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1. The homomorphism $f$ from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ iff $g = \gcd(m, n) > 1$.

   For $g = 1$ is the trivial case. Because the homomorphism is determined by $f(\bar{1})$. According to the Lagrange's theorem, the order of $f(\bar{1})$ should be divide by both $m$ and $n$. So the order of $f$ is 1.

   For $g > 1$, let $k = \frac{m}{g}$, we have that $f(\bar{a}) = \bar{ka}$ then $a = b + xn$ for some integer $x$ and then $f(\bar{a}) = k(b \bar{+} xn) = k\bar{b} + k\bar{xn} = f(\bar{b})$. Also, $kxn$ is divisible by $m$: $kxn = \frac{m}{g}xn = mx\frac{m}{g}$. Also, $f(\bar{a} + \bar{b}) = f(a \bar{+} b) = k(a \bar{+} b) = f(\bar{a}) + f(\bar{b})$.

2. i.

   a.

   Denote the hormophism from $G_i \to \prod_{i \in \mathrm{I}} \mathrm{G}_i$ be $H_i$ Let $e_G$ be the identity for $(G, *)$.
   Let $e_H$ be the identity for $(H, *_2)$.

   $(g, h) \circ (e_G, e_H) = (g *_1 e_G, h_{*2} e_H) = (g, h)$
   $(e_G, e_H) \cdot (g, h) = (e_G *_1 g, e_H *_2 h) = (g, h)$

   Thus $(e_G, e_H)$ is the identity of the group direct product. Thus the direct product of groups are groups.

   b.

   Suppose the groups here are finite groups. Denote the homophism $H_1$, $H_2$ ... $H_n$ are n normal subgroup of $G$.

   The direct sum is commutative up to isomorphism. of $G = \Sigma H_i$ we have $\forall g_i \text{ in } H_1, g_i' \text{ in } H_2$, we have $g_i * g_i' = g_i' * g_i$ and $\exists$ one $g_i$ and $g_i'$ s.t.
   $(g_i)_{i \in \mathrm{I}} * \left(g_i'\right)_{i \in \mathrm{I}} * \ldots (g_i^{n\prime})_{i \in \mathrm{I}} := \left(g_i *_i g_i' \ldots *_i g_i^{n\prime}\right)_{i \in \mathrm{I}}$ also, the cancellation of the sum in a quotient, so that $(\Sigma H_i + H_2)/H_2$ is isomorphic to $\Sigma H_i$

   From above, we have multiplication of elements in a direct sum is isomorphic to multiplication of the corresponding elements. Thus, from $a$, we have the direct sum of groups are groups.

   ii.

   By Lagrange theorem: [$S_3 : S_3'$]=2 and $S_3' \leq ker\phi$ So $S_3/ker\phi$ has order at most 2. Let $\phi$ be $S_3 \to \mathbb{Z}_6$. The possible kernels are $\{e\}$, $A_3$ and $S_3$.

   First $\{e\}$ is the trivial case but the order is 1, so just circle out.

   Then Check for $S_3$, then $S_3/S_3$ is identity, so you can map every element of $S_3$ to the identity of $\mathbb{Z}_6$ . $\phi(s) = 0, \forall s \in S_3$ and 0 is the identity of $\mathbb{Z}_6$.

   The last is $A_3$, the order of the factor group is 2, Using isomorphism theorem: $S_3/A_3 \simeq \phi(S_3)$. Then $\phi(S_3)$ is $\{3, 0\}$

   Thus all normal subgroups are $A_3 \text{ and } S_3$.

3. $\forall a \in \mathbb{Q}$ there exists some non-zero element (r) in $\mathbb{Z}$ s.t. $ra \in \mathbb{Z}$. But we know that every (a) can be written as $x/y$, where $x, y \in \mathbb{Z}$ thus we can always take $r = y$ and we canthen c onclude that every $\mathbb{Q}/\mathbb{Z}$ is a torsion group.

   Let $H$ be a non-zero subgroup of $\mathbb{Q}$ and let $0 \neq h = \frac{m}{n} \in H$, It's clear that $m = n \cdot h \in H$ and $\mathbb{Z} < H$ we have $h\mathbb{Q} \subseteq H$, so that $G/H$ is torsion, which $\simeq G$

4. First, we have $Aut(G)$ is $G/\mathbb{Z}(\mathbb{G})$ which is cylic. $\exists x \in G$, s.t. $G/\mathbb{Z}(\mathbb{G}) =< x\mathbb{Z}(G) >$

   We can wrote that $x = g^k h$ for some $k$ and some $h \in \mathbb{Z}(G)$. Given $x, y \in G$ we have $x = g^k h$ and $y = g^l h'$ Thus $xy = (g^k h)(g^l h') = g^k g^\ell h h' = \left(g^\ell h'\right)\left(g^k h\right) = yx$ so that $G$ is abelian. $\square$

5. Find the subgroup of $GL_2(\mathbb{R})$ generated by the matrices $A$.

   $C(\mathbb{U}_2(\not{C}/\mathbb{R})) = C\{A \in GL(\not{C}) \mid A^{-t} = A^{-1}\}$ which is $\simeq S' = \{\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \mid |z| = 1\}$

   So that where the matrix can be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.