1. Answer sheet yangyiwei 2018533218

   1. First. let the set of K's subgroup in G is Σ, the set of subgroups of G/N is $\mathit{\Gamma}$.
      $\forall G_1 \in \Sigma, \because \pi(G_1)$ is the image G_1 under $\pi \mid G_1 . \therefore f(G_1)$ is the subgroup of $\pi(G_1)$, which is $\pi(G_1) \in \mathit{\Gamma}$ where π is the natural homomorphism from G to G/N.

      Reversely, let $H_1 \in \mathit{\Gamma}$ The set of π's original image of H_1.
      $G_1 = \pi^{-1}(H_1) = \{x \in G \mid \pi(x) \in H_1\} \supseteq \{x \in G \mid \pi(x) = e', e' \text{ is the magnoid of H}\}$.

      And $\forall x, y \in G_1$,$\pi(xy^{-1}) = \pi(x)\pi(y)^{-1}$. So $xy^{[} - 1] \in G_1$ so G_1 is the subgroup of G. So $G_1 \in \Sigma$ π^{-1} is the injective from $\mathit{\Gamma}$ to Σ.

      From $\pi(\pi^{-1}(H_1)) = H_1$, we have $\pi \cdot \pi^{-1} = id_\Sigma$. Let $G_1 \in \Sigma$, we have $G_1 \subseteq \pi^{-1}(\pi(G_1))$. if $u \in \pi^{-1}(\pi(G_1))$, we have $v \in G_1$ s.t. $\pi(u) = \pi(v)$. So that
      $uv^{-1} \in K \subseteq G_1 \to u \in G \to \pi^{-1}(\pi(G_1)) = G_1 \to \pi^{-1}\pi = id_{\mathit{\Gamma}} . \therefore$ π establish a one on one injection form K to G/N

      Then let $G_1 \in \Sigma$ and $G_1 \triangleleft G$ if $a \in G, x \in G$, we have
      $\pi(x)\pi(a)\pi^{-1}(x) = f(xax^{-1}) \in f(G_1)$. Notice that π is surjective, $\pi(G_1) \triangleleft H$, reversely, if $H_1 \in \mathit{\Gamma}, H_1 \triangleleft H$ and let $b \in f^{-1}(H_1), y \in G$ we have $f(yby^{-1}) = f(y)f(b)f(y)^{-1} \in H_1$. so that $yby^{-1} \in f^{-1}(H_1)$ so that $f^{-1}(H_1) \triangleleft G$, thus the subgroup is injected to subgroup.

$$\begin{tikzcd}
G \arrow[rr, "\pi"] \arrow[d, "\pi_1"'] \arrow[drr, "\pi_2\pi"] & & H \arrow[d, "\pi_2"] \\
G/G_1 & & G/N/\pi(G_1) \arrow[ll, "\overline{\pi}"']
\end{tikzcd}$$

      Finally, let $G \in \mathit{\Gamma}$ and $G_1 \triangleleft G$ from above we have $\pi(G_1) \triangleleft G/N$ Let π_2 be the natural isomorphism from G/N to $G/N/\pi(G_1)$, and π_1 be the natural isomorphism from G to G/G_1 . We have

      $ker(\pi'\pi) = \{x \in G \mid \pi_2\pi(x) = f(G_1)\}$ =$\{x \in G \mid \pi(x) \in \pi(G_1)\} = \pi^{-1}(\pi(G_1)) = G_1$ So that the figure is satisfied, and $G/H \simeq (G/N)/(H/N)$

   2.a.

   Suppose A and B are the adjacency matrices of G and its complement. Since G is self-complementary, there is a permutation matrix P such that $P^T AP = B$. Now if J is the all-ones matrix then $B = J - I - A$ and $P^T JP = J$, so we find that $(P^2)^T AP^2 = A$. This implies that P^2 represents an automorphism of G. And $P^2 \neq I$, thus $Auf(G)$ is a group.

   b.

$\{(3,4),(2,4,3),(1,3)(2,4),(1,2)(3,4)\}$

c.

$\{I,(13)(24),(12)(34),(14)(23)\}$

3. $D_n = \mathrm{Sym}(\{1,2,\ldots,\mathrm{n}\},\mathrm{s})$, where $n \geq 3$ and

$$s = \{\{1,2\},\ldots,\{i,i+1\},\ldots,\{1,n\},\{n,1\}\}$$

is called the dihedral group of degree n. This group contains the subgroup $C_n =< \sigma >$, where $\sigma = (12\cdots n)$. Since C_n is its own centralizer in S_n, any element τ of D_n which is not in C_n does not commute with σ. But then, if $\tau(1) = i$, we must have $\tau(2) = \sigma^{-1}(i)$; otherwise, $\tau(2) = \sigma(i)$ which implies inductively that $\tau(j) = \sigma^{j-1}(i)$ and hence that $\tau(j) = \sigma^{i-1}(j)$, i.e., that $\tau = \sigma^{i-1}$ It follows inductively that $\tau(j) = \sigma^{-j+1}(i) = \sigma^{i-j}(1)$. Thus $\tau(j) = i - j$ for $1 \leq j \leq i$ and $\tau(j) = n - j + i + 1$ for $i + 1 \leq j \leq n$ which imply that $\tau^2 = 1$. Moreover, any function τ defined in this way is in D_n. It follows that $|D_n| = 2n$ and hence that C_n is a normal subgroup of D_n. Thus D_n is generated by σ and any element τ not in C_n. Moreover, $\tau\sigma\tau^{-1} = \sigma^{-1}$. Hence $C_n \cong C_n \rtimes_\rho C_2$, where $\rho : C_2 \to \mathrm{Aut}(C_n)$ is the homomorphism defined by $\rho(\tau)(\sigma) = \sigma^{-1}$. It follows that D_n has the presentation $D_n =< \sigma, \tau \mid \sigma^n = 1, \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} >$. since any group having these generators and relations is of order at most $2n$. Indeed, the elements in such a group are of the form $\sigma^i\tau^j$ with $0 \leq i < n, 0 \leq j < 2$. The group D_n is also isomorphic to the group of symmetries of a regular n -gon.
If $a, b \in D_n$ with $o(a) = n, o(b) = 2$ and $b \notin < a >$, we have

$$D_n =< a, b \mid a^n = 1, b^2 = 1, bab^{-1} = a^{-1} >$$

$\therefore$ there is an automorphism ψ such that $\psi(\sigma) = a, \psi(\tau) = b$ and any automorphism of D_n is of this form. Thus $|\mathrm{Aut}(D_n)| = n\phi(n)$, where $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$.

4. Both F_n and $\mathrm{F}_n / [\mathrm{F}_n : \mathrm{F}_n]$ are left adjoint and preserve coproduts. we can have up to canonical isomorphism $(FX)^{\mathrm{ab}} = \left(F \coprod_{x\in X}\{x\}\right)^{\mathrm{ab}} = (*_{x\in X}F_1)^{\mathrm{ab}} = \bigoplus_{x\in X}\mathbb{Z}$ since $F_1 = \mathbb{Z}$