Solutions to Discrete Mathematics: An Open Introduction

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1 First Chapter

2 Sequences

2.1 Definitions

2.2 Arithmetic and Geometric Sequences

Investigate!

1. $a_n = 1, 5, 9, \ldots$ for $n = 0, 1, 2, \ldots \rightarrow a_n = 4n + 1$

2. $a_n = 2, 6, 18, \ldots$ for $n = 0, 1, 2, \ldots \rightarrow a_n = 2(3^n)$

3. $a_n = 1, 3, 6, 10, \ldots$ for $n = 1, 2, 3, 4 \ldots \rightarrow a_n = n(n + 1)/2$

Arithmetic Sequences

If the terms of a sequence differ by a constant, we say the sequence is arithmetic. If the initial term (a_0) of the sequence is a and the common difference is d, then we have,

Recursive definition: $a_n = a_{n-1} + d$ with $a_0 = a$.

Closed formula: $a_n = a + dn$

Insights: Because each term in the sequence differs by a constant, we could have guessed the general form of the closed formula by knowing that the derivative of the closed formula must be a constant. If the closed formula is a linear equation of the form $a_n = a + d_n$, then it has a derivative equal to d, which is a constant.

Example 2.2.1. Find the recursive definitions and closed formulas for the sequences below. Assume the first term listed is a_0.

1. $a_n = 2, 5, 8, 11, 14, \ldots$

   Solution: The recursive definition is $a_n = a_{n-1} + 3$ with $a_0 = 2$. The closed formula is $a_n = 2 + 3n$.

2. $a_n = 50, 43, 36, 29, \ldots$

   Solution: The recursive definition is $a_n = a_{n-1} - 7$ with $a_0 = 50$. The closed formula is $a_n = 50 - 7n$.

Geometric Sequences

A sequence is called ***geometric*** if the ratio between successive terms is constant. Suppose the initial term a_0 is a and the ***common ratio*** is r. Then we have,

Recursive definition: $a_n = ra_{n-1}$ with $a_0 = a$.

Closed formula: $a_n = a \cdot r^n$

Example 2.2.2. Find the recursive and closed formula for the sequences below. Again, the first term listed is a_0.

1. $a_n = 3, 6, 12, 24, 48, \ldots$

   Solution: This is a geometric sequence where the initial term is $a_0 = 3$ and the common ratio is $r = 2$. Therefore, the recursive formula is $a_n = 2a_{n-1}$ and the closed formula is $a_n = 3 \cdot 2^n$.

2. $a_n = 27, 9, 3, 1, 1/3, \ldots$

   Solution: This is a geometric sequence where the initial term is $a_0 = 27$ and the common ratio is $r = 1/3$. Therefore, the recursive formula is $a_n = (1/3)a_{n-1}$ and the closed formula is $a_n = 27(1/3)^n$.

Sums of Arithmetic and Geometric Sequences

Investigate!

1. Suppose that the candy machine currently holds exactly 650 Skittles, and every time someone inserts a quarter, exactly 7 Skittles come out of the machine.

   (a) If 20 quarters are inserted, then $20 \times 7 = 140$ Skittles would have come out of the machine. Hence $650 - 140 = 510$ Skittles will be left in the machine.

   (b) There will never be exactly zero skittles left in the machine because 650 is not a multiple of 7. This statement assumes that exactly 7 Skittles must come out when a quarter is put into the machine. That is, if there are less than 7 Skittles remaining in the machine, then nothing would come out.

2. $a_n = 7, 10, 13, 16\ldots$ for $n = 1, 2, 3\ldots \to a_n = 7 + 3(n-1)$. After 20 quarters are put into the machine, the number of skittles given out is $\sum_{n=1}^{20}(7 + 3(n-1))$. The arithmetic sum formula is $S_n = (a_1 + a_n)(n/2)$ and hence $(7 + 64)(20/2) = 710$ Skittles are given out.

3. $a_n = 4, 7, 12, 19, \ldots$ for $n = 1, 2, 3, \ldots \to a_n = n^2 + 3$. After 20 quarters are put into the machine, $\sum_{n=1}^{20}(n^2 + 3)$ Skittles would have come out. The sum of squares series is $S_n = 1, 5, 14, 30, \ldots = \sum_{n=1}^{N} n^2$. To get the sum of squares equation, we note that the difference between successive terms is $d_n = 4, 9, 16, 25\ldots$ and the difference of the difference between successive terms is $dd_n = 5, 7, 9\ldots$ and the difference of the difference of the difference of successive terms is $ddd_n = 2, 2, 2\ldots$ which is constant. Since the third derivative of the sum of squares equation is a constant, then the formula must be a cubic equation of the form $S_n = An^3 + Bn^2 + Cn + D$. Substituting values for S_n and solving the system of equations gives $S_n = (2n+1)(n+1)(n/6)$. Finally, we have $\sum_{n=1}^{20}(n^2+3) = \sum_{n=1}^{20} n^2 + 3\sum_{n=1}^{20} = (2(20) + 1)(20 + 1)(20/6) + 3(20) = 2930$.

Problem 2.6.1. This is the solution.

Problem 2.6.2. This is the solution.