Answersheet 15 - Yiwei Yang 2018533218

1

By Gauss' Lemma we have that the polynomial is irreducible over $\mathbb{Q}$ if and only if it's irreducible over $\mathbb{Z}$. Now the easiest way would be to prove that polynomial is irreducible over $\mathbb{Z}_2$, which would be enough.

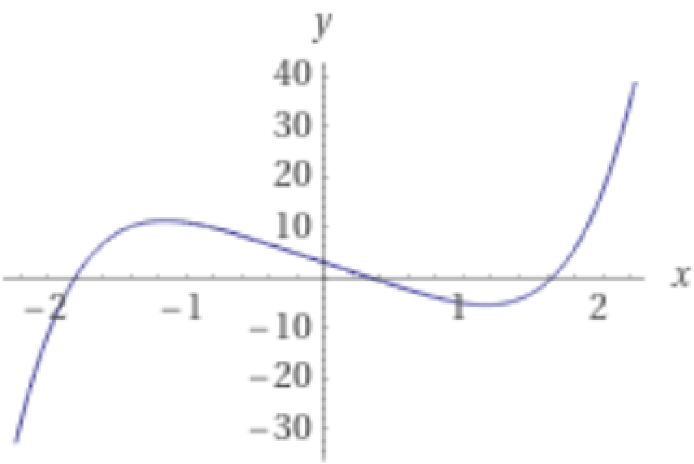
Assume it's reducible. As the polynomial has no roots over $\mathbb{Z}_2$, then the only possibility is if it's a product of polynomials of degree 2 and 3. So assume that:

$$x^5 - 9x + 3 = \left(x^3 + ax^2 + bx + c\right)\left(x^2 + dx + e\right) \quad \text{ over } \mathbb{Z}_2$$

Then multiply everything out and compare the factors. As $a, b, c, d, e \in \mathbb{Z}_2$, you only have two options. Eventually you will get that $ce = 3$ and $a + d = 0$, $ad + b = 0$, $cd + be = 3$. But this would imply that $c + bd + ea = a^2 - 9a + 3 = 1$ in $\mathbb{Z}_2$. But this is impossible, as it's the coefficient in front of x^2 and it should be 0 .
Hence the polynomial is irreducible over $\mathbb{Z}_2$ and eventually $\mathbb{Z}$ and $\mathbb{Q}$

2

1. Let $P = x^5 - 9x + 3$ and A=max{$|1|, |-9|, |3|$}=9. So that when $|x| > 1$, we have $|-9x + 1| < 9$. So that when $|x| \geq 10$, we have $|x|^5 > |-9x + 3|$. Thus $P(|x|) > 0$, $P(-|x|) < 0$ and $x \geq 10$. Thus P have 3 reel roots.



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2. a) Then to prove that it has 2 non-reel roots. For any real polynomial that degree greater than 0, it has non-reel roots.

Let $\alpha = p + qi$.
Let $p + qi$ be expressed in exponential form as $\alpha = re^{i\theta}$.
As $\alpha = re^{i\theta}$ satisfies $f(\alpha) = 0$, it follows that:

$$a_n r^n e^{ni\theta} + a_{n-1} r^{n-1} e^{(n-1)i\theta} + \cdots + a_1 re^{i\theta} + a_0 = 0$$

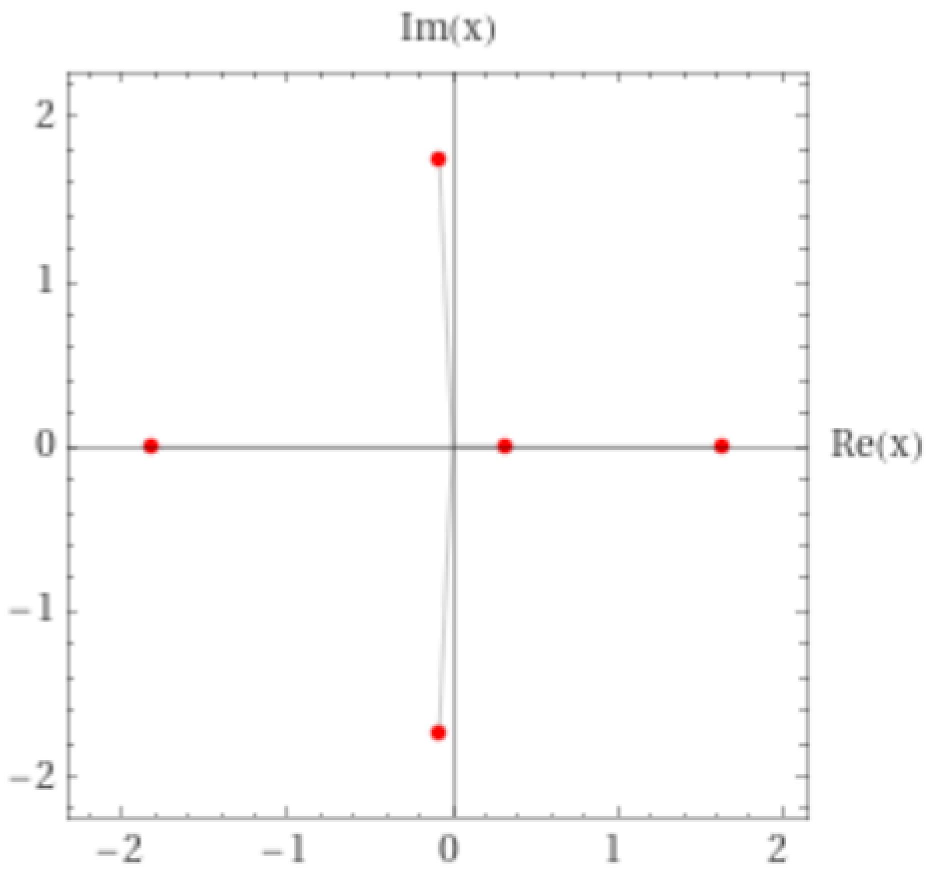
Taking the conjugate of both sides:

$$a_n r^n e^{-ni\theta} + a_{n-1} r^{n-1} e^{-(n-1)i\theta} + \cdots + a_1 re^{-i\theta} + a_0 = 0$$

it follows that $\bar{\alpha} = p - qi$ is also a root of f.
If any of the a_k are complex, then the conjugate of f is not the same polynomial as f.
Therefore there's multiple of 2 conjugate non-reel roots.

b) since $P \in \mathbb{Q}[x]$ So $P = m_{\mathbb{Q}}(x_1)(x) \to [\mathbb{Q}(x_1) : \mathbb{Q}] = degP = 5$. Because of Tower Law, $[L : \mathbb{Q}] = [L : \mathbb{Q}(x_1)][\mathbb{Q}(x_1) : \mathbb{Q}]$ Thus $5 = [\mathbb{Q}(x_1) : \mathbb{Q}]|[L : \mathbb{Q}]$

Since $|Gal(L/\mathbb{Q})| = [L : \mathbb{Q}]$, $5|[Gal(L/Q)]$



Thus there's 5 roots, 2 are non-reel, 3 are reel.

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3

if $p \in P$ and $p|\ |G| < \infty$ then $\exists a \in G$ s.t. $|a| = p$. So $\exists a \in Gal(L/\mathbb{Q})$ s.t. $|a| = p$. By restricting on $\{x_1 . . x_5\}$, this means $\exists \sigma$ 5-cycle$\in S_5$ s.t. $a|\{x_1, . . . x_5\}$=σ.

And $b \in Gal(L/\mathbb{Q})$ a+bi $\xrightarrow{b}$ a-bi is a $\mathbb{Q}$-automorphism. let b$|\{x_1, . . . x_5\}$ $x_4 \to x_5$. and $x_5 \to x_4$. with transposition $\tau = (x_4, x_5)$.

Claim: every permutation in S_5 is attained.

Proof: They are $\sigma\tau\sigma^{-1}, \sigma^2\tau\sigma^{-2}, \sigma^3\tau\sigma^{-3}, \sigma^4\tau\sigma^{-4}$ and their products are within the group where τ is a 5-cycle, τ is a transposition. the num of all elements are 120.

Conclustion: $Gal(L/K) \simeq S_5$

4

Observe that all 3 -cycles are conjugate in S_5. Moreover, the normalizer of a 3-cycle contains both odd and even permutations. For example the normalizer of $(1,2,3)$ contains $(4,5)$ which is an odd permutation. So for any $g \in S_5, g(1,2,3)g^{-1} = g(4,5)(1,2,3)(4,5)^{-1}g^{-1}$. So any S_5 -conjugate of $(1,2,3)$ can be realized in A_5 (since for any $g \in S_5$ either $g \in A_5$ or $g(4,5) \in A_5$) Thus $[S_5, S_5] = A_5$

Observe that all permutations which contain two 2-cycles are conjugate in S_5. Moreover, the normalizer of such a permuation contains both odd and even permutations. For example, the normalizer of $(1,2)(3,4)$ contains $(1,2)$ which is odd. So the same argument as in the previous paragraph shows that all permutations which contain two 2-cycles are conjugate in A_5.
Because the normalizer of a 5 -cycle is contained in A_5, it follows that not all 5 -cycles are conjugate in A_5. Because A_5 has index 2 in S_5 there are 2 conjugacy classes of 5 -cycles in A_5 and they have the same cardinality, $5!/(5*2) = 12$ each.

Thus $[A_5, A_5] = A_5$