Answersheet 14 - Yiwei Yang 2018533218

1

First, we have $[F_3(T,S):F_3[X,Y]]=9$. because T and S is algebraic no relation, so is X and Y. X and Y are the prime element of $F[T,S]$. So X^3-T^3 is the irreducible element of F. So $[F_3(T):F_3[X,Y]]=3$

Also, we have $[F_3(T)(S):F_3[X,Y]]=3$ So $[F_3(T,S):F_3[X,Y]]=9.$

If there exist a element, say θ in F_3 to F is the element. So that $\theta=\frac{f(T,S)}{g(T,S)}$. $f(T,S),g(T,S)\in F_3(T,S)$ So that $\theta^3=\frac{f(T^3,S^3)}{g(T^3,S^3)}\in F$. Which $[F(\theta):F]\leq 3<9$. ※

2

Let E is the splitting field of $x^7-9\in\mathbb{Q}[x]$. We have the roots in $\mathbb{C}$: $X=(-3)^{\frac{2}{7}}$,$-i3^{\frac{2}{7}}$, $-(-1)^{\frac{3}{7}}3^{\frac{2}{7}}$,$X=(-1)^{4/7}3^{2/7}$
$X=-(-1)^{5/7}3^{2/7}$

And let $\alpha=^7\sqrt{9},\epsilon=cos\frac{2\pi}{7}+i\ sin\frac{2\pi}{7}$. We have

$$Irr(\alpha,\mathbb{Q})=x^7-9=\prod_{k=0}^{7}(x-\alpha\epsilon^k)$$

$$Irr(\epsilon,\mathbb{Q})=\prod_{k=0}^{7}(x-\epsilon^k)$$

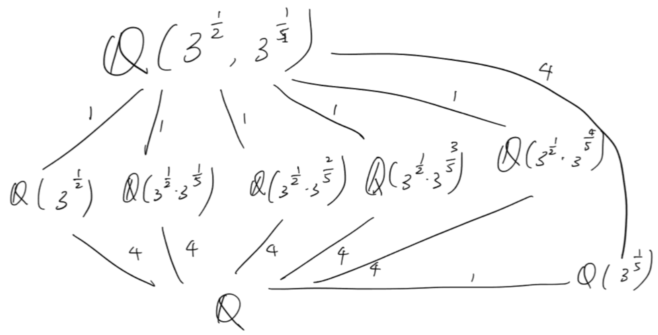
Notice that let $E=\mathbb{Q}(\alpha,\epsilon)$,$1\notin\{\frac{\epsilon^k-\epsilon}{\alpha(1-\epsilon^i)}|1\leq i,k\leq 4\}$ So that Let $\theta=\alpha+\epsilon$, we have $E=\mathbb{Q}(\theta)$,

Let E_1 be one of the automorphism of E, which have the relation of $E\subset E_1\subset\mathbb{Q}$. $\eta(3^{\frac{2}{7}})=\alpha^k3^{\frac{2}{7}},\ k=0-6$ and $\eta(\alpha)=\alpha^k,\ k=1-6$. Let $E_1=\eta(k=2)$. Because E_1 is the splitting field of $\{Irr(\alpha,\mathbb{Q})|\alpha\in E\}$ and $Irr(\alpha,\mathbb{Q})$ is the polynomial of $\mathbb{Q}$. Thus E_1 is the splitting field of $\{Irr(\alpha,\mathbb{Q})\in E[x]|\alpha\in E\}$, So that $E_1|\mathbb{Q}$ are normal.

3

The fixed field of $\{e\}$ is K, while the fixed field of $\langle\sigma,\tau\rangle=\mathrm{Gal}(K/\mathbb{Q})$ is $\mathbb{Q}$ by condition (3) of the
characterization of Galois extensions.
For the other fixed fields we can either compute the action explicitly on a basis (which is straightforward, if tedious) or try to identify elements of K that might generate some of these fields, and then exploit the Galois action. For example, observe that σ stabilizes $3^{1/5}$, and since the fixed field corresponding to σ must have degree 4 over $\mathbb{Q}$, it must be equal to $\mathbb{Q}\left(3^{1/5}\right)$.
Notice that $\langle\sigma\rangle$ is normal in the Galois group, and indeed $\mathbb{Q}\left(\zeta_3\right)$ is Galois over $\mathbb{Q}$. Likewise, we can see that τ stabilizes $3^{1/2}$, and since the fixed field of τ must have degree 3 over $\mathbb{Q}$, it must be $\mathbb{Q}\left(3^{1/2}\right)$.

The normal hull is as follows



4

Intersections of subgroups correspond to joins of fields, and joins of subgroups correspond to intersections of fields.

Proof: Suppose that H_1 and H_2 are subgroups of F with respective fixed fields E_1 and E_2.Then any element in $H_1 \cap H_2$ fixes both E_1 and E_2 hence fixes everything in $E_1 E_2$ (since the elements of the composite field are rational functions of elements of E_1 and E_2).Conversely, any automorphism fixing $E_1 E_2$ must in particular fix both E_1 and E_2 hence be contained in $H_1 \cap H_2$. Thus, $H_1 \cap H_2 = E_2$ corresponds to $E_1 E_2$.

Similarly, $E_1 \cap E_2$ is fixed by any element in H_1 or H_2, hence also by any word in such elements, so $\langle H_1, H_2 \rangle$ fixes $E_1 \cap E_2$ Inversely, if σ is any automorphism that does not fix $E_1 \cap E_2$, then for any $h \in H_1 \cup H_2$ we see that σh also does not fix $E_1 \cap E_2$Then by an easy induction argument on the word length, we see that σ cannot be written as a word in $\langle H_1, H_2 \rangle$. Thus, $\langle H_1, H_2 \rangle$ corresponds to $E_1 \cap E_2 = F$.

Thus $|E_1|F| = |E_1 E_2|E_2|$.