Lecture 8
- Phasor

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Outline

- Sinusoidal signals

- Phasor

Sinusoidal Signal (Current or Voltage)

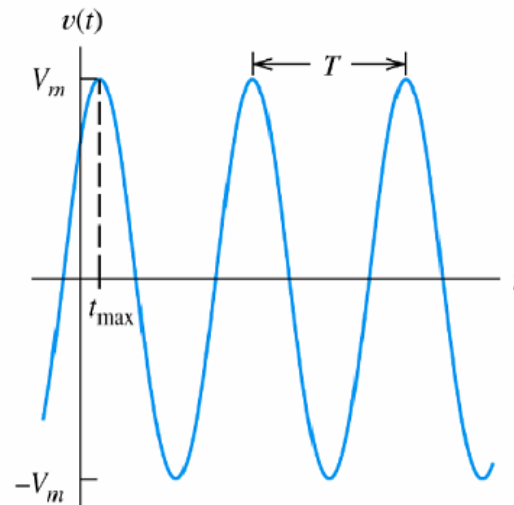$$v(t) = V_m \cos(\omega t + \theta)$$

V_m is the **peak value**

ω is the **angular frequency** in radians per second

$(\omega t + \theta)$ is the **phase angle**
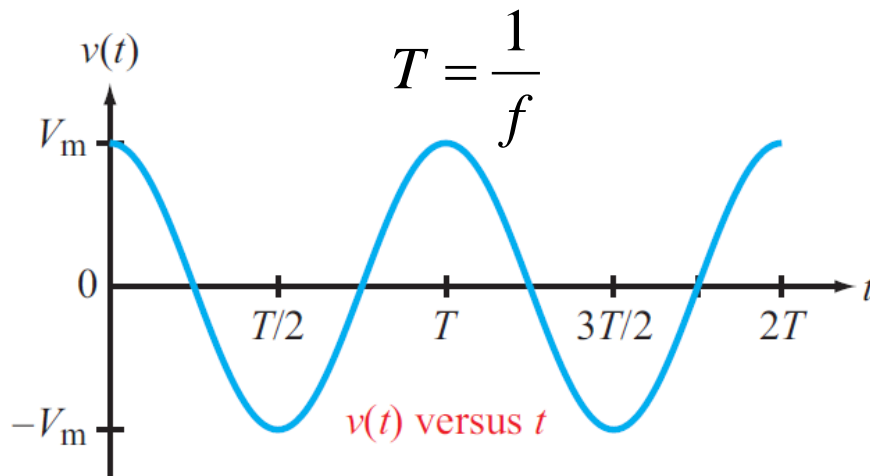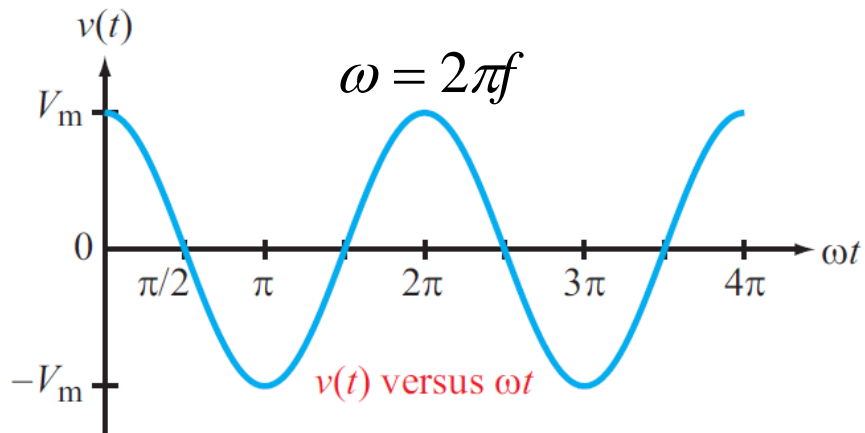
T is the **period**

$$f = \frac{1}{T} \qquad \omega = 2\pi f$$

Sinusoidal Signals

$$v(t) = V_m \cos(\omega t + \phi)$$

$$\omega = 2\pi f$$

$v(t)$ versus ωt

$$T = \frac{1}{f}$$

$v(t)$ versus t

Useful relations

$$\sin x = \pm \cos(x \mp 90°)$$
$$\cos x = \pm \sin(x \pm 90°)$$
$$\sin x = -\sin(x \pm 180°)$$
$$\cos x = -\cos(x \pm 180°)$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$2\sin x \sin y = \cos(x - y) - \cos(x + y)$$
$$2\sin x \cos y = \sin(x + y) + \sin(x - y)$$
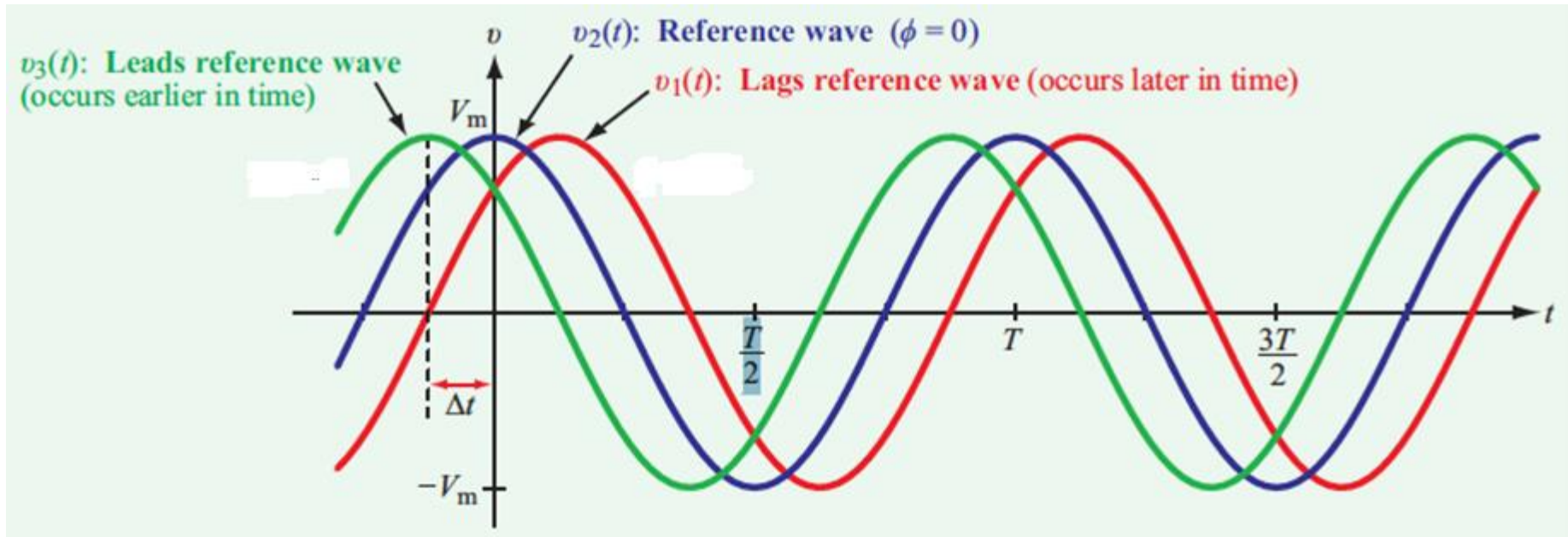$$2\cos x \cos y = \cos(x + y) + \cos(x - y)$$

Phase Lead/Lag

$$V_{\mathrm{m}} \cos \frac{2\pi t}{T} \qquad V_{\mathrm{m}} \cos \left(\frac{2\pi t}{T} - \frac{\pi}{4} \right) \qquad V_{\mathrm{m}} \cos \left(\frac{2\pi t}{T} + \frac{\pi}{4} \right)$$


[Source: Berkeley]
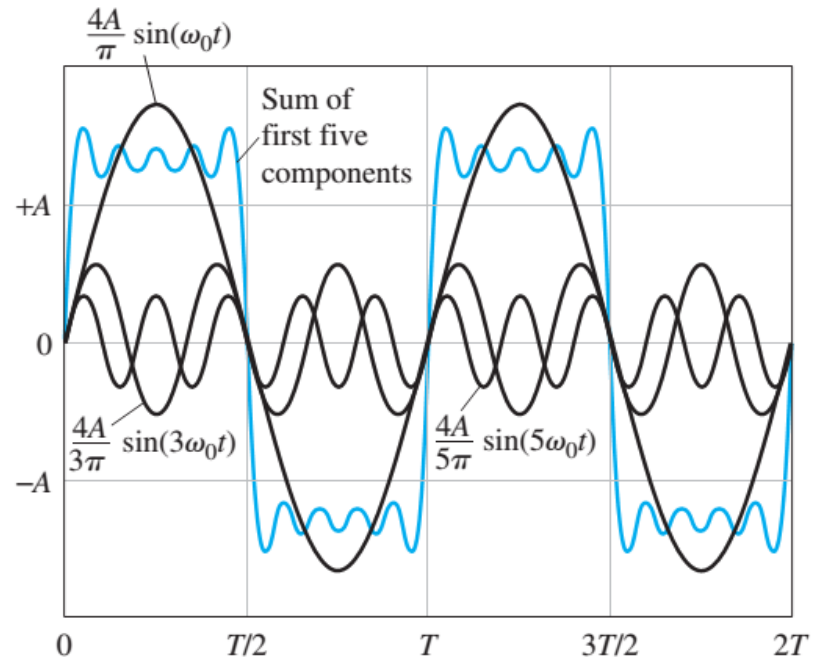
Why Sinusoids?

- Numbers of natural phenomenon are sinusoidal in nature.
  - Motion of a pendulum, vibration of a string, ripples on ocean surface

- A very easy signal to generate and transmit
  - Dominant form of signal in communication/electric power industries
  - In the late 1800's there was a battle between proponents of DC and AC. AC won out due to its efficiency for long distance transmission.

- Lastly, they are very easy to handle mathematically.
  - Derivative and integral are also sinusoids.

- Through *Fourier analysis*, any practical periodic function can be represented as sum of sinusoids.

Representing a Square Wave as a Sum of Sinusoids

$$\frac{4A}{\pi}\sin(\omega_0 t)$$

Sum of first five components
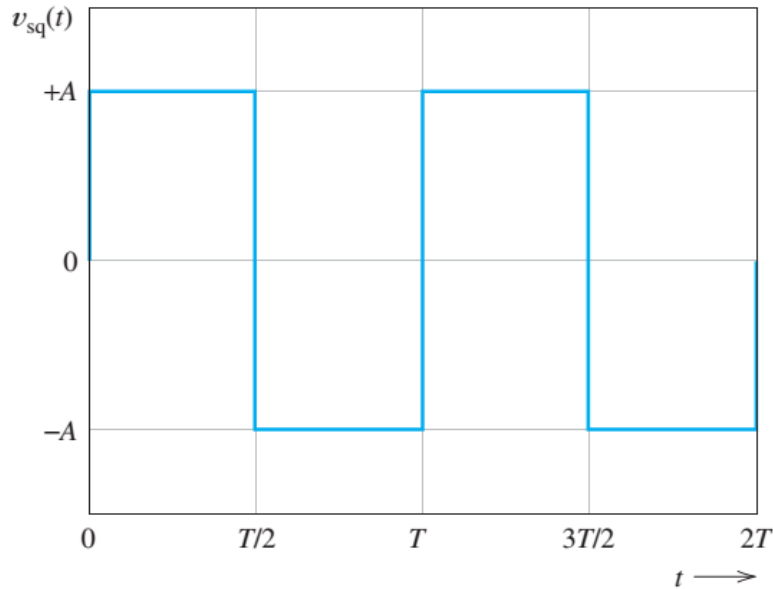
$$\frac{4A}{3\pi}\sin(3\omega_0 t)$$

$$\frac{4A}{5\pi}\sin(5\omega_0 t)$$

(a) Periodic square wave

(b) Several of the sinusoidal components and the sum of the first five components

$$v_{\text{sq}}(t) = \frac{4A}{\pi}\sin(\omega_0 t) + \frac{4A}{3\pi}\sin(3\omega_0 t) + \frac{4A}{5\pi}\sin(5\omega_0 t) + \cdots$$

$$\omega_0 = 2\pi/T$$

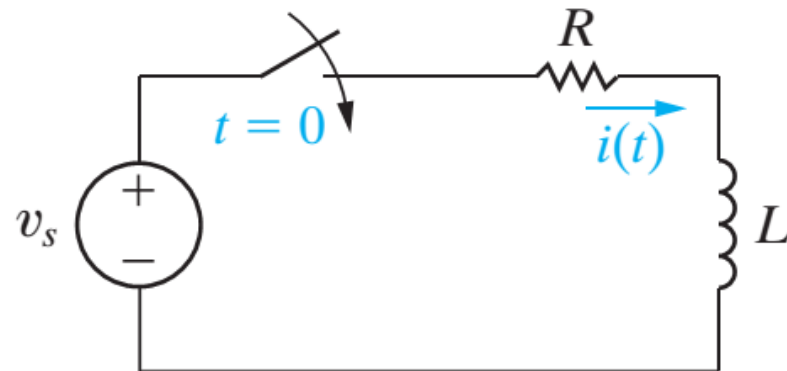# The Sinusoidal Response

$v_s = V_m \cos(\omega t + \phi)$, $i(0^-) = 0$.
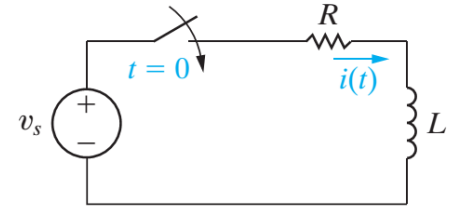
Find $i(t)$, $t \geq 0$.

$$L \frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

Ordinary differential equation

Sinusoidal Steady-State Response

$$v_s = V_m \cos(\omega t + \phi)$$

$$i = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)$$
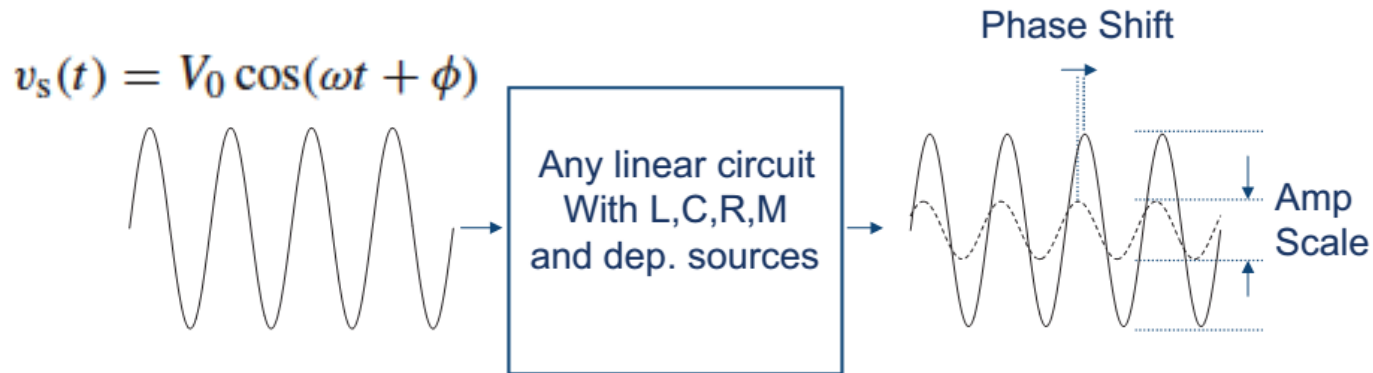
Transient response   Steady-state response

- Steady-state solution is sinusoidal
- Response frequency = source frequency
- Magnitude & phase of response differs from that of source

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The Magic of Sinusoids

$$v_s(t) = V_0 \cos(\omega t + \phi)$$

Any linear circuit
With L,C,R,M
and dep. sources

Phase Shift

Amp
Scale

- When a linear, time invariant (LTI) circuit is excited by a sinusoid, it's output is a sinusoid at the *same* frequency.
  - Only the magnitude and phase (<u>initial phase angle</u>) of the output differ from the input.
- In order to find a steady-state voltage or current, all we need to know is its <u>magnitude</u> and its <u>initial phase angle relative</u> to the source.
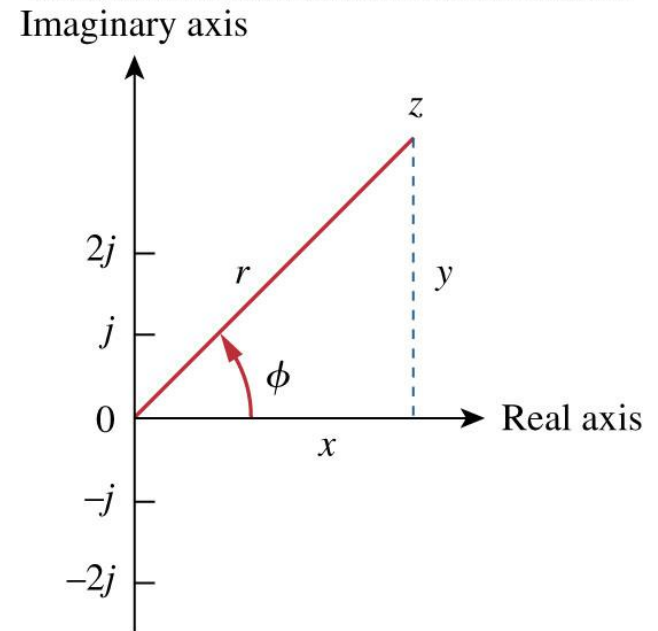
Complex Numbers

- A powerful method for representing sinusoids is the phasor.

- A complex number z can be represented in *rectangular form* as:

$$z = x + jy \qquad \begin{aligned} \text{Re}(z) &= x \\ \text{Im}(z) &= y \end{aligned}$$

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- It can also be written in *polar* or *exponential* form as:

$$z = r \angle \phi = re^{j\phi}$$

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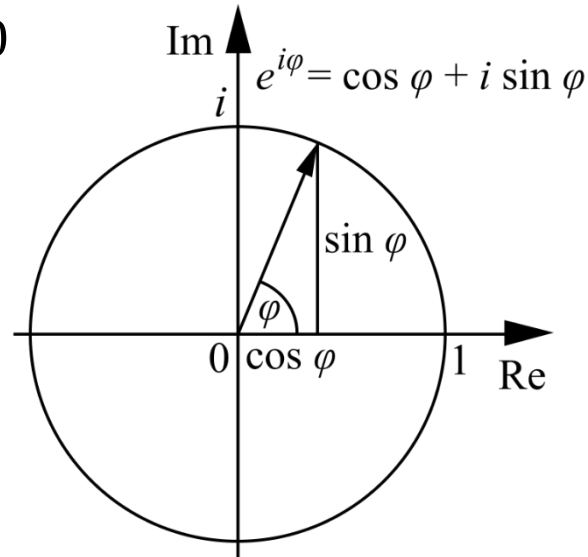
Euler's Formula

Euler's Identities $e^{j\pi} + 1 = 0$

$$\boxed{e^{j\varphi} = \cos \varphi + j \sin \varphi}$$

$$\cos \varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}$$

$$\sin \varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2}$$

$$\left| e^{j\varphi} \right| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$$

$e^{i\varphi} = \cos \varphi + i \sin \varphi$

$$1 = 1e^{j0} = 1\angle 0°$$

$$j = 1e^{j\frac{\pi}{2}} = 1\angle 90°$$

Exponential Form of a complex number Z

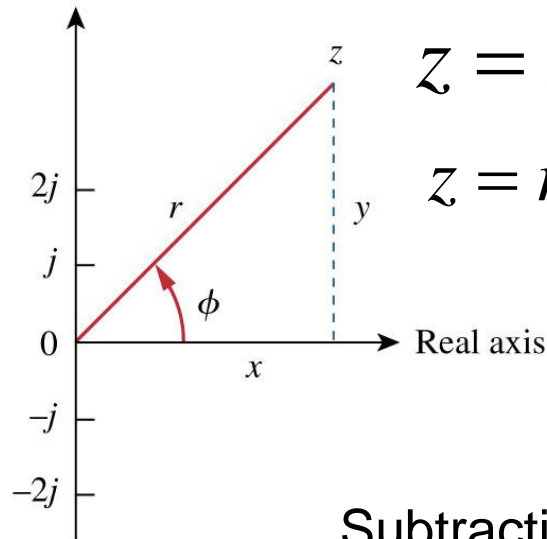$$\boldsymbol{Z} = |\boldsymbol{Z}|e^{j\varphi} = ze^{j\varphi} = z\angle\varphi$$

Source: https://en.wikipedia.org/wiki/Euler%27s_identity

Arithmetic With Complex Numbers

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$$z = x + jy$$

$$z = r\angle\phi = re^{j\phi}$$

Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 \angle(\phi_1 + \phi_2)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle(\phi_1 - \phi_2)$$

Reciprocal

$$\frac{1}{z} = \frac{1}{r} \angle(-\phi)$$

Square Root

$$\sqrt{z} = \sqrt{r} \angle(\phi / 2)$$

Complex Conjugate

$$z^* = x - jy = r\angle -\phi = re^{-j\phi}$$

Example

- Evaluate these complex numbers

(a) $(40\underline{/50°} + 20\underline{/-30°})^{1/2}$

(b) $\dfrac{10\underline{/-30°} + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Exercise

• Evaluate the following complex numbers

(a) $[(5 + j2)(-1 + j4) - 5\underline{/60°}]^*$

(b) $\dfrac{10 + j5 + 3\underline{/40°}}{-3 + j4} + 10\underline{/30°} + j5$

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Relations for Complex Numbers

Euler's Identity: $e^{j\theta} = \cos\theta + j\sin\theta$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad\qquad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\mathbf{z} = x + jy = |\mathbf{z}|e^{j\theta} \qquad\qquad \mathbf{z}^* = x - jy = |\mathbf{z}|e^{-j\theta}$$

$$x = \mathfrak{Re}(\mathbf{z}) = |\mathbf{z}|\cos\theta \qquad\qquad |\mathbf{z}| = \sqrt[+]{\mathbf{z}\mathbf{z}^*} = \sqrt[+]{x^2 + y^2}$$

$$y = \mathfrak{Im}(\mathbf{z}) = |\mathbf{z}|\sin\theta \qquad\qquad \theta = \tan^{-1}(y/x)$$

$$\mathbf{z}^n = |\mathbf{z}|^n e^{jn\theta} \qquad\qquad \mathbf{z}^{1/2} = \pm|\mathbf{z}|^{1/2}e^{j\theta/2}$$

$$\mathbf{z}_1 = x_1 + jy_1 \qquad\qquad \mathbf{z}_2 = x_2 + jy_2$$

$$\mathbf{z}_1 = \mathbf{z}_2 \text{ iff } x_1 = x_2 \text{ and } y_1 = y_2 \qquad \mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$\mathbf{z}_1\mathbf{z}_2 = |\mathbf{z}_1||\mathbf{z}_2|e^{j(\theta_1+\theta_2)} \qquad\qquad \frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{|\mathbf{z}_1|}{|\mathbf{z}_2|}e^{j(\theta_1-\theta_2)}$$

$$-1 = e^{j\pi} = e^{-j\pi} = 1\angle\pm180°$$

$$j = e^{j\pi/2} = 1\angle90° \qquad\qquad -j = e^{-j\pi/2} = 1\angle{-90°}$$

$$\sqrt{j} = \pm e^{j\pi/4} = \pm\frac{(1+j)}{\sqrt{2}} \qquad\qquad \sqrt{-j} = \pm e^{-j\pi/4} = \pm\frac{(1-j)}{\sqrt{2}}$$

[Source: Berkeley]

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Phasor

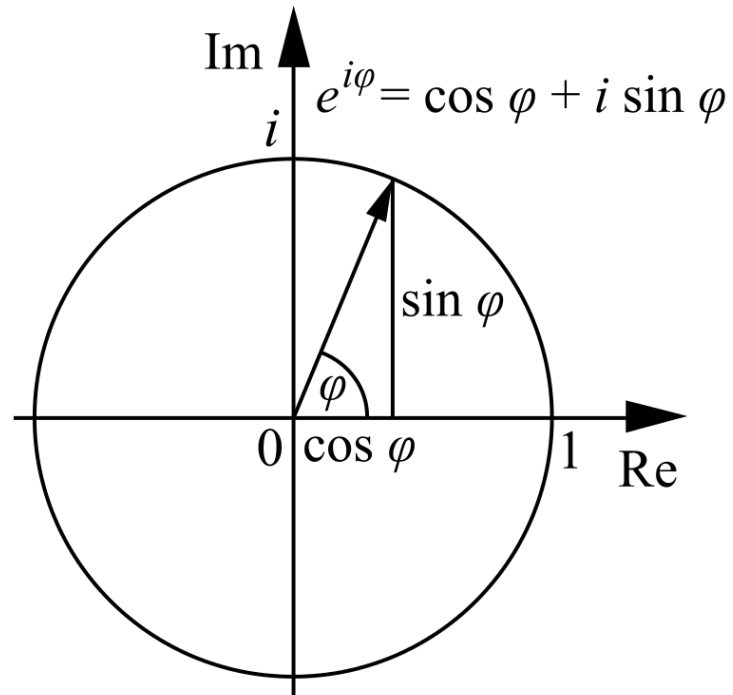- The idea of phasor representation is based on Euler's identity:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

- From this we can represent a sinusoid as the <u>real component</u> of a vector in the complex plane.

$$\cos\phi = \text{Re}\left(e^{j\phi}\right)$$

$$\sin\phi = \text{Im}\left(e^{j\phi}\right)$$

Similarly, $v(t) = V\cos(\omega t + \phi) = \text{Re}\left\{V e^{j\phi} e^{j\omega t}\right\}$

$e^{i\varphi} = \cos\varphi + i\sin\varphi$

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Phasor:

$$v(t) = V\cos(\omega t + \phi) = \text{Re}\left\{Ve^{j\phi}e^{j\omega t}\right\} = \text{Re}\left(\mathbf{V}e^{j\omega t}\right) \qquad \boxed{\mathbf{V} = Ve^{j\phi}}$$

Complex representation of the magnitude and phase of a sinusoid
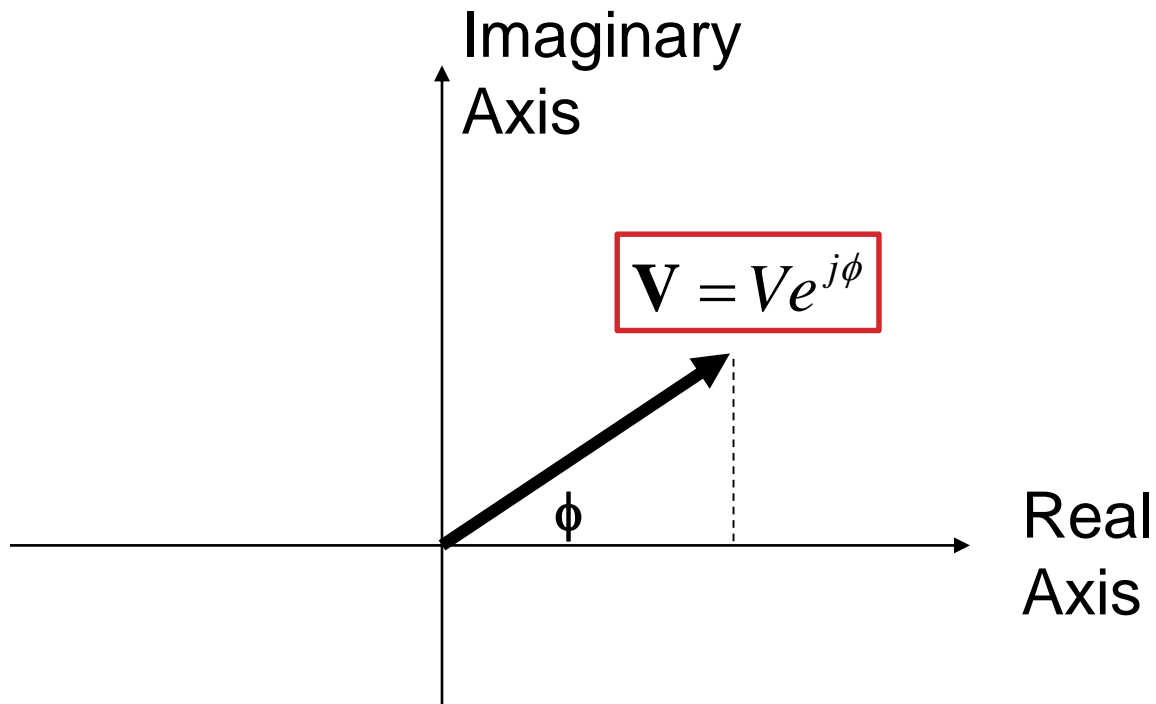
Example

- Transform these sinusoids to phasors

(a) $i = 6 \cos(50t - 40°)$ A
(b) $v = -4 \sin(30t + 50°)$ V

Example

- Find the sinusoids represented by these phasors

$$\mathbf{I} = -3 + j4 \text{ A}$$

$$\mathbf{V} = j8e^{-j20°} \text{ V}$$

$$\mathbf{V} = -25\underline{/40°} \text{ V}$$
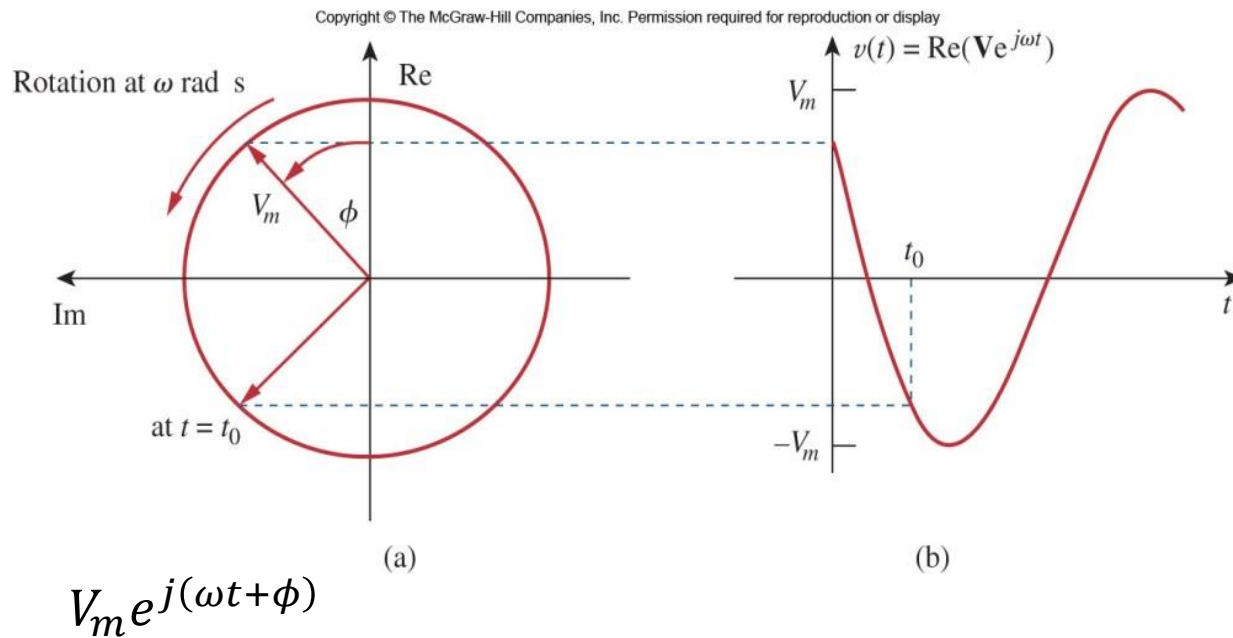
$$\mathbf{I} = j(12 - j5) \text{ A}$$

Phasors

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)

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Rotation at ω rad s

Re

Im

V_m

ϕ

at $t = t_0$

(a)

$v(t) = \mathrm{Re}(\mathbf{V}e^{j\omega t})$

V_m

t_0

$-V_m$

t

(b)

$$V_m e^{j(\omega t + \phi)}$$

Sinusoid-Phasor Transformation

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation)   (Phasor-domain representation)

- Applying a derivative to a phasor yields:

$$\frac{dv}{dt} \quad \Leftrightarrow \quad j\omega V$$

(Time domain)   (Phasor domain)

- Applying an integral to a phasor yields:

$$\int v \, dt \quad \Leftrightarrow \quad \frac{V}{j\omega}$$

(Time domain)   (Phasor domain)

Example

- Use the phasor approach, determine the current $i(t)$ in the circuit described by the integrodifferential equation

$$4i + 8 \int i \, dt + 3\frac{di}{dt} = 50 \cos(2t + 75°)$$

Time Domain - Phasor Domain Transformation

$x(t)$		$\mathbf{X}$
$A \cos \omega t$	$\longleftrightarrow$	A
$A \cos(\omega t + \phi)$	$\longleftrightarrow$	$A e^{j\phi}$
$-A \cos(\omega t + \phi)$	$\longleftrightarrow$	$A e^{j(\phi \pm \pi)}$
$A \sin \omega t$	$\longleftrightarrow$	$A e^{-j\pi/2} = -jA$
$A \sin(\omega t + \phi)$	$\longleftrightarrow$	$A e^{j(\phi - \pi/2)}$
$-A \sin(\omega t + \phi)$	$\longleftrightarrow$	$A e^{j(\phi + \pi/2)}$
$\dfrac{d}{dt}(x(t))$	$\longleftrightarrow$	$j\omega \mathbf{X}$
$\dfrac{d}{dt}[A \cos(\omega t + \phi)]$	$\longleftrightarrow$	$j\omega A e^{j\phi}$
$\displaystyle\int x(t)\, dt$	$\longleftrightarrow$	$\dfrac{1}{j\omega} \mathbf{X}$
$\displaystyle\int A \cos(\omega t + \phi)\, dt$	$\longleftrightarrow$	$\dfrac{1}{j\omega} A e^{j\phi}$

[Source: Berkeley]

It is much easier to deal with exponentials in the phasor domain than sinusoidal relations in the time domain.
- You just need to track magnitude/phase, knowing that everything is at frequency ω.

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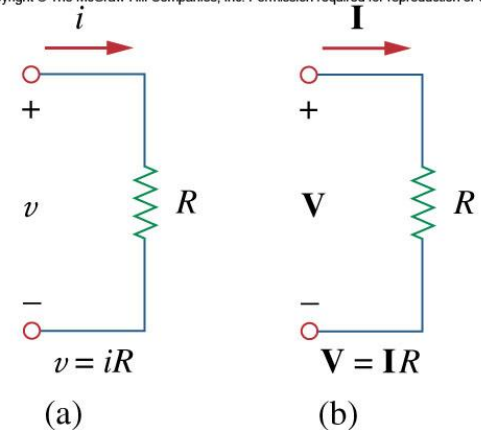
Phasor Relationships for Resistors

- For the resistor, the voltage and current are related via Ohm's law. As such, the voltage and current are in phase with each other.

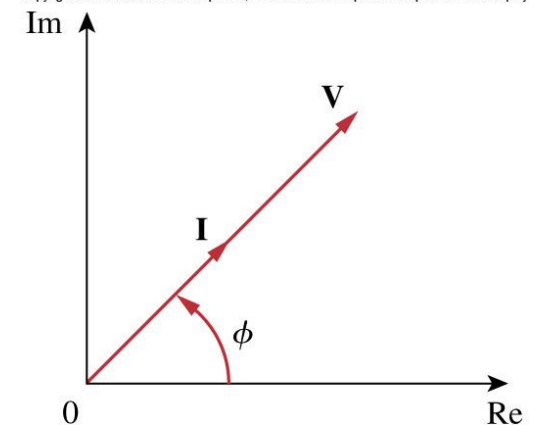$$i = I_m \cos(\omega t + \phi) \qquad \mathbf{I} = I_m \angle \phi$$

$$v = R I_m \cos(\omega t + \phi) \qquad \mathbf{V} = R I_m \angle \phi$$

$$\boxed{\mathbf{V} = R \mathbf{I}}$$

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$v = iR$

$\mathbf{V} = \mathbf{I}R$

(a)   (b)

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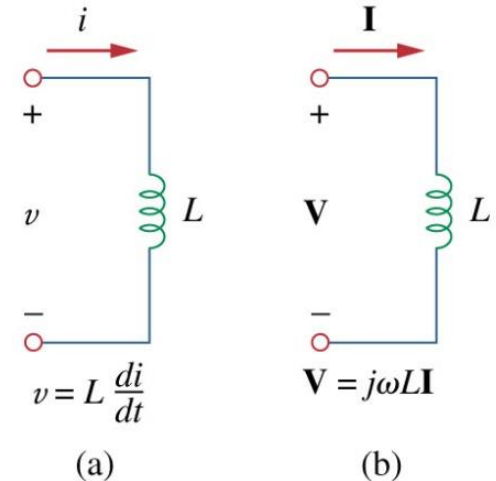School of Information Science and Technology

Phasor Relationships for Inductors

$$i = I_m \cos(\omega t + \phi) \qquad \mathbf{I} = I_m \angle \phi$$

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$
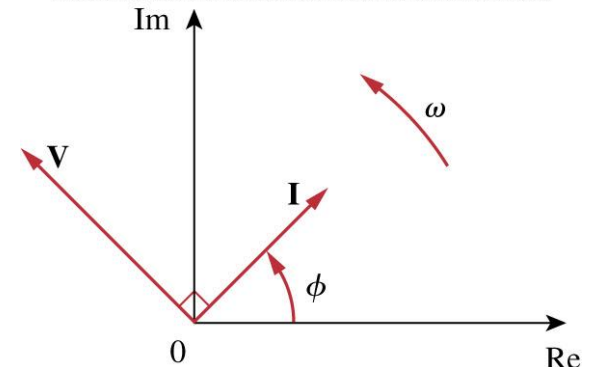
$$= \omega L I_m \cos(\omega t + \phi + 90°)$$

$$\mathbf{V} = \omega L I_m \angle \phi + 90° = \omega L I_m e^{j(\phi+90°)} = \omega L e^{j90°} \cdot I_m e^{j\phi} = j\omega L \cdot \mathbf{I}$$

$$\boxed{\mathbf{V} = j\omega L\mathbf{I}}$$

- The voltage leads the current by 90° (phase shift = 90°)

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Phasor Relationships for Capacitors

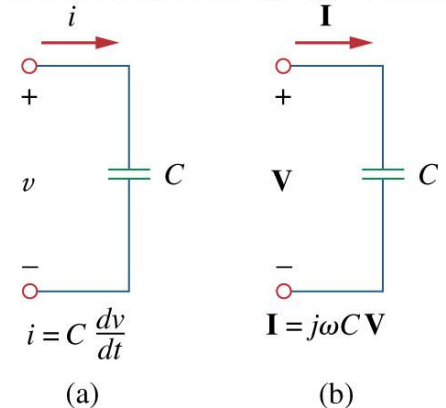$$v = V_m \cos(\omega t + \phi) \qquad \mathbf{V} = V_m \angle \phi$$

$$i = C \frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi)$$

$$= \omega C V_m \cos(\omega t + \phi + 90°)$$

$$\mathbf{I} = \omega C V_m \angle \phi + 90° = \cdots = j\omega C \cdot \mathbf{V}$$

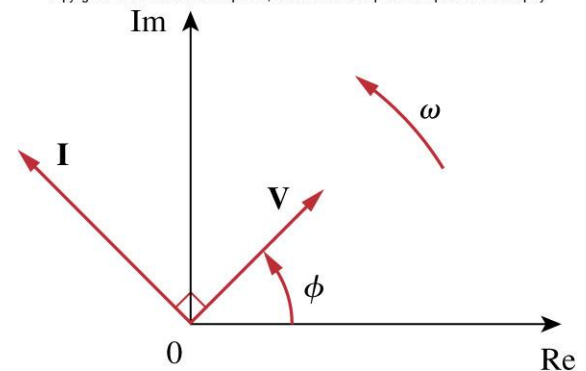Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display

$$i = C \frac{dv}{dt}$$

$$\mathbf{I} = j\omega C \mathbf{V}$$

(a)   (b)

$$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

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- The voltage lags the current by 90°.

Impedance

- The voltage-current relations for R, L and C elements are

$$\mathbf{V} = R\mathbf{I} \qquad \mathbf{V} = j\omega L\mathbf{I} \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = R \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

- Phasors allow us to express the relationship between current and voltage using a formula like Ohm's law:

$$\mathbf{V} = \mathbf{I}\,\mathbf{Z} \quad \text{or} \quad \mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

- **Z** is called impedance, measured in ohms.
  - Impedance is not a phasor! But it is (often) a complex number.
  - Impedance depends on the frequency ω.

Admittance

Admittance is simply the inverse of impedance, unit: Simens.

$$Y = \frac{I}{V}$$

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$Y = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$Y = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$Y = j\omega C$

Summary of *R, L, C*

Property	R	L	C
v–i	$v = Ri$	$v = L \dfrac{di}{dt}$	$i = C \dfrac{dv}{dt}$
V–**I**	$\mathbf{V} = R\mathbf{I}$	$\mathbf{V} = j\omega L\mathbf{I}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$
Z	R	$j\omega L$	$\dfrac{1}{j\omega C}$
dc equivalent	R	Short circuit	Open circuit
High-frequency equivalent	R	Open circuit	Short circuit
Frequency response	$|\mathbf{Z_R}|$, R	$|\mathbf{Z_L}|$, ωL	$|\mathbf{Z_C}|$, $1/\omega C$

[Source: Berkeley]