isometric linear isomorphism from $(C^0([a,b]))^*$ to $V([a,b])$.

Remark 1. $\exists$ a much more general version of Riesz representation. X is compact Hausdorff space. $(C^0(X))^*$ can be identified with a space of nice Borel measures on X. For $X=[a,b]$, the integrals can also be represented as R-S integrals. $((\alpha(x)=\mu([a,b]), \mu([c,d])=\alpha(d)-\alpha(c))$ (See Rudin's "Real and Complex Analysis", Chapter 2.)

Proof.

Let $\Lambda \in (C^0([a,b]))^*$. Want: find $\alpha \in V([a,b])$ s.t.

$$\Lambda f = \int f \mathrm{d}\alpha \ \forall f \in C^0([a,b])$$

We can "extract" α by using characteristic functions:

$$\Lambda(\chi_{[a,x]}) = \int_a^b \chi_{[a,x]} \mathrm{d}\alpha = \int_a^x \mathrm{d}\alpha = \alpha(x)$$

Problem: $\chi_{[a,x]}$ is not continuous! We use Hahn-Banach to extend Λ. $C^0([a,b]) \subseteq C_b([a,b])$, where $C_b([a,b])$ is the space of all bounded functions on $[a,b]$. $\exists$ an extension $\tilde{\Lambda} \in C_b([a,b])^*$ of Λ with $\|\tilde{\Lambda}\| = \|\Lambda\|$. Define $\alpha(x) = \tilde{\Lambda}(\chi_{[a,x]})$ for $x \in (a,b]$ where $\alpha(a)=0$.

Claim. α is of bounded variation and $T_\alpha(a,b) \leq \|\Lambda\|$.

Proof.

Let $P=\{x_0,...,x_n\}$ be a partition of $[a,b]$. For each $j, \exists \theta_j$ s.t.

$$|\alpha(x_j)-\alpha(x_{j-1})| = e^{i\theta_j}(\alpha(x_j)-\alpha(x_{j-1}))$$

$$\Rightarrow \sum_{j=1}^n |\alpha(x_j)-\alpha(x_{j-1})| = \sum_{j=1}^n e^{i\theta_j}(\alpha(x_j)-\alpha(x_{j-1}))$$

$$= e^{i\theta_1}\alpha(x_1) + \sum_{j=2}^n e^{i\theta_j}(\alpha(x_j)-\alpha(x_{j-1}))$$

$$= e^{i\theta_1}\tilde{\Lambda}(\chi_{[a,x_1]}) + \sum_{j=2}^{n} e^{i\theta_j}(\tilde{\Lambda}(\chi_{[a,x_j]}) - \tilde{\Lambda}(\chi_{[a,x_{j-1}]}))$$

$$= \tilde{\Lambda}(e^{i\theta_1}\chi_{[a,x_1]}) + \sum_{j=2}^{n} \tilde{\Lambda}(e^{i\theta_j}\chi_{(x_{j-1},x_j]})$$

$$= \tilde{\Lambda}(e^{i\theta_1}\chi_{[a,x_1]} + \sum_{j=2}^{n} e^{i\theta_j}\chi_{(x_{j-1},x_j]})$$

$\forall x \in [a,b], \exists$ a unique interval $[a,x_i]$ of $(x_{j-1},x_j]$ that contains x. So $h(x) = e^{i\theta_{j_0}}$ for some j_0. $|h(x)| = 1\ \forall x \in [a,b]$. $\Rightarrow \|h\|_\infty = 1$.

$$\sum_{j=1}^{n} |\alpha(x_j) - \alpha(x_{j-1})| \leq \|\tilde{\Lambda}\|\|h\|_\infty = \|\Lambda\|$$

$\Rightarrow T_\alpha(a,b) \leq \|\Lambda\|$. $\Rightarrow \alpha$ is of bounded variation. □

Define $\Psi : (C^0([a,b]))^* \to V([a,b])$ by $\Psi(\Lambda) = \tilde{\alpha}$, where $\tilde{\alpha}$ is the right-continuous modification of α with $\tilde{\alpha(a)} = 0$. Then $T_{\tilde{\alpha}}(a,b) = T_\alpha(a,b) \leq \|\Lambda\|$.(The "=" is quite subtle, need to check) $\Rightarrow T_{\Psi(\Lambda)}(a,b) \leq \|\Lambda\|\ \forall \Lambda \in (C^0([a,b]))^*$.

Claim. $\Lambda f = \Lambda_{\tilde{\alpha}} f := \int f \mathrm{d}\tilde{\alpha}\ \forall f \in C^0([a,b])$.

If Claim holds, recalling that we defined

$$\Phi(\alpha) = \Lambda_\alpha = \int \cdot \mathrm{d}\alpha$$

then

$$\Phi(\Psi(\Lambda)) = \Phi(\tilde{\alpha}) = \Lambda_{\tilde{\alpha}} = \Lambda$$

Φ is surjective. Moreover, $T_\alpha(a,b) \leq \|\Phi(\alpha)\| \leq T_\alpha(a,b)$. $\Rightarrow \Phi$ is an isometry. Ψ is an isometric linear isomorphism.

Proof of Claim.

Let $\epsilon > 0$, and let $f \in C^0([a,b])$. f is R-S integrable w.r.t. $\alpha, \exists \delta_1 > 0$ s.t.

$$\left| \int f \mathrm{d}\alpha - \sum_{j=1}^{n} f(t_j)(\alpha(x_j) - \alpha(x_{j-1})) \right| < \epsilon$$

whenever mesh$(P) < \delta_1$. Similarly to above, we can write

$$\sum_{j=1}^{n} f(t_j)(\alpha(x_j) - \alpha(x_{j-1})) = \tilde{\Lambda}\left(f(t_1)\chi_{[a,x_j]} + \sum_{j=2}^{n} f(t_j)\chi_{(x_j,x_{j-1}]} \right) = \tilde{\Lambda}\tilde{f}$$

$$\Rightarrow \left| \int f \mathrm{d}\alpha - \tilde{\Lambda}(\tilde{f}) \right| < \epsilon ...(*)$$

On the other hand, f is uniformly continuous on $[a,b]$. $\exists \delta_2 > 0$ s.t.

$$|f(x) - f(y)| < \epsilon\ \forall x,y \in [a,b] \text{ with } |x-y| < \delta_2$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Let P be a partition s.t. mesh$(P) < \delta$. Then

$$f(x) - \tilde{f}(x) = f(x)\left(\chi_{[a,x_1]}(x) + \sum_{j=2}^{n} \chi_{(x_{j-1},x_j]}(x)\right) - \tilde{f}(x)$$

$$= (f(x) - f(t_1))\chi_{[a,x_1]}(x) + \sum_{j=2}^{n}(f(x) - f(t_j))\chi_{(x_{j-1},x_j]}(x)$$

So $\exists j_0$ s.t.

$$|f(x) - \tilde{f}(x)| = |f(x) - f(t_{j_0})|$$

$$\Rightarrow \|f - \tilde{f}\|_\infty < \epsilon$$

$$|\tilde{\Lambda}f - \tilde{\Lambda}\tilde{f}| \leq \|\tilde{\Lambda}\|\|f - \tilde{f}\|_\infty < \epsilon\|\Lambda\|$$

$$\left|\int f \mathrm{d}\alpha - \tilde{\Lambda}f\right| \leq \left|\int f \mathrm{d}\alpha - \tilde{\Lambda}(\tilde{f})\right| + |\tilde{\Lambda}(\tilde{f}) - \tilde{\Lambda}f|$$

Taking $\epsilon \to 0$,

$$\Lambda_{\tilde{\alpha}}f = \Lambda_\alpha f = \int f \mathrm{d}\alpha = \Lambda f$$

This proves the claim. □

□

Reflexive Spaces

If X is normed space then so is X^*. We can consider X^{**}. $1 < p < \infty, (\ell^p)^{**} \simeq \ell^p$. $(c_0)^{**} \simeq (\ell^1)^* \simeq \ell^\infty$. It turns out that any vector in X can be viewed as a vector in X^{**}.

Proposition 1. For each $x \in X$, we define a function $\tilde{x}$ on X^*. (that is, $\tilde{x} \in L(X^*, \mathbb{C})$) by

$$\tilde{x}(\Lambda) = \Lambda x \ \forall \Lambda \in X^*$$

Then $\tilde{x} \in X^{**}, \|\tilde{x}\| = \|x\|$. The mapping $J : x \mapsto \tilde{x}$ is an isometric linear map from X to X^{**}, called the canonical identification.

Proof.

Clearly J is linear.

$$|\tilde{x}(\Lambda)| = |\Lambda x| \leq \|x\|\|\Lambda\|$$

$$\Rightarrow \tilde{x} \in X^{**} \text{ and } \|\tilde{x}\| \leq \|x\|$$

If $x = 0$, the equality holds. If $x \neq 0$, by Hahn-Banach, $\exists \Lambda_0 \in X^*$ s.t. $\Lambda_0 x = \|x\|$ and $\|\Lambda_0\| = 1$.

$$\Rightarrow \|x\| = \Lambda_0 x = \tilde{x}(\Lambda_0) \leq \|\tilde{x}\|\|\Lambda_0\| = \|\tilde{x}\|$$

So J is an isometry. □

Definition 1. A normed space is called **reflexive** if J is an isometric linear isomorphism.

Remark 2.
- To show a normed space is reflexive we only need to show that J is surjective.
- Reflexive $\Rightarrow X \simeq X^{**}$.
- " $\Leftarrow$ " is not true in general. $\exists$ a nonreflexive Banach space X s.t. $\exists$ an isometric linear isomorphism from X to X^{**}.(Of course cannot be J.)

Proposition 2. Let $1 < p < \infty$. Then ℓ^p is reflexive.

Proof.

Recall: For each $\Lambda \in (\ell^p)^*, \exists! y^\Lambda \in \ell^q$ s.t.

$$\Lambda x = \sum_{j=1}^{\infty} y_j^\Lambda x_j \ \forall x \in \ell^p$$

Let $\sigma \in (\ell^p)^{**}$. Want to find $z \in \ell^p$ s.t. $\tilde{z} = \sigma$. Define

$$\sigma_1 y^\Lambda := \sigma\Lambda$$

where $\sigma_1 \in (\ell^q)^*$. σ_1 is bounded. Since $(\ell^q)^* \simeq \ell^p, \exists z \in \ell^p$ s.t.

$$\sigma_1 y^\Lambda = \sum_{j=1}^{\infty} y_j^\Lambda z_j \ \forall y^\Lambda \in \ell^q$$

In other words,

$$\sigma\Lambda = \sum_{j=1}^{\infty} y_j^\Lambda z_j$$

The canonical identification of z is given by

$$\tilde{z}(\Lambda) = \Lambda z = \sum_{j=1}^{\infty} y_j^\Lambda z_j$$

$\Rightarrow \tilde{z} = \sigma \Rightarrow \ell^p$ is reflexive. ☐

Similarly, L^p is reflexive for $1 < p < \infty$. $p = 1$? General Banach spaces?

Proposition 3. A reflexive space is a Banach space.

Proof. $X \simeq X^{**}$, which is complete. ☐

In particular, $(C^0([a,b]), \|\cdot\|_{L^p})$ is not reflexive for $1 \le p < \infty$.

Proposition 4. If X^* is separable, then X is also separable.

Proof.

Since X^* is separable,

$$\{\Lambda \in X^* : \|\Lambda\| = 1\} \text{ is also separable.}$$

Pick a countable dense subset $\{\Lambda_1, \Lambda_2, ...\}$. For each $k, \exists x_k$ with $\|x_k\| = 1$ s.t. $|\Lambda_k x_k| \ge \frac{1}{2}$. Let

$$E = \text{span}\{x_1, ...\}$$

Then E is separable.

Claim. $E = X$.

Proof.

Suppose not. Pick $x_0 \in X \setminus E$. By Hahn-Banach, $\exists \Lambda_0 \in X^*$ s.t. $\Lambda_0 = 0$ on E, $\|\Lambda_0\| = 1$. Since $\{\Lambda_1, \Lambda_2, ...\}$ are dense, $\exists k_0$ s.t.

$$\|\Lambda_0 - \Lambda_{k_0}\| < \frac{1}{4}$$

Then

$$\frac{1}{2} \leq |\Lambda_{k_0} x_{k_0}| = |(\Lambda_{k_0} - \Lambda_0) x_{k_0}| \leq \|\Lambda_{k_0} - \Lambda_0\| \|x_{k_0}\| < \frac{1}{4}$$

A contradiction. □

□

ℓ^1 is not reflexive. If it was, then $(\ell^1)^{**} = (\ell^\infty)^* \simeq \ell^1$. ℓ^1 separable $\Rightarrow (\ell^\infty)^*$ is separable $\Rightarrow \ell^\infty$ separable, a contradiction.

Bounded linear operators

Let X, Y be normed spaces over $\mathbb{C}$. A linear operator $T : X \to Y$ is bounded if it maps a bounded set in X to a bounded set in Y. Check: A linear operator is bounded $\Leftrightarrow$ it is continuous.

$$\mathcal{B}(X, Y) = \{\text{bounded linear operators from } X \text{ to } Y\}$$

$X^* = \mathcal{B}(X, \mathbb{C})$ If $X = Y$, we write $\mathcal{B}(X) = \mathcal{B}(X, X)$. We can also define a norm on $\mathcal{B}(X, Y)$: For $T \in \mathcal{B}(X, Y)$, define its operator norm by

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Tx\|_Y = \sup_{\|x\|_X \leq 1} \|Tx\|_Y$$

Check:

- $\|\cdot\|$ is a norm.
- If $T \in \mathcal{B}(X, Y), S \in \mathcal{B}(Y, Z)$ then $ST \in \mathcal{B}(X, Z)$ and $\|ST\| \leq \|S\| \|T\|$.

When $X = Y = Z$, there is a multiplicative structure on $\mathcal{B}(X)$. $\mathcal{B}(X)$ is an algebra.

Proposition 5. Let $T \in \mathcal{B}(X, Y)$. Suppose that $\exists M > 0$ s.t.

(a) $\|Tx\| \leq M\|x\| \; \forall x \in D$, where D is dense in X.

(b) $\exists$ a nonzero sequence (x_n) in D s.t.

$$\frac{\|Tx_k\|}{\|x_k\|} \to M$$

Then $M = \|T\|$.

Proof.

Clearly(?), $\|T\| \leq M$.

$$M = \lim_{k\to\infty} \frac{\|Tx_k\|}{\|x_k\|} \leq \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \|T\|$$

□

In linear algebra, we want to solve nonhomogeneous system.

$$Ax = b$$

where A is a $m \times n$ matrix and $b \in \mathbb{R}^m$. The Fredholm alternative states that either this system is unique solvable, or the homogeneous system

$$A^\dagger y = 0$$

has a nonzero solution y. Moreover, the nonhomogeneous system is solvable $\Leftrightarrow$ $b \perp y$ $\forall$ solutions y of the homogeneous system. Question: What happens in infinite dimensional spaces?