Analysis 2 W3-1

fat

March 5, 2024

Recall: X a normed space. $L(X, \mathbb{C}) =$ space of all linear functionals from X to $\mathbb{C}$.

Proposition 1. $\Lambda \in L(X, \mathbb{C})$.

(a) Λ is bounded $\Leftrightarrow \exists C > 0$ s.t. $|\Lambda x| \leq C\|x\| \; \forall x \in X$.

(b) Λ is continuous $\Leftrightarrow \Lambda$ is continuous at one point.

(c) Λ is continuous $\Leftrightarrow \Lambda$ is bounded.

Proof.

We proved (a).

(b). We only show the other "$\Leftarrow$". Suppose that Λ is continuous at x_0. Fix $x \in X$. Let (x_n) be a sequence in X that converges to x. Define $y_n = x_n - x + x_0$. Then $y_n \to x_0$. So $\Lambda y_n \to \Lambda x_0 \Rightarrow \Lambda x_n - \Lambda x \to 0 \Rightarrow \Lambda x_n \to \Lambda x$. So Λ is continuous at x.

(c). Suppose that Λ is not bounded. $\exists M > 0$ and a sequence (x_n) in X s.t. $\|x_n\| \leq M$ but $|\Lambda x_n| \to \infty$. May assume $\Lambda x_n \neq 0 \; \forall n$. Define $y_n = x_n / |\Lambda x_n|$. Then $\|y_n\| \to 0$, but $|\Lambda y_n| = 1$ for all n. Λ is not continuous. On the other hand, suppoes that Λ is bounded. By (a), Λ is Lipschitz $\Rightarrow \Lambda$ is continuous.

□

$X^* =$ Set of all bounded linear functionals on X, called the Dual space/topological dual. $X^* \subset L(X, \mathbb{C})$. If X is finite dimensional, then $X^* = L(X, \mathbb{C})$. If X is infinite dimensional, that $X^* \subset L(X, \mathbb{C})$ but $X^* \neq L(X, \mathbb{C})$. Why? Let B be a Hamel basis for X. Normalize the vectors in B s.t. all have norm 1. Pick $\{x_1, x_2, x_3, ...\}$ from B. Define

$$\begin{cases} \Lambda x_i = i & \forall i \\ \Lambda x = 0 & \text{if } x \in B \setminus \{x_1, ...\} \end{cases}$$

Extend Λ linearly.($\forall v \in X, v = a_1 v_1 + \ldots + a_n v_n$ where $v_1, ..., v_n \in B$. $\Lambda v = a_1 \Lambda v_1 + \ldots + a_n \Lambda v_n$.) Then Λ is unbounded.

Proposition 2. Let X be a normed space and $\Lambda \in X^*$. Define

$$\|\Lambda\| = \sup_{x \neq 0} \frac{|\Lambda x|}{\|x\|}$$

Then $\|\cdot\|$ is a norm on X^* called the operator norm.

Remark 1. We can define

$$\|\Lambda\| = \sup_{\|x\|=1} |\Lambda x| = \sup_{\|x\| \leq 1} |\Lambda x|$$

From definition, we have $|\Lambda x| \leq \|\Lambda\| \|x\| \; \forall x \in X$. Exercise: Verify this.

Proof.

We only prove the Δ-ineq. Let $\Lambda_1, \Lambda_2 \in X^*$ and let $x \in X$ with $\|x\| = 1$. Then

$$|(\Lambda_1 + \Lambda_2)(x)| \leq |\Lambda_1 x| + |\Lambda_2 x| \leq \|\Lambda_1\| + \|\Lambda_2\|$$

Take sup over all $\|x\| = 1$, done. $\square$

Proposition 3. If X is a normed space, then X^* is a Banach space.

Proof.

Let (Λ_k) be a Cauchy sequence in X^*. Fix $\epsilon > 0$. $\exists K$ s.t. $\|\Lambda_k - \Lambda_l\| < \epsilon \; \forall k, l \geq K$. For $x \in X$,

$$|\Lambda_k x - \Lambda_l x| \leq \|\Lambda_k - \Lambda_l\| \|x\| < \epsilon \|x\| \; \forall k, l \geq K \ldots (*)$$

So for each $x \in X, (\Lambda_k x)$ is a Cauchy sequence in $\mathbb{C}$. $\lim_{k \to \infty} \Lambda_k x$ exists $\forall x \in X$. Define $\Lambda x = \lim_{k \to \infty} \Lambda_k x$. Check: Λ is linear. Let $l \to \infty$ in $(*)$.

$$|\Lambda_k x - \Lambda x| \leq \epsilon \|x\| \; \forall k \geq K \ldots (!)$$

$$|\Lambda x| \leq (\epsilon + \|\Lambda_K\|) \|x\|$$

This holds for all $x \in X$, so $\Lambda \in X^*$. Finally, we show $\Lambda_k \to \Lambda$ in norm. from (!), $\|\Lambda_k - \Lambda\| \leq \epsilon \; \forall k \geq K$. So $\Lambda_k \to \Lambda$. $\square$

Example 1.

Proposition 4. Let $1 \leq p < \infty$. The dual of l^p is isometrically isomorphic to l^q where $q = \frac{p}{p-1}$.

Proof.

We only prove the case when $p > 1$. Define $e_j = (0, \ldots 0, 1, 0, 0, \ldots)$ where only the j^{th} position is 1. Define $\Phi : (l^p)^* \to l^q$ as follows: For $\Lambda \in (l^p)^*$, define

$$\Phi(\Lambda) = (\Lambda e_1, \Lambda e_2, \ldots)$$

We first verify that $\Phi(\Lambda) \in l^q$. Write $\Phi(\Lambda) = (a_1, a_2, ...,)$. $\exists \theta_j$ s.t. $a_j = e^{i\theta_j}|a_j|$. Define

$$a^N = (e^{-i\theta_1}|a_1|^{q-1}, ..., e^{-i\theta_N}|a_N|^{q-1}, 0, 0, ...)$$

Then $a^N \in l^p$ $\forall N$. ($a^N = \sum_{j=1}^N e^{-i\theta_j}|a_j|^{q-1}e_j$).

$$|\Lambda a^N| = \left|\sum_{j=1}^N e^{-i\theta_j}|a_j|^{q-1}\Lambda e_j\right|$$

$$= \left|\sum_{j=1}^N e^{-i\theta_j}|a_j|^{q-1}a_j\right| = \sum_{j=1}^N |a_j|^q$$

$$\|a^N\|_{l^p} = \left(\sum_{j=1}^N \left(|a_j|^{q-1}\right)^p\right)^{\frac{1}{p}} = \left(\sum_{j=1}^N |a_j|^q\right)^{\frac{1}{p}}$$

Recall: $|\Lambda x| \leq \|\Lambda\|\|x\|$. So we have

$$\sum_{j=1}^N |a_j|^q \leq \|\Lambda\| \left(\sum_{j=1}^N |a_j|^q\right)^{\frac{1}{p}}$$

$$\Rightarrow \left(\sum_{j=1}^N |a_j|^q\right)^{\frac{1}{q}} \leq \|\Lambda\|$$

Let $N \to \infty$, we have

$$\|\Phi(\Lambda)\|_{l^q} \leq \|\Lambda\|$$

That is, $\Phi : (l^p)^* \to l^q$. Next, we show that Φ is onto. We will construct the inverse explicitly. For each $a \in l^q$, we define $\Psi(a) = \Lambda_a$, where $\Lambda_a \in (l^p)^*$ is given by $\Lambda_a x = \sum_{j=1}^\infty a_j x_j$ $\forall x \in l^p$. By Hölder's inequality,

$$|\Lambda_a x| \leq \|a\|_{l^q}\|x\|_{l^p} \ \forall x \in l^p$$

So $\Lambda_a x$ is well-defined, and $\Lambda_a \in (l^p)^*$. Taking sup over all x with $\|x\|_{l^p} = 1$, we have

$$\|\Psi(a)\| = \|\Lambda_a\| \leq \|a\|_{l^q}$$

Claim. $\Phi(\Psi(a)) = a$ $\forall a \in l^q$.

Proof.

Let $a \in l^q$.

$$\Phi(\Psi(a)) = (\Psi(a)e_1, \Psi(a)e_2, ...)$$

$$= (a_1, a_2, ...) = a$$

$\square$

Finally, we show Φ is an isometry. Recall:

$$\|\Phi(\Lambda)\|_{l^q} \leq \|\Lambda\|$$

We also have$\Psi(\Phi(\Lambda)) = \Lambda \ \forall \Lambda \in (l^p)^*$. (Why? $\Psi(\Phi(\Lambda)) = \Psi(\Lambda e_1, \Lambda e_2, ...) = \Lambda_{(\Lambda e_1, \Lambda e_2, ...)}$

$$\forall x \in l^p, \Lambda_{(\Lambda e_1, \Lambda e_2, ...)} x = \sum_{j=1}^{\infty} \Lambda e_j x_j = \Lambda \left(\sum_{j=1}^{\infty} x_j e_j \right) = \Lambda x$$

) Thus,

$$\|\Lambda\| = \|\Psi(\Phi(\Lambda))\| \leq \|\Phi(\Lambda)\|_{l^q} \leq \|\Lambda\|$$

$$\Rightarrow \|\Phi(\Lambda)\|_{l^q} = \|\Lambda\|$$

So Φ is an isometry. □

Fact:$(L^p)^* = L^q$ is $1 \leq p < \infty, q = \frac{p}{p-1}$. What is $(C^0([a,b]))^*$? We will need the Hahn-Banach theorem.

Definition 1. Let X be a vector space (over $\mathbb{C}$). A function $p : X \to [0, \infty]$ is called **subadditive** if $p(x+y) \leq p(x) + p(y) \ \forall x, y \in X$. $p : X \to [0, \infty]$ is **positively homogeneous** if

$$p(\alpha x) = \alpha p(x) \ \forall x \in X, \ \forall \alpha \geq 0$$

A subadditive, positively homogeneous function is also called a **gauge** or a **Minkowski function**.

Example 2.

- If X is a normed space, then $p(x) = c\|x\|$ $(c > 0)$ is a gauge.
- Let C be a convex set containing O in X. Define

$$p_C(x) = \inf\{\alpha > 0 : x \in \alpha C\}$$

Also, define $p_C(x) = \infty$ if no such α exists.

Claim. p_C is a gauge

Proof.

Positive homogenity is easy. Let $x, y \in X$. If $p_C(x) = \infty$ or $p_C(y) = \infty$, there is nothing to prove. Assume $p_C(x), p_C(y) < \infty$. Fix $\epsilon > 0$. $\exists \alpha, \beta > 0$ s.t. $p_C(x) > \alpha - \epsilon$ and $p_C(y) > \beta - \epsilon$ and $\frac{x}{\alpha} \in C, \frac{y}{\beta} \in C$.

$$\frac{x+y}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta} \cdot \frac{x}{\alpha} + \frac{\beta}{\alpha+\beta} \cdot \frac{y}{\beta} \in C$$

Hence, $p_C(x+y) \leq p_C(x) + p_C(y)$. □

Theorem 1 (Hahn-Banach)**.** Let X be a vector space and let p be a gauge on X. Suppose that Y is a proper subspace of X. Let $\Lambda \in L(Y, \mathbb{C})$ satisfy

$$Re(\Lambda x) \leq p(x) \ \forall x \in Y$$

Then $\exists$ an extension $\tilde{\Lambda}$ of Λ to $L(X, \mathbb{C})$ s.t.

$$Re(\tilde{\Lambda} x) \leq p(x) \ \forall x \in X$$

We will assume our vector spaces are over $\mathbb{R}$ first. The $\mathbb{C}$ case follows from the $\mathbb{R}$ case (we will see).

Lemma 1 (One-step extension)**.** Let $\Lambda \in L(Y, \mathbb{R})$ s.t. $\Lambda x \leq p(x) \ \forall x \in Y$, ans let $x_0 \in X \setminus Y$. $\exists$ an extension Λ_1 of Λ on Y' = the space spanned by x_0 and Y s.t. $\Lambda_1 x \leq p(x) \ \forall x \in Y'$.

Proof.

Every $x \in Y'$ is of the form $x = y + cx_0$ for some $y \in Y$ and $c \in \mathbb{R}$. Any linear functional Λ_1 extending Λ satisfies

$$\Lambda_1 x = \Lambda_1(y + cx_0) = \Lambda_1 y + c\Lambda_1 x_0$$

$$= \Lambda y + c\Lambda_1 x_0$$

Conversely, by assigning any value to $\Lambda_1 x_0$, one obtains an extension of Λ to Y'. However, what we need is to choose an appropriate $\Lambda_1 x_0$ s.t. $\Lambda_1 x \leq p(x) \ \forall x \in Y'$. To see such a choice is possible, we focus on the case that $c = \pm 1$. Need:

$$\Lambda y \pm \Lambda_1 x_0 = \Lambda_1(y \pm x_0) \leq p(y \pm x_0)$$

$$\Leftrightarrow \Lambda_1 x_0 \leq p(y + x_0) - \Lambda y \text{ and } \Lambda y - p(y - x_0) \leq \Lambda_1 x_0 \ \forall y \in Y$$

$$\Leftrightarrow \forall y, z \in Y, \Lambda x - p(z - x_0) \leq \Lambda_1 x_0 \leq p(y + x_0) - \Lambda y$$

Let

$$\alpha = \sup_{z \in Y} (\Lambda z - p(z - x_0))$$

$$\beta = \inf_{y \in Y} (p(y + x_0) - \Lambda y)$$

If $\alpha \leq \beta$, we set $\Lambda_1 x_0$ as any value between α and β

Claim. This gives the desired extension.

Proof.

Need $\Lambda_1 x \leq p(x) \ \forall x \in Y'$. For $c > 0$,

$$\Lambda_1(y \pm cx_0) = \Lambda y \pm c\Lambda_1 x_0$$

$$= c\left(\Lambda\left(\frac{y}{c} \pm \Lambda_1 x_0\right)\right) = c\left(\Lambda\left(\frac{y}{c} \pm x_0\right)\right) \leq cp\left(\frac{y}{c} \pm x_0\right) = p(y \pm cx_0)$$

Remains to show $\alpha \leq \beta$. Only need to show

$$\Lambda z - p(z - x_0) \leq p(y + x_0) - \Lambda y \ \forall y, z \in Y$$

$$\Leftrightarrow \Lambda(y + z) \leq p(y + x_0) + p(z - x_0)$$

$y + z \in Y$

$$\Lambda(y + z) \leq p(y + z) = (y + x_0 + z - x_0) \leq p(y + x_0) + p(z - x_0)$$

□

□