Answer sheet 9 - Yiwei Yang 2018533218

1

i

We first have that I, F is the ultra filter, so we have 2 scenarios $N \subset U, N \in F$ or $M/\ N \in F$. (every set is either large or co-large)

$\rightarrow F$ is ultrafiler, so that there exists $N \subset U$ such taht $N \notin F\ M/\mathbb{N} \notin F$. Since $F \neq \emptyset$ So for $G \in F$, we have $G \subset N$ or M or both empty.

we have $\emptyset \notin F$ So there;s no $H_1, H_2 \in F\ H_1 \subset N$ or $M/\ N$ Suppose the latter, let $B = \{N \cap H | H \in F\}$, so that $\emptyset \in A$. But $F' = F \cup B \cup \{N\}$ does not contain $\emptyset$ and $F \subset F'$ which is contradiction. So $\square$

$\leftarrow (M/\ N) \cap N = \emptyset$ and F' does not exists. So F is ultra filter.

ii

Let $\mathcal{F} = \{N \subset M | m_0 \in N\}$. We have $\forall N \in F, m_0 \in N$, so $\emptyset \notin F'$

let $N_1, N_2 \in F, m_0 \in (N_1 \cap N_2)$ we have $(N_1 \cap N_2) \in F$ so F is a filter. and for $H \in P(U)/\ F, \{m_0\} \in F, H \cup \{m_0\} = \emptyset$ so

iii

Take X to be any infinite set, choose N a countable subset of X and enumerate N as a sequence, $N = \{x_n\}_{n \in \mathcal{N}}$. Set $B_1 = N$ and $B_n = N/\left\{x_1, \ldots, x_{n-1}\right\}, n \in \mathcal{N}$
Then $\mathcal{B} = \{B_n : n \in \mathcal{N}\}$ is a filter base that can be extended to a non principal ultrafilter on X. $F \subset F'$. F is ultrafilter.

iv

Let $R = \mathcal{F}(I, \mathbb{R})$ and let $U = \left\{f^{-1}(0) : f \in J\right\}$. First, note that if $f \in J$, then we can multiply f by a function to assume f only takes the values 0 and 1 without changing its zero set (just multiply by a function that is $1/f$ when f doesn't vanish). So, $A \in U$ iff the function $1_{I \backslash A}$ which is 1 on $I \backslash A$ and 0 on A is in J.
We now see easily that U is a filter: it is closed under supersets since J is closed under multiplication by elements of R, and it is closed under intersections since $1_{I \backslash A} + 1_{I \backslash B}$ vanishes only on $A \cap B$. To see that it is an ultrafilter, it suffices to show that for any $A \subseteq I$, exactly one of $e = 1_A$ and $f = 1_{I \backslash A}$ is in J. This follows from the fact that $e + f = 1$ and $ef = 0$: since $e + f = 1$, they cannot both be in J, and since $ef = 0$, at least one must be in J since J is prime.

2

Since R has no zero-divisor. The closure under the calculation is trivial, and the associativity under plus and mul is satisfied.

For $(\mathbb{Q}(\mathbb{R}), +)$ we have $[(O_R, I_R)]$ is the unit and $\forall (r, s) \in \mathbb{Q}(\mathbb{R})$, $(-r, s) + (r, s) := (-rs + rs, ss) =$$(O_R, SS) = (O_R, I_R)$, which is invertible.

Also, for $\forall(r_i, s_i) \in \mathbb{Q}(\mathbb{R}), i \in \{1,2,3\}$ we have
$(r_3, s_3)((r_1, s_1) + (r_2, s_2)) = (r_3, s_3)(r_1 s_2 + r_2 s_1, s_1 s_2)$
$= (r_3(r_1 s_2 + s_1 r_2), s_1 s_2 s_3) = (r_1 r_2, s_3 s_1) + (r_3 r_2, s_3 s_2) = (r_3, s_3)(r_1, s_1) + (r_3, s_3)(r_2, s_2)$ the distribution holds, too.

So, it's a ring and the commutativity of $\mathbb{Q}(\mathbb{R}), \cdot$ is deducted from $(R, \cdot)$.

To prove that (O_R, I_R) is not invertible, $\forall(r_1, s_1), (r_2, s_2) \neq (O_R, I_R)$ we have
$(r_1, s_1) \cdot (r_2, s_2) = (r_1 r_2, s_1 s_2) = (O_R, I_R)$. $r_1 r_2 = O_R$. so that $r_1 = O_R$ $r_2 = O_R$.

$\mathbb{Q}(\mathbb{R})$ has no zero-divisor, we have (I_R, I_R) is unit and (O_R, I_R) is not invertible. Thus $(\mathbb{Q}(\mathbb{R}), +, \cdot)$ is a field.

3

i

The closure of $+, \cdot$ is trivial.

As for associativity, in $(R_W, +)$, it follows directly from the associativity of + in R and W. Now
$\forall(r_i, w_i) \in R_W$
$(r_3, w_3) * ((r_1, w_1) * (r_2, w_2)) = (r_1 r_2 r_3, r_3(r_1 w_2 + r_2 w_1) + r_1 r_2 w_3) = ((r_1, w_1) + (r_2, w_2)) * (r_3, w_3)$
and $(O, O_W), (I, O_W)$ are respectively units for + and *.
$\forall(r, w) \in R_W$ $(r, w) + (-r, -w) = (O, O_w)$, so every element's invertible.

commutativity is from (R,+) (W,+)

The distribution is $\forall(r_1, w_1)(r_2, w_2)(r_3, w_3) \in R_W$, we ahve $(r_3, w_3) * ((r_1, w_1) + (r_2, w_2)) =$
$(r_3, w_3)(r_1 w_2 + r_2 w_1, w_1 w_2) = (r_3(r_1 w_2 + w_1 r_2), w_1 w_2 w_3) = (r_1 r_2, w_3 w_1) + (r_3 r_2, w_3 w_2) =$
$((r_1, w_1) + (r_2, w_2)) * (r_3, w_3)$

Thus, because $(O, O_w) \neq (I, I_W)$. $(R_W, +, *)$ is a non-zero communitative unitary ring.

ii

The R_W's maximal ideal is $m \leftrightarrow d(R_W) \geq \dim R_W \leftrightarrow R_W \text{ is finite demensional.}$ W is finite dimensional as R-vector space. d is depth, h is height

Suppose the maximal prime is m, deduction proof on $d(R_W)$

$d(R_W) = 0$ $\forall$ big enough n $l(R_W/m^n)$ is constant. So that there exist $m^{n+1} = m^n$ so that $m^n = 0$
So that dim R_W=0

If $d(A) > 0$. let $p_0 \subset p_1 \cdots p_r$ be R_W's prime ideal chain. let $x \in p_1/\,p_0$, π is the homomorphism from R_W to $R'_W = R_W/\,p_0$. So $x' = \pi(x) \neq 0$ R'_W is integer ring. So that
$d(R'_W/<x\prime>) \leq d(R'_W) - 1$ if m' is the maximal ideal, $R'_W/\,m'^n$ is the homomorphism image of $R_W/\,m^n$. So $l(R_W/\,m)^n \geq l(R'_W/\,m'^n)$ So that $d(R_W) \geq d(R'_W)$ and
$d(R'_W/<x\prime>) \leq d(R_W) - 1$.

So that the length of $d(R'_W/<x\prime>)$ is no bigger than $d(R_W) - 1$, but p1...pr construct a chain of length $r - 1$. So that dim $R_W \leq d(R_W)$

4

Let height p be the sup of the length r of prime ideal chain $p_0 \subset p_1 \cdots p_r = p$ whose destination is p and depth p be the sup of the length s of prime ideal chain $p = p_0 \subset p_1 \cdots p_s$ who starts with p

We have height p = dim R_p, depth p = dim $R/\ p$ because $S^{-1}p_0 \subset S^{-1}p_1 \cdots S^{-1}p_r = S^{-1}p, S = R/\ p$ and $p/\ p = q_o/\ p \subset q_1/\ p \dots q_s/\ p$ are the prime ideal chain of $S^{-1}R$ and $R/\ p$

For a as maximal prime of d-dim commutative non-zero unitary ring R, for all p_i , we have $0 \leq$ height $p_i \leq$ height $a =$ dim R. $\forall t, 0 \leq t \leq d$ exists prime ideal p_i such that height $p_i = t$ height $p_i = d$ if and only if $p_i = a$

Because $p_0 \subset \cdots p_r = p \subset a$ and R have prime ideal chain $p_0 \subset \cdots p_{dim(R)}$