Introduction to Algebra II, Field Theory

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Contents

1 Basics

1.1 Characteristic

Definition 1 (Characteristic)**.** Let F be a field. The characteristic of F is defined by

$$\operatorname{char} F := \begin{cases} \min\{n \in \mathbb{N} : n \cdot 1_F = \underbrace{1_F + \cdots + 1_F}_{n} = 0\} & \text{if such } n \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 1. $\operatorname{char} F$ is a either a prime or 0.

Proof.

Suppose $p = \operatorname{char} F = ab$ for some $a, b \in \mathbb{N}_{\geq 1}$. Then $p1_F = (ab)1_F = (a1_F)(b1_F) = 0$. Since F is a integral domain, $(a1_F) = 0$ or $(b1_F) = 0$, which contradicts with the minimality of p. □

1.2 Field extensions

Definition 2 (Field Extension)**.** K is a **field extension** of F if K is a field containing a subfield F, denoted by K/F.

Examples:

1. $\mathbb{C}/\mathbb{R}/\mathbb{Q}$ ($\mathbb{C}$ is a field extension of $\mathbb{R}$ and $\mathbb{R}$ is a field extension of $\mathbb{Q}$)
2. For any squarefree integer $D \neq 1$, $\mathbb{C}/\mathbb{Q}(\sqrt{D})/\mathbb{Q}$.

我們有 $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ 作為 $\mathbb{R}$ 的 field extension。對於任意 $a + bi \in \mathbb{C}$，我們可以將其視為以 $a, b \in \mathbb{R}$ 作為係數，$\{1, i\}$ 作為基底而得到的一個向量。事實上，可以觀察到若 K/F，則 $(K, F, +, \cdot)$ 是一個向量空間，其中加法使用兩個 K 的元素，而乘法為 K 中元素與 F 元素的係數積。

Definition 3 (Degree)**.** If K/F, the **degree** $[K : F]$ is defined by the dimension of K as an F-vector space.

$$[K : F] = \dim_F(K)$$

Examples:

1. $[\mathbb{C} : \mathbb{R}] = 2$ since $\mathbb{C}/\mathbb{R}$ has a basis $\{1, i\}$
2. $[\mathbb{Q}(\sqrt{D}) : \mathbb{Q}] = 2$ since $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$ has a basis $\{1, \sqrt{D}\}$
3. $[\mathbb{R} : \mathbb{Q}] = \infty$, since $\dim_{\mathbb{Q}}(\mathbb{R}) = \infty$.

Theorem 1 (Degree 的 Chain Rule)**.** If $L/K/F$. Then $[L : F] = [L : K][K : F]$

Proof.

Let $[L : K] = m$, $K : F = n$ (both finite) and

$$\alpha_1, \ldots, \alpha_m \text{ be basis of } L/K$$

$$\beta_1, \ldots, \beta_n \text{ be basis of } K/F$$

這邊就直接猜 L/F 的一組基底是

$$C = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \{\alpha_i \beta_j\}$$

- C span L/K

  對於任意 $x \in L$，存在 $a_1, \ldots, a_m \in K$ 使得

  $$x = \sum_{i=1}^{m} a_i \alpha_i$$

  而其中的係數 α_i 又可以用 F 中的係數表示

  $$a_i = \sum_{j=1}^{n} b_{ij} \beta_j \quad b_{ij} \in F$$

  所以

  $$x = \sum_{i,j} b_{ij} \boxed{\alpha_i \beta_j}$$

- C is linearly independent

  若存在 $b_{ij} \in F$ 使得

  $$\sum_{i,j} b_{ij} \alpha_i \beta_j = 0$$

  只要寫成

  $$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} b_{ij} \beta_j \right) \alpha_i = 0$$

  因為 α_i 在 L/K 中是線性獨立的，所以

  $$\left(\sum_{j=1}^{n} b_{ij} \beta_j \right) = 0 \quad \forall i$$

  再一次，因為 β_j 在 K/F 中是線性獨立的，所以：

  $$b_{ij} = 0 \quad \forall i, j$$

所以 C 是 L/F 上的基底，且

$$[L:F] = |C| = mn = [L:K][K:F]$$

無窮的證明暫略..... □

Definition 4 (Subfield generated by elements)**.** 假定 F 是一個 field，K/F。並且令：

$$\alpha_1 \dots \alpha_n \in K$$

由於 subfield 的交集還是 subfield，所以若令 $\mathcal{J}$ 為 K 中「同時包含 F 與 $\alpha_1 \dots \alpha_n$ 的 subfield 形成的搜集」，也就是：

$$\mathcal{J} = \{J \subseteq K \mid K/J, \text{ and } J/F \quad \text{ and } \alpha_1 \dots \alpha_n \in J\}$$

則所有這樣的 subfield 形成的交集：

$$\bigcap_{J \in \mathcal{J}} J$$

仍然會是一個 K 中同時包含 F 與 $\alpha_1 \dots \alpha_n$ 的 subfield。且這是所有「K 中同時包含 F 與 $\alpha_1 \dots \alpha_n$ 的 subfield」中最小的 subfield，稱為 subfield generated by $\alpha_1 \dots \alpha_n$ over F，並且記成：

$$F(\alpha_1, \alpha_2 \dots \alpha_n) = \bigcap_{J \in \mathcal{J}} J$$

如果只有一個 α，則 $F(\alpha)$ 成為一個 simple extension，α 為一個 primitive element。

現在我們來看這個 extension $E = F(\alpha_1, \alpha_2 \dots \alpha_n)$ 實際上長什麼樣子。因為 $\alpha_1, \dots, \alpha_n \in E$ 而 E 有加法和乘法的封閉性，所以任何以 F 中元素為係數的 $\alpha_1, \dots, \alpha_n$ 的多項式都在 E 裡面。而 E 也有除法的封閉性 (因為乘法有逆)，所以應該包含所有這些多項式的分式：

$$F(\alpha_1, \dots, \alpha_n) \supseteq \left\{ \frac{f(\alpha_1, \dots, \alpha_n)}{g(\alpha_1, \dots, \alpha_n)} : f, g \in F[\alpha_1, \dots, \alpha_n], g \neq 0 \right\}$$

可以驗證由這些多項式分式的蒐集應該也是一個 field，所以也是 $\mathcal{J}$ 中的元素，但 E 又是其中最小的，所以"$\subseteq$" 也成立，於是等號成立。

1.3 Prime subfield

Definition 5 (Prime Subfield)**.** The prime subfield P of a field F is the minimal subfield of F containing 1_F:

$$P = \bigcap_{\substack{F/S \\ 1_F \in S}} S$$

i.e. the subfield generated by 1_F.

我們能夠定義一個 natural ring homomorphism。如果 $\operatorname{char} F = p > 0$，考慮 $\phi : \mathbb{Z} \to F, \phi(a) = a1_F$。$\phi$ 的 kernel 是

$$\ker\phi = \{a \in \mathbb{Z} : \phi(a) = a \cdot 1_F = 0_F\} = \{a \in \mathbb{Z} : p|a\} = p\mathbb{Z}$$

ϕ 的 image 就是 F 的 prime subfield。所以由 1st theorem of isomorphism 我們有 $P \cong \mathbb{Z}/p\mathbb{Z}$。

如果 $\operatorname{char} F = 0$，考慮 $\phi : \mathbb{Q} \to P, \phi(a/b) = (a1_F)(b1_F)^{-1}$。因為 $b \neq 0$ 所以 $(b1_F) \neq 0_F$，ϕ 是 well-defined。ϕ 是可逆的：

$$\phi^{-1}((a1_F)(b1_F)^{-1}) = \frac{a}{b}$$

所以 ϕ 是一個 ring isomorphism，$P \cong \mathbb{Q}$。可以總結如下

Proposition 2. Let P be prime subfield of a field F.

- If $\operatorname{char} F > 0$, then $P \cong \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$
- If $\operatorname{char} F = 0$, then $P \cong \mathbb{Q}$

1.4 Simple Extension

Definition 6 (Simple Extension)**.** If K/F, we say that K is a simple extension if $K = F(\alpha)$ for some $\alpha \in K$.

一個 extension 是否 simple 並不顯然，即使我們使用兩個以上的元素 extent 也可能得到一個 simple extension，比如 $\mathbb{Q}[\sqrt{2}, \sqrt{3}] = \mathbb{Q}[\sqrt{2} + \sqrt{3}]$（作業題）。

Definition 7 (algebraic/transcendental)**.** Let K/F be a field extension. An element $\alpha \in K$ is said to be **algebraic over** F is α is a root of some nonzero polynomial over F. If no such polynomials exist, then we say α is **transcendental** over F. If every element of K is algebraic over F, then we say K is an **algebraic extension of** F.

Theorem 2. Let $p(x)$ be an irreducible polynomial in $F[x]$. Then there is an extension field K s.t. $p(x)$ has a root in K.

Proof.

Let $K = F[x]/(p(x))$. By Prop 15 of Chapter 9, K is a field. It contains F as a subfield. (To be more rigorous, K contains a subfield $\{a + (p(x)) : a \in F\}$ which is isomorphic to F.) It's clear $\alpha = x + (p(x)) \in K$ is a root of $p(x)$. $(p(\alpha) = p(x) + (p(\alpha)) = 0 + (p(x)))$ □

Proposition 3 (Minimal Polynomial: 以特定元素為根的多項式存在唯一的最小元素)**.** Assume α is algebraic over F. Then $\exists!$ monic irredubcible polynomial $m_{\alpha,F}(x) \in F[x]$ s.t.

$$\begin{cases} \alpha \text{ is a root of } m_{\alpha,F}(x) \\ \text{a polynomial } f(x) \text{ has } \alpha \text{ as a root } \Leftrightarrow m_{\alpha,F}(x)|f(x) \end{cases}$$

The polynomial $m_{\alpha,F}(x)$ is called the **minimal polynomial** of α over F. We define the **degree** of α over F to be $deg(m_{\alpha,F}(x))$.

Proof.

Let $I_\alpha := \{f(x) \in F[x], f(\alpha) = 0\}$. It's straightforward to check that I is an ideal of $F[x]$. By the assuumption that α is algebraic over F, $I_\alpha \neq \{0\}$. Let $p(x)$ be a polynomial s.t. $I_\alpha = (p(x))$. Check $p(x)$ is irreducible. Assume $p(x) = a(x)b(x), a(x), b(x) \in F[x]$. We want to show that one of $a(x), b(x)$ is a unit, i.e. one of $a(x), b(x)$ is a nonzero constant polynommial. Now we have

$$a(\alpha)b(\alpha) = p(\alpha) = 0$$

$$\Rightarrow a(\alpha) = 0 \text{ or } b(\alpha) = 0$$

If $a(\alpha) = 0$, then $a(x) \in I_\alpha = (p(x))$

$$\Rightarrow p(x)|a(x)$$

$$deg(a(x)) \geq deg(p(x)) \geq deg(a(x))$$

$$\Rightarrow deg(a(x)) = deg(p(x)) = deg(b(x)) = 0$$

$\Rightarrow b(x)$ is a nonzero constant polynomial, i.e. $b(x) \in F[x]^\times$. Likewise, if $b(\alpha) = 0$, then $a(x) \in F[x]^\times$. This proves that $p(x)$ is irreducible. Set

$$m_{\alpha,F}(x) = \frac{1}{(\text{leading coefficients of } p(x))} p(x)$$

Then $m_{\alpha,F}(x)$ is the polynomial with the claimed properties. □

Theorem 3. Given a simple extension $F(\alpha)$ of F, where α is algebraic over F with minimal polynomial $m(x)$. Then

$$F(\alpha) \cong F[x]/(m(x))$$

Proof.

Consider the surjective ring homomorphism (evalutation at α) $\psi : F[x] \to F(\alpha)$, $p(x) \mapsto p(\alpha)$. The kernel is the set of polynomials having α as a root, which is simple the ideal $(m(x))$ of multiples of m. By the first isomorphism theorem

$$F(\alpha) \cong F[x]/(m(x))$$

□

Corollary 1 (Extension as a vector space)**.** If α is algebraic over F, then

- $[F(\alpha), F] = \deg m_{\alpha,F} =: n$
- $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a basis of $F(\alpha)$ over F
- $F(\alpha) = F[\alpha]$

Proof.

Let $n = \deg m$, then the coset representatives are the remainders in the division by $m(x)$,

$$F[x]/(m(x)) = \{a_0 + a_1 x + \bar{\ldots} a_{n-1}x^{n-1} : a_i \in F\}$$

Equivalently, this says that the set $1, \bar{x}, \ldots, \bar{x}^{n-1}$ forms an F-basis for $F(x)/(m(x))$. Applying the isomorphism to $F(\alpha)$ shows that the set $\{1, \cdots, \alpha^{n-1}\}$ is an F-basis for $F(\alpha)$. Therefore, we have $[F(\alpha) : F] = n$. Furthermore, we see immediately that $F(\alpha) = F[a]$. □

Thus, for example, if K is a finite extension of $\mathbb{F}_p$. Let $n = |K : \mathbb{F}_p|$ and $\{\alpha_1, \ldots, \alpha_n\}$ be a basis of $K/\mathbb{F}_p$, then

$$K = \{a_1\alpha_1 + \cdots + a_n\alpha_n : a_j \in \mathbb{F}_p\}$$

so $|K| = p^n$. Since every finite field F has a prime subfield P isomorphic to $\mathbb{F}_p$, the cardinality of a finite field must be a prime power.

Corollary 2. (Algebraic Equivalence) If α and β are two elements in K/F have the same minimal polynomials, then the fields $F(\alpha)$ and $F(\beta)$ are isomorphic as fields. Explicitly, there is an isomorphism $\phi : F(\alpha) \to F(\beta)$ that fixes F (i.e. sends every element in F to itself) and sends α to β

Proof.

Let $m(x)$ be the common minimal polynomials. $F(\alpha)$ and $F(\beta)$ are both isomorphic to $F[x]/(m(x))$. Thus $F(\alpha)$ and $F(\beta)$ are isomorphic. □

Proposition 4 (Algebraic iff Finite Degree)**.** α is algebraic over $F \Leftrightarrow [F(\alpha) : F] < \infty$

Proof.

" $\Rightarrow$ " is clear. Conversely, suppose that $[F(\alpha) : F] = n < \infty$. Consider $1, \alpha, \ldots, \alpha^n$, the former $n-1$ terms form a basis of $F(\alpha)$ over F. So $\alpha^n = a_0 + a_1\alpha + \cdots + a_n\alpha^n$ for some $a_i \in F$. So $a_0 + a_1\alpha + \cdots - a_n\alpha^n = 0$. Thus α is algebraic over F □

Corollary 3 (Finite-degree Extensions are Algebraic)**.** If K/F is a finite extension, then K/F is an algebraic extension. (The converse is not true in general)

1.5 Algebraic Extension

Theorem 4 (Finite Algebraic Extensions)**.** If K/F is a field extension with $K = F(\alpha_1, \ldots, \alpha_n)$, then

$$K/F \text{ is algebraic} \iff \text{each of the the } \alpha_i \text{ are algebraic over } F$$

In this case,

$$[K : F] \leq \prod_{i=1}^{n}[F(\alpha_i) : F]$$

and every element of K is a polynomial with coefficients from F in the α_i

Proof.

("⇒"): Prove the negation. If any of the α_i is transcendental over F then K is not algebraic over F.

("⇐"): Suppose the α_i are algebraic. For each i we have $F(\alpha_1, \ldots, \alpha_i) = F(\alpha_1, \ldots, \alpha_{i-1})(\alpha_i)$. Since a simple extension is algebraic if and only if it has finite degree. By the chain rule we have

$$[K : F] = \prod_{i=1}^{n} [F(\alpha_1, \ldots, \alpha_i) : F(\alpha_1, \ldots, \alpha_{i-1})]$$

So $[K : F]$ is also finite, K/F is algebraic.

Now, consider the minimal polynomial $m(x)$ of α_i over F and the minimal polynomial $m'(x)$ of α_i over $F(\alpha_1, \ldots, \alpha_{i-1})$. Since $m(x)$ is also a polynomial in $F(\alpha_1, \ldots, \alpha_{i-1})$ having α_i as a root, by properties of minimal polynomials we see that $m'(x)|m(x)$, so

$$[F(\alpha_1, \ldots, \alpha_i) : F(\alpha_1, \ldots, \alpha_{i-1})] = \deg m' \leq \deg m = [F(\alpha_i) : F]$$

Taking the product from $i = 1$ to n yields

$$[K : F] = \prod_{i=1}^{n} [F(\alpha_i) : F]$$

□

More explicitly, every element of $K = F(\alpha_1, \ldots, \alpha_n)$ is an F-linear combination of elements of the form $\alpha_1^{c_1} \ldots \alpha_n^{c_n}$, where each c_i is an integer with $0 \leq c_i \leq [F(\alpha_i) : F]$

$$F(\alpha_1, \ldots, \alpha_n) = \{b_{1\ldots n} \alpha_1^{c_1} \ldots \alpha_n^{c_n} : b_{1\ldots n} \in F,\ c_i \in \mathbb{Z},\ 0 \leq c_i \leq [F(\alpha_i) : F]\}$$

Theorem 5 (Towers of Algebraic Extensions)**.** If L/K is an algebraic extension, and K/F is an algebraic extension, then L/F is an algebraic extension.

Proof.

These results are obvious if the extensions have finite degree. the content is when one of the extensions has infinite degree (but is still algebraic).

Consider any $\alpha \in L$. Since L/K is algebraic, α is algebraic over K and is the root of some polynomial $p(x) = a_0 + a_1 x + \ldots a_n x^n \in K[x]$. Since K/F is also algebraic, each of the a_i are algebraic over F, so the extension $E = F(a_0, \ldots, a_n)$ has finite degree over F.

Furthermore, $|E(\alpha) : E| < \infty$ because α is the root of a nonzero polynomial in $E[x]$. Thus, since $|E(\alpha) : E|$ and $|E/F|$ are both finite, so does $|E(\alpha) : F|$. So α is a root of a polynomial of finite degree over F, so α is algebraic over F. This holds for all $\alpha \in L$, so L is algebraic over F. □

1.6 Composite Field

1.7 Example of Extensions

By using our results on simple and composite extensions, along with the chain rule of field degrees, we can often say a great deal about extensions of small degree. First, we can characterize quadratic extensions:

Proposition 5 (Quadratic Extensions)**.** F: field with char $\neq 2$; K/F: a quadratic extension (i.e. $[K:F]=2$). Then $K=F(\alpha)$ for any $\alpha \in K \setminus F$.

Proof.

If $\alpha \in K \setminus F$, then the set $\{1, \alpha\}$ is basis for K/F since $[K:F]=2$. Thus $K=F(\alpha)$ □

Determine the degree of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$

1.8 Algebraic Elements

2 Splitting fields and algebraic closures

3 Separable Extensions

4 Cyclomotic Polynomials and Extensions

Definition 8 (nth root of unity)**.** Let $\zeta_n = e^{2\pi i/n}$ and μ_n denote the group of nth root of unity over $\mathbb{Q}$.

$$\mu_n = \{1, \zeta_n, \ldots, \zeta_n^{n-1}\}$$

這些元素就是 $x^n - 1 \in \mathbb{Q}[x]$ 的所有根。$\xi \notin \mathbb{Q}$，可以用他 extent 出 $x^n - 1$ 的 splitting field $E = \mathbb{Q}(\xi)$。

Definition 9 (Primitive)**.** $\zeta \in \mu_n$ is primitive if $\langle \zeta \rangle = \mu_n$, i.e. if $\zeta = \zeta_n^k$ where $(k,n)=1$.

Remark 1. 可以 recall 一些 group theory 的東西:

- $\mathbb{Z}/n\mathbb{Z}$ 和 μ_n 之間有一個 isomorphism: $a \mapsto \zeta_n^a$，且 μ_n 中有 $\varphi(n)$ 個 primitive 的元素。
- $|\zeta_n^i| = n/(n,i)$

Definition 10 (Cyclomotic Polynomial)**.** Define the nth cyclotomic polynomial $\Phi_n(x)$ to be the polynomial whose roots are the primitive nth roots of unity:

$$\Phi_n(x) := \prod_{\zeta \text{ primitive } \in \mu_n} (x - \zeta) = \prod_{\substack{1 \leq k < n \\ (k,n)=1}} (x - \zeta_n^k) \in E[x]$$

so $\deg \Phi_n = \varphi(n)$

現在我們來看一個包含完整的 root of unity 的 $x^n - 1$ 如何拆分成 Cyclomotic Polynomials: 如果 $n = p$ 是質數，則

$$x^p - 1 = (x-1)(x^{p-1} + x^{p-2} + \cdots + 1) = (x-1)\Phi_p(x)$$

如果 n 不是質數，比如

$$x^4 - 1 = (x-1)(x+1)(x^2+1)$$

$$x^6 - 1 = (x^3-1)(x^3+1) = (x-1)(x+1)(x^2+x+1)(x^2-x+1)$$

$$x^n - 1 = \prod_{\zeta \in \mu_n} (x - \zeta)$$

可以將 μ_n 中的元素以它們的 order d(為 n 的因數) 分類:

$$x^n - 1 = \prod_{d|n} \prod_{\substack{\zeta \in \mu_n \\ |\zeta| = d}} (x - \zeta)$$

因為 $\mu_d = \{1, \zeta_d, \ldots, \zeta_d^{d-1}\}$ 中元素的 order 為 $|\zeta_d^i| = d/(d,i)$。所以第二個連乘相當於把 μ_d 中的 primitive 元素收集起來:

$$x^n - 1 = \prod_{d|n} \prod_{\substack{1 \le k < d \\ (k,d)=1}} (x - \zeta_d^k) = \prod_{d|n} \Phi_d(x)$$

在 Definition ??中我們是在 $E[X]$ 中定義的，現在我們證明他實際上在 $\mathbb{Z}[X]$ 中而且是 monic。

Lemma 1. $\Phi_n(x) \in \mathbb{Z}[x]$ and is monic.

Proof.

We already have

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

We now prove by induction on n.

$$n - 1 \Rightarrow \Phi_1(x) = x - 1$$

Assume the statement holds until $n - 1$. Now

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{d|n, d \neq n} \Phi_d(x)} \in \mathbb{Z}[x]$$

where the above fraction polynomial is in $\mathbb{Z}[x]$ because both the numerator and the denominator are monic. ☐

用以上 Lemma 的證明方式我們也能遞迴地找出各個 Cyclotomic polynomials:

- $\Phi_1 = x - 1$
- $\Phi_2 = x + 1$
- $x^3 - 1 = \Phi_1 \Phi_3$, so $\Phi_3 = x^2 + x + 1$
- $x^4 - 1 = \Phi_1 \Phi_2 \Phi_4$, so $\Phi_4 = x^2 + 1$
- $x^6 - 1 = \Phi_1 \Phi_2 \Phi_3 \Phi_6$, so $\Phi_6 = x^2 - x + 1$
- $\Phi_8 = x^4 + 1$
- $\Phi_9 = x^6 + x^3 + 1$
- $\Phi_{10} = x^4 - x^3 + x^2 - x + 1$
- $\Phi_{12} = x^4 - x^2 + 1$

更進一步，我們還能發現它們 irreducible

Theorem 6. $\Phi_n(x)$ is irreducible over $\mathbb{Q}$

Proof.

Let $f(x) = m_{\zeta_n,\mathbb{Q}}(x)$. Then $f(x)|\Phi_n(x)$. We'll show that $f(x) = \Phi_n(x)$. This implies $\Phi_n(x)$ is irreducible over $\mathbb{Q}$. Proving "$f(x) = \Phi_n(x)$" $\Leftrightarrow$ "$\forall k$ such that $(k,n) = 1$, $f(\zeta_n^k) = 0$". So it suffices to show that $\forall k, (k,n) = 1$, $f(\zeta_n^k) = 0$. ($\forall$ primitive nth roots ζ of unity, $f(\zeta) = 0$.)

We first prove the case $k = p$ is a prime. Write $\Phi_n(x) = f(x)g(x)$. (By Gauss's lemma, $f, g \in \mathbb{Z}[x]$.) We have

$$\Phi_n(\zeta_n^p) = 0$$

Suppose that $f(\zeta_n^p) \neq 0$, then $g(\zeta_n^p) = 0$. $\Rightarrow \zeta_n$ is a root of $g(x^p)$. $\Rightarrow f(x)|g(x^p)$. Say $g(x^p) = f(x)h(x)$. Now consider the reduction modulo p. Say $g(x) = a_m x^m + \cdots + a_n$. Since $\overline{a_m}^p = \overline{a_m}$ $\forall a_m \in \mathbb{Z}$.($\overline{a_n}$ = residue class of a_n modulo p.) We have

$$\overline{g(x^p)} = \overline{a_m}x^{pm} + \cdots + \overline{a_0}$$

$$= \overline{a_m}x^{pm} + \cdots \overline{a_0}^p$$

$$= (\overline{a_m}x^m + \cdots + \overline{a_0})^p = \overline{g(x)}^p$$

Therefore

$$\overline{g(x)}^p = \overline{f(x)h(x)}$$

Since $(\mathbb{Z}/p\mathbb{Z})[x]$ is a UFD, this implies that $GCD(\overline{f(x)}, \overline{g(x)}) \neq 1$. $\Rightarrow \overline{\Phi_n(x)} = \overline{f(x)g(x)}$ has a repeated root. However we can show that $\overline{\Phi_n(x)}$ has no repeated roots (which will be proved below). This yields a contradiction. Thus we must have $f(\zeta_n^p) = 0$.

We will now show the claim. We'll show that $\overline{x^n - 1}$ has no repeated roots. Then since $\overline{\Phi_n(x)}|\overline{x_n - 1}, \overline{\Phi_n(x)}$ does not have a repeated root either. Here $\mathrm{D}(\overline{x^n - 1}) = \overline{nx^{n-1}}$. Since $p \nmid n, \bar{n} \neq \bar{0}$. We have

$$\overline{x^n - 1} = (\overline{n^{-1}x})\mathrm{D}(\overline{x^n - 1}) - 1$$

$$\Rightarrow (\overline{x^n - 1}, \mathrm{D}(\overline{x^n - 1})) = 1$$

$\Rightarrow \overline{\Phi_n(x)}$ has no repeated roots. $\square$