Answer sheet 1 - Yiwei Yang 2018533218

1. i.

   proof: Denote set G as the powerset of set M. We have $|G|=2^{|M|}$. This indicates if M is infinite, G is infinite. Take $g_i, g_j \in G, \forall g_i, g_j \in G$, we have $g_i \Delta g_j=(g_i \cup g_j)-(g_i \cap g_j) \in G$.

   1) $g_i \Delta g_i=\emptyset$, every element in this group is its own inverse.

   2) $g_i \Delta g_j=(g_i \cup g_j)-(g_i \cap g_j)=g_j \Delta g_i$ which is connectivity

   3) $(g_i \Delta g_j) \Delta g_k=((g_i \cup g_j)-(g_i \cap g_j)) \cup g_k$
   $-((g_i \cup g_j)-(g_i \cap g_j)) \cap g_k=g_i \cup((g_j \cup g_k)-(g_j \cap g_k))-g_i \cap((g_j \cup g_k)-(g_j \cap g_k))=$
   $g_i \Delta(g_j \Delta g_k)$ which is associative.

   4) The empty set is the identity of the group

   Thus $(P(M), \Delta)$ is abelian.□

   ii.

   proof: 1) $((a, b) *(c, d)) *(e, f)=(a, b d) *(e, f)=(a, b d f)=(a, b) *((c, d) *(e, f))$

   2) $(a, b) *(1,1)=(a, b)$, while $(1,1) *(a, b)=(1, b)$

   Thus $(\mathbb{R} \times \mathbb{R}, *)$ is non-abelian semigroup.

   The set of right- & left- units can be $\{(1, t),(k, t^{-1}) | k, t \in \mathbb{R}\}$

2. $G=\{a^n=a, \forall n \geq 1\}$

3. $\because l_a, r_a$ are bijections. Assume that for any elements a,b in G, we can find x,y in G such that $a * x=b, y * a=b$.

   We are looking for a neutral element e. This satisfies $g_0 e=g_0, g_0 \in G$. By assumption, there is some e with $g e=g, \forall g \in G$. By assumption we may write $g=h g_0$ for some h. Then, we have $g e=(h g_0) e=h(g_0 e)=h g_0=g$. and $\therefore g g^{-1}=e$

   $\therefore (G, *)$ is a group. □

4. Let T be the set of all $n \in \mathbb{N}_{>0}$ s.t. $a_{f(1)} * \cdots * a_{f(n)}=a_1 * \cdots * a_n$
   holds for all sequences $\langle a_k\rangle_{1 \leq k \leq n}$ of n elements of S which satisfy:
   $\forall i, j \in[1 \ldots n]: a_i * a_j=a_j * a_i$
   for every permutation $f: \mathbb{N}_n \rightarrow \mathbb{N}_n$.

   There are 3 cases to consider

   1. $f(m+1)=m+1$.
      $a_{f(1)} * . . a_{f(m+1)}=(a_{f(1)} * . . a_{f(m)}) * a_{f(m+1)}=(a_1 * . . a_m) * a_{m+1} \xlongequal{\text{ind}} a_1 * . . a_{m+1}$
   2. $f(1)=m+1$.
      $a_{f(1)} * . . a_{f(m+1)}=a_{f(1)} *(a_{f(2)} * . . a_{f(m+1)}) \xlongequal{\text{inc}} a_{m+1} *(a_{f'(1)} * . . a_{f'(m)}) \xlongequal{\text{def}} a_{m+1} *(a_1 * . . . a_m)$
      $\xlongequal{\text{commutative}}(a_1 * . . . a_m) * a_{m+1}=a_1 * . . . a_m * a_{m+1}$. Here f' is defined as
      $\forall k \in[1 \ldots m]: f'(k)=f(k+1)$

3. $f(r) = m+1$ for some $r \in [2..m]$.
   $a_{f(1)} \circ \cdots \cdot a_{f(m+1)} = \left(a_{f(1)} \circ \cdots \circ a_{f(r-1)}\right) \cdot \left(a_{f(r)} \circ \left(a_{f(r+1)} \circ \cdots \cdot a_{f(m+1)}\right)\right)$
   $= \left(a_{f''(1)} \circ \cdots \circ a_{f''(r-1)}\right) \circ \left(a_{f''(m+1)} \circ \left(a_{f''(r)} \circ \cdots \circ a_{f''(m)}\right)\right)$
   $= \left(a_{f''(1)} \circ \cdots \circ a_{f''(r-1)}\right) \circ \left(\left(a_{f''(r)} \circ \cdots \circ a_{f''(m)}\right) \circ a_{f''(m+1)}\right) = a_{f''(1)} \circ \cdots \cdot a_{f''(m+1)}$
   Here f'' is defomed as $\forall k \in \mathbb{N}_{m+1} : \begin{cases} \sigma(k) & : k \in [1 \dots r-1] \\ \sigma(k+1) & : k \in [r \dots m] \\ m+1 & : k = m+1 \end{cases}$

5. i.

proof: $\leftarrow$ if $(M, *)$ is a semigroup and t has an inverse. We have $t^{-1} * b = t^{-1}tb = b, \quad b * t^{-1} = btt^{-1} = b. \therefore \ (\mathrm{M}, \odot)$ is a monoid.

$\rightarrow$: Suppose $(\mathrm{M}, \odot)$ is a monoid, Suppose 1 is a monoid of M, and a is the monoid of $(M, *)$, we have $1 = a * 1 = at1 = at, \quad 1 = 1 * a = 1ta = ta$. Thus $a = t^{-1}$ which is the inverse of $(M, *)$. $\square$

ii.

proof: $\leftarrow$: $(M, *)$ is a group, $\because a \odot b = atb$ we have $a'(a \odot b) = (b \odot a)a', \forall a, b \in M$(1) ,$a'$ is the one-dimention mapping from M to M. And we have $\forall a, b, c \in G, a'ac = c = b'bc$. From (1), we have $a'a = a'acc' = b'bc' = b'b$. Let $a' \odot a = e$, then e is the left monoid of G. a' Is the left inverse of a. $\therefore G$ is group.

$\rightarrow$: The process of above prove is reversable. $\square$