Analysis 2 W1-1

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1 Differentiation and Integration

Recall that if f is continuous, then $F(x) = \int_a^x f(t)dt$ is differentiable. f Lebesgue integrable?

The derivative of F is the limit

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt \tag{1}$$

$$= \frac{1}{|I|} \int_I f(t)dt \tag{2}$$

where $I = (x, x+h)$. The average should $\to f(x)$ as $|I| \to 0$. But this would not hold ofr all x if f is just Lebesgue integrable. (f is just defined a.e.).

Want: Understand the set of x s.t. the convergence holds.

Problem: Given f in $L^1(\mathbb{R}^d)$, for which x one has

$$\lim_{m(B)\to 0, x\in B} \frac{1}{m(B)} \int_B f(y)dy = f(x)$$

If f is continuous, this holds for all $x \in \mathbb{R}^d$. In general, need to understand how the averages of f behave.

2 Hardy-Littlewood maximal function

Definition 1. If $f \in L^1(\mathbb{R}^d)$, we define its **maximal function** f^* by

$$f^*(x) = \sup_{B\ni x} \frac{1}{m(B)} \int_B |f(y)|dy \tag{3}$$

Claim. f^* is measurable.

Proof. Wil show $\{f^* > \alpha\}$ is open. Let $y \in \{f^* > \alpha\}$. Then exists ball $B \in y$ s.t.

$$\frac{1}{m(B)} \int_B |f(t)|dt > \alpha$$

If x is sufficiently close to y, then $x \in B$. $\Rightarrow f^*(x) > \alpha$. $\Rightarrow \{f^* > \alpha\}$ is open.

Theorem 1. For all $\alpha > 0$,

$$m(\{x \in \mathbb{R}^d : f^*(x) > \alpha\}) \leq \frac{3^d}{\alpha}||f||_{L^1(\mathbb{R}^d)}$$

Some discussions:

- We have $f^*(x) < \infty$ for a.e. x:

  Let $\alpha \to \infty$ on both sides. $RHS \to 0$, $LHS \to m(\{x : f^*(x) = \infty\})$.

- For any $f \in L^1(\mathbb{R}^d)$, for any $\alpha > 0$,

$$m(\{x \in \mathbb{R}^d : |f(x)| > \alpha\}) \leq \frac{1}{\alpha}||f||_{L^1} \tag{4}$$

  Proof. Chebyshev/Markov inequality

$$\int_{\mathbb{R}^d} |f| = \int_{\{|f|>\alpha\}} |f| + \int_{\{|f|<\alpha\}} |f| \leq \int_{\{|f|>\alpha\}} \alpha = \alpha m(\{|f| > \alpha\})$$

  □

  Will show later that $f^*(x) \leq |f(x)|$ for a.e. x. So the inequality in Theorem is sharper in general. In fact, $f^*(x) \notin L^1$ in general.

- f^* is of weak L^1.

□

Definition 2. A function g is of **weak** L^1 if

$$\sup_{\alpha>0} \alpha m(\{x \in \mathbb{R}^d : |g(x)| > \alpha\}) < \infty$$

$L^1 \Rightarrow$ weak L^1, converse does not hold.

Notation: If $B = B(x, r)$ is a ball and if $c > 0$, we write $cB = B(x, cr)$.

Lemma 1 (Vitali)**.** Suppose that $\mathcal{B} = \{B_1, ..., B_N\}$ is a finite collection of open balls in $\mathbb{R}^d$, exists a disjoint subcollection $\{B_{i_1}, ..., B_{i_k}\}$ of $\mathcal{B}$ s.t.

$$\bigcup_{l=1}^{N} B_l \in \bigcup_{j=1}^{k} 3B_{i_j}$$

and hence

$$m(\bigcup_{l=1}^{N} B_l) \leq 3^d \sum_{j=1}^{k} m(B_{i_j})$$

Proof. Observation: if B and B' are balls that intersect, and if $B' \leq$ radius of B, then $B' \subset 3B$. Pick a ball B_{i_1} in $\mathcal{B}$ with maximal radius. Remove the ball B_{i_1} and any ball that intersects B_{i_1} from $\mathcal{B}$. All balls we removed are contained in $3B_{i_1}$.In the remaining collection of balls, pick B_{i_2} with maximal radius. Repeat the same procedure. □

Proof of Theorem. Write $E_\alpha = \{x \in \mathbb{R}^d : f^*(x) > \alpha\}$. If $x \in E_\alpha$, $\exists$ a ball $B_x \ni x$ s.t.

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)|dy > \alpha$$

$$\Leftrightarrow m(B_x) < \frac{1}{\alpha} \int_{B_x} |f(y)|dy$$

Let $K \subset E_\alpha$ be compact. Since $K \subset \bigcup_{x\in E_\alpha} B_x$, exists a finite subcover of K, say $K \subset \bigcup_{l=1}^N B_l$. By Vitali's covering lemma, exists a disjoint subcollection of balls $B_{i_1}, ..., B_{i_k}$ s.t.

$$m(\bigcup_{l=1}^N B_l) \leq 3^d \sum_{i=1}^k m(B_{i_j})$$

$$\Rightarrow m(K) \leq m(\bigcup_{l=1}^N B_l) \leq 3^d \sum_{j=1}^k m(B_{i_j}) \leq \frac{3^d}{\alpha} \sum_{j=1}^k \int_{B_{i_j}} |f(y)|dy$$

$$\overset{\text{disjointness}}{=} \frac{3^d}{\alpha} \int_{\bigcup_{j=1}^k B_{i_j}} |f(y)|dy \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)|dy = \frac{3^d}{\alpha} ||f||_{L^1}$$

So $m(K) \leq \frac{3^d}{\alpha} ||f||_{L^1}$ for all compact $K \subset E_\alpha$.

Fact: $m(E_\alpha) = \sup\{m(K) : K \subset E_\alpha \text{compact}\}$. (inner regularity). So $m(E_\alpha) \leq \frac{3^d}{\alpha} ||f||_{L^1}$. □

Theorem 2 (Lebesgue differentiation)**.** If $f \in L^1(\mathbb{R}^d)$, then for a.e. x,

$$\lim_{m(B)\to 0, B\ni x} \frac{1}{m(B)} \int_B f(y)dy = f(x)$$

Proof. For $\alpha > 0$, define

$$E_\alpha = \{x : \limsup_{m(B)\to 0, x\in B} |\frac{1}{m(B)} \int_B f(y)dy - f(x)| > \alpha\}$$

Will show $m(E_\alpha) = 0 \forall \alpha > 0$ If this holds, $\bigcup_{n=1}^\infty E_{\frac{1}{n}}$ has measure zero. This implies the theorem. Fix $\alpha > 0$. Recall for each $\epsilon > 0$, exists a continuous g of compact support s.t. $||f - g||_{L^1} < \epsilon$. Continuity of g

$$\Rightarrow \lim_{m(B)\to 0, x\in B} \frac{1}{m(B)} \int_B g(y)dy = g(x) \forall x$$

$$\frac{1}{m(B)} \int_b f(y)dy - f(x) = [\frac{1}{m(B)} \int_B (f(y) - g(y))dy] + [\frac{1}{m(B)} \int_B g(y)dy - g(x)] + [g(x) - f(x)]$$

Where the first term is $(f - g)^*(x)$ and the second term goes to 0 since g is continuous.

$$\Rightarrow \limsup_{m(B)\to 0, x\in B} |\frac{1}{m(B)} \int_B f(y)dy - f(x)| \leq (f - g) * (x) + |g(x) - f(x)|$$

Let $F_\alpha = \{x : (f - g)^*(x) > \frac{\alpha}{2}\}, G_\alpha = \{x : |g(x) - f(x)| > \frac{\alpha}{2}\}$. Observe: $E_\alpha \subset F_\alpha \bigcup G_\alpha$.

$$m(G_\alpha) \leq \frac{2}{\alpha} ||g - f||_{L^1} \leq \frac{2\epsilon}{\alpha}$$

$$m(F_\alpha) \leq \frac{2 \cdot 3^d}{\alpha} ||f - g||_{L^1} \leq \frac{2 \cdot 3^d \epsilon}{\alpha}$$

Observe: $E_\alpha \subset F_\alpha \bigcup G_\alpha$.

$$m(G_\alpha) \leq \frac{2}{\alpha} ||g - f||_{L^1} \leq \frac{2\epsilon}{\alpha} \text{(Chebyshev)}$$

$$m(F_\alpha) \leq \frac{2 \cdot 3^d}{\alpha} ||f - g||_{L^1} \leq \frac{2 \cdot 3^d}{\alpha} \epsilon \text{(Theorem)}$$

$$\Rightarrow m(E_\alpha) \leq C\epsilon$$

Let $\epsilon \to 0$, we get $m(E_\alpha) = 0$. □

Corollary 1. One has $f^*(x) \leq |f(x)|$ for a.e. x.

Proof. Apply the Lebesgue diff theorem to $|f|$. □

We cam weaken the assumption a bit.

Definition 3. A measurable function f on $\mathbb{R}^d$ is **locally integrable** if for every ball B, the function $f\chi_B$ is integrable.

$$L^1_{loc}(\mathbb{R}^d) = \{\text{locally integrable functions}\}$$

The Lebesgue differentiation theorem holds for locally integrable functions. $f(x) = x$ is locally integrable but not integrable.

Definition 4. If $E \subset \mathbb{R}^d$ is measurable and $x \in \mathbb{R}^d$, we say that x is a **density point** of E if

$$\lim_{m(B)\to 0, x \in B} \frac{m(B \bigcap E)}{m(B)} = 1$$

Intuition: If x is a density point, then $m(B \bigcap E) \sim m(B)$. E covers a large portion of B.

Corollary 2 (Lebesgue density theorem)**.** Suppose that $E \subset \mathbb{R}^d$ is measurable. Then a.e. $x \in E$ is a density point of E, a.e. $x \notin E$ is not a density point of E.

Proof. Apply the Lebesgue diff theorem to χ_E. □

Definition 5. Let $f \in L^1_{loc}(\mathbb{R}^d)$. The **Lebesgue set** of f is the set of all points $x \in \mathbb{R}^d$ s.t. $|f(x) < \infty$ and

$$\lim_{m(B)\to 0, x \in B} \frac{1}{m(B)} \int_B |f(y) - f(x)| dy = 0$$

Write the set $Leb(f)$.

Corollary 3. If $f \in L^1_{loc}(\mathbb{R}^d)$, then $x \in Leb(f)$ for a.e. x.

Proof. For each $r \in \mathbb{Q}$, there exists E_r of measure zero s.t.

$$\frac{1}{m(B)} \int_B |f(y) - r| dy = |f(x) - r|$$

for all $x \notin E_r$. Then $E \equiv \bigcup_{r\in\mathbb{Q}} E_r$ is also of measure zero. Suppose that $x \notin E$ and $|f(x)| < \infty$. Let $\epsilon > 0$ be given. Then $\exists r \in \mathbb{Q}$ s.t. $|f(x) - r| < \frac{\epsilon}{2}$.

$$\frac{1}{m(B)} \int_B |f(y) - f(x)|dx \leq \frac{1}{m(B)} \int_B |f(y) - r|dy + \frac{1}{m(B)} \int_B |r - f(x)|dy$$

$$\Rightarrow \limsup_{m(B)\to 0, x\in B} \int_B |f(y) - f(x)|dx \leq 2|f(x) - r| < \epsilon$$

$$\Rightarrow x \in Leb(f)$$

$E^c \subset Leb(f)$, E has measure zero. So a.e. point $x \in Leb(f)$. □

Remark 1. $f \in L^1(\mathbb{R}^d)$ are equivalence classes. But the set of points where the limit $\lim_{m(B)\to 0, x\in B} \frac{1}{m(B)} \int_B f(y)dy$ exists is independent of the representation of f chosen. But $Leb(f)$ depends on the representation function f.

3 Approximations to the identity

Definition 6. A family of functions $(K_\delta)_{\delta>0}$ is called an **approximation to the identity** if they are integrable and $\exists A \in \mathbb{R}$ s.t.

(a) $\int K_\delta = 1$,

(b) $|K_\delta(x)| \leq A\delta^{-d}$ for all $\delta > 0$ and all $x \in \mathbb{R}^d$,

(c) $|K_\delta(x)| \leq \frac{A\delta}{|x|^{d+1}}$ for all $\delta > 0$ and all $x \in \mathbb{R}^d$.

Observations:

$\int |K_\delta| \leq A'$.

Proof. Recall: $\exists c > 0$ s.t. $\forall \epsilon > 0$,

$$\int_{\{|x|\geq\epsilon\}} \frac{dx}{|x|^{d+1}} \leq \frac{C}{\epsilon}$$

$$|K_\delta| = \int_{\{|x|<\delta\}} |K_\delta| + \int_{\{|x|\geq\delta\}} |K_\delta|$$

$$\leq A\delta^{-d} \int_{\{|x|<\delta\}} 1 + A\delta \int_{\{|x|\geq\delta\}} \frac{dx}{|x|^{d+1}} = A'$$

□

For every $\eta > 0$,

$$\lim_{\delta\to 0} \int_{\{|x|\geq\eta\}} |K_\delta| = 0$$

Proof.

$$\int_{\{|x|\geq\eta\}} |K_\delta| \leq A\delta \int_{\{|x|\geq\eta\}} \frac{dx}{|x|^{d+1}} \leq \frac{C\delta}{\eta} \overset{\delta\to 0}{\to} 0$$

□

An example:

$$K_\delta(x) = \frac{1}{\delta^d}\phi(\frac{x}{\delta})$$

ϕ is a nonnegative bounded function in $\mathbb{R}^d$ s.t. $\int \phi = 1$.
The mapping

$$f \to f * K_\delta$$

converges to the identity map $f \to f$ as $\delta \to 0$ in various senses. As $\delta \to 0$, K_δ converges to the "Dirac delta function" D, informally defined by

$$D(x) = \begin{cases} \infty & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x \neq 0 \end{cases}$$

and $\int D(x)dx = 1$. Doesn't make sense when we think of D as an integrable function. Can think of

$$f * D = \int f(x-y)D(y)dy = 0(y \neq 0)$$

The mass of D is concentrated at 0. Intuitively, $(f * D)(x) = f(x)$. (Think of D as the identity element for convolutions.) Can be rigorously defined, but we need abstract measure theory and functional analysis.

Theorem 3. If $(K_\delta)_{\delta>0}$ is an approximation to the identity and $f \in L^1(\mathbb{R}^d)$, then

$$(f * K_\delta)(x) \to f(x) \forall x \in Leb(f)$$

In particular, the convergence holds for a.e. x.

Lemma 2. Suppose that $f \in L^1(\mathbb{R}^d)$ and $x \in Leb(f)$. For $r > 0$, define

$$A(r) = \frac{1}{r^d}\int_{\{|y|<r\}} |f(x-y) - f(x)|dy$$

then A is continuous in r and $A(r) \to 0$ as $r \to 0$. Moreover, $\exists M > 0$ s.t. $A(r) \leq M \ \forall r > 0$.

Proof.

Recall: If $f \in L^1(\mathbb{R}^d)$, then $\forall\epsilon > 0$, $\exists\delta > 0$ s.t. $m(E) < \delta \Rightarrow \int_E |f| < \epsilon$. (Absolute continuity). Check that A is continuous by using this. If $x \in Leb(f)$,

$$\lim_{m(b)\to 0, x\in B} \frac{1}{m(B)}\int_B |f(y) - f(x)|dy = \lim_{r\to 0}\frac{1}{c_d r^d}\int_{\{|y-x|\leq r\}} |f(y) - f(x)|dy$$

$$= \lim_{r\to 0}\frac{1}{c_d r^d}\int_{\{|z|\leq r\}} |f(x-z) - f(x)|dz = 0$$

This, together with continuity, shows that A is bounded for $0 < r \leq 1$. Why is A bounded for $r > 1$?

$$A(r) = \frac{1}{r^d}\int_{\{|y|<r\}} |f(x-y) - f(x)|dy$$

$$\leq \frac{1}{r^d}\int_{\{|y|<r\}} |f(x-y)|dy + \frac{1}{r^d} m(\{|y| < r\})|f(x)| \leq \frac{1}{r^d}||f||_{L^1} + c_d|f(x)|$$

So $A(r)$ is bounded.

$\square$