Abstract

A random walk $\{X_n\}_{n\geq 1}\}$ on the line, having steps scaled by n^{-a} $(0 < a < 1)$ of signs controlled towards 0 does converge to 0. The re-scaled walk $Z_n = n^a X_n$ converges in distribution to a random variable Z. Unlike the case of walks with identically distributed steps, the distribution of Z is not normal, and depends on the distribution of the steps.

1 A random walk problem

Vincent Granville is a data scientist and blogger. In the blog "Interesting probability problem for serious geeks" he poses the following problem:

"Let's start with $X(1) = 0$, and define $X(k)$ recursively as folows, for $k > 1$:

$$X(k) = \begin{cases} X(k-1) + \frac{U(k)}{k^a} \text{ if } X(k-1) < 0 \\ \\ X(k-1) - \frac{U(k)}{k^a} \text{ if } X(k-1) \geq 0 \end{cases}$$

and let's define $U(k), Z(k)$ and Z as follows:

$$Z(k) = k^a \ X(k)$$

$$Z = \lim_{k\to\infty} Z(k)$$

$$U(k) = V(k)^b$$

where the $V(k)$'s are deviates from *independent* uniform variables on $[0, 1]$ [...].

Prove that if $0 < a < 1$, then $X(k)$ converges to 0 as k increases. Under the same condition, prove that the limiting distribution Z

- always exists,
- always takes values between -1 and $+1$, with $\min(Z) = -1$, and $\max(Z) = +1$,
- is symmetric, with mean and median equal to 0
- and does not depend on a, but only on b."

In the blog, Grandville discusses special cases. In particular, the case with:

$$\{U(k)\}_k \text{ are iid with density } f_U \tag{1}$$

$$essential\ support\{f_U\} \subseteq [0,1] \tag{2}$$

$$\text{there is } \epsilon > 0 \text{ such that } [0,\epsilon] \subset essential\ support\{f_U\} \tag{3}$$

From formal considerations, Grandville conjectures that Z has density f_Z, where f_Z solves the Fredholm integral equation:

$$f_Z(z) = \int_0^1 \{f_U(x+z) + f_U(x-z)\}\ f_Z(x)\ dx. \tag{4}$$

Note that $Z(1) = 0$, $Z(2) = -U(2)$, and so $Z(k)$ has density with respect to Lebesgue measure for $k \geq 2$, and the density f_k of $Z(k)$ satisfies:

$$f_2(z) = f_U(-z)$$

and

$$f_k(z) = \int_{-\infty}^{+\infty} f_{k-1}(x)\ f_U\left(\left(x\left(\frac{k}{k-1}\right)^a - z\right)\ sign(x)\right)\ dx$$

for $k > 2$.

Using simulations, Grandville found the shape of the limit density for special cases of f_U. He proposed an explicit solution to the integral equation (4), which he verified. He then proposed the form of a solution for the general case. As it turns, Grandville ansatz is correct.

Lemma. *Let $F_U(z) = \int_{-\infty}^z f_U(x)dx$ be the cumulative distribution function of f_U. Then*

$$f_Z(z) = \frac{1}{2} \cdot \frac{1 - F_U(|z|)}{1 - \int_0^1 F_U(x)dx} \tag{5}$$

solves the integral equation (4).

Proof. By linearity of (4), it is enough to show that

$$1 - F_U(|z|) = \int_0^1 \big(f_U(x+z) + f_U(x-z)\big)\ \big(1 - F_U(x)\big)\ dx.$$

Since both sides of this equation involve even functions of z, it is enough to show the equality for $z \geq 0$.

For $z \geq 0$,

$$\begin{aligned}
&\int_0^1 \big(f_U(x+z) + f_U(x-z)\big)\ \big(1 - F_U(x)\big)\ dx \\
&= \int_0^1 f_U(x-z)\big(1 - F_U(x)\big)\ dx + \int_0^1 f_U(x+z)\big(1 - F_U(x)\big)\ dx \\
&= \int_0^1 f_U(x-z)\big(1 - F_U(x)\big)\ dx - \int_0^1 \big(1 - F_U(x+z)\big)'\big(1 - F_U(x)\big)\ dx \\
&= \int_0^1 f_U(x-z)\big(1 - F_U(x)\big)\ dx \\
&\quad - \big(1 - F_U(x+z)\big)\big(1 - F_U(x)\big)\Big|_{x=0}^{x=1} - \int_0^1 \big(1 - F_U(x+z)\big)f_U(x)\ dx
\end{aligned}$$

But $1 - F_U(1) = 0$, and $1 - F_U(0) = 1$, so,

$$-\big(1 - F_U(x+z)\big)\big(1 - F_U(x)\big)\Big|_{x=0}^{x=1} = 1 - F_U(z).$$

Also, since $f_U(x) = 0$ for $x \notin [0,1]$, we have:

$$\begin{aligned}
&\int_0^1 \big(1 - F_U(x+z)\big)f_U(x)\ dx = \int_{-\infty}^{\infty} \big(1 - F_U(x+z)\big)f_U(x)\ dx \\
&= \int_{-\infty}^{\infty} \big(1 - F_U(t)\big)f_U(t-z)\ dt = \int_0^1 \big(1 - F_U(t)\big)f_U(t-z)\ dt
\end{aligned}$$

The last equality holds on account of the fact that if $t > 1$, then $1 - F_U(t) = 0$, and if $t < 0$, then $f_U(t-z) = 0$.

Therefore,

$$\begin{aligned}
&\int_0^1 \big(f_U(x+z) + f_U(x-z)\big)\ \big(1 - F_U(x)\big)\ dx \\
&= \int_0^1 f_U(x-z)\big(1 - F_U(x)\big)\ dx \\
&\quad + \big(1 - F_U(z)\big) - \int_0^1 \big(1 - F_U(t)\big)f_U(t-z)\ dt \\
&= 1 - F_U(z)
\end{aligned}$$

$\square$

The next result is a justification of why our later work cannot be restricted to working on a compact interval. Let W be one of the kernels transforming f_{k-1} into f_k, i.e. a kernel of the form:

$$W(x,z) = f_U((x\rho - z)\ \ sign(x))$$

with $\rho \geq 1$. We will write W_k for the kernel with $\rho = (\frac{k}{k-1})^a$, which is used to transform f_{k-1} into f_k.

Lemma. *Let $M > 0, \rho \geq 1$ be such that $M\rho \geq 1$, and assume essential support$\{f_U\} = [0,1]$.*

For $g \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, let

$$\mathcal{W}(g)(z) = \int_{-\infty}^{+\infty} g(x) f_U((x\rho - z)\ \ sign(x))\ \ dx.$$

Then if g has compact support, so does $\mathcal{W}(g)$, more precisely:

a) If supp$(g) \subseteq [-M,M]$ then supp$(\mathcal{W}(g)) \subseteq [-M\rho, M\rho]$

b) The bounds on the support are tight in the sense that:

$$(-M\rho, M\rho) \subseteq \bigcup_{g\in L^1, supp(g)\subseteq[-M,M]} supp(\mathcal{W}(g)) \subseteq [-M\rho, M\rho]$$

Proof. Suppose $W(x,z) \neq 0$ and $|x| \leq M$ for some x, z.

If $0 \leq x \leq M$, then, by definition of W and assumption about the support of f_U we get $0 \leq x\rho - z \leq 1$, so

$$-M\rho \leq -1 \leq x\rho - 1 \leq z \leq x\rho \leq M\rho$$

so $|z| \leq M\rho$.

If $-M \leq x < 0$, then a similar argument yields: $0 \leq |x|\rho + z \leq 1$, so

$$-M\rho \leq -|x|\rho \leq z \leq 1 - |x|\rho \leq 1 \leq M\rho$$

and again, $|z| \leq M\rho$. Therefore, if *supp*$(g) \subseteq [-M,M]$ then

$$\mathcal{W}(g)(z) = \int_{-M}^{M} g(x) W(x,z)\ dx = 0 \text{ for } |z| > M\rho$$

i.e. *supp*$(\mathcal{W}(g)) \subseteq [-M\rho, M\rho]$.

Let $\mathcal{A} = \bigcup_{g\in L^1, supp(g)\subseteq[-M,M]} supp(\mathcal{W}(g))$. Part a) shows $\mathcal{A} \subseteq [-M\rho, M\rho]$. To show the other inclusion, take $g = \delta_r$ (the distribution concentrated at $x = r$), with $r \in (0, M)$. We have

$$\mathcal{W}(g)(z) = f_U(r\rho - z),$$

whose support is $[r\rho-1, r\rho]$. By approximating δ_r under the integral with g in L^1, we conclude $(r\rho-1, r\rho) \subseteq \mathcal{A}$. Since this holds for all $r \in (0, M)$, $(-1, M\rho) = \bigcup_{r\in(0,M)}(r\rho-1, r\rho) \subseteq \mathcal{A}$.

Next, taking $g = \delta_{-r}$, conclude $(-r\rho, 1-r\rho) \subseteq \mathcal{A}$, and $(-M\rho, 1) = \bigcup_{r\in[0,M]}(-r\rho, 1-r\rho) \subseteq \mathcal{A}$. This shows b). $\square$

We include the above Lemma to underscore two points that complicates a naïve approach to the problem:

i. There is no bounded interval $[-M, M]$ with $M \geq 1$ such that the integral transformations $g \mapsto \mathcal{W}_k(g)(z) = \int_{-M}^{M} g(x)W_k(x, z)\ dx$, $(g \in L^1(\mathbb{R})), k = 2, 3, \ldots$ map the space of L^1 functions supported in $[-M, M]$ to itself. Thus, we cannot fix a bounded interval that contains a priory the support of the family $\{f_k\}_k \geq 1$, and use compactness of integral transformations.

ii) For the limiting case (with $k = \infty$) with $\rho = 1$, we get that $\mathcal{W}$ maps $L^1[-M, M]$ (with $M \geq 1$) to itself. We will come back to this point when analyzing the distribution of Z.

There are some unrefereed resources that seem to suggest that because the operator that maps $g \in L^1(\mathbb{R})$ to

$$\mathcal{W}(g)(z) = \int_{-\infty}^{+\infty} g(x) f_U((x\rho - z)\ \ sign(x))\ dx$$

has an integrand kernel, it must be compact. This is not the case, since the spectrum of $\mathcal{W}$ in $L^1(\mathbb{R})$ is not discrete. We will show this in later sections.

We will show:

Theorem. *With X, Z, U, a, f_k as above, under assumptions (3) about the support of f_U,*

a) $\lim_{k\to\infty} X(k) = 0$ *a.s.*

b) $\overline{\lim}_{k\to\infty}|Z(k)| \leq 1$ *a.s.*

c) the solution f_Z to (4) given in (5) is unique, i.e.: there is one and only one pdf solution of (4)

d) $f_k \to f_Z$ *in* $L^1(\mathbb{R})$.

e) if $f_U \in L^p([0, 1])$ for some $p > 1$ then there is $k_0 = k_0(p)$ such that $f_k \in L^\infty(\mathbb{R})$ for $k \geq k_0$ and $f_k \to f_Z$ in $L^\infty(\mathbb{R})$

f) $Z(k) \to_d Z$ *(convergence in distribution).*

2 The path $\{X(k)\}_{k\geq 1}$

2.1 Stopping times

Let's consider some stopping times associated to the random path $\{X(k)\}_k \geq 1$. By definition,

$$X(2) = -U(1) < 0$$

so, to get to $X(3)$ we move in the direction towards 0 by adding a positeve term:

$$X(3) = X(2) + \frac{U(3)}{3^a}$$

and keep on adding terms until the path crosses 0 and changes sign, i.e.:

$$X(k) = X(2) + \sum_{j=3}^{k} \frac{U(j)}{j^a}$$

as long as $X(k) < 0$. We will show that eventually, $X(k) \geq 0$, at which time instead of adding, we force the path towards 0 by subtracting terms until the sign changes again, and so on. We define:

$$T_1 = T(1) = 2 \tag{6}$$

and for $k > 1$, as long as $T(k-1)$ is finite,

$$T_k = T(k) = \inf \left\{ j > T_{k-1} \;\middle|\; \sum_{s=1+T_{k-1}}^{j} \frac{U(s)}{s^a} > |X(T_{k-1})| \right\}. \tag{7}$$

Equivalently, as long as these stopping times are finite,

$$\begin{aligned}
T_2 = T(2) = &\text{ first step } j > T_1 \text{ such that } X(j) > 0 \\
T_3 = T(3) = &\text{ first step } j > T_2 \text{ such that } X(j) < 0 \\
&\dots \\
T_k = T(k) = &\text{ first step } j > T_{k-1} \text{ such that } (-1)^k X(j) > 0
\end{aligned}$$

and we have

$$X(j) \;=\; X(T_k) - (-1)^k \sum_{s=1+T_k}^{j} \frac{U(s)}{s^a} \text{ for } T_k < j \leq T_{k+1} \tag{8}$$

and

$$sign(X(j)) = (-1)^{k-1} \text{ for } T_{k-1} \leq j < T_k \;(k > 2). \tag{9}$$

In particular,

$$X(T_k) \;=\; X(T_{k-1}) + (-1)^k \sum_{s=1+T_{k-1}}^{T_k} \frac{U(s)}{s^a} \;=\; (-1)^k \left(\sum_{s=1+T_{k-1}}^{T_k} \frac{U(s)}{s^a} - |X(T_{k-1})| \right)$$

and so

$$\begin{aligned} |X(T_k)| \;&=\; \sum_{s=1+T_{k-1}}^{T_k} \frac{U(s)}{s^a} - |X(T_{k-1})| \;=\; \left(\sum_{s=1+T_{k-1}}^{T_k - 1} \frac{U(s)}{s^a} - |X(T_{k-1})| \right) + \frac{U(T_k)}{{T_k}^a} \\ &\leq \frac{U(T_k)}{{T_k}^a} \leq \frac{1}{{T_k}^a}. \end{aligned} \tag{10}$$

Since $T_1 = 2 < T_2 < ..$ (and they are integers),

$$T_k \geq k + 1. \tag{11}$$

Let

$$\mathcal{F}_k = \sigma\text{-algebra generated by } U(1), ..., U(k). \tag{12}$$

These σ-algebras form a filtration $\mathcal{F}_*$ of the underlying probability space in which $U(1), U(2), \ldots$ are defined.

Lemma 2.1. *For $k \geq 1$, T_k is a stopping time of the filtration $\mathcal{F}_*$.*

Proof. T_k is the step when the k^{th} sign change occurs, so the event $(T_k \leq s)$ is true iff there are k or more sign changes in the sequence $X(1), \ldots, X(s)$, which depend on $U(1), ..., U(s)$, i.e.: $(T_k \leq s) \in \mathcal{F}_s$. □

Remarks The process $U(1), U(2), \ldots$ is (trivially) a stationary Markov process. The strong Markov property applied to this case implies that if T is a stopping time of $\mathcal{F}_*$, finite a.s, then

i. $U(T+1), U(T+2), \ldots$ are independent i.i.d distributed with the same distribution as $U(1), U(2), \ldots$

ii. $U(T+1), U(T+2), \ldots$ are independent of $\mathcal{F}_T$, the filtration stopped at T.

2.2 Asymptotic properties of the stopping times

Lemma 2.2. *Let $U(k), X(k), Z(k)$ as in the first section, $T(k)$ as in (7) , f_U satisfies assumptions (1) to (3). Let $\mu = E(U(1)) = \int_0^1 x\ f_U(x)\ dx$. Then,* *i.* *For every* $j \geq 1$, T_j *is finite a.s.*

ii. *For* $j, N \geq 1$, *with* $\Delta T_{j+1} = T_{j+1} - T_j$,

$$E\left(\Delta T_{j+1} > N | \mathcal{F}_{T_j}\right) \ \leq \ exp\left(\mu \frac{(1+T_j)^a}{T_j^a} - \frac{\mu^2 (1+T_j)^a}{2} \sum_{s=1}^{N} \frac{1}{(s+T_j)^a}\right)$$

iii. *There is a constant* $\gamma > 0$ *(*$\gamma = 3$ *works) such that, regardless of the density* f_U, $\overline{\lim}_{j\to\infty} \mu^2 \frac{(T_{j+1}-T_j)}{T_j(\frac{\ln j}{j})} \leq \gamma$ *almost surely.*

iv. $\lim_{j\to\infty} \frac{T_{j+1}}{T_j} = 1$ *a.s.*

Proof. We proceed inductively. If $k = 1$, then $T_1 = 2$, and the statement holds trivially. Suppose T_j is finite a.s.. Let P_j be conditional probability given $\mathcal{F}_{T_j}$. We have

$$\Delta T_{j+1} = T_{j+1} - T_j > k \text{ iff } \sum_{s=1}^{k} \frac{U(s+T_j)}{(s+T_j)^a} \leq |X(T_j)|,$$

therefore, taking conditional probabilities:

$$P_j\left(\Delta T_{j+1} > k\right) \leq P_j\left(\sum_{s=1}^{k} \frac{U(s+T_j)}{(s+T_j)^a} \leq |X(T_j)|\right).$$

Since T_j and $X(T_j)$ are $\mathcal{F}_{T_j}$ measurable, and $U(T_j+1), \ldots$ are iid conditional on $\mathcal{F}_{T_j}$, with the same distribution as $U(1), \ldots$, if we apply Markov's inequality we get for any $\lambda > 0$:

$$\begin{aligned} P_j\left(\sum_{s=1}^{k} \frac{U(s+T_j)}{(s+T_j)^a} \leq |X(T_j)|\right) &\leq e^{\lambda |X(T_j)|} E_j\left\{e^{-\lambda(\sum_{s=1}^{k} \frac{U(s+T_j)}{(s+T_j)^a})}\right\} \\ &= e^{\lambda |X(T_j)|} \prod_{s=1}^{k} E_j\left\{e^{-\lambda \frac{U(s+T_j)}{(s+T_j)^a}}\right\}, \end{aligned}$$

where E_j is the conditional expectation given $\mathcal{F}_{T_j}$.

Then we have

$$E_j\left\{e^{-\lambda \frac{U(s+T_j)}{(s+T_j)^a}}\right\} = e^{-\lambda \frac{\mu}{(s+T_j)^a}} E_j\left\{e^{-\lambda \frac{U(s+T_j)-\mu}{(s+T_j)^a})}\right\}.$$

Applying Hoeffding's Lemma to $-\lambda \frac{U(s+T_j)-\mu}{(s+T_j)^a}$, which has mean 0 and is bounded in absolute value by $\frac{\lambda}{(s+T_j)^a}$, we get

$$E_j\left\{e^{-\lambda \frac{U(s+T_j)-\mu}{(s+T_j)^a}}\right\} \leq exp\left(\frac{1}{2} \frac{\lambda^2}{(s+T_j)^{2a}}\right).$$

Therefore:

$$
\begin{aligned}
P_j\left(\Delta T_{j+1}>k\right) &\leq exp\left(\lambda|X(T_j)|-\lambda\mu\sum_{s=1}^{k}\frac{1}{(s+T_j)^a}+\frac{\lambda^2}{2}\sum_{s=1}^{k}\frac{1}{(s+T_j)^{2a}}\right)\\
&\leq exp\left(\lambda\frac{1}{T_j^a}-\lambda\mu\sum_{s=1}^{k}\frac{1}{(s+T_j)^a}+\frac{\lambda^2}{2}\sum_{s=1}^{k}\frac{1}{(s+T_j)^{2a}}\right) \qquad (13)
\end{aligned}
$$

First, let's chose λ in (13) so that

$$
\frac{\lambda^2}{2}\sum_{s=1}^{k}\frac{1}{(s+T_j)^{2a}}\leq\frac{\lambda\mu}{2}\sum_{s=1}^{k}\frac{1}{(s+T_j)^a}
$$

For example,

$$
\lambda=\mu(1+T_j)^a. \qquad (14)
$$

With this choice, each term in the sums above satisfies the inequality

$$
\lambda\frac{1}{(s+T_j)^{2a}}\leq\mu\frac{1}{(s+T_j)^a},
$$

so we get

$$
P_j\left(\Delta T_{j+1}>k\right) \;\leq\; exp\left(\mu\frac{(1+T_j)^a}{T_j^a}-\frac{\mu^2(1+T_j)^a}{2}\sum_{s=1}^{k}\frac{1}{(s+T_j)^a}\right). \qquad (15)
$$

This shows part ii).

Taking limits when $k\to\infty$,

$$
P_j(\Delta T_{j+1}=+\infty)=0, \qquad (16)
$$

and taking unconditional probabilities (averaging over T_j):

$$
P(\Delta T_{j+1}=+\infty)=0, \qquad (17)
$$

so T_{j+1} is finite a.s. This completes the induction step to show part i).

Going back to (15), and taking k of the form $k_j = R\,(1+T_j)\frac{\ln(j)}{j}$, with $R>0$ we have:

$$
\begin{aligned}
\sum_{s=1}^{k_j}\frac{1}{(s+T_j)^a}\geq\int_1^{k_j+1}\frac{1}{(x+T_j)^a}\,dx&=\frac{(k_j+1+T_j)^{1-a}-(1+T_j)^{1-a}}{1-a})\\
&= (1+T_j)^{1-a}\frac{(1+R\frac{\ln(j)}{j})^{1-a}-1}{1-a}) \;=\; (1+T_j)^{1-a}(R\frac{\ln(j)}{j})(1+o(1))\\
&\geq (1+j)(1+T_j)^{-a}(R\frac{\ln(j)}{j})(1+o(1)) \;=\; (1+T_j)^{-a}R\ln(j)(1+o(1))
\end{aligned}
$$

and so

$$P_j\left(\Delta T_{j+1} > k_j\right) \;\leq\; exp\Big(\mu(1+\frac{1}{T_j})^a - \frac{\mu^2}{2}R\ln(j)(1+o(1))\Big)$$
$$\leq exp(\mu(\frac{3}{2})^a)\frac{1}{j^{\frac{R\mu^2(1+o(1))}{2}}}.$$

Taking unconditional probabilities:

$$P\left(\Delta T_{j+1} > k_j\right) \;\leq\; exp(\mu(\frac{3}{2})^a)\frac{1}{j^{\frac{R\mu^2(1+o(1))}{2}}}.$$

Adding these inequalities:

$$\sum_{j\geq 1} P\left(\Delta T_{j+1} > k_j\right) \;\leq\; \sum_{j\geq 1} exp(\mu(\frac{3}{2})^a)\frac{1}{j^{\frac{R\mu^2(1+o(1))}{2}}} < \infty \tag{18}$$

if R is sufficiently large. By Borel-Cantelli,

$$P\left(\Delta T_{j+1} > k_j \text{ infinitely often }\right) = 0, \tag{19}$$

i.e.: for each particular realization of the process $\{X(k)\}_{k\geq 1}$, there is N_0 (depending on the particular realization) such that

$$\Delta T_{j+1} \leq R\;(1+T_j)\frac{\ln(j)}{j} \tag{20}$$

for $j \geq N_0$.

Going back to (18) with $R = \frac{2}{\mu^2}(1+\epsilon)$ we get:

$$\overline{\lim}_{j\to\infty}\; \mu^2\frac{(T_{j+1}-T_j)}{T_j(\frac{\ln j}{j})} \leq 3.$$

This shows iii).

From (20), for j large:

$$0 \leq \frac{\Delta T_{j+1}}{T_j} \leq R\;(1+\frac{1}{T_j})\frac{\ln(j)}{j}$$

so

$$\lim_{j\to\infty}\frac{\Delta T_{j+1}}{T_j} = 0.$$

This shows iv).

$\square$

2.3 Convergence of $\{X(k)\}_k$ and $\{Z(k)\}_k$

Lemma 2.3. *Let $U(k), X(k), Z(k)$ as in the first section, $T(k)$ as in (7) , f_U satisfies assumptions (1) to (3). Let $\mu = E(U(1)) = \int_0^1 x\ f_U(x)\ dx$. Then,*

i. $\lim_{j\to\infty} X(j) = 0$ *almost surely*

ii. $\overline{\lim}_{j\to\infty} |Z(j)| \leq 1$ *almost surely*

iii. There exists $M_0, \gamma_0, \ldots, \gamma_3$ that depend only on μ and a such that, if $j \geq \gamma_0$, $M \geq M_0$ then

$$P(|Z(j)| \geq M) \ \leq \ exp(\gamma_3 - \ M^{(1-a)/a}\ (1+j)^{1-a}\gamma_2).$$

Proof. Since $X(j) = \frac{Z(j)}{j^a}$, i) follows from ii). To show ii), let $j \geq 2$ and let

$$k = \text{ number of sign changes in the sequence } X(1), X(2), \ldots, X(j-1)$$

Since $X(1), X(2), \ldots$ changes sign at $T(1), T(2), \ldots$, we have

$$k = \min\{s \mid T_s \geq j\} - 1,$$

so $T_k < j \leq T_{k+1}$.

If $j < T_{k+1}$, then $sign(X(j)) = (-1)^k$ and, as in (8),

$$\begin{aligned} X(j) &= X(T_k) - (-1)^k \sum_{s=1+T_k}^{j} \frac{U(s)}{s^a} \\ &= (-1)^k \left(|X(T_k)| - \sum_{s=1+T_k}^{j} \frac{U(s)}{s^a} \right). \end{aligned}$$

and

$$|X(j)| \ = \ |X(T_k)| - \sum_{s=1+T_k}^{j} \frac{U(s)}{s^a} \qquad \leq \ |X(T_k)| \leq \max\{|X(T_k)|, |X(T_{k+1})|\}.$$

If $j = T_{k+1}$, then trivially $|X(j)| \leq \max\{|X(T_k)|, |X(T_{k+1})|\}$. Whether $j = T_{k+1}$, or $j < T_{k+1}$, we have $|X(j)| \leq \max\{|X(T_k)|, |X(T_{k+1})|\}$. Now apply the bounds in (10),

$$|X(j)| \leq \max\{\frac{1}{T_k^a}, \frac{1}{T_{k+1}^a}\} = \frac{1}{T_k^a}$$

and

$$|Z(j)| = j^a |X(j)| \leq \frac{j^a}{T_k^a} \leq \frac{j^a}{T_{k+1}^a}\frac{T_{k+1}^a}{T_k^a} \leq \frac{T_{k+1}^a}{T_k^a} \tag{21}$$

From Lemma 2.2 we have $\lim_{j\to\infty}\frac{T_{j+1}}{T_j}=1$ a.s., so

$$\overline{\lim}_{j\to\infty}|Z(j)|\leq 1 a.s..$$

This shows ii).

From ii), we get that $\lim_{j\to\infty}P(|Z(j)|\geq M)=0$ for any $M>1$. We want to quantify the rate of convergence to justify the simulation based observation made by Grandville that $\{Z(j)\}_{j\geq 1}$ converges to a distribution supported on $[-1,1]$ at a high rate. The idea is to exploit (21), since

$$(|Z(j)|\geq M)\subset(\frac{T_{k+1}^a}{T_k^a}\geq M)=(\Delta T_{k+1}\geq T_k(M^{1/a}-1)).$$

Note that $k=k(j)$ is a random variable. It can be used to establish point-wise results, but since $T(k(j))$ is not a stopping time of the filtration (12), we need some extra care when using conditional expectations.

Let $M>1$. For the remainder of the proof, in order to lighten the notation, we will use γ_1,γ_2, etc. to denote positive constants that depend only on M,μ,a, but not always the same constant from line to line.

Let $j\geq\gamma_0>1$; with γ_0 to be determined. We have:

$$\begin{aligned}
(|Z(j)|\geq M) &\subseteq \bigcup_{k\geq 1}(|Z(j)|\geq M \text{ and } T_k\leq j<T_{k+1}) \\
&\subseteq \bigcup_{k\geq 1}(\frac{T_{k+1}^a}{T_k^a}\geq M \text{ and } T_k\leq j<T_{k+1}) \\
&= \bigcup_{k\geq 1}(\Delta T_{k+1}\geq T_k(M^{1/a}-1) \text{ and } T_k\leq j<T_{k+1}) \\
&\subseteq \bigcup_{k\geq 1}(\Delta T_{k+1}>\max\{1,T_k\ \Gamma_1,j-T_k\} \text{ and } T_k\leq j),
\end{aligned}$$

where $\Gamma_1=M^{1/a}-2$, and we take $\Gamma_1>2$ by requiring M to be large enough. Then

$$\begin{aligned}
P(|Z(j)|\geq M) &= \sum_{k\geq 1}P\Big(\Delta T_{k+1}>\max\{T_k\ \Gamma_1,j-T_k,1\} \text{ and } T_k\leq j\Big) \\
&= E\left(\sum_{k\geq 1}P_k\Big(\Delta T_{k+1}>\max\{T_k\ \Gamma_1,j-T_k,1\} \text{ and } T_k\leq j\Big)\right)
\end{aligned}$$

where P_k is conditional probability given $\mathcal{F}_{T_k}$. The event $(T_k\leq j)$ is $\mathcal{F}_{T_k}$-measurable, therefore

$$\begin{aligned}
&P_k\Big(\Delta T_{k+1}\geq\max\{T_k\ \Gamma_1,j-T_k,1\} \text{ and } T_k\leq j\Big) \\
&= P_k\Big(\Delta T_{k+1}\geq\max\{T_k\ \Gamma_1,j-T_k,1\}\Big)I(T_k\leq j)
\end{aligned}$$

From Lemma 2.2, relabeling T_j as T_k, and with $N = floor(\max\{T_k\ \Gamma_1, j - T_k, 1\})$:

$$P_k(\Delta T_{k+1} > N) \ \leq \ exp\left(\mu\frac{(1+T_k)^a}{T_k^a} - \frac{\mu^2(1+T_k)^a}{2}\sum_{s=1}^{N}\frac{1}{(s+T_k)^a}\right).$$

But

$$\sum_{s=1}^{N}\frac{1}{(s+T_k)^a} \quad \geq \int_1^{N+1}\frac{1}{(s+T_k)^a}\,ds \ = \ \frac{(N+T_k+1)^{1-a}-(1+T_k)^{1-a}}{1-a}.$$

If $\max\{T_k\ \Gamma_1, j - T_k, 1\} = 1$, then $T_k\ \Gamma_1 \leq 1$, and $j \leq 1 + T_k \leq 1 + 1/\Gamma_1$ which is impossible if $j >> 1$.

If $\max\{T_k\ \Gamma_1, j - T_k, 1\} = T_k\ \Gamma_1$, then $N = floor(T_k\Gamma_1)$, and

$$(N+T_k+1)^{1-a}-(1+T_k)^{1-a} \ \geq \ \gamma_2(\Gamma_1+1)^{1-a}(1+T_k)^{1-a}.$$

Also $j - T_k \leq T_k\ \Gamma_1$, so

$$(1+T_k) \geq T_k \geq \frac{1}{1+\Gamma_1}\ j \geq \gamma_3\ (1+j)$$

and

$$(N+T_k+1)^{1-a}-(1+T_k)^{1-a} \ \geq \ \gamma_2(\Gamma_1+1)^{1-a}(1+j)^{1-a}.$$

If $\max\{T_k\ \Gamma_1, j - T_k, 1\} = j - T_k \geq T_k\ \Gamma_1$, then $N = j - T_k$, and

$$(N+T_k+1)^{1-a}-(1+T_k)^{1-a} = (j+1)^{1-a}-(1+T_k)^{1-a} \geq (j+1)^{1-a}-(1+\frac{j}{1+\Gamma_1})^{1-a} \ \geq \ \gamma_3(\Gamma_1+1)^{1-a}(1+$$

In any case, regardless of the value of $\max\{T_k\ \Gamma_1, j - T_k, 1\}$

$$\sum_{s=1}^{N}\frac{1}{(s+T_k)^a} \ \geq \ \gamma_2(\Gamma_1+1)^{1-a}(1+j)^{1-a},$$

therefore,

$$P_k(\Delta T_{k+1} > N) \ \leq \ exp\left(\gamma_1 - \gamma_2(1+T_k)^a\ (\Gamma_1+1)^{1-a}(1+j)^{1-a}\right). \tag{22}$$

Since $k \leq 1 + T_k$, we have

$$P_k(\Delta T_{k+1} > N) \ \leq \ exp\left(\gamma_1 - \gamma_2 k^a\ (\Gamma_1+1)^{1-a}\ (1+j)^{1-a}\right), \tag{23}$$

and so (recall $\Gamma_1 + 1 = M^{1/a} - 1$):

$$\begin{aligned}
P(|Z(j)| \geq M) \;&\leq\; E\left(\sum_{k\geq 1} exp\left(\gamma_1 - \gamma_2 k^a \; M^{(1-a)/a} \; (1+j)^{1-a}\right) I(T_k \leq j)\right) \\
&\leq\; \sum_{k\geq 1} exp\left(\gamma_1 - \gamma_2 k^a \; M^{(1-a)/a} \; (1+j)^{1-a}\right) \\
&\leq\; exp(\gamma_3 - \; M^{(1-a)/a} \; (1+j)^{1-a}\gamma_2).
\end{aligned}$$

This shows iii).

□

Remark In the next section we will show that $Z(j)_{j\geq 1}$ converges in probability to a distribution supported on the interval $[-1, 1]$. We will show elsewhere that the process $Z(j)_{j\geq 1}$ visits the neighbourhoods of any point in $(-1, 1)$ infinitely often.

3 The spectrum of W

Let $U(k), X(k), Z(k)$ be as in the Introduction, f_U satisfies assumptions.

Definition Let $C([A, B])$ be the space of (real valued) continuous functions on $[A, B]$ with the usual *supremum* norm, and let $C_{00}([A, B]) = \{f \in C([A, B]) | f(A) = f(B) = 0\}$ be the subspace of $C([A, B])$ consisting of those functions that vanish at the boundary of $[A, B]$.

Let $W_k : C(\mathbb{R}) \longmapsto C(\mathbb{R})$ be defined as

$$W_k(g)(x) = \int_{-\infty}^{+\infty} g(s) \; f_U\left(\left(s\left(\frac{k}{k-1}\right)^a - x\right) \; sign(s)\right) \; ds. \tag{24}$$

Here $k = 3, 4, \ldots, \infty$.

We will write W for the limit case $k = \infty$. In this section we go over properties of the spectrum of W. Many of the statements apply to more general integral operators. Some are linked to the form of W.

Since $f_U \in L^1([0, 1])$, the integrals in the definition of W_k is convergent, and W_k maps continuous functions into continuous functions. By abuse of notation, we will use W_k, W for the restrictions and co-restrictions of these maps to various subspaces; the domains and co-domains should be clear from the context.

We can extend an element $g \in C_{00}([A, B])$ to an element of $C(\mathbb{R})$ or $C_{00}([A_1, B_1])$ if $[A, B] \subset [A_1, B_1]$ by defining g as 0 outside $[A, B]$; we will not distinguish g from its extensions.

The operator W_k is closely related to convolutions. Indeed, we can write

$$\begin{aligned} W_k(g) &= \int_{-\infty}^{+\infty} g(s)\ f_U\left(\left(s\left(\frac{k}{k-1}\right)^a - x\right)\ sign(s)\right) ds \\ &= \int_{-\infty}^{0} \left(\frac{k-1}{k}\right)^a g(\left(\frac{k}{k-1}\right)^a s)\ f_U((x-s))\, ds \\ &+ \int_{0}^{\infty} \left(\frac{k-1}{k}\right)^a g(\left(\frac{k}{k-1}\right)^a s)\ f_U((s-x))\, ds \end{aligned}$$

Thus, we can write:

$$W_k(g) = A_k(g) * f_U + B_k(g) * \mathcal{O}(f_U) \tag{25}$$

where

$$\begin{aligned} A_k(g)(s) &= \left(\frac{k-1}{k}\right)^a g(\left(\frac{k}{k-1}\right)^a s) I(s \le 0) \\ B_k(g)(s) &= \left(\frac{k-1}{k}\right)^a g(\left(\frac{k}{k-1}\right)^a s) I(s \ge 0) \\ \mathcal{O}(f_U)(s) &= f_U(-s) \end{aligned}$$

Note that (25) can be used to define W_k in L^p spaces.

3.1 W restricted to $C_\infty([-M, M])$

For any $r > 0$, let $B(0, r)$ be the open disc in the complex plane centered at 0 and of radius r.

Lemma. *f_U satisfies the assumptions in section 1, W as in (24), $M \geq 1$. Then,*

i. $W(C([-M, M])) \subseteq C_\infty([-M, M])$*, in particular, W is an endomorphism of* $C_\infty([-M, M])$

ii. $W : C([-M, M]) \longmapsto C_\infty([-M, M])$ *is compact*

iii. *W is order preserving: if $g \in C(\mathbb{R})$ satisfies $g(x) \geq 0$ for all x, then $W(g)(x) \geq 0$ for all x*

iv. $\int_{-\infty}^{+\infty} W(g)(x)\ dx = \int_{-\infty}^{+\infty} g(s)\ ds$ *for all $g \in L^\infty([-M, M])$ (g extended as 0 outside $[-M.M]$)*

v. *Let $\varphi \in C_\infty([-M, M])$, $\varphi \in Ker(\mathbb{I} - W)$, then*

a. $|\varphi| \in Ker(\mathbb{I} - W)$

b. $\varphi(-x) \in Ker(\mathbb{I} - W)$

c. if $\varphi(0) = 0$ *then* $\varphi = 0$

vi. The spectrum of W *satisfies*

$$1 \in spec\left(W\Big|_{C_{00}([-M,M])}\right) \subset \overline{B(0,1)}$$

vii. (Perron) There is an eigenvector ϕ_1 *of* W *corresponding to the eigenvalue* 1 *such that*

a. $\phi_1(x) \geq 0$

b. $\int \phi_1 = 1$

c. 1 *is a simple eigenvalue (i.e. of multiplicity 1)*

Moreover, ϕ_1 *is supported on* $[-1, 1]$.

viii. If $\lambda \in spec\left(W\Big|_{C_{00}([-M,M])}\right)$*, and* $|\lambda| = 1$ *then* $\lambda = 1$

ix. If $1 \leq M_1 < M_2$ *then*

$$spec(W\Big|_{C_{00}([-M_1,M_1])}) \subseteq spec(W\Big|_{C_{00}([-M_2,M_2])})$$

Remark Some comments before going through the proof. Part vii.c) is a consequence of the assumption that f_U doesn't have too many zeros on $[0, \epsilon]$ for some $\epsilon > 0$, as per the assumption (3) about f_U. This assumption is needed to ensure simplicity of 1, and the spectral gap, and it is standard in the context of Perron's theorem. Conceivably, the assumptions on f_U can be relaxed, but without a strong way of identifying the weak limit of $\{Z(k)\}_{k\geq1}$, one would expect the proof of weak convergence will be more delicate.

Proof. Most of these are basic facts of integral operators with a stochastic kernel, and we will skip some details.

If $g \in C([-M, M]$, then, by part a) of the lemma in the introduction, $supp(W(g)) \subseteq [-M, M]$. Since $W(g)$ is continuous everywhere, and vanishes outside [-M.M], by continuity, $W(g)$ vanishes at $\pm M$, i.e. $W(g) \in C_{00}([-M, M])$. This shows i).

Part ii) is a standard result of integral operators.

Part iii) is immediate, since, in case g is non-negative, the integrand in (24) is non-negative. We encountered W in the context of writing the kernel mapping a probability density to another probability density. Part iv) is basically a re-statement of that fact.

Part iv) can be re-stated as follows. $C_{00}([-M,M])'$, the dual of $C_{00}([-M,M])$ consists of finitely additive measures. By abuse of notation, let dx be the Lebesgue measure on $[-M,M]$, as element of $C_{00}([-M,M])'$ in the canonical way. Then iv) says that $W'(dx) = dx$.

For v.a), let $\varphi = W(\varphi)$.

$$\begin{aligned} |\varphi(x)| &= \left| \int_{-\infty}^{+\infty} \varphi(s)\ f_U((s-x)\ \ sign(s))\ ds \right| \\ &\leq \int_{-\infty}^{+\infty} |\varphi(s)|\ f_U((s-x)\ \ sign(s))\ ds \\ &= W(|\varphi|)(x) \end{aligned}$$

Integrating over x and using iv), be get

$$\int |\varphi|\ dx \leq \int W(|\varphi|)\ dx = \int |\varphi|\ dx$$

so the first inequality is an equality, and $|\varphi(x)| = W(|\varphi|)(x)$ a.e., but since both are continuous functions, then theay are equal everywhere.

v.b) follows from a simple change of variables in the integral.

For part v.c), let $\varphi \in Ker(\mathbb{I} - W)$ be such that $\varphi(0) = 0$. Then $\varphi_1(x) = \frac{|\varphi(x)| + |\varphi(-x)|}{2}$ is also in $Ker(\mathbb{I} - W)$, it is non-negative, symmetric, and vanishes at 0. Using the definition of W and these properties of φ_1

$$0 = \varphi_1(0) = \int_0^1 \varphi_1(s)\ 2\ f_U(s)\ ds.$$

but $f_U(s) > 0$ for almost all points (by the assumption *essential support*$f_U = [0,1]$), therefore $\varphi_1(s) = 0$ for $s \in [0,1]$.

Let $A = \sup\{x \in [0,M] \mid \varphi_1(s) = 0 fors \in [0,x]\}$. So far we know $1 \leq A \leq M$. If $A < M$ then

$$0 = \varphi_1(A) = \int_A^M \{f_U(x+A) + f_U(x-A)\}\ \varphi_1(x)\ dx,$$

but $f_U(x+A) = 0$ for $x > 0$, since $A \geq 1$, so

$$0 = \varphi_1(A) = \int_A^M\ f_U(x-A)\ \varphi_1(x)\ dx,$$

then $f_U(x-A)\ \varphi_1(x) =$ for all $x \in [A, \min\{1,M\}]$, where $f_U(x-A) > 0$ almost everywhere, so $\varphi_1(x) =$ for all $x \in [A, \min\{1,M\}]$, which contradicts the definition of A. It follows that $A = M$, and $\varphi_1 = 0$ and so $\varphi = 0$.

For part vi), notice that from iv), $W'(dx) = dx$, so 1 is an eigenvalue of W' with eigenvector dx, but $spec(W') = spec(W)$, so $1 \in spec(W)$.

If $\lambda \in spec(W)$, $\lambda \neq 0$, then, since W is compact, there is an eigenvector ψ corresponding to λ, normalized so that $\int |\psi|\ dx = 1$. Then

$$\begin{aligned} |\lambda| \ &= \ \left|\lambda \int \psi\ dx\right| = \left|\int W(\psi)\ dx\right| \\ &\leq \ \int |W(\psi)|\ dx \ \leq \ \int W(|\psi|)\ dx = \int |\psi|\ dx = 1 \end{aligned}$$

For vii), since 1 is an eigenvalue, it has an eigenvector ϕ, which we can normalize so that $\int |\phi|\ dx = 1$. By part v), $\phi_1 = |\phi|$ is also an eigenvector corresponding to the eigenvalue 1.

If ϕ_2 is another eigenvector corresponding to the eigenvalue 1, then $\phi_2 - \frac{\phi_2(0)}{\phi_1(0)}\phi_1 \in Ker(\mathbb{I} - W)$ and vanishes at 0. By part v.c), $\phi_2 = \frac{\phi_2(0)}{\phi_1(0)}\phi_1$, so $Ker(\mathbb{I} - W) =< \phi_1 >$, i.e.: 1 has geometric multiplicity 1.

If $(\mathbb{I} - W)^2 H = 0$, then $(\mathbb{I} - W)H \in Ker(\mathbb{I} - W)$, so $(\mathbb{I} - W)H = \alpha\phi_1$ for some α. But $0 = \int(\mathbb{I} - W)H\ dx = \alpha \int \phi_1\ dx = \alpha$, so $(\mathbb{I} - W)H = 0$. This shows 1 has algebraic multiplicity 1.

For part viii), let $\lambda \in spec(W)$, $|\lambda| = 1$, and let ϕ be an eigenvector normalized as $\int |\phi|\ dx = 1$. We have

$$|\psi(x)| = |W(\psi)(x)| \leq W(|\psi|)(x)$$

and since $\int |\psi| = 1 = \int W(|\psi|)$, then the inequality is an equality a.e., and $|\psi(x)| \in Ker(\mathbb{I} - W)$, so $|\psi(x)| = \alpha\varphi_1$ for some α, and om account of the L^1 normalization, $\alpha = 1$. We can write $\psi(x) = \epsilon(x)\varphi_1(x)$ with ϵ continuous where $\varphi_1(x) > 0$, and

$|\epsilon(x)| = 1$ for all x. Then, for any x:

$$\begin{aligned}
\varphi_1(x) &= \overline{\lambda}\ \overline{\epsilon(x)} \int_{-M}^{M} \epsilon(s)\ \varphi_1(s)\ f_U((s-x)sign(s))\ ds \\
&= \int_{-M}^{M} \overline{\lambda}\ \overline{\epsilon(x)}\ \epsilon(s)\ \varphi_1(s)\ f_U((s-x)sign(s))\ ds \\
&\leq \int_{-M}^{M} \Re\Big\{\overline{\lambda}\ \overline{\epsilon(x)}\ \epsilon(s)\Big\}\ \varphi_1(s)\ f_U((s-x)sign(s))\ ds \\
&\leq \int_{-M}^{M} \varphi_1(s)\ f_U((s-x)sign(s))\ ds = \varphi_1(x)
\end{aligned}$$

so

$$\Re\Big\{\overline{\lambda}\ \overline{\epsilon(x)}\ \epsilon(s)\Big\} = 1 \text{ where } \varphi_1(s)\ f_U((s-x)sign(s)) > 0$$

Taking $x > 0$ small, $s = -x$, $\varphi_1(s)\ f_U((s-x)sign(s)) = \varphi_1(-x)\ f_U(2x) > 0$ for a.e. x near 0, so, for a.e. small $x > 0$

$$\begin{aligned}
&\Re\Big\{\overline{\lambda}\ \overline{\epsilon(x)}\ \epsilon(-x)\Big\} = 1 \\
&\overline{\lambda}\ \overline{\epsilon(x)}\ \epsilon(-x)\ =\ 1 \\
&\lambda = \epsilon(x)\overline{\epsilon(-x)}.
\end{aligned}$$

But ϵ is continuous at 0, so letting $x \to +0$, we get $\lambda = |\epsilon(0)|^2 = 1$

For ix), if $\lambda \neq 0$ is in $spec(W\Big|_{C_{00}([-M_1,M_1])})$, and ϕ is a corresponding eigenvalue, then, extending ϕ as 0 outside $[-M_1, M_1]$ we have:

$$\begin{aligned}
W\Big|_{C_{00}([-M_2,M_2])}(\phi)(x) &= \int_{-M_2}^{M_2} \phi(s) f_U((s-x)sign(s))\ ds \\
&= \int_{-M_1}^{M_1} \phi(s) f_U((s-x)sign(s))\ ds
\end{aligned}$$

because $\phi(s) = 0$ for $|s| \geq M_1$.

If $|x| \leq M_1$ then

$$\int_{-M_1}^{M_1} \phi(s) f_U((s-x)sign(s))\ ds = W\Big|_{C_{00}([-M_1,M_1])}(\phi)(x) = \lambda\ \phi(x)$$

If $M_1 < x \leq M_2$ then

$$\int_{-M_1}^{M_1} \phi(s) f_U((s-x)sign(s))\ ds = \int_{0}^{M_1} \phi(-s) f_U(s+x) + \phi(s) f_U(s-x)\ ds = 0$$

because under the integral, $s+x > 0+M_1 \geq 1$, so $f_U(s+x) = 0$, and $s-x \leq M_1-x < 0$, so $f_U(s-x) = 0$. Similarly, the integral is 0 when $-M_2 \leq x < -M_1$. But the extension of ϕ is also 0 at x such that $M_1 < |x| \leq M_1$, so

$$W\Big|_{C_{00}([-M_2,M_2])}(\phi)(x) = \lambda\ \phi(x)$$

for all $x \in [-M_2, M_2]$, hence $\lambda \in spec(W\Big|_{C_{00}([-M_2,M_2])})$ ☐

Corollary 3.1. *(Spectral gap) There is r, $0 < r < 1$ such that*

$$spec(W) \subset \overline{B(0,r)} \cup \{1\}.$$

Proof. 1 is the only point of $spec(W)$ at the boundary of $B(0,1)$, and 0 is the only accumulation point of $spec(W)$. ☐

3.2 W restricted to $L^1[-M, M]$

W can be extended to several spaces; in particular, if $M \geq 1$, considering that the integral

$$\int_{-M}^{M}\int_{-M}^{M} |g(s)|\ f_U((s-x)sign(s))\ ds\ dx$$

is finite for all $g \in L^1[-M, M]$, we can extend W to $L^1[-M, M]$ as

$$W(g)(x) = \int_{-M}^{M} g(s)\ f_U((s-x)sign(s))\ ds \tag{26}$$

for those x for which the integrand is in L^1, i.e.: for a.e. x. This defines a map from $L^1[-M, M]$ to itself.

We explore the spectrum of the resulting operator. When working in C_{00} we have access to the pointwise definition of $W(g)(x)$. In L^1, we will make use of duality.

Lemma. *Let $M \geq 1$; let W be defined by (26); let ϕ_1 be the eigenvector of $W\Big|_{C_{00}([-1,1])}$ corresponding to the eigenvalue 1, normalized as $\phi_1 \geq 0$, $\int \phi_1 = 1$. Then*

i. $W\Big|_{L^1[-M,M]}$ *is compact.*

ii. There is r*,* $0 < r < 1$ *such that*

$$spec(W\Big|_{L^1[-M,M]}) \subset \overline{B(0,r)} \cup \{1\}.$$

iii. $Ker(W\Big|_{L^1[-M,M]} - 1) =< \phi_1 > .$

Before going into the proof, let's mention that ii) is the statement of existence of the spectral gap for the Markov operator W, and iii) is equivalent to existence and uniqueness of the invariant measure (invariant for the Markov chain defined by W). One would expect, given the form of kernel of W, that it would be easy to show that if $h \in L^1[-M,M]$ satisfies $W(h) = h$, then h is continuous, and reduce the problem of identifying h to the case of continuous eigenvectors, however, it

Proof. For i), replace f_U by $H \in L^1[0,1]$, H continuous. The operator with f_U replaced by H is compact in L^1, and converges (in the operator norm for endomorphisms of $L^1[-M,M]$) to W as $H \longrightarrow f_U$ in L^1. So W is compact.

For ii), note that

$$\begin{aligned} spec(W\Big|_{L^1[-M,M]}) &= spec(W^*\Big|_{L^\infty[-M,M]}) \subseteq spec(W^*\Big|_{C_{00}([-M,M])^*}) \\ &= spec(W\Big|_{C_{00}([-M,M])}) \subset \overline{B(0,r)} \cup \{1\} \end{aligned}$$

because, respectivly: Schauder's theorem, $L^\infty[-M,M] \subseteq C_{00}([-M,M])^*$, Schauder's theorem again, and the Lemma in the previous section.

Let $\lambda \in spec(W\Big|_{L^1[-M,M]})$, $\lambda \neq 0$. From Fredholm theory, $dim(Ker(W-\lambda)) = co{-}dim(Im(W-\lambda))$, so we can find $h_1, \ldots, h_d$ that form a basis of $Ker\left(W\Big|_{L^1[-M,M]} - \lambda\right)$, and:

$$\begin{aligned} L^1[-M,M] &= < h_1, \ldots, h_d > \oplus V \text{ for some subspace } V \\ L^1[-M,M] &= < \omega_1, \ldots, \omega_d > \oplus Im(W-\lambda) \text{ for some } \omega_1, \ldots, \omega_d \\ &\omega_1, \ldots, \omega_d \text{ are linearly independent} \\ (W-\lambda) &: V \longrightarrow Im(W-\lambda) \text{ is an isomorphism} \end{aligned}$$

Applying Ritz lemma, we can find $\omega_1, \ldots, \omega_d$ in $C_{00}([-M,M])$.

Let $\zeta_1, \ldots, \zeta_d \in L^1[-M,M]^* = L^\infty[-M,M]$ be defined as

$$\begin{aligned} \zeta_i(\omega_j) &= \delta_{i,j} = 1 \text{ if } i = j, \text{ 0 otherwise} \\ \zeta_i &= 0 \text{ on } Im(W-\lambda). \end{aligned}$$

The fact that $\zeta_1, \ldots, \zeta_d$ are continuous on $L^1[-M, M]$ follows from the decomposition of L^1. Note that

$$\begin{aligned}
0 &= \; <\zeta_i, h> \text{ for all } h \in Im(W-\lambda) \\
\text{iff } 0 &= \; <\zeta_i, (W-\lambda)(g)> \text{ for all } g \in L^1[-M, M] \\
\text{iff } 0 &= \; <(W^* - \lambda)\zeta_i, g> \text{ for all } g \in L^1[-M, M] \\
\text{iff } 0 &= (W^* - \lambda)\zeta_i \\
\text{iff } \zeta_i &\in Ker(W^*\Big|_{L^\infty} - \lambda)
\end{aligned}$$

Note that the inclusion map $\iota : C_{00}([-1,1]) \longrightarrow L^1[-M, M]$ implies a dual inclusion map $\iota^* : L^1[-M, M]^* \longrightarrow C_{00}([-1,1])^*$, so (identifying elemnts of $L^1[-M, M]^*$ with their images under ι^*), we can consider $\zeta_1, \ldots, \zeta_d$ as elements of $C_{00}([-1,1])^*$. Also, (ignoring ι^* in the notation) $\zeta_1, \ldots, \zeta_d \in Ker\left(W^*\Big|_{C_{00}([-M,M])^*} - \lambda\right)$.

$\zeta_1, \ldots, \zeta_d$ are linearly independent in $C_{00}([-M, M])^*$, because, if for some scalars $\alpha_1, \ldots$ we have

$$0 = \sum_i \alpha_i \; \zeta_i \text{ in } C_{00}([-M, M])^*$$

then evaluating 0 and $\sum_i \alpha_i \; \zeta_i$ at $\omega_j \in C_{00}([-M, M])^*$, we get $0 = \alpha_j$. In particular:

$$\begin{aligned}
d = dim\left(Ker\left(W\Big|_{L^1[-M,M]} - \lambda\right)\right) &= dim\left(Ker\left(W^*\Big|_{L^\infty[-M,M]} - \lambda\right)\right) \\
\leq dim\left(Ker\left(W^*\Big|_{C_{00}([-M,M])^*} - \lambda\right)\right) &= dim\left(Ker\left(W\Big|_{C_{00}([-M,M])} - \lambda\right)\right).
\end{aligned}$$

We apply this inequality to the case $\lambda = 1$, which is in $spec\left(W\Big|_{L^1[-M,M]} - 1\right)$, and for which we know simplicity of eigenvalue 1 to get

$$\begin{aligned}
1 &\leq dim\left(Ker\left(W\Big|_{L^1[-M,M]} - 1\right)\right) \\
&\leq dim\left(Ker\left(W\Big|_{C_{00}([-M,M])} - 1\right)\right) = 1
\end{aligned}$$

Therefore, since $\phi_1 \in Ker\left(W\Big|_{L^1[-M,M]} - 1\right)$, iii) follows. $\square$

3.3 W on $C_\infty(\mathbb{R})$ and $L^1(\mathbb{R})$

There are some un-refereed sources that seem to suggest that, because W is an integral operator involving a compactly supported function, then it is automatically compact. W is close to a convolution operator, and so it cannot be compact when acting on the entire line. It turns out that part of the range of the Fourier transform of f_U is contained in the spectrum of W. To settle any doubt, we include the result in this section.

Since f_U has compact support, the Fourier transform of f_U given by $\hat{f}_U(z) = \int_0^1 e^{-2\pi izs} f_u(s)\ ds$, is an entire function of $z \in \mathbb{C}$. For $\xi \in \mathbb{R}$, small:

$$\begin{aligned}\hat{f}_U(\xi) &= \int_0^1 (1 - 2\pi i\xi s - 2\pi^2\xi^2 s^2 + \dots) f_u(s)\ ds \\ &= 1 - 2\pi i\xi\mu - 2\pi^2\xi^2(\mu^2 + \sigma^2) + \dots\end{aligned}$$

so the parabola $x = 1 - \frac{1}{2}\left(\frac{\mu^2+\sigma^2}{\mu^2}\right) y^2$ osculates the curve $\hat{f}_U(\mathbb{R})$ at $x = 1$, $y = 0$. In particular, the range of $\hat{f}_U$ has lots of points (it is not discrete).

The fact that $supp(f_U) \subseteq [0, \infty)$ implies that $|\hat{f}_U(z)| \leq 1$ if the imaginary part of z satisfies $\Im(z) \leq 0$.

Lemma. *Let W as in (24), i.e.:*

$$W(g)(x) = \int_{-\infty}^{+\infty} g(s)\ f_U((s-x)\ \ sign(s))\ \ ds.$$

where the we take $g \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$. Let ϕ_1 be the eigenvector of W in $C_\infty([-1,1])$ corresponding to the eigenvalue 1, normalized so that $\phi_1 \geq 0$, $\int \phi_1 = 1$. Let $\mathbb{C}^- = \{z \in \mathbb{C} | \Im(z) \leq 0\}$ be the closed lower half complex plane. Then:

i. $W : C(\mathbb{R}) \longmapsto C(\mathbb{R})$ *is not compact*

ii. $\hat{f}_U(\mathbb{C}^-) \subseteq spec\left(W\Big|_{C(\mathbb{R})}\right)$

iii. $W : C_\infty(\mathbb{R}) \longmapsto C_\infty(\mathbb{R})$ *is not compact*

iv. $W : L^1(\mathbb{R}) \longmapsto L^1(\mathbb{R})$ *is not compact*

v. $\hat{f}_U(\mathbb{C}^-) \subseteq spec(W\Big|_{L^1(\mathbb{R})})$

vi. $Ker\left(W\Big|_{L^1(\mathbb{R})} - 1\right) =< \phi_1 >$, *and 1 has algebraic multiplicity 1 as eigenvalue of* $W\Big|_{L^1(\mathbb{R})}$.

Proof. Let $M \geq 2$, $z \in \mathbb{C}^-$. Let

$$A_M(x) = \begin{cases} 1 & \text{if } |x| \leq M \\ 0 & \text{if } |x| \geq M+1 \\ \textit{piecewise linear} & \text{in between} \end{cases}$$

and $g_M(x) = e^{-2\pi i |x| z} A_M(x)$. We have

$$\begin{aligned} W(g_M)(x) &= \int_{-M-1}^{M+1} e^{-2\pi i |s| z} A_M(s)\ f_U((s-x)\ \ sign(s))\ ds \\ &= \int_0^{M+1} e^{-2\pi i s z} A_M(s) \Big(f_U(s-x) + f_U(s+x) \Big)\ ds = W(g_M)(-x), \end{aligned}$$

i.e.: $W(g_M)$ is even.

Since g_M is supported on $[-M-1, M+1]$, then $W(g_M)(x) = 0 = \hat{f}_U(z) g_M(x)$ when $|x| \geq M+1$.

If $1 \leq x \leq M-1$, then $f_U(s+x) = 0$ for any $s \geq 0$, and so

$$\begin{aligned} W(g_M)(x) &= \int_0^{M+1} e^{-2\pi i s z} A_M(s) f_U(s-x)\ ds \\ &= \int_{-x}^{M+1-x} e^{-2\pi i (s+x) z} A_M(s+x) f_U(s)\ ds \\ &= e^{-2\pi i x z} \int_0^1 e^{-2\pi i s z} A_M(s+x) f_U(s)\ ds, \end{aligned}$$

but, in the integrand, $s + x \leq 1 + M - 1 = M$, so $A_M(s+x) = 1$, and

$$W(g_M)(x) = e^{-2\pi i x z} \int_0^1 e^{-2\pi i s z} f_U(s)\ ds = e^{-2\pi i x z} \hat{f}_U(z) = \hat{f}_U(z) g_M(x).$$

Therefore

$$W(g_M)(x) - \hat{f}_U(z) g_M(x) = 0$$

if $1 \leq x \leq M-1$. Since $W(g_M)$ and g_M are even, it follows that $W(g_M)(x) - \hat{f}_U(z) g_M(x) = 0$ if $1 \leq |x| \leq M-1$ or $M+1 \leq |x|$. In any case, $|W(g_M)(x) - \hat{f}_U(z) g_M(x)| \leq 2$ for all x.

i follows from ii, since compact operators have discrete spectra, and $\hat{f}_U(\mathbb{C}^-)$ is not discrete.

To show ii, let $z \in \mathbb{C}^-$.

If $\hat{f}_U(z) \in spec\left(W\Big|_{C_{00}([-1,1])} \right)$ then there is $\rho \in C_{00}([-1,1]) \subset C(\mathbb{R})$, $\rho \neq 0$, such that $(W - \hat{f}_U(z))(\rho) = 0$. Thus, $\hat{f}_U(z) \in spec\left(W\Big|_{C(\mathbb{R})} \right)$.

If $\hat{f}_U(z) \notin spec\left(W\Big|_{C_{\infty}([-1,1])}\right)$. Taking $g(x) = e^{-2\pi i|x|z}$, which is in $C(\mathbb{R})$, and applying the previous computation letting $M \leftarrow \infty$, we get $\Delta_z = W(g) - \hat{f}_U(z)g$ is in $C_0([-1,1])$, and since $\hat{f}_U(z) \notin spec\left(W\Big|_{C_{\infty}([-1,1])}\right)$, we can write $\Delta_z = (W - \hat{f}_U(z))(\rho_z)$ for some $\rho_z \in C_0([-1,1])$, then

$$(W - \hat{f}_U(z))(g - \rho_z) = 0$$

and $g - \rho_z \neq 0$, since ρ_z has compact support and g does not. This shows $\hat{f}_U(z) \in spec\left(W\Big|_{C(\mathbb{R})}\right)$.

If $z_o \in \mathbb{C}^-$ and $\hat{f}_U(z_o) \neq 0$ is in $spec\left(W\Big|_{C_0([-1,1])}\right)$, using the analyticity of $\hat{f}_U$ and the discreteness of $spec\left(W\Big|_{C_0([-1,1])}\right)$ away from 0 , we can find a sequence z_j in $\mathbb{C}^-$ such that $z_j \leftarrow z_o$, $\hat{f}_U(z_j) \notin spec\left(W\Big|_{C_0([-1,1])}\right)$, hence, from our previous result, $\hat{f}_U(z_j) \in spec\left(W\Big|_{C(\mathbb{R})}\right)$, so passing to the limit, $\hat{f}_U(z_o) \in spec\left(W\Big|_{C(\mathbb{R})}\right)$. 0 is an accumulation point of $\hat{f}_U\Big|_{\mathbb{R}}$, so it is in $spec\left(W\Big|_{C_0([-1,1])}\right)$, which cover the case $\hat{f}_U(z_o) = 0$.

To show iii, assume $W : C_{\infty}(\mathbb{R}) \longmapsto C_{\infty}(\mathbb{R})$ is compact. Pick any z such that $\hat{f}_U(z) \neq 0$. Since the sequence $\{g_M | M = 2, 3, ...\}$ is contained in $C_0(\mathbb{R})$ and it is bounded, the sequence $\{W(g_M) | M = 2, 3, ...\}$ would contained a convergent subsequence. Any accumulation point G of $\{W(g_M)\}$ is of the form $G(x) = e^{-2\pi i|x|\xi}\hat{f}_U(z)$ for $1 \leq |x|$, hence not in $C_0(\mathbb{R})$, a contradiction.

iv follows from v, because compact operators have discrete spectrum, and $\hat{f}_U(\mathbb{C}^-)$ is not discrete.

To show v, let $\Im(z) < 0$, and such that $\hat{f}_U(z) \neq 0$. In part ii we showed that there is an eigenvector of W in $C(\mathbb{R})$ of the form $V(x) = e^{-2\pi i|x|z} - \rho_z(x)$ with $\rho_z \in C_0([-1,1])$, therefore V is integrable, so $\hat{f}_U(z) \in spec(W\Big|_{L^1(\mathbb{R})})$. By continuity we get $\hat{f}_U(\mathbb{C}^-) \subseteq spec(W\Big|_{L^1(\mathbb{R})})$.

To show vi, recall ϕ_1 is continuous and is supported on $[-1,1]$, so $\phi_1 \in L^1(\mathbb{R})$, which shows the inclusion $< \phi_1 > \subseteq Ker\left(W\Big|_{L^1(\mathbb{R})} - 1\right)$.

For the other inclusion, let $g \in Ker\left(W\Big|_{L^1(\mathbb{R})} - 1\right)$, $g \neq 0$. We will show that

g has compact support, then, applying the result of section (3.2) we conclude $g \in < \phi_1 >$. If

$$g(x) = \int_{-\infty}^{+\infty} g(s)\ f_U\Big((s-x)sign(s)\Big)\ ds \tag{27}$$

(in $L^1(\mathbb{R})$) then

$$|g(x)| \leq \int_{-\infty}^{+\infty} |g(s)|\ f_U\Big((s-x)sign(s)\Big)\ ds \tag{28}$$

and since both sides have the same integral over $\mathbb{R}$ we conclude that equality holds in (28) a.e., so $|g| \in Ker\left(W\Big|_{L^1(\mathbb{R})} - 1\right)$. Let $G(x) = |g(x)| + |g(-x)|$, then $G \in Ker\left(W\Big|_{L^1(\mathbb{R})} - 1\right)$, $G \geq 0$, $G \neq 0$, and G is even. Moreover, G has compact support iff g has compact support, so we can work with G.

Let $V(a) = \int_a^{+\infty} G(x)\ dx$. Since G is non-negative and integrable, V is continuous and non-increasing. We claim that $V(a) = V(1)$ for all $a \geq 1$. Indeed, given $a \geq 1$,

$$\int_a^{+\infty} G(x)\ dx = \int_a^{+\infty} \int_0^{+\infty} G(s)\ \Big[f_U(s-x) + f_U(s+x)\Big]\ ds\ dx$$

But if $x \geq a \geq 1$ and $u \geq 0$, then $f_U(s+x) = 0$, so

$$\begin{aligned}
\int_a^{+\infty} G(x)\ dx &= \int_a^{+\infty} \int_0^{+\infty} G(s)\ f_U(s-x)\ ds\ dx \\
&= \int_a^{+\infty} \int_{-x}^{+\infty} G(h+x)\ f_U(h)\ dh\ dx \\
&= \int_a^{+\infty} \int_0^{1} G(h+x)\ f_U(h)\ dh\ dx \\
&= \int_0^{1} f_U(h) \int_a^{+\infty} G(h+x)\ dx\ dh = \int_0^1 f_U(h) \int_{a+h}^{+\infty} G(x)\ dx\ dh \\
&\leq \int_0^1 f_U(h) \int_a^{+\infty} G(x)\ dx\ dh = \int_a^{+\infty} G(x)\ dx
\end{aligned}$$

therefore, the above inequality is an equality, and we get

$$\int_{a+h}^{+\infty} G(x)\ dx = \int_a^{+\infty} G(x)\ dx$$

a.e. $h \in [0,1]$ with respect to the measure $f_U\ dh$, in particular,

$$V(a+h) = V(a)$$

for some $h > 0$. Applying this result to $a = 1$ and using the monotonicity of V, we conclude that $V(1) = V(1+h)$ for all $h \in [0, \epsilon)$ for some $\epsilon > 0$.

If $V(a) < V(1)$ for some $a > 1$, take $a_0 = \inf\{a > 1 | V(a) < V(1)\}$ and apply the result to a_0. We have $V(1) = V(a_0)$, and so $V(1) = V(a_0 + h)$ for all small $h > 0$, but also, $V(a) < V(1)$ for some points $a > a_0$. A contradiction.

Since $\lim_{a\to\infty} V(a) = 0$, we conclude $V(1) = 0$, and G is supported on $[-1, 1]$, which completes the proof of $Ker\left(W\Big|_{L^1(\mathbb{R})} - 1\right) =< \phi_1 >$.

We can show that 1 is an algebraically simple eigenvalue of $W\Big|_{L^1(\mathbb{R})}$ as before, because is $h \in L^1(\mathbb{R})$ satisfies $(W-1)^2(h) = 0$, then $g = (W-1)(h) \in Ker\left(W\Big|_{L^1(\mathbb{R})} - 1\right) =< \phi_1 >$, so $(W - 1)(h) = g = \lambda\phi_1$ for some constant λ. Integrating over $\mathbb{R}$ we get $0 = \int(W - 1)(h) = \lambda \int \phi_1 = \lambda$, hence $(W - 1)(h) = 0$. This completes the proof of vi. $\square$

3.4 Iterates of W on $C_{00}([-M, M])$ and $L^1[-M, M]$

We want to make parallel statements about W acting on $C_{00}([-M, M])$ and on $L^1[-M, M]$. We describe decompositions of these spaces that are akin to an invariant manifold description of the dynamics generated by iterates of W.

Lemma 3.2. *Let φ_1 be the eigenfunction of $C_{00}([-1, 1])$ that is non-negative, normalized so that $\int \varphi_1\, ds = 1$. Let $M > 1$. For $\mathbb{B}$ either of $C_{00}([-M, M])$ or $L^1[-M, M]$, let $S_1 =< \varphi_1 >$ be the one-dimensional space spanned by φ_1 in $\mathbb{B}$ and let $S_2 = Ker(dx) = \{g \in \mathbb{B} \mid \int_{-M}^{M} g\, dx = 0\}$. Then*

i. S_1, S_2, $C_{00}([-M_0, M_0]$ with $1 \le M_0 < M$ are invariant subspaces of W

ii. $\mathbb{B}$ is isomorphic to the direct sum $< \varphi_1 > \oplus Ker(dx)$

iii. There is r , $0 < r < 1$ such that

$$spec(W\Big|_{S_2}) \subset \overline{B(0, r)}$$

iv. There is a norm $\mathcal{N}$ on $\mathbb{B}$, equivalent to the standard norm in $\mathbb{B}$, such that $W\Big|_{S_2}$ is a contraction

Proof. Part i) follows from the facts that ϕ_1 is an eigenvalue, W preserves integrals, and W does not increase the support of elements in $C_{00}([-M_0, M_0]$.

Let

$$\pi_1 : \mathbb{B} \longmapsto S_1$$
$$\pi_1(g) = (\int g\varphi_1)\ \varphi_1$$

and

$$\pi_2 : \mathbb{B} \longmapsto S_2$$
$$\pi_2(g) = g - (\int g\varphi_1)\ \varphi_1.$$

The map $\pi_1 \oplus \pi_2$ is (well defined), linear, continuous (giving S_1, S_2 the metric induced by the norm in $\mathbb{B}$), invertible. This shows ii).

For iii, $spec(W\Big|_{S_2})$ is compact, so, with the possible exception of 0, the spectrum coincides with the point spectrum, eigenvectors of $W\Big|_{S_2}$ are eigenvectors of W, so $spec(W\Big|_{S_2}) \subseteq spec(W) \subseteq \{1\} \cup \overline{B(0,r)}$; but $1 \notin spec(W\Big|_{S_2})$, else "the" eigenvector of W corresponding to 1 would be in S_2, hence

$$spec(W\Big|_{S_2}) \subseteq \overline{B(0,r)}.$$

For part iv), let $\mathcal{W} = W|_{S_2}$, which is an endomorphism of S_2. By part iii), The map $(\mathbb{I} - z\mathcal{W})$ is invertible for $|z| < 1/r$, so $(\mathbb{I} - z\mathcal{W})|_{S_2}^{-1} = \sum_{j\geq 0} z^j \mathcal{W}^j$ converges for $|z| < 1/r$. In particular:

$$\overline{\lim}_{n\to\infty} \|\mathcal{W}^n\|^{1/n} \leq r$$

where $\|\ \|$ is the operator norm for automorphisms of $\mathbb{B}$ restricted and co-restricted to S_2. Therefore, if r_1 satisfies $r < r_1 < 1$, there is B such that $\|\mathcal{W}^n\| \leq B\ r_1^n$, and there is z such that $r_1 < z < 1$.

Let if $\phi \in \mathbb{B}$. Write $\phi = \alpha\varphi_1 + g$ with $g \in S_2$. Consider

$$\mathcal{N}(\phi) =: |\alpha|\|\varphi_1\| + \sum_{j\geq 0} z^{-j}\|\mathcal{W}^j g\|.$$

Since

$$\sum_{j\geq 0} z^{-j}\|\mathcal{W}^j g\| \ \leq \ \sum_{j\geq 0} z^{-j}\|\mathcal{W}^j\|\ \|g\| \ \leq \ \sum_{j\geq 0} z^{-j} B r_1^j\ \|g\| = \frac{B}{1 - \frac{r_1}{z}}\|g\| \ < \ \infty,$$

the series in the definition of $\mathcal{N}(\phi)$ is convergent.

$\|g\| = z^{-j}\|\mathcal{W}^j\ g\|\Big|_{j=0} \leq \mathcal{N}(g)$, form where it follows that $\mathcal{N}$ is equivalent to the norm $\|\|$.

$\mathcal{N}$ satisfies the properties of a norm.

Finally,

$$\mathcal{N}(\mathcal{W}(g)) \ = \ \sum_{j\geq 0} z^{-j}\|\mathcal{W}^{j+1} g\| \ = \ z \sum_{j\geq 1} z^{-j}\|\mathcal{W}^j g\| \ \leq \ z\mathcal{N}(g).$$

Since $z < 1$, we get that $\mathcal{W}$ restricted to S_2 is a contraction. $\square$

4 Weak convergence of $\{Z(j)\}_{j\geq 1}$

In this section we complete the proof of the Theorem stated in the Section 1. Parts a), b) are dealt with in Lemma 2.3. Part c) is proved in Section 3.3.

Lemma 4.1. *Part d of the Theorem stated in the Section 1 holds.*

Proof. To show d), i.e.: $f_k \to f_Z$ in $L^1(\mathbb{R})$, first choose $M_0 > 2$ as in Lemma 2.3, so that the tail probabilities of Z_j are suitably bounded for j sufficiently large. By modifying γ_2 and γ_3 we may have that there are constants C, γ_2 that depend only on μ and a such that:

$$P(|Z(j)| \geq M) \leq C\ exp(-\gamma_2 M^{(1-a)/a} j^{1-a})$$

for $M \geq M_0$ and $j \geq 1$.

For such value M_0, by the result in section 3.4, there is a norm $\mathcal{N}$ on $L^1[-M_0, M_0]$ that is equivalent to the standard L^1 norm under which the operator W restricted to a suitable subspace is a contraction. Explicitly, there is $\lambda \in (0,1)$ such that for any $g \in L^1[-M_0, M_0]$, if $\int g(x)dx = 0$, then $\mathcal{N}(W(g)) \leq \lambda \mathcal{N}(g)$.

Let us consider the following modification of the operator W_k defined in (24). To lighten the notation, let $K_k(x,s) = f_U\big(\big(s\left(\frac{k}{k-1}\right)^a - z\big)\ \ sign(s)\big)$. For $g \in L^1[-M_0, M_0]$, define $V_k(g)$ as:

$$V_k(g)(z) = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} g(s) K_k(x,s) ds & \text{if } |z| \leq M_0 \\ 0 & \text{if } |z| > M_0 \end{cases}$$

We claim that $V_k \to W$ as operators on $L^1[-M_0, M_0]$. Indeed, if $g \in L^1[-M_0, M_0]$, then for $z \in [-M_0, M_0]$,

$$V_k(g)(z) - W(g)(z) = \int_{|x|\leq M_0} g(x)\big[K_k(x,s) - K_\infty(x,s)\big]dx$$

so

$$\begin{aligned} &\int_{|z|\leq M_0} |V_k(g) - W(g)| dz \\ &\leq \int_{|x|\leq M_0} |g(x)| \int_{|z|\leq M_0} \big|\big|K_k(x,s) - K_\infty(x,s)\big| dz dx \\ &\leq \int_{|x|\leq M_0} |g(x)| dx \sup_{|\xi|\leq M_0((\frac{k}{k-1})^a - 1)} \int \big|f_U(\xi + z) - f_U(z)\big| dz \end{aligned}$$

thus,

$$\|V_k - W\|_{Hom(L^1)} \leq \sup_{|\xi|\leq M_0((\frac{k}{k-1})^a-1)} \int \big|f_U(\xi+z) - f_U(z)|dz.$$

Therefore, there is $\lambda_1 \in (0,1)$ and k_0 such that for any $g \in L^1[-M_0, M_0]$, if $\int g(x)dx = 0$, and $k \geq k_0$ then $\mathcal{N}(V_k(g)) \leq \lambda_1 \mathcal{N}(g)$.

Let $\Delta_k = f_k - \phi_1 - (\int_{|x|\leq M_0} f_k - 1) = f_k - \phi_1 + (\int_{|x|>M_0} f_k) = f_k - \phi_1 + P(|Z_k| > M_0)$. We seek an inductive bound of Δ_k. We are interest in the a bound for $f_k - \phi_1$, but in order to use the contraction property of V_k, we subtract a constant from $f_k - \phi_1$.

We have

$$\Delta_{k+1} = f_{k+1}(z) - \phi_1(z) + \int_{|x|>M_0} f_{k+1}$$

$$= \int f_k(x)K_k(x,s)dx - \int \phi_1(x)K_\infty(x,s)dx + \int_{|x|>M_0} f_{k+1}$$

$$= \int_{|x|\leq M_0} \big(f_k(x) - \phi_1(x)\big)K_k(x,s)dx + \int_{|x|\leq M_0} \phi_1(x)K_k(x,s)dx$$

$$+ \int_{|x|>M_0} f_k(x)K_k(x,s)dx$$

$$- \int \phi_1(x)K_\infty(x,s)dx + \int_{|x|>M_0} f_{k+1}$$

$$= \int_{|x|\leq M_0} \big(f_k(x) - \phi_1(x)\big)K_k(x,s)dx$$

$$+ \int_{|x|>M_0} f_k(x)K_k(x,s)dx$$

$$+ \int \phi_1(x)\big(K_k(x,s) - K_\infty(x,s)\big)dx$$

$$+ \int_{|x|>M_0} f_{k+1}$$

$$= \int_{|x|\leq M_0} \Delta_k(x)K_k(x,s)dx$$

$$+ \Big(\int_{|x|>M_0} f_k\Big)\Big(\int_{|x|\leq M_0} K_k(x,s)dx\Big)$$

$$+ \int_{|x|>M_0} f_k(x)K_k(x,s)dx$$

$$+ \int \phi_1(x)\big(K_k(x,s)-K_\infty(x,s)\big)dx$$

$$+ \int_{|x|>M_0} f_{k+1} = I + II + III + IV + V$$

We bound each term separately.

$I = \int_{|x|\leq M_0} \Delta_k(x)K_k(x,s)dx = V_k(\Delta_k)$, so $\mathcal{N}(I) \leq \lambda_1 \mathcal{N}(\Delta_k)$

$II = \big(\int_{|x|>M_0} f_k\big)\big(\int_{|x|\leq M_0} K_k(x,s)dx\big) = P(|Z(k)| \geq M_0)W_k(I_{|x|\leq M_0})$, so

$$\begin{aligned}\int_{|z|\leq M_0} II(z)dz &\leq P(|Z(k)|\geq M_0)\int W_k(I_{|x|\leq M_0})\\ &= P(|Z(k)|\geq M_0)\int I_{|x|\leq M_0}) = 2\ M_0 P(|Z(k)|\geq M_0)\\ &\leq C\ M_0 exp(-\gamma_2 M_0^{(1-a)/a}k^{1-a})\end{aligned}$$

$III = \int_{|x|>M_0} f_k(x)K_k(x,s)dx = W_k(f_k I_{(|x|>M_0)})$, therefore

$$\int_{|z|\leq M_0} III(z)dz \leq \int f_k I_{(|x|>M_0)} = P(|Z_k|\geq M_0) \leq C\ exp(-\gamma_2 M_0^{(1-a)/a}k^{1-a}).$$

$IV = \int \phi_1(x)\big(K_k(x,s)-K_\infty(x,s)\big)dx$, terefore

$$\begin{aligned}\int_{|z|\leq M_0} IV(z)dz &\leq \int \phi_1(x)\int_{|z|\leq M_0} \big|K_k(x,s)-K_\infty(x,s)\big|dsdx\\ &\leq \sup_{|\xi|\leq M_0((\frac{k}{k-1})^a-1)} \int \big|f_U(\xi+s)-f_U(s)\big|ds.\end{aligned}$$

$V = \int_{|x|>M_0} f_{k+1} = P(|Z_{k+1}| \geq M_0) \leq C\ exp(-\gamma_2 M_0^{(1-a)/a}(k+1)^{1-a})$, therefore

$$\int_{|z|\leq M_0} V(z)dz \leq C\ M_0 exp(-\gamma_2 M_0^{(1-a)/a}(k+1)^{1-a})$$

Putting all these five inequalities together, using the equivalence of $\mathcal{N}$ and the usual L^1 norm, we get (C represents a constant, not allways the same from occurrence to occurrence)

$$\begin{aligned}
&\mathcal{N}(\Delta_{k+1}) \leq \lambda_1 \mathcal{N}(\Delta_k) + C\ M_0 exp(-\gamma_2 M_0^{(1-a)/a} k^{1-a}) \\
&\quad + C \sup_{|\xi|\leq M_0((\frac{k}{k-1})^a - 1)} \int \big| f_U(\xi+s) - f_U(s) \big| ds + C\ M_0 exp(-\gamma_2 M_0^{(1-a)/a}(k+1)^{1-a} \\
&\leq \lambda_1 \mathcal{N}(\Delta_k) + C\ M_0 exp(-\gamma_2 M_0^{(1-a)/a} k^{1-a}) \\
&\quad + C \sup_{|\xi|\leq M_0((\frac{k}{k-1})^a - 1)} \int \big| f_U(\xi+s) - f_U(s) \big| ds \\
&= \lambda_1 \mathcal{N}(\Delta_k) + \epsilon_k
\end{aligned}$$

where

$$\epsilon_k = C\ M_0 exp(-\gamma_2 M_0^{(1-a)/a} k^{1-a}) + C \sup_{|\xi|\leq M_0((\frac{k}{k-1})^a - 1)} \int \big| f_U(\xi+s) - f_U(s) \big| ds \to 0$$

Thus, we have obtained an inequality of the form $A_{k+1} \leq \lambda_1 A_k + \epsilon_k$, with $0 < \lambda_1 < 1$, and $\epsilon_k \to 0$. This implies that $A_k \to 0$. Indeed, consider that,

$$\frac{A_{k+1}}{\lambda_1^{k+1}} \leq \frac{A_k}{\lambda_1^k} + \frac{\epsilon_k}{\lambda_1^{k+1}}$$

so, for $q < p$:

$$\frac{A_p}{\lambda_1^p} \leq \frac{A_q}{\lambda_1^q} + \sum_{j=q}^{p-1} \frac{\epsilon_j}{\lambda_1^{j+1}}$$

then

$$\begin{aligned}
A_p &\leq \lambda_1^p \frac{A_q}{\lambda_1^q} + \sum_{j=q}^{p-1} \epsilon_j \lambda_1^{p-j-1} \\
&\leq \lambda_1^p \frac{A_q}{\lambda_1^q} + \sup_{j\geq q} \epsilon_j \sum_{j=q}^{p-1} \lambda_1^{p-j-1} \\
&\leq \lambda_1^p \frac{A_q}{\lambda_1^q} + \sup_{j\geq q} \epsilon_j \sum_{j=0}^{\infty} \lambda_1^j \\
&= \lambda_1^p \frac{A_q}{\lambda_1^q} + \sup_{j\geq q} \epsilon_j \frac{1}{1-\lambda_1}.
\end{aligned}$$

so $\int_{|z|\leq M_0} |\Delta_p(z)|dz \to 0$ as $p\to\infty$.

Now part d) is trivial:

$$\int |f_k - \phi_1| = \int_{|z|\leq M_0} |f_k - \phi_1| + \int_{|z|>M_0} |f_k - \phi_1|$$

but ϕ_1 is supported on $[-1,1]$, so $\int_{|z|>M_0} |f_k - \phi_1| = \int_{|z|>M_0} f_k = P(|Z_k| > M_0) \to 0$ as $k\to\infty$, and

$$\begin{aligned}\int_{|z|\leq M_0} |f_k - \phi_1| &= \int_{|z|\leq M_0} |\Delta_k - P(|Z_k| > M_0)| \\ &\leq \int_{|z|\leq M_0} |\Delta_k| + 2M_0\ P(|Z_k| > M_0) \to 0\end{aligned}$$

□

Lemma 4.2. *Part f) of the Theorem stated in the Section 1 holds.*

Proof. Part f) is a simple consequence of d). For any Borel set $B\subset\mathbb{R}$, $P(Z_k\in B) = \int_B f_k \to \int_B \phi_1 = P(Z\in B)$. □

Lemma 4.3. *Part e) of the Theorem stated in the Section 1 holds.*

Proof. Let $f_U \in L^p([0,1])$ for some $p>1$. Choose N minimal positive integer such that $p\geq 1+\frac{1}{N-1}$. Since $f_U\in L^1([0,1])$, then $f_U\in L^{p_1}([0,1])$ where $\frac{1}{p_1} = 1-\frac{1}{N}$.

We claim that for $j=1,\ldots,N$, f_j, the probability density function of Z_j, is in L^{p_j} where $\frac{1}{p_j} = 1-\frac{j}{N}$. Indeed, $Z_1 = -U_1$, so $f_1(s) = f_U(-s)\in L^{p_1}$ by assumption. Now we proceed inductively. If $f_j\in L^{p_j}$ for some $1\leq j<N$, then $f_{j+1} = W_{j+1}(f_j)$. Recall W_{j+1} can be written as in (25) in terms of convolutions, so $f_{j+1} = A_{j+1}(f_j) * f_U + B_{j+1}(f_j) * \mathcal{O}(f_U)$. We apply Young's inequality to conclude $\|f_{j+1}\|_r \leq C\|f_j\|_{p_j}\|f_U\|_{p_1}$ where

$$\frac{1}{r} \;=\; \frac{1}{p_j} + \frac{1}{p_1} - 1 \;=\; 1-\frac{j}{N} + 1 - \frac{1}{N} - 1 \;=\; 1-\frac{j+1}{N}$$

which completes the induction.

In particular, when $j=N$ we get $\|f_j\|_\infty \leq C\|f_U\|_{p_1}^j$ for a constant C that only depends on p_1.

For $j = N+1, \ldots$, $\|f_j\|_{infty} = \|W_j(f_{j-1})\|_{infty} \leq \|f_{j-1}\|_{infty}$, thus, the family $\{f_j\}_{j>N}$ is uniformly bounded. Moreover, the family is equicontinuous on compact subsets of $\mathbb{R}$, the continuity modulus dominated by the L^1 continuity modulus of f_U. Therefore the family is relatively compact on intervals, and since it converges in L^1 to ϕ_1, it converges to ϕ_1 uniformly on any baunded interval. To show convergence over the entire line we need only uniform bounds of the tails.

Let M_0 as in Lemma 2.3. For $j > N$, and $|z| > 3M_0$ we have

$$
\begin{aligned}
0 \leq f_j(z) &= \int f_{j-1}(x) f_U\big((x\rho_j - z)sign(x)\big)dx \\
&= \int_{|x-z|\leq 1} \rho_j^{-1} f_{j-1}(\rho_j^{-1}x) f_U\big((x-z)sign(x)\big)dx \\
&\leq \left(\int_{|x-z|\leq 1} \left[\rho_j^{-1} f_{j-1}(\rho_j^{-1}x)\right]^{q_1} dx\right)^{\frac{1}{q_1}} \left(\int f_U(t)^{p_1} dt\right)^{\frac{1}{p_1}}
\end{aligned}
$$

where we wrote ρ_j for $\left(\frac{j}{j-1}\right)^a$ to simplify the notation, and $\frac{1}{q_1} = 1 - \frac{1}{p_1} = \frac{1}{N}$, so $q_1 = N$.

In the integral, $|x - z| \leq 1$, so $|x| \geq |z| - 1 \geq 3M_0 - 1 > 2M_0$, then

$$
\begin{aligned}
&\left(\int_{|x-z|\leq 1} \left[\rho_j^{-1} f_{j-1}(\rho_j^{-1}x)\right]^{q_1} dx\right)^{\frac{1}{q_1}} \\
&\leq \left(\rho_j^{-1}\|f_{j-1}\|_\infty\right)^{\frac{N-1}{N}} \left(\int_{|x|\geq \rho_j^{-1}2M_0} f_{j-1}(x)dx\right)^{\frac{1}{N}} \\
&\leq \|f_N\|_\infty^{\frac{N-1}{N}} P(|Z_{j-1}| \geq \rho_j^{-1}2M_0)^{\frac{1}{N}} \\
&\leq \ C\|f_N\|_\infty^{\frac{N-1}{N}} exp\left(-\gamma_5 M_0^{\frac{1-a}{a}} j^{1-a}\right)
\end{aligned}
$$

where we made use of $\rho_j^{-1} < 1$, $\|f_j\|_\infty \leq \|f_N\|_\infty$ for $j \geq N$, the bound for the tail probability of Z_{j-1} given in Lemma 2.3, C and γ_5 are constants that depend only on p_1, a and $\mu = E(U_1)$.

There $f_j \to 0$ uniformly in the complement of $[-3M_0, 3M_0]$

□

REFERENCES
TBC