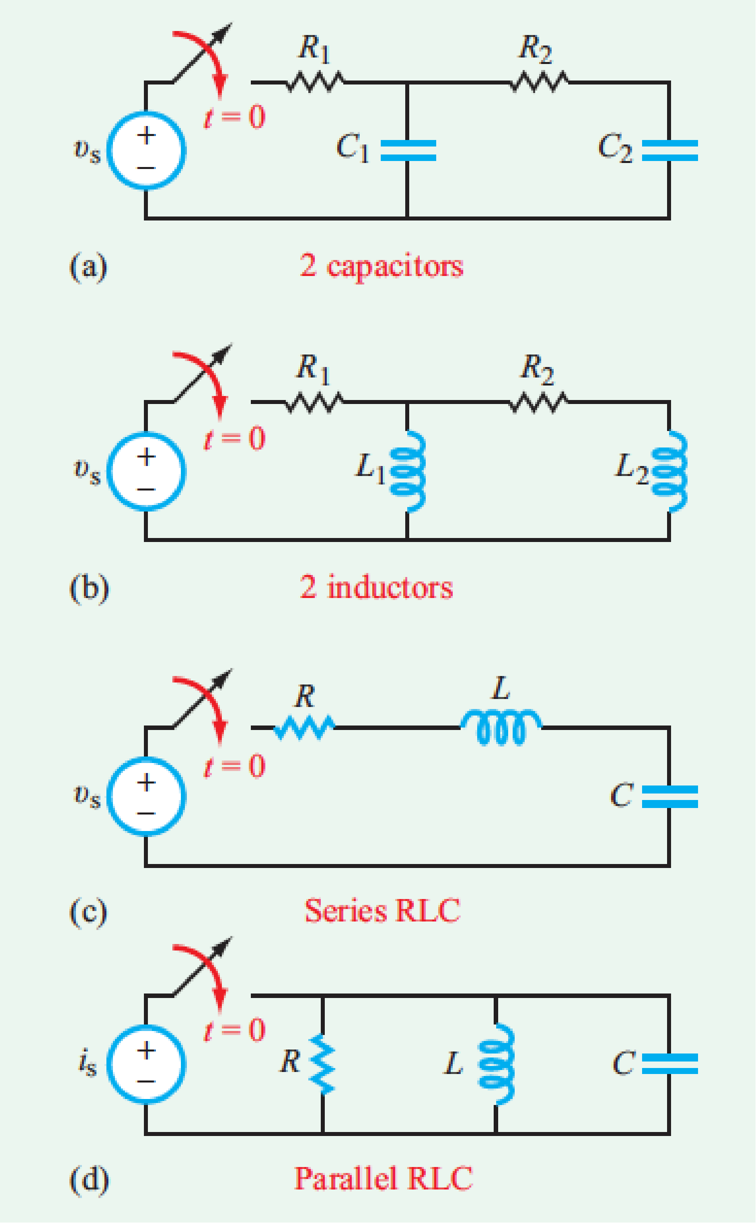
Lecture 7

- Second-Order Circuits

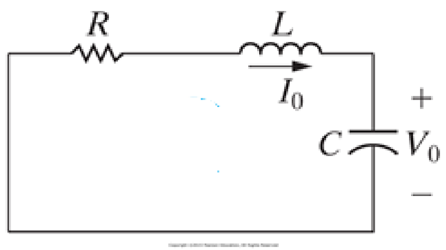
Second-Order Circuits

- Two energy storage elements
- Analysis: Determine voltage or current as a function of time
- Initial/final values of voltage/current, *and their derivatives are* needed

A second order circuit is characterized by a second order differential equation.

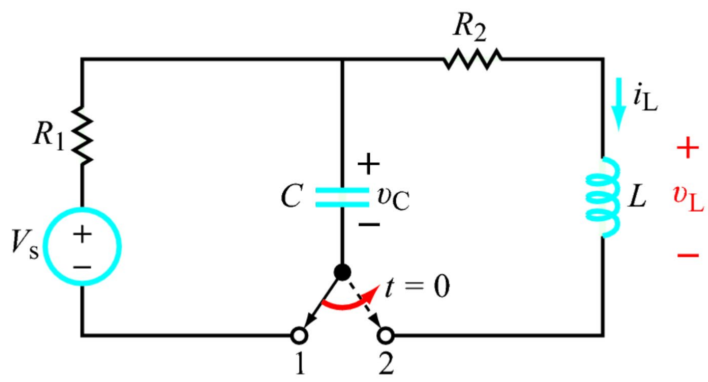


Source-Free Series RLC



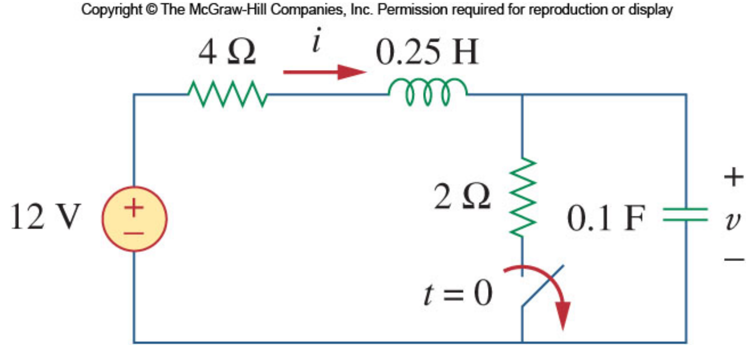
$$\frac{d^2v}{d^2t}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{\mathrm{LC}}v=0$$

Initial and Final Conditions



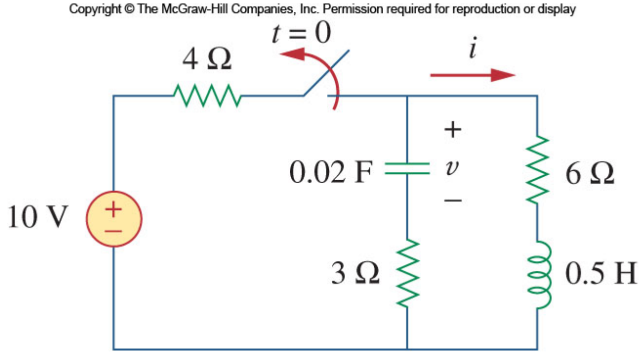
Example

- The switch has been closed for a long time. It is open at $t = 0$. Find
  - $i(0^+)$, $v(0^+)$
  - $di(0^+)/dt$, $dv(0^+)/dt$
  - $i(\infty)$, $v(\infty)$



Exercise

- Assume the circuit has reached steady state at $t = 0^-$. Find
  - $i(0^+)$, $v(0^+)$
  - $di(0^+)/dt$, $dv(0^+)/dt$
  - $i(\infty)$, $v(\infty)$



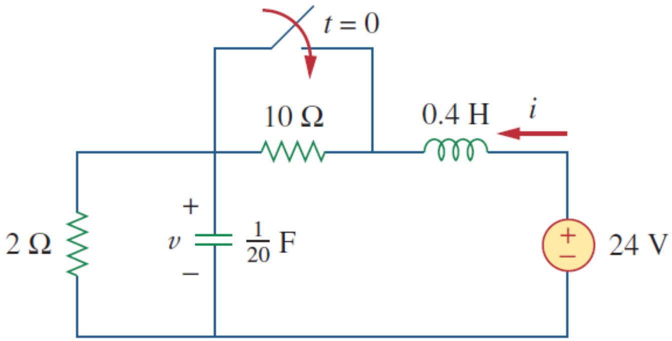


Figure 8.4
For Practice Prob. 8.1.

For the circuit in Fig. 8.7, find: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$, (b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, $dv_R(0^+)/dt$, (c) $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$.

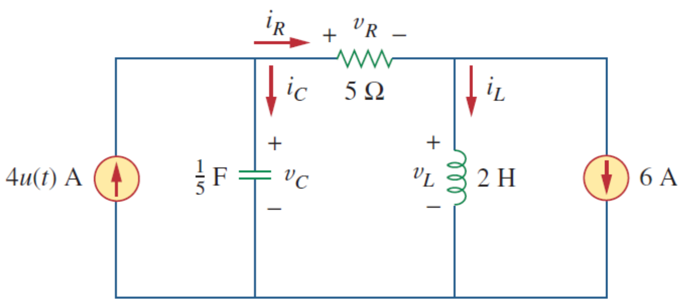
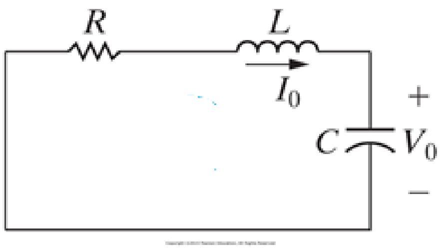


Figure 8.7
For Practice Prob. 8.2.

Source-Free Series RLC



$$\frac{d^2v}{d^2t} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{\mathrm{LC}}v = 0$$

$$\alpha = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

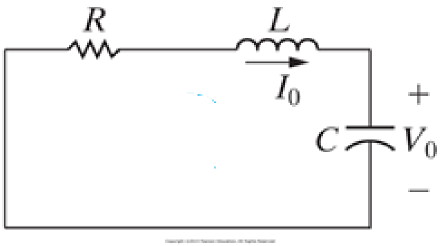
[Source: Berkeley]

- $s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

$$\alpha = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

Case 1: Overdamped ($\alpha > \omega_0$)

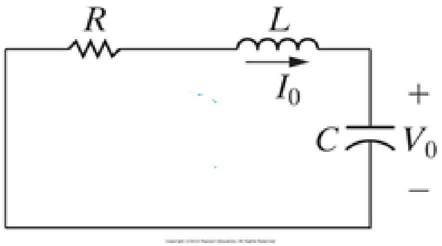


$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

$$A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Case 2: Critically Damped ($\alpha=\omega_0$)



$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{\mathrm{LC}}v=0 \qquad s=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

$$\alpha=\frac{R}{2L} \qquad \omega_0=\frac{1}{\sqrt{LC}}$$

$$s_1=s_2=-\alpha=-\frac{R}{2L}$$

Case 2: Critically Damped ($\alpha=\omega_0$)

Go back to

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{\mathrm{LC}}v=0 \qquad \alpha=\frac{R}{2L} \quad \omega_0=\frac{1}{\sqrt{LC}}$$

- When $\alpha=\omega_0=R/2L$, the equation becomes

$$\frac{d^2v}{dt^2}+2\alpha\frac{dv}{dt}+\alpha^2v=0$$

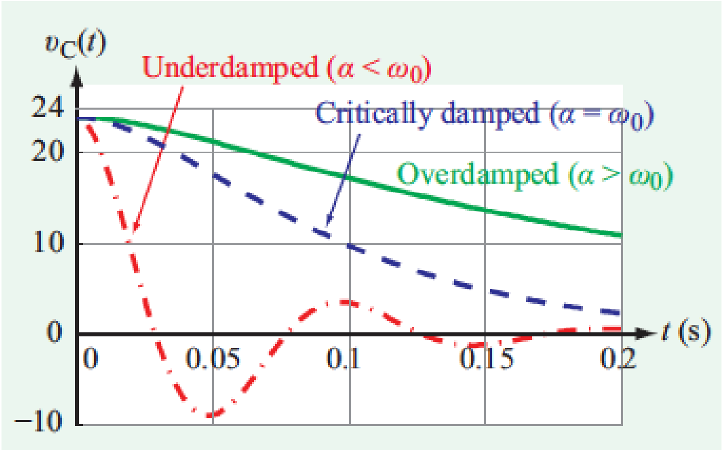
$$\Downarrow$$

$$\frac{d}{dt}\left(\frac{dv}{dt}+\alpha v\right)+\alpha\left(\frac{dv}{dt}+\alpha v\right)=0$$

- $v(t)=(A_1t+A_2)e^{-\alpha t}$

Case 2: Critically Damped ($\alpha=\omega_0$)

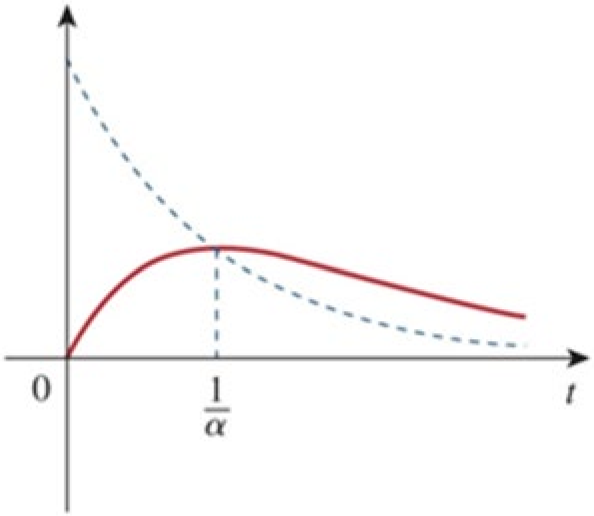
$$v(t) = (A_1 t + A_2)e^{-\alpha t}$$



(If $A_2 = 0, A_1 = 1$)

A typical critically damped response is shown

Why maximize at $t = \frac{1}{\alpha}$?



Case 3: Underdamped ($\alpha<\omega_0$)

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

where $j = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.

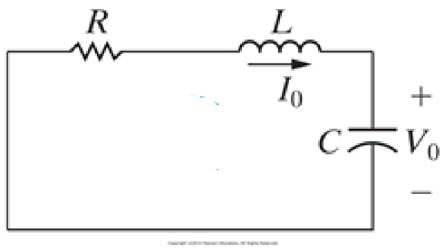
- ω_0 is often called the <u>undamped natural frequency.</u>
- ω_d is called the <u>damped natural frequency.</u>

The natural response

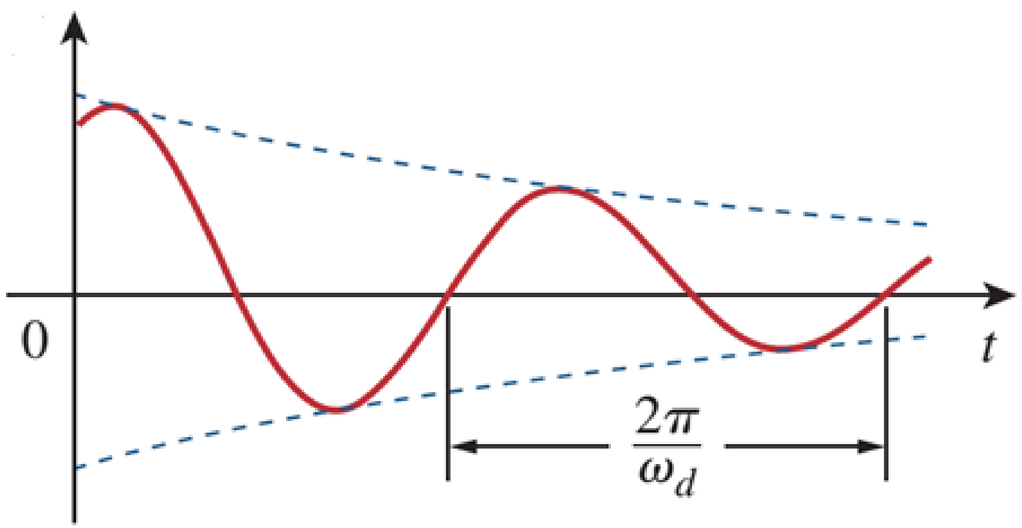
$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

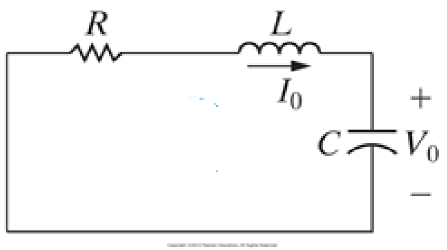
$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$



- Exponential $e^{-\alpha t}$ * Sine/Cosine term
  - Exponentially damped, time constant = $1/\alpha$
  - Oscillatory, period $T = \frac{2\pi}{\omega_d}$



Properties of Series RLC Network



- Behavior captured by damping
  - Gradual loss of the initial stored energy
  - α determines the rate of damping
    - $\alpha > \omega_0$ (i.e., $R > 2\sqrt{\frac{L}{C}}$), overdamped

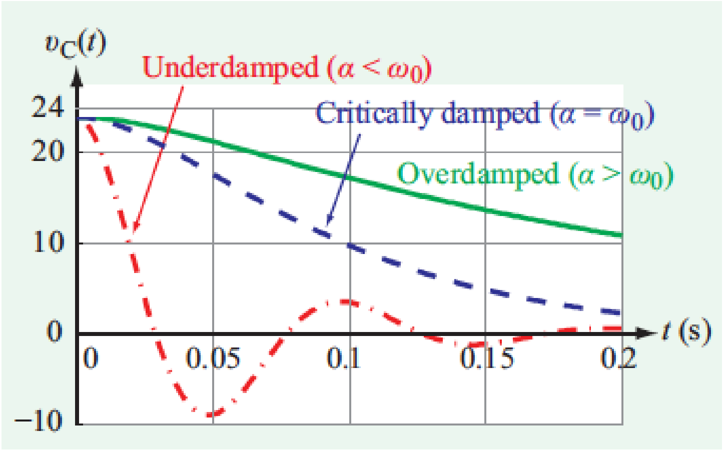
$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

    - $\alpha = \omega_0$ (i.e., $R = 2\sqrt{\frac{L}{C}}$), critically damped

$$v(t) = (A_1 t + A_2) e^{-\alpha t}$$

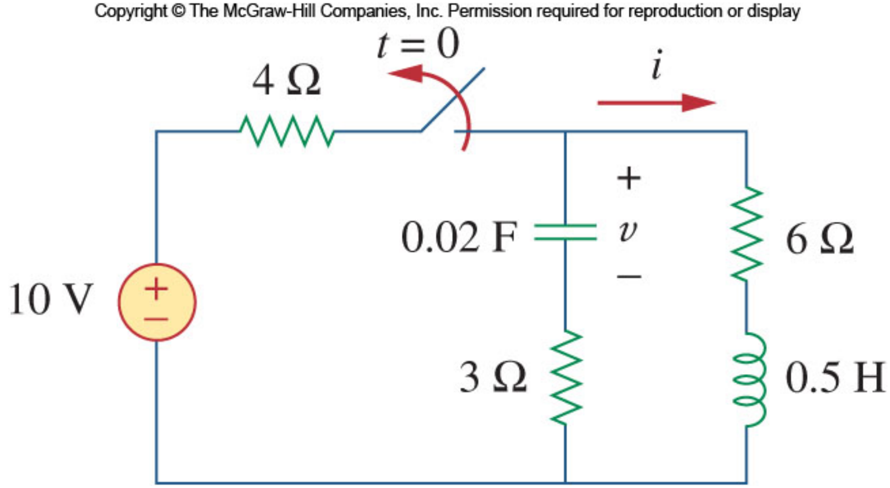
    - $\alpha < \omega_0$ (i.e., $R < 2\sqrt{\frac{L}{C}}$), underdamped

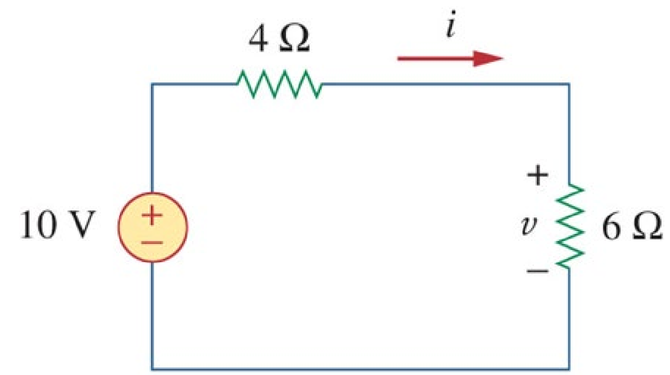
$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

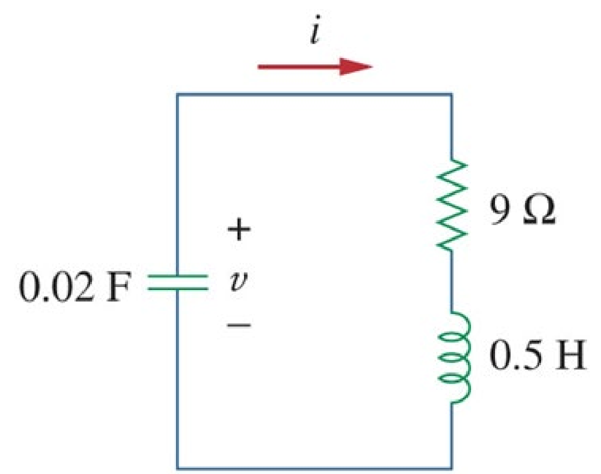


Example

- Find $v(t)$ &*i(t)* in the circuit below. Assume the circuit has reached steady state at $t = 0^{-}$.



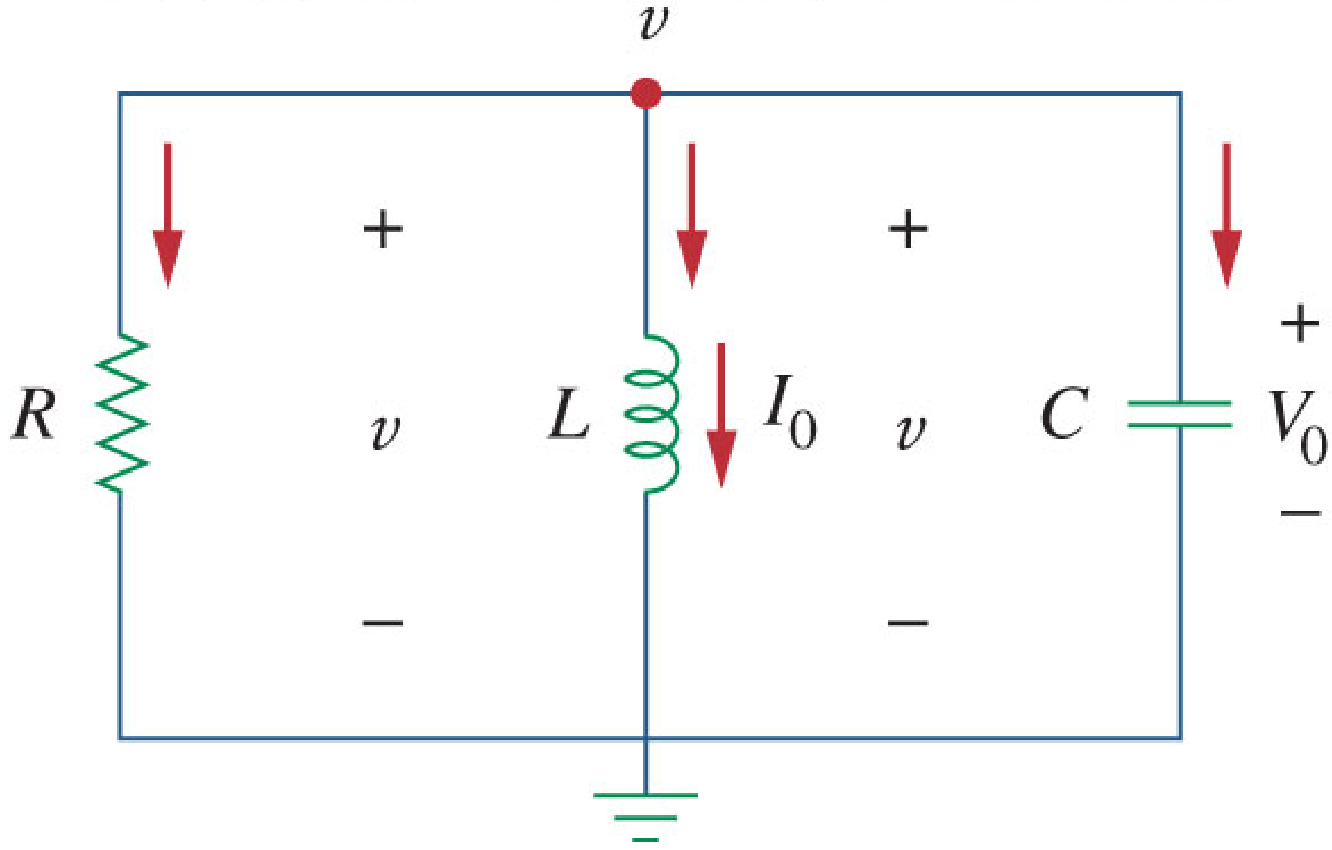




$$\alpha = \frac{R}{2L} = 9 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = 10$$

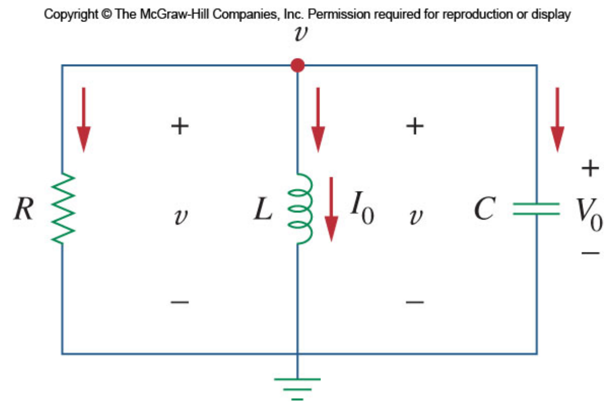
$$s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2} = -9 \mp j4.359$$

Source-Free Parallel RLC Network



Source-Free Parallel RLC Network

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$



- The characteristic equation is:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

From this, we can find the roots of the characteristic equation to be:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

As in last time, there are three scenarios to consider.

Three Damping Cases

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \qquad \alpha = \frac{1}{2RC} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

- For the overdamped case, the roots are real and negative,

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

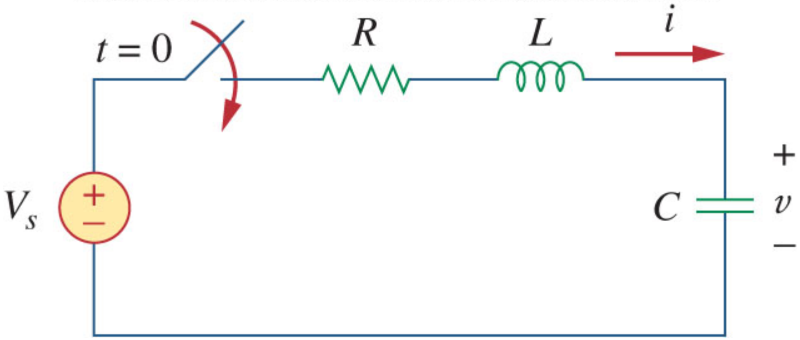
- For critically damped, the roots are real and equal

$$v(t) = (A_2 + A_1 t)e^{-\alpha t}$$

- In the underdamped case, the roots are complex

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

Step Response of a Series RLC Circuit



$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{v}{LC}=\frac{V_s}{LC}$$

- The solution

$$v(t)=v_t(t)+v_{ss}(t)$$

The complete solutions for the three conditions of damping are:

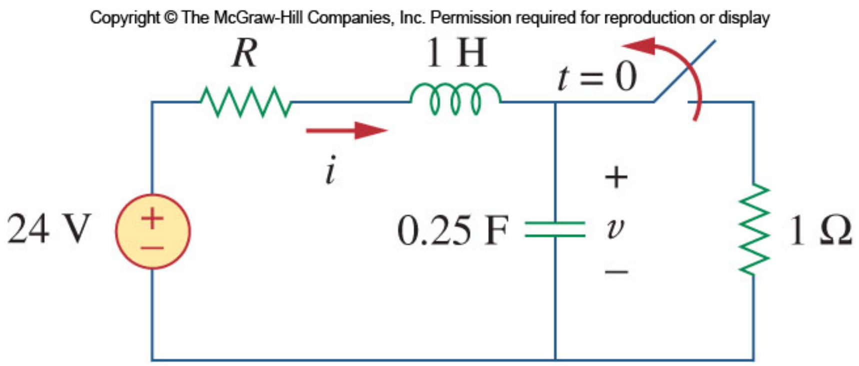
- $v(t) = V_s + (A_1 e^{s_1 t} + A_2 e^{s_2 t})$ (Overdamped)

- $v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t}$ (Critically Damped)

- $v(t) = V_s + (A_1 \cos\omega_d t + A_2 \sin\omega_d t)e^{-\alpha t}$ (Underdamped)

Example

- Find $v(t)$ and $i(t)$ for $t > 0$.
  Consider three cases:
  - $R = 5\Omega$
  - $R = 4\Omega$
  - $R = 1\Omega$



When $R = 5\Omega$,

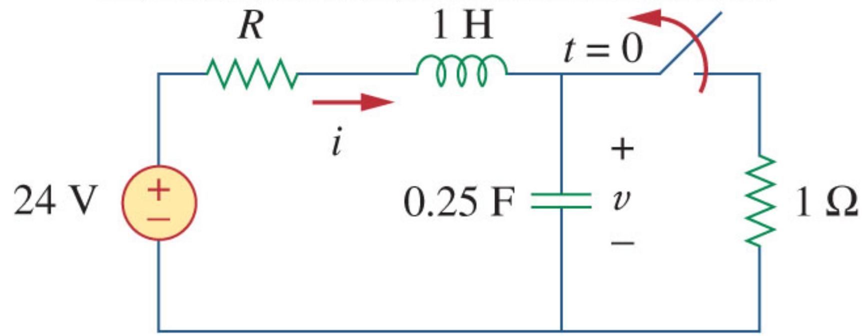
- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 4A = C\frac{dv(0)}{dt},\ v(0) = 4V,\ \frac{dv(0)}{dt} = 16$$

- For $t > 0$, switch open, a series RLC network

$$\alpha = \frac{R}{2L} = 2.5,\ \omega_0 = \frac{1}{\sqrt{LC}} = 2,\ s_{1,2} = -1, -4 \quad \text{Overdamped.}$$

$$v(t) = v_{ss} + (A_1 e^{-t} + A_2 e^{-4t})$$



When $R = 4\Omega$,

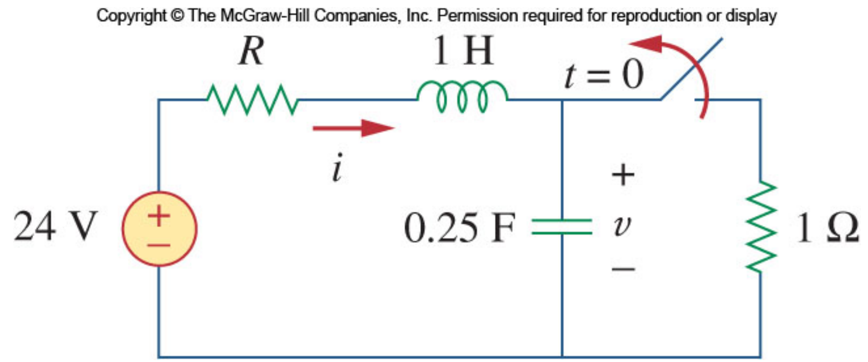
- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 4.8A = C\frac{dv(0)}{dt}, \;\; v(0) = 4.8V, \;\; \frac{dv(0)}{dt} = 19.2$$

- For $t > 0$, switch open, a series RLC network

$$\alpha = \frac{R}{2L} = 2, \;\; \omega_0 = \frac{1}{\sqrt{LC}} = 2, \;\; s_{1,2} = -2$$ Critically damped

$$v(t) = v_{ss} + (A_1 + A_2 t)e^{-2t}$$



When $R = 1\Omega$,

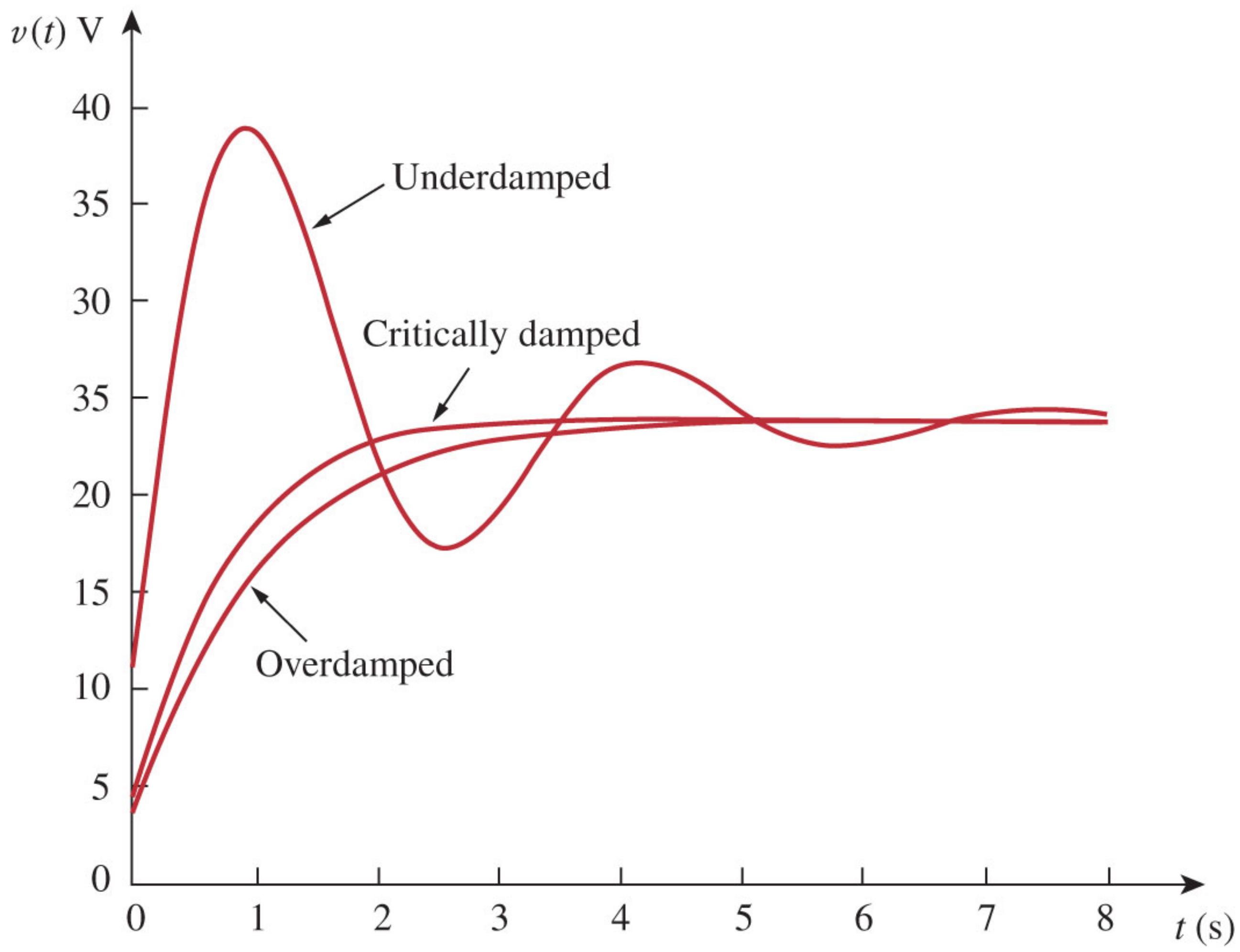
- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 12A = C\frac{dv(0)}{dt}, \ \ v(0) = 12V, \ \ \frac{dv(0)}{dt} = 48$$

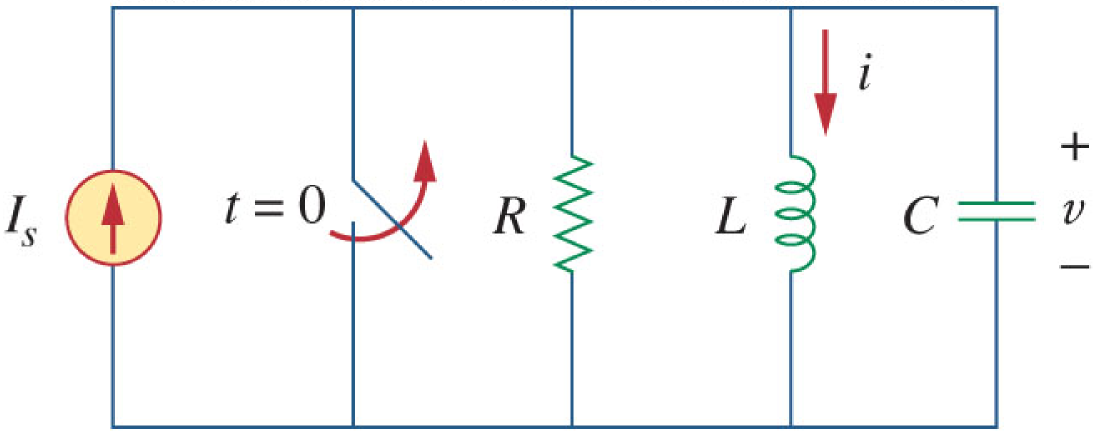
- For $t > 0$, switch open, a series RLC network

$$\alpha = \frac{R}{2L} = 0.5, \ \ \omega_0 = \frac{1}{\sqrt{LC}} = 2, \ \ s_{1,2} = -0.5 \mp j1.936 \quad \text{Underdamped}$$

$$v(t) = v_{ss} + (A_1 \cos 1.936t + A_2 \sin 1.936t)e^{-0.5t}$$


v(t) V
40
35
30
35
20
15
10
5
0
Underdamped
Critically damped
Overdamped
0
1
2
3
4
5
6
7
8
t (s)

Step Response of a Parallel RLC Circuit



So we get

$$\frac{d^2 i}{d^2 t} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

Step Response of a Parallel RLC Circuit

$$\frac{d^2i}{d^2t}+\frac{1}{RC}\frac{di}{dt}+\frac{i}{LC}=\frac{I_s}{LC}$$

- As in the series RLC case, the response is a combination of transient and steady state responses:

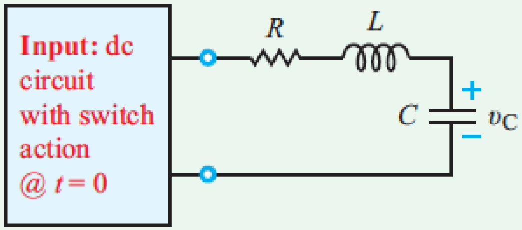
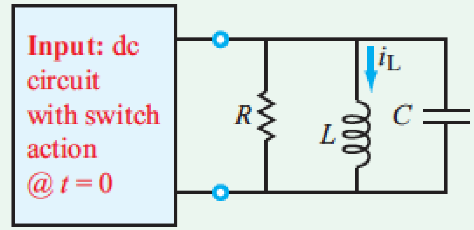
$$i(t)=I_s+A_1e^{\tau_1 t}+A_2e^{\tau_2 t} \quad \text{(Overdamped)}$$

$$i(t)=I_s+\left(A_1+A_2t\right)e^{-\alpha t} \quad \text{(Critally Damped)}$$

$$i(t)=I_s+\left(A_1\cos\omega_d t+A_2\sin\omega_d t\right)e^{-\alpha t} \quad \text{(Underdamped)}$$

Here the variables A_1 and A_2 are obtained from the initial conditions, *i(0)* and *di(0)/dt*.

: Step response of RLC circuits for $t \geq 0$.

Series RLC	Parallel RLC
	
Total Response	**Total Response**
Overdamped $(\alpha > \omega_0)$ $v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + v_C(\infty)$ $A_1 = \dfrac{\frac{1}{C}\, i_C(0) - s_2[v_C(0) - v_C(\infty)]}{s_1 - s_2}$ $A_2 = \left[\dfrac{\frac{1}{C}\, i_C(0) - s_1[v_C(0) - v_C(\infty)]}{s_2 - s_1}\right]$	**Overdamped** $(\alpha > \omega_0)$ $i_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + i_L(\infty)$ $A_1 = \dfrac{\frac{1}{L}\, v_L(0) - s_2[i_L(0) - i_L(\infty)]}{s_1 - s_2}$ $A_2 = \left[\dfrac{\frac{1}{L}\, v_L(0) - s_1[i_L(0) - i_L(\infty)]}{s_2 - s_1}\right]$
Critically Damped $(\alpha = \omega_0)$ $v_C(t) = (B_1 + B_2 t)e^{-\alpha t} + v_C(\infty)$ $B_1 = v_C(0) - v_C(\infty)$ $B_2 = \frac{1}{C}\, i_C(0) + \alpha[v_C(0) - v_C(\infty)]$	**Critically Damped** $(\alpha = \omega_0)$ $i_L(t) = (B_1 + B_2 t)e^{-\alpha t} + i_L(\infty)$ $B_1 = i_L(0) - i_L(\infty)$ $B_2 = \frac{1}{L}\, v_L(0) + \alpha[i_L(0) - i_L(\infty)]$
Underdamped $(\alpha < \omega_0)$ $v_C(t) = e^{-\alpha t}(D_1 \cos\omega_d t + D_2 \sin\omega_d t) + v_C(\infty)$ $D_1 = v_C(0) - v_C(\infty)$ $D_2 = \dfrac{\frac{1}{C}\, i_C(0) + \alpha[v_C(0) - v_C(\infty)]}{\omega_d}$	**Underdamped** $(\alpha < \omega_0)$ $i_L(t) = e^{-\alpha t}(D_1 \cos\omega_d t + D_2 \sin\omega_d t) + i_L(\infty)$ $D_1 = i_L(0) - i_L(\infty)$ $D_2 = \dfrac{\frac{1}{L}\, v_L(0) + \alpha[i_L(0) - i_L(\infty)]}{\omega_d}$

Auxiliary Relations

$$\alpha = \begin{cases} \dfrac{R}{2L} & \text{Series RLC} \\ \dfrac{1}{2RC} & \text{Parallel RLC} \end{cases} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

***8.21** Calculate $v(t)$ for $t > 0$ in the circuit of Fig. 8.75.

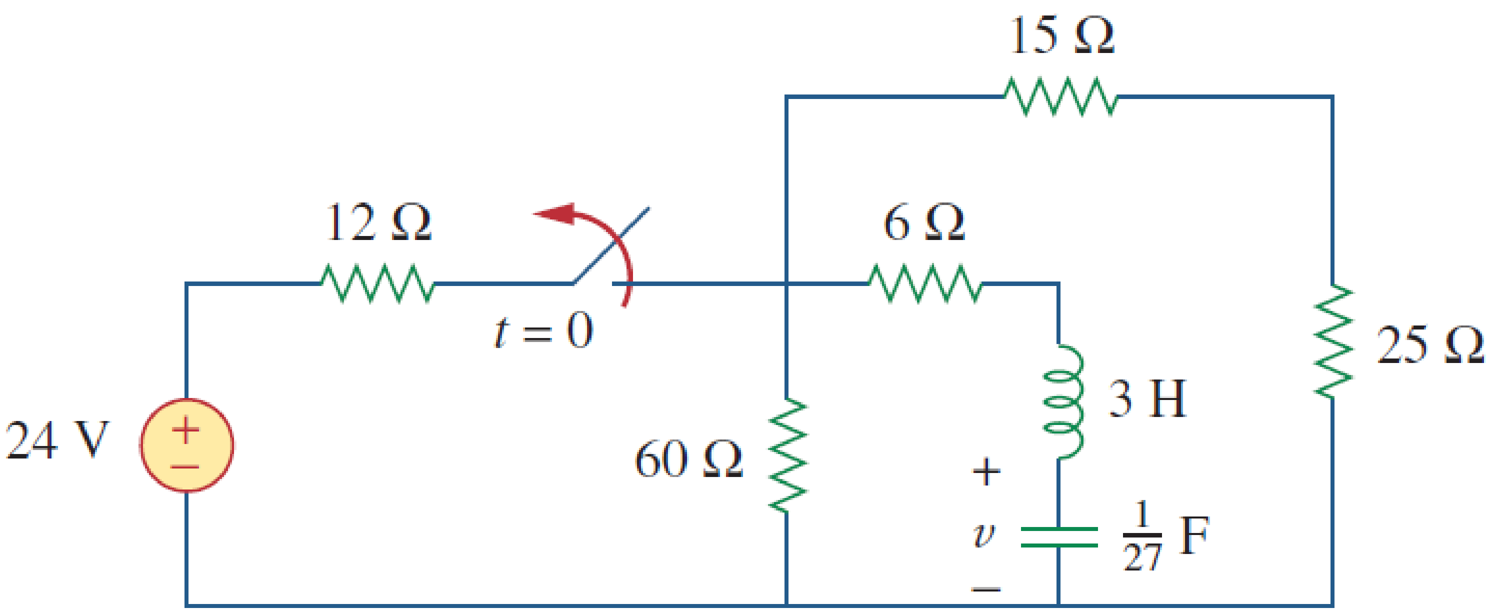


Figure 8.75
For Prob. 8.21.

8.49 Determine $i(t)$ for $t > 0$ in the circuit of Fig. 8.96.

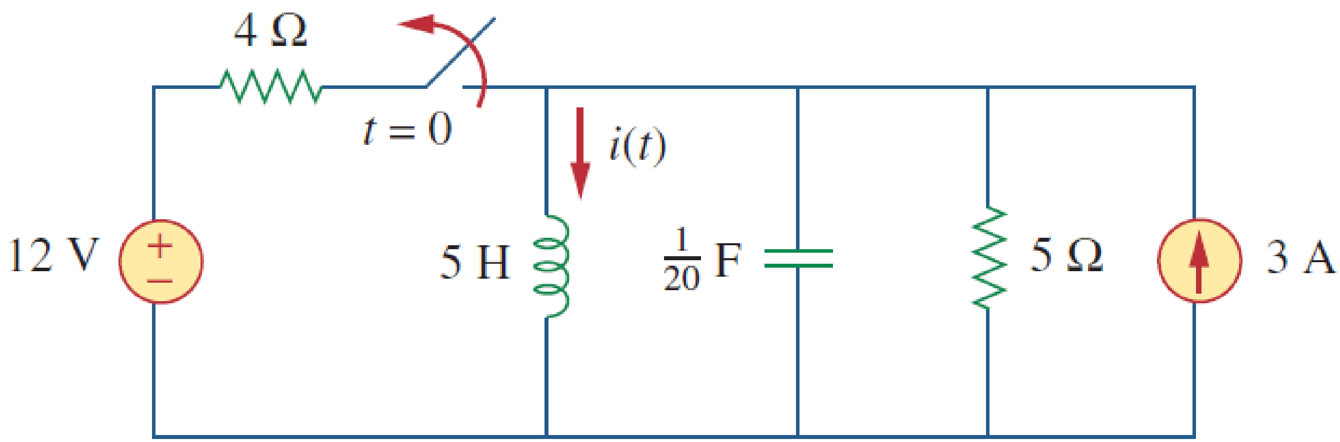


Figure 8.96
For Prob. 8.49.

General Second-Order Circuits

- The principles of the approach to solving the series and parallel forms of RLC circuits can be applied to general second order circuits:
  1. First determine the initial conditions, *x(0)* and *dx(0)/dt*.
  2. Find the equation
  3. The total response = transient response + steady-state response.

$$x(t) = x_t(t) + x_{ss}(t)$$

General solution for second-order circuits for $t \geq 0$.

$x(t)$ = unknown variable (voltage or current)

Differential equation: $x'' + ax' + bx = c$

Initial conditions: $x(0)$ and $x'(0)$

Final condition: $x(\infty) = \dfrac{c}{b}$

$$\alpha = \frac{a}{2} \qquad \omega_0 = \sqrt{b}$$

Overdamped Response $\alpha > \omega_0$

$$x(t) = [A_1 e^{s_1 t} + A_2 e^{s_2 t} + x(\infty)]$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$A_1 = \frac{x'(0) - s_2[x(0) - x(\infty)]}{s_1 - s_2} \qquad A_2 = -\left[\frac{x'(0) - s_1[x(0) - x(\infty)]}{s_1 - s_2}\right]$$

Critically Damped $\alpha = \omega_0$

$$x(t) = [(B_1 + B_2 t)e^{-\alpha t} + x(\infty)]$$

$$B_1 = x(0) - x(\infty) \qquad B_2 = x'(0) + \alpha[x(0) - x(\infty)]$$

Underdamped $\alpha < \omega_0$

$$x(t) = [D_1 \cos \omega_d t + D_2 \sin \omega_d t + x(\infty)]e^{-\alpha t}$$

$$D_1 = x(0) - x(\infty) \qquad D_2 = \frac{x'(0) + \alpha[x(0) - x(\infty)]}{\omega_d}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$x(t)$ = unknown variable (voltage or current)

Differential equation: $x'' + ax' + bx = c$

Initial conditions: $x(0)$ and $x'(0)$

Final condition: $x(\infty) = \dfrac{c}{b}$

$$\alpha = \frac{a}{2} \qquad \omega_0 = \sqrt{b}$$

Overdamped Response $\alpha > \omega_0$

$$x(t) = [A_1 e^{s_1 t} + A_2 e^{s_2 t} + x(\infty)]\, u(t)$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$A_1 = \frac{x'(0) - s_2[x(0) - x(\infty)]}{s_1 - s_2} \qquad A_2 = -\left[\frac{x'(0) - s_1[x(0) - x(\infty)]}{s_1 - s_2}\right]$$

Critically Damped $\alpha = \omega_0$

$$x(t) = [(B_1 + B_2 t)e^{-\alpha t} + x(\infty)]\, u(t)$$

$$B_1 = x(0) - x(\infty) \qquad B_2 = x'(0) + \alpha[x(0) - x(\infty)]$$

Underdamped $\alpha < \omega_0$

$$x(t) = [D_1 \cos\omega_d t + D_2 \sin\omega_d t + x(\infty)]e^{-\alpha t}\, u(t)$$

$$D_1 = x(0) - x(\infty) \qquad D_2 = \frac{x'(0) + \alpha[x(0) - x(\infty)]}{\omega_d}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

[Important]

1.This table works well when c is a constant, as $x(\infty)$ is actually a particular solution of the equation.

2. While for c is a function of time (t), such as c= 5t; c= t^2+3; you should also be able to solve the equation (Requirement of the course).

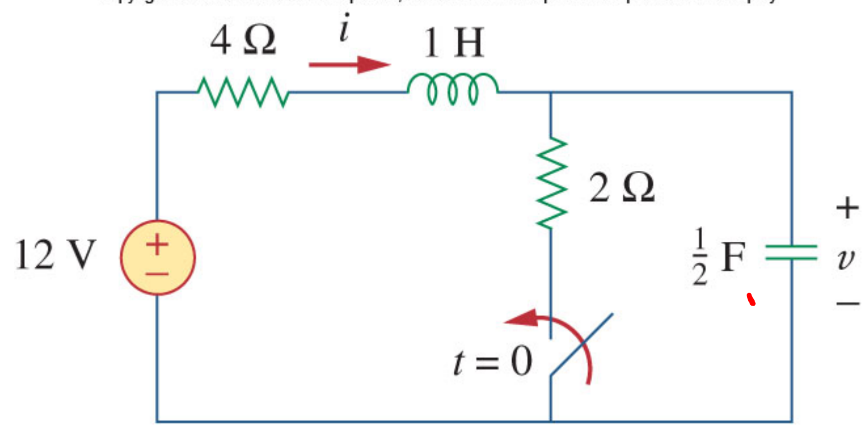
General RLC Circuits

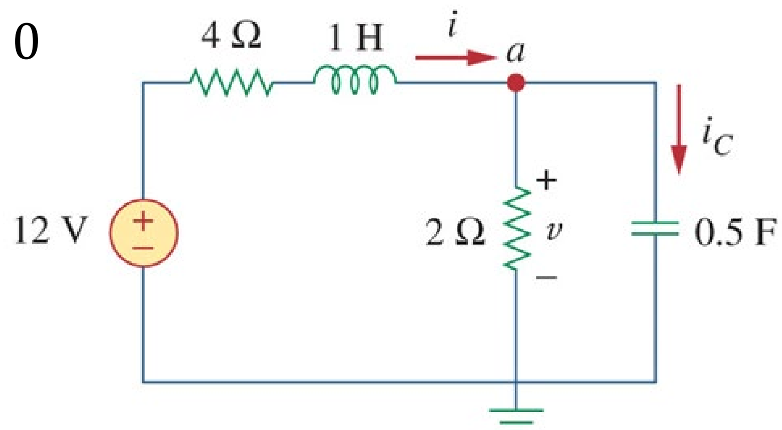
- Find the complete response v for $t > 0$ in the circuit.

1. Initial conditions

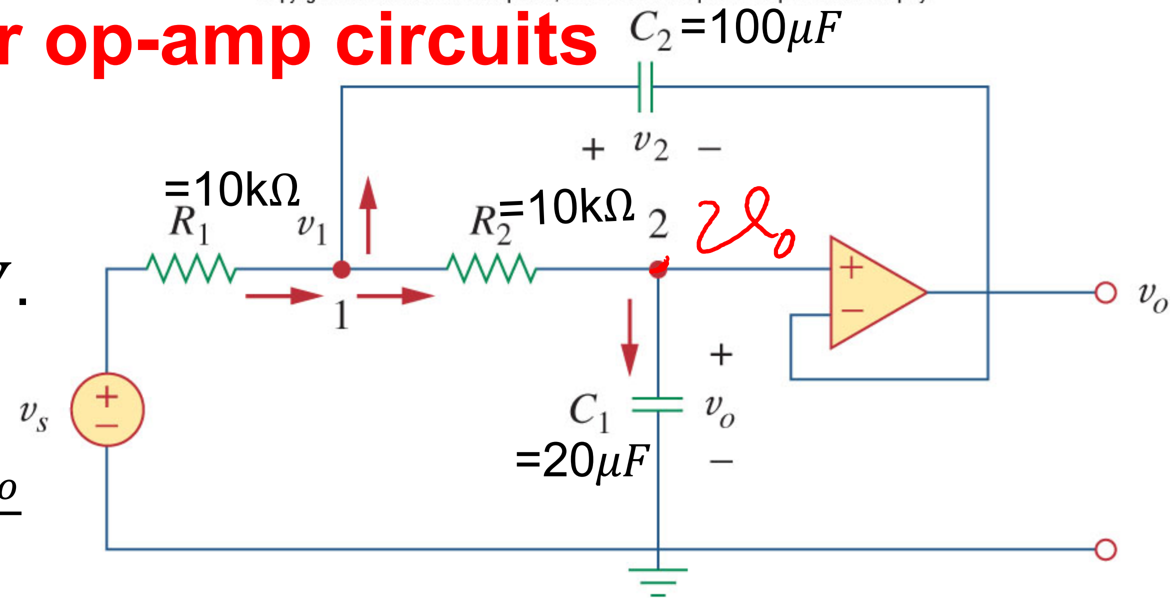
$$v(0^+) = v(0^-) = 12V,\ i(0^+) = i(0^-) = 0$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{-12/2}{0.5} = -12\ V/s$$





Example of 2nd-order op-amp circuits



- Find v_o for $t > 0$ when $v_s = 10u(t)mV$.

KCL at node 1:

$$\frac{v_s - v_1}{R_1} = C_2\frac{dv_2}{dt} + \frac{v_1 - v_o}{R_2}$$

KCL at node 2:

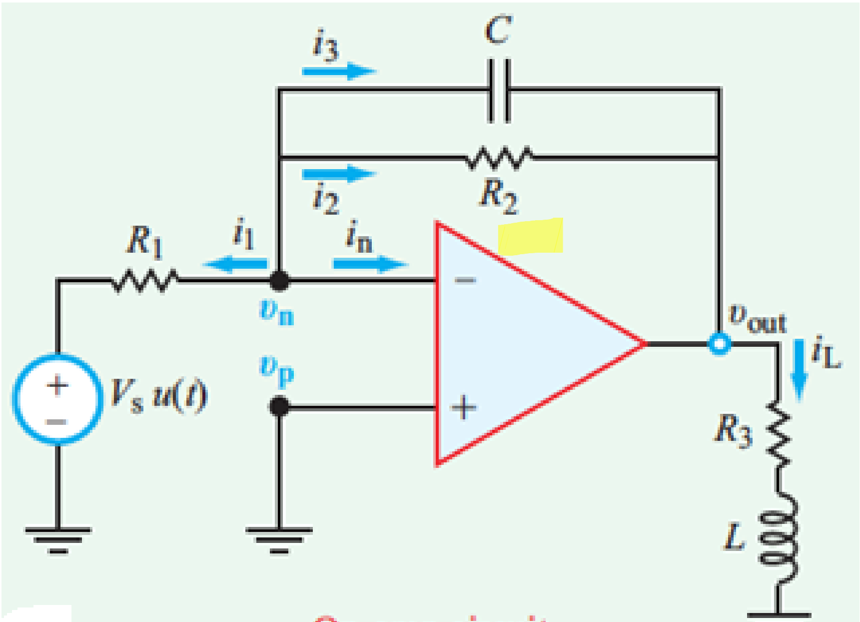
$$C_1\frac{dv_o}{dt} = \frac{v_1 - v_o}{R_2}$$

$$v_1 - v_2 = v_o$$

$$\Rightarrow \quad \frac{d^2v_o}{dt^2} + \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)\frac{dv_o}{dt} + \frac{v_o}{R_1R_2C_1C_2} = \frac{v_s}{R_1R_2C_1C_2}$$

Initial conditions: $v_o(0^+) = 0,\ C_1\frac{dv_o(0^+)}{dt} = \frac{v_1(0^+) - v_o(0^+)}{R_2} = \frac{v_2(0^+)}{R_2} = 0$

Example



Op-amp circuit

$$\frac{R_3}{R_2} i_L + \left(\frac{L}{R_2} + R_3 C\right) \frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = -\frac{V_s}{R_1}$$

$$i_L(\infty) = \frac{v_{out}(\infty)}{R_3} = -\frac{R_2 V_s}{R_1 R_3} = -1 \text{ mA}$$

$$i_L(0) = i_L(0^-) = 0 \quad i_L'(0) = \frac{1}{L} v_L(0) = 0.$$

2nd-order Oscillation Circuit