Analysis 2 W5-2

fat

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1 Spectrum

Recall: $\mathcal{B}(X) = \mathcal{B}(X, X)$. If X is a Banach space then $\mathcal{B}(X)$ is a Banach space. I is the identity map from X to X, $I \in \mathcal{B}(X)$.

Definition 1. $\lambda \in \mathbb{C}$ is called an **eigenvalue** of T is $\exists$ nonzero $x \in X$ called an **eigenvector**, such that $Tx = \lambda x$.

We will focus on the case that X is a Banach space and T is bounded. Open mapping theorem $\Rightarrow$

$$T - \lambda I \text{ invertible } \Leftrightarrow T - \lambda I \text{ bijective}$$

Definition 2. $\lambda \in \mathbb{C}$ is called a **regular value** for T if $T - \lambda I$ is invertible.

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is a regular value for } T\}$$

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

where $\rho(T)$ is called the **resolvent set** and $\sigma(T)$ the **spectrum** of T.

If λ is an eigenvalue of T then $\lambda \in \sigma(T)$. $(T - \lambda I$ is not injective.$)$ In finite dimensions, $S : X \to X$ injective $\Leftrightarrow S$ is surjective. NOT TRUE in infinite dimensions.

Example 1. Let $X = C^0([0, 1])$. Consider

$$Tf(x) = xf(x) \ \forall x \in [0, 1], f \in C^0([0, 1]).$$

$T \in \mathcal{B}(C^0([0, 1])), \|T\| \leq 1$. If λ is an eigenvalue of T and φ is an eigenvector/eigenfunction, theen

$$x\varphi(x) = \lambda\varphi(x) \ \forall x \in [0, 1]$$

$$\Rightarrow \varphi = 0$$

T does not have any eigenvalue! For $\lambda \in \mathbb{C} \setminus [0, 1]$, the inverse of $T - \lambda I$ is

$$Sf(x) = \frac{f(x)}{x - \lambda}$$

Can check: $S \in \mathcal{B}(C^0([0, 1]))$. If $\lambda \in [0, 1]$, then the inverse of $T - \lambda I$ does not exist. Conclusion: T has no eigenvalues, but $\sigma(T) = [0, 1], \rho(T) = \mathbb{C} \setminus [0, 1]$.

What can we say about $\sigma(T)$?

Proposition 1 (Lemma)**.** Let $T \in \mathcal{B}(X)$, where X is a Banach space. Then $I - T$ is invertible if $\|T\| < 1$. (Want to prove that $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$.)

Proof.

By assumption, $\exists r \in (0,1)$ such that $\|T\| \leq r$.

$$\|T^k\| \leq \|T\|^k \leq r^k$$

$\Rightarrow \sum_k T^k$ converges in X (similar to the Weierstrass M-test).

$$(I-T)\sum_{k=0}^{\infty} T^k = \lim_{n\to\infty}\left((I-T)\sum_{k=0}^{n} T^k\right) = \lim_{n\to\infty}(I - T^{n+1}) = I$$

Similaraly,

$$\sum_{k=0}^{\infty} T^k (I-T) = I$$

So $I - T$ is invertible with inverse $\sum_{k=0}^{\infty} T^k$. □

Corollary 1. Suppose that $T \in \mathcal{B}(X)$. Then $\sigma(T)$ is compact in $\mathbb{C}$. In fact, $|\lambda| \leq \|T\|\ \forall \lambda \in \sigma(T)$.

Proof.

Let $\lambda \notin \sigma(T)$ ($\lambda \in \rho(T)$). Then $T - \lambda I$ is invertible. Want: If $\lambda' \approx \lambda$ then $T - \lambda' I$ is also invertible. Let $r = \|(T-\lambda I)^{-1}\|^{-1}$. Consider $S \in B(T - \lambda I, r)$, a ball in $\mathcal{B}(X)$.

$$\|I - (T-\lambda I)^{-1}S\| = \|(T-\lambda I)^{-1}(T - \lambda I - S)\|$$

$$\leq \|(T-\lambda I)^{-1}\|\|T - \lambda I - S\| < \frac{1}{r}\cdot r = 1$$

By lemma, $(T-\lambda I)^{-1}S$ is invertible $\Rightarrow S$ is invertible $\Rightarrow \rho(T)$ is open $\Rightarrow \sigma(T)$ is closed. If $|\lambda| > \|T\|$, then $I - \lambda^{-1}T$ is invertible. ($\|T/\lambda\| < 1$ and by lemma.) $I - \lambda^{-1}T$ invertible $\Rightarrow \lambda I - T$ is invertible $\Rightarrow \lambda \in \rho(T)$. If $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$. $\Rightarrow \sigma(T)$ is bounded $\Rightarrow \sigma(T)$ is compact. □

Is $\sigma(T) \neq \phi\ \forall T \in \mathcal{B}(X)$? Yes, but the proof relies on complex analysis.

Theorem 1 (Liouville's Theorem)**.** If $f : \mathbb{C} \to \mathbb{C}$ is complex analytic, and if f is bounded then f is a constant.

Theorem 2. Let $T \in \mathcal{B}(X)$, X Banach space over $\mathbb{C}$. Then

(a) For each $\Lambda \in \mathcal{B}(X)^*$, the function $\varphi(\lambda) = \Lambda(\lambda I - T)^{-1}$ is analytic in $\rho(T)$.

(b) $\sigma(T) \neq \phi$.

Proof.

(a) Fix $\lambda_0 \in \rho(T)$. It suffices to show that we can represent φ as a power series in λ around λ_0.

$$\lambda I - T = \lambda_0 I - T + (\lambda - \lambda_0)I$$

$$= (I + (\lambda - \lambda_0)(\lambda_0 I - T)^{-1})(\lambda_0 I - T)$$

$$" \Rightarrow "(\lambda I - T)^{-1} = (\lambda_0 I - T)^{-1}(I + (\lambda - \lambda_0)(\lambda_0 I - T)^{-1})^{-1}$$

Consider

$$(\lambda_0 I - T)^{-1} \sum_{k=0}^{\infty} (-1)^k (\lambda_0 I - T)^{-k} (\lambda - \lambda_0)^k$$

$\|(\lambda - \lambda_0)(\lambda_0 I - T)^{-1}\| < 1$ happens when

$$|\lambda - \lambda_0| < \|(\lambda_0 I - T)^{-1}\|^{-1}$$

For such a λ, the power series converges, and it converges to $(\lambda I - T)^{-1}$.

For $\Lambda \in \mathcal{B}(X)^*$,

$$\varphi(\lambda) = \sum_{k=0}^{\infty} (-1)^k \Lambda((\lambda_0 I - T)^{-k-1})(\lambda - \lambda_0)^k$$

This series converges when $|\lambda - \lambda_0| < \|(\lambda_0 I - T)^{-1}\|^{-1}$. So φ is analytic at λ_0.

(b) We first show that $\varphi(\lambda) \to 0$ as $|\lambda| \to \infty$ for any $\Lambda \in \mathcal{B}(X)^*$. We expand φ "at ∞".

$$(\lambda I - T)^{-1} = \frac{1}{\lambda}(I - \lambda^{-1}T)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \lambda^{-k} T^k$$

This series makes sense when $|\lambda| > \|T\|$.

$$|\varphi(\lambda)| = |\Lambda(\lambda I - T)^{-1}| \leq \frac{1}{|\lambda|} \sum_{k=0}^{\infty} |\lambda|^{-k} |\Lambda T^k|$$

Note that

$$|\lambda|^{-k} |\Lambda T^k| \leq \frac{\|\Lambda\| \|T^k\|}{|\lambda|^k} \leq \|\Lambda\| \left(\frac{\|T\|}{\lambda} \right)^k < \|\Lambda\|$$

Thus

$$|\varphi(\lambda)| \leq \frac{C\|\Lambda\|}{|\lambda|} \to 0 \text{ as } |\lambda| \to \infty$$

If $\sigma(T) = \phi$, then φ is analytic over $\mathbb{C}$. Moreover, it is bounded. By Liouville's theorem, $\varphi(\lambda) = 0 \ \forall \lambda \in \mathbb{C}$. $\Rightarrow \Lambda(\lambda I - T)^{-1} = 0 \ \forall \lambda \in \mathbb{C}, \ \forall \Lambda \in \mathcal{B}(X)^*$. By Hahn-Banach,

$$(\lambda I - T)^{-1} = 0 \ \forall \lambda \in \mathbb{C}$$

Impossible! So $\sigma(T) \neq \phi$.

□

Definition 3. Let $T \in \mathcal{B}(X)$. The **spectral radius** of T is

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$$

We know $0 \leq r(T) \leq \|T\|$.

Theorem 3 (Spectral Radius Formula)**.** If X is a Banach space over $\mathbb{C}$ and if $T \in \mathcal{B}(X)$, then

$$r(T) = \limsup_{n\to\infty} (\|T^n\|)^{\frac{1}{n}}$$

Remark 1. Can replace $\limsup_{n\to\infty}$ by $\lim_{n\to\infty}$.

$$\|T^{m+n}\| \leq \|T^m\|\|T^n\|$$

$$\log\|T^{m+n}\| \leq \log\|T^m\| + \log\|T^n\|$$

$$\lim_{n\to\infty} \frac{\log\|T^n\|}{n} = \inf_{n\geq 1} \frac{\log\|T^n\|}{n}$$

By Fekete's lemma, $\lim_{n\to\infty}(\|T^n\|)^{\frac{1}{n}}$ exists and equals $\inf_{n\geq 1}(\|T^n\|)^{\frac{1}{n}}$.

Proof of Theorem.

For $|\lambda| > \limsup_{n\to\infty}(\|T^n\|))^{\frac{1}{n}}, \exists \delta \in (0,1)$ and n_0 such that

$$|\lambda|(1-\delta) > (\|T^n\|)^{\frac{1}{n}} \ \forall n \geq n_0$$

$$\Leftrightarrow \frac{\|T^n\|}{|\lambda|^n} < (1-\delta)^n \ \forall n \geq n_0$$

$\Rightarrow (1/\lambda)\sum_{k=0}^{\infty} \lambda^{-k}T^k = (\lambda I - T)^{-1}$ converges. $\Rightarrow \lambda \in \rho(T)$. $\Rightarrow r(T) \leq \limsup_{n\to\infty}(\|T^n\|)^{\frac{1}{n}}$. On the other hand, if $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$. For $\Lambda \in \mathcal{B}(X)^*, \varphi(\lambda) = \Lambda(\lambda I - T)^{-1} = \sum_{k=0}^{\infty} \frac{\Lambda T^k}{\lambda^{k+1}}$ holds $\forall |\lambda| > \|T\|$. φ is analytic on $\rho(T)$. $\Rightarrow \varphi(\lambda)$ is analytic if $|\lambda| > r(T)$. So $\varphi(\lambda) = \sum_{k=0}^{\infty}$ still holds $\forall |\lambda| > r(T)$. $\Rightarrow$ for each such λ, $\left(\frac{\Lambda T^k}{\lambda^{k+1}}\right)_k$ is bounded for each $\Lambda \in \mathcal{B}(X)^*$.

$$\frac{\Lambda T^k}{\lambda^{k+1}} = \frac{\tilde{T^k}\Lambda}{\lambda^{k+1}}$$

where $\tilde{T^k} \in \mathcal{B}(X)^{**}$ is the canonical identification. We have $\left(\frac{\tilde{T^k}}{\lambda^{k+1}}\right)_k$ is bounded pointwise. By the uniform boundedess principle, $\exists M > 0$ such that

$$\frac{\|T^k\|}{|\lambda|^{k+1}} = \left\| \frac{\tilde{T^k}}{\lambda^{k+1}} \right\| \leq M \ \forall k$$

$$\Rightarrow \limsup_{n\to\infty}(\|T^n\|)^{\frac{1}{n}} \leq |\lambda|$$

$$\Rightarrow \limsup_{n\to\infty}(\|T^n\|)^{\frac{1}{n}} \leq r(T)$$

□

2 Hilbert Space

A Hilbert space is a Banach space whose norm is induced by an inner product.

Example 2. • ℓ^2 is an inner product space with

$$\langle x, y\rangle = \sum_{j=1}^{\infty} x_j \overline{y_j}$$

This is well-defined by Cauchy-Schwarz. Can also define sequences over integers:

$$\ell^2(\mathbb{Z}) = \{x = (x_n)_{n\in\mathbb{Z}} : \sum_{n=-\infty}^{\infty} |x_n|^2 < \infty\}$$

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x_n \overline{y_n}$$

- The inner product of $L^2(\mathbb{R})$ is given by

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\mathrm{d}x$$

We say two vectors x, y are orthogonal if $\langle x, y\rangle = 0$. Define a norm by using inner product:

$$\|x\| = \sqrt{\langle x, x\rangle}$$

Check:

- $(x, y) \mapsto \langle x, y\rangle$ is continuous from $X \times X$ to $\mathbb{C}$.
- If $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, and if $(X', \|\cdot\|')$ is a completion of $(X, \|\cdot\|)$, where $\|\cdot\|$ is the norm induced by $\langle\cdot,\cdot\rangle$, then $\exists$ an inner product $\langle\cdot,\cdot\rangle'$ on X' such that it induces $\|\cdot\|'$.

A complete inner product space is called a Hilbert space.

Proposition 2 (Parallelogram law)**.** If $\|\cdot\|$ is induced by an inner product then $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

Example 3. (a) On $\mathbb{C}^n, n \geq 2, \|\cdot\|_[$ is induced from an inner product $\Leftrightarrow p = 2$. Take $x = (1, 1, 0, ..., 0)$ and $y = (1, -1, 0, ..., 0)$. $\|x\|_p = 2^{1/p} = \|y\|_p$. $\|x+y\|_p = 2 = \|x-y\|_p$. Parallelogram law holds only when $p = 2$.

(b) The sup norm on $C^0([0, 1])$ is not from an inner product. Take $f(x) = 1, g(x) = x$.

Remark 2. Polarization identity:

$$\mathrm{Re}\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2), \mathrm{Im}\langle x, y\rangle = \frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2)$$

Theorem 4. Let K be a nonempty closed, convex, proper subset of a Hilbert space. Let $x_0 \in X \setminus K$. Then $\exists! x^* \in K$ such that

$$\|x_0 - x^*\| = \inf_{x\in K} \|x - x_0\|$$

Proof.

Let (x_n) be a minimizing sequence:

$$\|x_0 - x_n\| \to \inf_{x\in K} \|x - x_0\| =: d$$

Claim. (x_n) is Cauchy.

Proof.

$$\begin{aligned}
\|x_n - x_m\|^2 &= \|x_n - x_0 + x_0 - x_m\|^2 \\
&\overset{//\text{-gram law}}{=} -\|x_n - x_0 + x_m - x_0\|^2 + 2(\|x_n - x_0\|^2 + \|x_m - x_0\|^2) \\
&= -4\left\|\frac{x_m + x_n}{2} - x_0\right\|^2 + 2(\|x_n - x_0\|^2 + \|x_m - x_m\|^2) \\
&\leq -4d^2 + 2(\|x_n - x_0\|^2 + \|x_m - x_0\|^2)
\end{aligned}$$

Since $(x_m + x_n)/2 \in K$,

$$\to -4d^2 + 2(d^2 + d^2) = 0$$

□

So $x^* = \lim_{n\to\infty} x_n \in K$ exists. Norm is continuous $\Rightarrow$ $d = \|x_0 - x^*\|$. Uniqueness: exercise. □