# A controlled random walk with decreasing step-sizes

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**Abstract**

A random walk $\{X_n\}_{n \geq 1}$ on the line, having steps scaled by $n^{-a}$ ($0 < a < 1$) of signs controlled towards 0 does converge to 0. The re-scaled walk $Z_n = n^a X_n$ converges in distribution to a random variable $Z$. Unlike the case of walks with identically distributed steps, the distribution of $Z$ is not normal, and depends on the distribution of the steps.

## 1  A random walk problem

Vincent Granville is a data scientist and blogger. In the blog "Interesting probability problem for serious geeks" he poses the following problem:

"Let's start with $X(1) = 0$, and define $X(k)$ recursively as folows, for $k > 1$:

$$X(k) = \begin{cases} X(k-1) + \frac{U(k)}{k^a} \text{ if } X(k-1) < 0 \\ \\ X(k-1) - \frac{U(k)}{k^a} \text{ if } X(k-1) \geq 0 \end{cases}$$

and let's define $U(k), Z(k)$ and $Z$ as follows:

$$Z(k) = k^a \ X(k)$$

$$Z = \lim_{k \to \infty} Z(k)$$

$$U(k) = V(k)^b$$

where the $V(k)$'s are deviates from *independent* uniform variables on $[0, 1]$ [...].

Prove that if $0 < a < 1$, then $X(k)$ converges to 0 as $k$ increases. Under the same condition, prove that the limiting distribution $Z$

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- always exists,

- always takes values between $-1$ and $+1$, with $\min(Z) = -1$, and $\max(Z) = +1$,

- is symmetric, with mean and median equal to $0$

- and does not depend on a, but only on b."

In the blog, Grandville discusses special cases. In particular, the case with:

$$\{U(k)\}_k \text{ are iid with density } f_U \tag{1}$$

$$essential\ support\{f_U\} \subseteq [0, 1] \tag{2}$$

$$\text{there is } \epsilon > 0 \text{ such that } [0, \epsilon] \subset essential\ support\{f_U\} \tag{3}$$

From formal considerations, Grandville conjectures that $Z$ has density $f_Z$, where $f_Z$ solves the Fredholm integral equation:

$$f_Z(z) = \int_0^1 \{f_U(x + z) + f_U(x - z)\}\ f_Z(x)\ dx. \tag{4}$$

Note that $Z(1) = 0$, $Z(2) = -U(2)$, and so $Z(k)$ has density with respect to Lebesgue measure for $k \geq 2$, and the density $f_k$ of $Z(k)$ satisfies:

$$f_2(z) = f_U(-z)$$

and

$$f_k(z) = \int_{-\infty}^{+\infty} f_{k-1}(x)\ f_U\left(\left(x\left(\frac{k}{k-1}\right)^a - z\right)\ sign(x)\right)\ dx$$

for $k > 2$.

Using simulations, Grandville found the shape of the limit density for special cases of $f_U$. He proposed an explicit solution to the integral equation ( 4 ), which he verified. He then proposed the form of a solution for the general case. As it turns, Grandville ansatz is correct.

**Lemma.** *Let $F_U(z) = \int_{-\infty}^z f_U(x)dx$ be the cumulative distribution function of $f_U$. Then*

$$f_Z(z) = \frac{1}{2} \cdot \frac{1 - F_U(|z|)}{1 - \int_0^1 F_U(x)dx} \tag{5}$$

*solves the integral equation (4).*

*Proof.* By linearity of (4), it is enough to show that

$$1 - F_U(|z|) = \int_0^1 \big(f_U(x+z) + f_U(x-z)\big)\big(1 - F_U(x)\big)\ dx.$$

Since both sides of this equation involve even functions of $z$, it is enough to show the equality for $z \geq 0$.

For $z \geq 0$,

$$\int_0^1 \big(f_U(x+z) + f_U(x-z)\big)\big(1 - F_U(x)\big)\ dx$$

$$= \int_0^1 f_U(x-z)\big(1 - F_U(x)\big)\ dx + \int_0^1 f_U(x+z)\big(1 - F_U(x)\big)\ dx$$

$$= \int_0^1 f_U(x-z)\big(1 - F_U(x)\big)\ dx - \int_0^1 \big(1 - F_U(x+z)\big)'\big(1 - F_U(x)\big)\ dx$$

$$= \int_0^1 f_U(x-z)\big(1 - F_U(x)\big)\ dx$$

$$- \big(1 - F_U(x+z)\big)\big(1 - F_U(x)\big)\Big|_{x=0}^{x=1} - \int_0^1 \big(1 - F_U(x+z)\big)f_U(x)\ dx$$

But $1 - F_U(1) = 0$, and $1 - F_U(0) = 1$, so,

$$-\big(1 - F_U(x+z)\big)\big(1 - F_U(x)\big)\Big|_{x=0}^{x=1} = 1 - F_U(z).$$

Also, since $f_U(x) = 0$ for $x \notin [0,1]$, we have:

$$\int_0^1 \big(1 - F_U(x+z)\big)f_U(x)\ dx = \int_{-\infty}^{\infty} \big(1 - F_U(x+z)\big)f_U(x)\ dx$$

$$= \int_{-\infty}^{\infty} \big(1 - F_U(t)\big)f_U(t-z)\ dt = \int_0^1 \big(1 - F_U(t)\big)f_U(t-z)\ dt$$

The last equality holds on account of the fact that if $t > 1$, then $1 - F_U(t) = 0$, and if $t < 0$, then $f_U(t-z) = 0$.

Therefore,

$$\int_0^1 \big(f_U(x+z) + f_U(x-z)\big)\big(1 - F_U(x)\big)\ dx$$

$$= \int_0^1 f_U(x-z)\big(1 - F_U(x)\big)\ dx$$

$$+ \big(1 - F_U(z)\big) - \int_0^1 \big(1 - F_U(t)\big)f_U(t-z)\ dt$$

$$= 1 - F_U(z)$$

$\square$

The next result is a justification of why our later work cannot be restricted to working on a compact interval. Let $W$ be one of the kernels transforming $f_{k-1}$ into $f_k$, i.e. a kernel of the form:

$$W(x, z) = f_U((x\rho - z) \; sign(x))$$

with $\rho \geq 1$. We will write $W_k$ for the kernel with $\rho = (\frac{k}{k-1})^a$, which is used to transform $f_{k-1}$ into $f_k$.

**Lemma.** *Let $M > 0, \rho \geq 1$ be such that $M\rho \geq 1$, and assume essential support$\{f_U\} = [0, 1]$.*
   *For $g \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, let*

$$\mathcal{W}(g)(z) = \int_{-\infty}^{+\infty} g(x)f_U((x\rho - z) \; sign(x)) \; dx.$$

*Then if $g$ has compact support, so does $\mathcal{W}(g)$, more precisely:*

   *a) If $supp(g) \subseteq [-M, M]$ then $supp(\mathcal{W}(g)) \subseteq [-M\rho, M\rho]$*

   *b) The bounds on the support are tight in the sense that:*

$$(-M\rho, M\rho) \subseteq \bigcup_{g \in L^1, supp(g) \subseteq [-M, M]} supp(\mathcal{W}(g)) \subseteq [-M\rho, M\rho]$$

*Proof.* Suppose $W(x, z) \neq 0$ and $|x| \leq M$ for some $x, z$.
   If $0 \leq x \leq M$, then, by definition of $W$ and assumption about the support of $f_U$ we get $0 \leq x\rho - z \leq 1$, so

$$-M\rho \leq -1 \leq x\rho - 1 \leq z \leq x\rho \leq M\rho$$

so $|z| \leq M\rho$.
   If $-M \leq x < 0$, then a similar argument yields: $0 \leq |x|\rho + z \leq 1$, so

$$-M\rho \leq -|x|\rho \leq z \leq 1 - |x|\rho \leq 1 \leq M\rho$$

and again, $|z| \leq M\rho$. Therefore, if $supp(g) \subseteq [-M, M]$ then

$$\mathcal{W}(g)(z) = \int_{-M}^{M} g(x)W(x, z) \; dx = 0 \text{ for } |z| > M\rho$$

i.e. $supp(\mathcal{W}(g)) \subseteq [-M\rho, M\rho]$.

Let $\mathcal{A} = \bigcup_{g \in L^1, supp(g) \subseteq [-M,M]} supp(\mathcal{W}(g))$. Part a) shows $\mathcal{A} \subseteq [-M\rho, M\rho]$. To show the other inclusion, take $g = \delta_r$ (the distribution concentrated at $x = r$), with $r \in (0, M)$. We have

$$\mathcal{W}(g)(z) = f_U(r\rho - z),$$

whose support is $[r\rho - 1, r\rho]$. By approximating $\delta_r$ under the integral with $g$ in $L^1$, we conclude $(r\rho - 1, r\rho) \subseteq \mathcal{A}$. Since this holds for all $r \in (0, M)$, $(-1, M\rho) = \bigcup_{r \in (0,M)} (r\rho - 1, r\rho) \subseteq \mathcal{A}$.

Next, taking $g = \delta_{-r}$, conclude $(-r\rho, 1 - r\rho) \subseteq \mathcal{A}$, and $(-M\rho, 1) = \bigcup_{r \in [0,M]} (-r\rho, 1 - r\rho) \subseteq \mathcal{A}$. This shows b). $\qquad \square$

We include the above Lemma to underscore two points that complicates a naïve approach to the problem:

i. There is no bounded interval $[-M, M]$ with $M \geq 1$ such that the integral transformations $g \mapsto \mathcal{W}_k(g)(z) = \int_{-M}^{M} g(x) W_k(x, z) \, dx$, $(g \in L^1(\mathbb{R})), k = 2, 3, \ldots$ map the space of $L^1$ functions supported in $[-M, M]$ to itself. Thus, we cannot fix a bounded interval that contains a priory the support of the family $\{f_k\}_k \geq 1$, and use compactness of integral transformations.

ii) For the limiting case (with $k = \infty$) with $\rho = 1$, we get that $\mathcal{W}$ maps $L^1[-M, M]$ (with $M \geq 1$) to itself. We will come back to this point when analyzing the distribution of $Z$.

There are some unrefereed resources that seem to suggest that because the operator that maps $g \in L^1(\mathbb{R})$ to

$$\mathcal{W}(g)(z) = \int_{-\infty}^{+\infty} g(x) f_U((x\rho - z) \ sign(x)) \ dx$$

has an integrand kernel, it must be compact. This is not the case, since the spectrum of $\mathcal{W}$ in $L^1(\mathbb{R})$ is not discrete. We will show this in later sections.

We will show:

**Theorem.** *With $X, Z, U, a, f_k$ as above, under assumptions (3) about the support of $f_U$,*

a) $\lim_{k \to \infty} X(k) = 0$ a.s.

b) $\overline{\lim}_{k \to \infty} |Z(k)| \leq 1$ a.s.

c) *the solution $f_Z$ to (4) given in (5) is unique, i.e.: there is one and only one pdf solution of (4)*

d) $f_k \to f_Z$ in $L^1(\mathbb{R})$.

e) *if $f_U \in L^p([0,1])$ for some $p > 1$ then there is $k_0 = k_0(p)$ such that $f_k \in L^\infty(\mathbb{R})$ for $k \geq k_0$ and $f_k \to f_Z$ in $L^\infty(\mathbb{R})$*

f) *$Z(k) \to_d Z$ (convergence in distribution).*

# 2 The path $\{X(k)\}_{k \geq 1}$

## 2.1 Stopping times

Let's consider some stopping times associated to the random path $\{X(k)\}_k \geq 1$. By definition,

$$X(2) = -U(1) < 0$$

so, to get to $X(3)$ we move in the direction towards 0 by adding a positeve term:

$$X(3) = X(2) + \frac{U(3)}{3^a}$$

and keep on adding terms until the path crosses 0 and changes sign, i.e.:

$$X(k) = X(2) + \sum_{j=3}^{k} \frac{U(j)}{j^a}$$

as long as $X(k) < 0$. We will show that eventually, $X(k) \geq 0$, at which time instead of adding, we force the path towards 0 by subtracting terms until the sign changes again, and so on. We define:

$$T_1 = T(1) = 2 \tag{6}$$

and for $k > 1$, as long as $T(k-1)$ is finite,

$$T_k = T(k) = \inf \left\{ j > T_{k-1} \;\middle|\; \sum_{s=1+T_{k-1}}^{j} \frac{U(s)}{s^a} > |X(T_{k-1})| \right\}. \tag{7}$$

Equivalently, as long as these stopping times are finite,

$$
\begin{aligned}
T_2 = T(2) &= \text{ first step } j > T_1 \text{ such that } X(j) > 0 \\
T_3 = T(3) &= \text{ first step } j > T_2 \text{ such that } X(j) < 0 \\
&\cdots \\
T_k = T(k) &= \text{ first step } j > T_{k-1} \text{ such that } (-1)^k X(j) > 0
\end{aligned}
$$

and we have

$$X(j) = X(T_k) - (-1)^k \sum_{s=1+T_k}^{j} \frac{U(s)}{s^a} \text{ for } T_k < j \leq T_{k+1} \tag{8}$$

and
$$sign(X(j)) = (-1)^{k-1} \text{ for } T_{k-1} \leq j < T_k \ (k > 2). \tag{9}$$

In particular,

$$X(T_k) = X(T_{k-1}) + (-1)^k \sum_{s=1+T_{k-1}}^{T_k} \frac{U(s)}{s^a} = (-1)^k \left( \sum_{s=1+T_{k-1}}^{T_k} \frac{U(s)}{s^a} - |X(T_{k-1})| \right)$$

and so

$$|X(T_k)| = \sum_{s=1+T_{k-1}}^{T_k} \frac{U(s)}{s^a} - |X(T_{k-1})| = \left( \sum_{s=1+T_{k-1}}^{T_k-1} \frac{U(s)}{s^a} - |X(T_{k-1})| \right) + \frac{U(T_k)}{T_k^a}$$

$$\leq \frac{U(T_k)}{T_k^a} \leq \frac{1}{T_k^a}. \tag{10}$$

Since $T_1 = 2 < T_2 < ..$ (and they are integers),

$$T_k \geq k + 1. \tag{11}$$

Let
$$\mathcal{F}_k = \sigma\text{-algebra generated by } U(1), ..., U(k). \tag{12}$$

These $\sigma$-algebras form a filtration $\mathcal{F}_*$ of the underlying probability space in which $U(1), U(2), \ldots$ are defined.

**Lemma 2.1.** *For $k \geq 1$, $T_k$ is a stopping time of the filtration $\mathcal{F}_*$.*

*Proof.* $T_k$ is the step when the $k^{th}$ sign change occurs, so the event $(T_k \leq s)$ is true iff there are $k$ or more sign changes in the sequence $X(1), \ldots, X(s)$, which depend on $U(1), ..., U(s)$, i.e.: $(T_k \leq s) \in \mathcal{F}_s$.  □

**Remarks** The process $U(1), U(2), \ldots$ is (trivially) a stationary Markov process. The strong Markov property applied to this case implies that if $T$ is a stopping time of $\mathcal{F}_*$, finite a.s, then

  i. $U(T + 1), U(T + 2), \ldots$ are independent i.i.d distributed with the same distribution as $U(1), U(2), \ldots$

  ii. $U(T + 1), U(T + 2), \ldots$ are independent of $\mathcal{F}_T$, the filtration stopped at $T$.

## 2.2   Asymptotic properties of the stopping times

**Lemma 2.2.** *Let $U(k), X(k), Z(k)$ as in the first section, $T(k)$ as in (7) , $f_U$ satisfies assumptions (1) to (3). Let $\mu = E(U(1)) = \int_0^1 x \; f_U(x) \; dx$. Then,*

  *i. For every $j \geq 1$, $T_j$ is finite a.s.*

  *ii. For $j, N \geq 1$, with $\Delta T_{j+1} = T_{j+1} - T_j$,*

$$E\left(\Delta T_{j+1} > N | \mathcal{F}_{T_j}\right) \; \leq \; exp\left(\mu \frac{(1+T_j)^a}{T_j^a} - \frac{\mu^2(1+T_j)^a}{2} \sum_{s=1}^{N} \frac{1}{(s+T_j)^a}\right)$$

  *iii. There is a constant $\gamma > 0$ ($\gamma = 3$ works) such that, regardless of the density $f_U$,*
   $\varlimsup_{j \to \infty} \mu^2 \frac{(T_{j+1}-T_j)}{T_j(\frac{\ln j}{j})} \leq \gamma$ *almost surely.*

  *iv. $\lim_{j \to \infty} \frac{T_{j+1}}{T_j} = 1$ a.s.*

*Proof.* We proceed inductively. If $k = 1$, then $T_1 = 2$, and the statement holds trivially. Suppose $T_j$ is finite a.s.. Let $P_j$ be conditional probability given $\mathcal{F}_{T_j}$. We have

$$\Delta T_{j+1} = T_{j+1} - T_j > k \text{ iff } \sum_{s=1}^{k} \frac{U(s+T_j)}{(s+T_j)^a} \leq |X(T_j)|,$$

therefore, taking conditional probabilities:

$$P_j\left(\Delta T_{j+1} > k\right) \leq P_j\left(\sum_{s=1}^{k} \frac{U(s+T_j)}{(s+T_j)^a} \leq |X(T_j)|\right).$$

Since $T_j$ and $X(T_j)$ are $\mathcal{F}_{T_j}$ measurable, and $U(T_j+1), \ldots$ are iid conditional on $\mathcal{F}_{T_j}$, with the same distribution as $U(1), \ldots$, if we apply Markov's inequality we get for any $\lambda > 0$:

$$P_j\left(\sum_{s=1}^{k} \frac{U(s+T_j)}{(s+T_j)^a} \leq |X(T_j)|\right) \leq e^{\lambda|X(T_j)|} E_j\left\{e^{-\lambda(\sum_{s=1}^{k} \frac{U(s+T_j)}{(s+T_j)^a})}\right\}$$

$$= e^{\lambda|X(T_j)|} \prod_{s=1}^{k} E_j\left\{e^{-\lambda \frac{U(s+T_j)}{(s+T_j)^a}}\right\},$$

where $E_j$ is the conditional expectation given $\mathcal{F}_{T_j}$.

  Then we have

$$E_j\left\{e^{-\lambda \frac{U(s+T_j)}{(s+T_j)^a}}\right\} = e^{-\lambda \frac{\mu}{(s+T_j)^a}} E_j\left\{e^{-\lambda \frac{U(s+T_j)-\mu}{(s+T_j)^a})}\right\}.$$

Applying Hoeffding's Lemma to $-\lambda \frac{U(s+T_j)-\mu}{(s+T_j)^a}$, which has mean $0$ and is bounded in absolute value by $\frac{\lambda}{(s+T_j)^a}$, we get

$$E_j \left\{ e^{-\lambda \frac{U(s+T_j)-\mu}{(s+T_j)^a}} \right\} \leq exp\left( \frac{1}{2} \frac{\lambda^2}{(s+T_j)^{2a}} \right).$$

Therefore:

$$P_j\left(\Delta T_{j+1} > k\right) \leq exp\left( \lambda |X(T_j)| - \lambda\mu \sum_{s=1}^{k} \frac{1}{(s+T_j)^a} + \frac{\lambda^2}{2} \sum_{s=1}^{k} \frac{1}{(s+T_j)^{2a}} \right)$$

$$\leq exp\left( \lambda \frac{1}{T_j^a} - \lambda\mu \sum_{s=1}^{k} \frac{1}{(s+T_j)^a} + \frac{\lambda^2}{2} \sum_{s=1}^{k} \frac{1}{(s+T_j)^{2a}} \right) \quad (13)$$

First, let's chose $\lambda$ in (13) so that

$$\frac{\lambda^2}{2} \sum_{s=1}^{k} \frac{1}{(s+T_j)^{2a}} \leq \frac{\lambda\mu}{2} \sum_{s=1}^{k} \frac{1}{(s+T_j)^a}$$

For example,

$$\lambda = \mu(1+T_j)^a. \quad (14)$$

With this choice, each term in the sums above satisfies the inequality

$$\lambda \frac{1}{(s+T_j)^{2a}} \leq \mu \frac{1}{(s+T_j)^a},$$

so we get

$$P_j\left(\Delta T_{j+1} > k\right) \leq exp\left( \mu \frac{(1+T_j)^a}{T_j^a} - \frac{\mu^2(1+T_j)^a}{2} \sum_{s=1}^{k} \frac{1}{(s+T_j)^a} \right). \quad (15)$$

This shows part ii).

Taking limits when $k \to \infty$,

$$P_j(\Delta T_{j+1} = +\infty) = 0, \quad (16)$$

and taking unconditional probabilities (averaging over $T_j$):

$$P(\Delta T_{j+1} = +\infty) = 0, \quad (17)$$

so $T_{j+1}$ is finite a.s. This completes the induction step to show part i).

Going back to (15), and taking $k$ of the form $k_j = R (1+T_j) \frac{\ln(j)}{j}$, with $R > 0$ we have:

$$\sum_{s=1}^{k_j} \frac{1}{(s+T_j)^a} \geq \int_1^{k_j+1} \frac{1}{(x+T_j)^a} \, dx = \frac{(k_j + 1 + T_j)^{1-a} - (1+T_j)^{1-a}}{1-a})$$

$$= (1+T_j)^{1-a} \frac{(1 + R\frac{\ln(j)}{j})^{1-a} - 1}{1-a}) = (1+T_j)^{1-a}(R\frac{\ln(j)}{j})(1 + o(1))$$

$$\geq (1+j)(1+T_j)^{-a}(R\frac{\ln(j)}{j})(1+o(1)) = (1+T_j)^{-a} R \ln(j)(1+o(1))$$

and so

$$P_j (\Delta T_{j+1} > k_j) \leq exp\Big(\mu(1 + \frac{1}{T_j})^a - \frac{\mu^2}{2} R \ln(j)(1+o(1))\Big)$$

$$\leq exp(\mu(\frac{3}{2})^a) \frac{1}{j^{\frac{R\mu^2(1+o(1))}{2}}}.$$

Taking unconditional probabilities:

$$P (\Delta T_{j+1} > k_j) \leq exp(\mu(\frac{3}{2})^a) \frac{1}{j^{\frac{R\mu^2(1+o(1))}{2}}}.$$

Adding these inequalities:

$$\sum_{j\geq 1} P (\Delta T_{j+1} > k_j) \leq \sum_{j\geq 1} exp(\mu(\frac{3}{2})^a) \frac{1}{j^{\frac{R\mu^2(1+o(1))}{2}}} < \infty \qquad (18)$$

if $R$ is sufficiently large. By Borel-Cantelli,

$$P (\Delta T_{j+1} > k_j \text{ infinitely often }) = 0, \qquad (19)$$

i.e.: for each particular realization of the process $\{X(k)\}_{k\geq 1}$ , there is $N_0$ (depending on the particular realization) such that

$$\Delta T_{j+1} \leq R (1+T_j)\frac{\ln(j)}{j} \qquad (20)$$

for $j \geq N_0$ .

Going back to (18) with $R = \frac{2}{\mu^2}(1 + \epsilon)$ we get:

$$\overline{\lim}_{j\to\infty} \mu^2 \frac{(T_{j+1} - T_j)}{T_j(\frac{\ln j}{j})} \leq 3.$$

This shows iii).

From (20), for $j$ large:

$$0 \leq \frac{\Delta T_{j+1}}{T_j} \leq R \ (1 + \frac{1}{T_j}) \frac{\ln(j)}{j}$$

so

$$\lim_{j \to \infty} \frac{\Delta T_{j+1}}{T_j} = 0.$$

This shows iv).

$\square$

## 2.3   Convergence of $\{X(k)\}_k$ and $\{Z(k)\}_k$

**Lemma 2.3.** *Let* $U(k), X(k), Z(k)$ *as in the first section,* $T(k)$ *as in (7)* , $f_U$ *satisfies assumptions (1) to (3). Let* $\mu = E(U(1)) = \int_0^1 x \ f_U(x) \ dx$. *Then,*

i. $\lim_{j \to \infty} X(j) = 0$ *almost surely*

ii. $\overline{\lim}_{j \to \infty} |Z(j)| \leq 1$ *almost surely*

iii. *There exists* $M_0, \gamma_0, \ldots, \gamma_3$ *that depend only on* $\mu$ *and* $a$ *such that, if* $j \geq \gamma_0$, $M \geq M_0$ *then*

$$P(|Z(j)| \geq M) \ \leq \ exp(\gamma_3 - \ M^{(1-a)/a} \ (1 + j)^{1-a} \gamma_2).$$

*Proof.* Since $X(j) = \frac{Z(j)}{j^a}$, i) follows from ii). To show ii), let $j \geq 2$ and let

$$k = \ \text{number of sign changes in the sequence } X(1), X(2), \ldots, X(j-1)$$

Since $X(1), X(2), \ldots$ changes sign at $T(1), T(2), \ldots$, we have

$$k = \min\{s \mid T_s \geq j\} - 1,$$

so $T_k < j \leq T_{k+1}$.

If $j < T_{k+1}$, then $sign(X(j)) = (-1)^k$ and, as in (8),

$$\begin{aligned}
X(j) \ &= \ X(T_k) - (-1)^k \sum_{s=1+T_k}^{j} \frac{U(s)}{s^a} \\
&= \ (-1)^k \left( |X(T_k)| - \sum_{s=1+T_k}^{j} \frac{U(s)}{s^a} \right).
\end{aligned}$$

and

$$|X(j)| \; = \; |X(T_k)| - \sum_{s=1+T_k}^{j} \frac{U(s)}{s^a} \qquad \leq \; |X(T_k)| \leq \max\{|X(T_k)|, |X(T_{k+1})|\}.$$

If $j = T_{k+1}$, then trivially $|X(j)| \leq \max\{|X(T_k)|, |X(T_{k+1})|\}$. Whether $j = T_{k+1}$, or $j < T_{k+1}$, we have $|X(j)| \leq \max\{|X(T_k)|, |X(T_{k+1})|\}$. Now apply the bounds in (10),

$$|X(j)| \leq \max\{\frac{1}{T_k^a}, \frac{1}{T_{k+1}^a}\} = \frac{1}{T_k^a}$$

and

$$|Z(j)| = j^a |X(j)| \leq \frac{j^a}{T_k^a} \leq \frac{j^a}{T_{k+1}^a} \frac{T_{k+1}^a}{T_k^a} \leq \frac{T_{k+1}^a}{T_k^a} \tag{21}$$

From Lemma 2.2 we have $\lim_{j \to \infty} \frac{T_{j+1}}{T_j} = 1$ a.s., so

$$\overline{\lim}_{j \to \infty} |Z(j)| \leq 1 a.s..$$

This shows ii).

From ii), we get that $\lim_{j \to \infty} P(|Z(j)| \geq M) = 0$ for any $M > 1$. We want to quantify the rate of convergence to justify the simulation based observation made by Grandville that $\{Z(j)\}_{j \geq 1}$ converges to a distribution supported on $[-1, 1]$ at a high rate. The idea is to exploit (21), since

$$(|Z(j)| \geq M) \subset (\frac{T_{k+1}^a}{T_k^a} \geq M) = (\Delta T_{k+1} \geq T_k(M^{1/a} - 1)).$$

Note that $k = k(j)$ is a random variable. It can be used to establish point-wise results, but since $T(k(j))$ is not a stopping time of the filtration (12), we need some extra care when using conditional expectations.

Let $M > 1$. For the remainder of the proof, in order to lighten the notation, we will use $\gamma_1, \gamma_2$, etc. to denote positive constants that depend only on $M, \mu, a$, but not always the same constant from line to line.

Let $j \geq \gamma_0 > 1$; with $\gamma_0$ to be determined. We have:

$$
\begin{aligned}
(|Z(j)| \geq M) \;&\subseteq\; \bigcup_{k \geq 1} (|Z(j)| \geq M \text{ and } T_k \leq j < T_{k+1}) \\
&\subseteq\; \bigcup_{k \geq 1} (\frac{T_{k+1}^a}{T_k^a} \geq M \text{ and } T_k \leq j < T_{k+1}) \\
&=\; \bigcup_{k \geq 1} (\Delta T_{k+1} \geq T_k(M^{1/a} - 1) \text{ and } T_k \leq j < T_{k+1}) \\
&\subseteq\; \bigcup_{k \geq 1} (\Delta T_{k+1} > \max\{1, T_k \, \Gamma_1, j - T_k\} \text{ and } T_k \leq j),
\end{aligned}
$$

where $\Gamma_1 = M^{1/a} - 2$, and we take $\Gamma_1 > 2$ by requiring $M$ to be large enough. Then

$$P(|Z(j)| \geq M) \;=\; \sum_{k\geq 1} P\big(\Delta T_{k+1} > \max\{T_k\ \Gamma_1, j - T_k, 1\}\text{ and }T_k \leq j\big)$$

$$\;=\; E\left(\sum_{k\geq 1} P_k\big(\Delta T_{k+1} > \max\{T_k\ \Gamma_1, j - T_k, 1\}\text{ and }T_k \leq j\big)\right)$$

where $P_k$ is conditional probability given $\mathcal{F}_{T_k}$. The event $(T_k \leq j)$ is $\mathcal{F}_{T_k}$-measurable, therefore

$$P_k\Big(\Delta T_{k+1} \geq \max\{T_k\ \Gamma_1, j - T_k, 1\}\text{ and }T_k \leq j\Big)$$

$$\;=\; P_k\Big(\Delta T_{k+1} \geq \max\{T_k\ \Gamma_1, j - T_k, 1\}\Big)I(T_k \leq j)$$

From Lemma 2.2, relabeling $T_j$ as $T_k$, and with $N = floor(\max\{T_k\ \Gamma_1, j - T_k, 1\})$:

$$P_k(\Delta T_{k+1} > N) \;\leq\; exp\left(\mu\frac{(1+T_k)^a}{T_k^a} - \frac{\mu^2(1+T_k)^a}{2}\sum_{s=1}^{N}\frac{1}{(s+T_k)^a}\right).$$

But

$$\sum_{s=1}^{N}\frac{1}{(s+T_k)^a} \;\geq\; \int_1^{N+1}\frac{1}{(s+T_k)^a}\,ds \;=\; \frac{(N+T_k+1)^{1-a} - (1+T_k)^{1-a}}{1-a}.$$

If $\max\{T_k\ \Gamma_1, j - T_k, 1\} = 1$ , then $T_k\ \Gamma_1 \leq 1$, and $j \leq 1 + T_k \leq 1 + 1/\Gamma_1$ which is impossible if $j \gg 1$.

If $\max\{T_k\ \Gamma_1, j - T_k, 1\} = T_k\ \Gamma_1$, then $N = floor(T_k\Gamma_1)$, and

$$(N+T_k+1)^{1-a} - (1+T_k)^{1-a} \;\geq\; \gamma_2(\Gamma_1+1)^{1-a}(1+T_k)^{1-a}.$$

Also $j - T_k \leq T_k\ \Gamma_1$, so

$$(1+T_k) \geq T_k \geq \frac{1}{1+\Gamma_1}\ j \geq \gamma_3\ (1+j)$$

and

$$(N+T_k+1)^{1-a} - (1+T_k)^{1-a} \;\geq\; \gamma_2(\Gamma_1+1)^{1-a}(1+j)^{1-a}.$$

If $\max\{T_k\ \Gamma_1, j - T_k, 1\} = j - T_k \geq T_k\ \Gamma_1$, then $N = j - T_k$, and

$$(N+T_k+1)^{1-a}-(1+T_k)^{1-a} = (j+1)^{1-a}-(1+T_k)^{1-a} \geq (j+1)^{1-a}-(1+\frac{j}{1+\Gamma_1})^{1-a} \geq \gamma_3(\Gamma_1+1)^{1-a}(1+$$

In any case, regardless of the value of $\max\{T_k \ \Gamma_1, j - T_k, 1\}$

$$\sum_{s=1}^{N} \frac{1}{(s + T_k)^a} \geq \gamma_2(\Gamma_1 + 1)^{1-a}(1 + j)^{1-a},$$

therefore,

$$P_k(\Delta T_{k+1} > N) \leq exp\left(\gamma_1 - \gamma_2(1 + T_k)^a \ (\Gamma_1 + 1)^{1-a}(1 + j)^{1-a}\right). \qquad (22)$$

Since $k \leq 1 + T_k$, we have

$$P_k(\Delta T_{k+1} > N) \leq exp\left(\gamma_1 - \gamma_2 k^a \ (\Gamma_1 + 1)^{1-a} \ (1 + j)^{1-a}\right), \qquad (23)$$

and so (recall $\Gamma_1 + 1 = M^{1/a} - 1$):

$$P(|Z(j)| \geq M) \leq E\left(\sum_{k\geq1} exp\left(\gamma_1 - \gamma_2 k^a \ M^{(1-a)/a} \ (1 + j)^{1-a}\right) I(T_k \leq j)\right)$$

$$\leq \sum_{k\geq1} exp\left(\gamma_1 - \gamma_2 k^a \ M^{(1-a)/a} \ (1 + j)^{1-a}\right)$$

$$\leq exp(\gamma_3 - \ M^{(1-a)/a} \ (1 + j)^{1-a}\gamma_2).$$

This shows iii).

$\square$

**Remark** In the next section we will show that $Z(j)_{j\geq1}$ converges in probability to a distribution supported on the interval $[-1, 1]$. We will show elsewhere that the process $Z(j)_{j\geq1}$ visits the neighbourhoods of any point in $(-1, 1)$ infinitely often.

# 3   The spectrum of $W$

Let $U(k), X(k), Z(k)$ be as in the Introduction, $f_U$ satisfies assumptions.

**Definition** Let $C([A, B])$ be the space of (real valued) continuous functions on $[A, B]$ with the usual *supremum* norm, and let $C_\omega([A, B]) = \{f \in C([A, B])|f(A) = f(B) = 0\}$ be the subspace of $C([A, B])$ consisting of those functions that vanish at the boundary of $[A, B]$.

Let $W_k : C(\mathbb{R}) \longmapsto C(\mathbb{R})$ be defined as

$$W_k(g)(x) = \int_{-\infty}^{+\infty} g(s) \ f_U\left(\left(\left(s \left(\frac{k}{k-1}\right)^a - x\right) \ sign(s)\right) \ ds. \qquad (24)$$

Here $k = 3, 4, \ldots, \infty$.

We will write $W$ for the limit case $k = \infty$. In this section we go over properties of the spectrum of $W$. Many of the statements apply to more general integral operators. Some are linked to the form of $W$.

Since $f_U \in L^1([0,1])$, the integrals in the definition of $W_k$ is convergent, and $W_k$ maps continuous functions into continuous functions. By abuse of notation, we will use $W_k$, $W$ for the restrictions and co-restrictions of these maps to various subspaces; the domains and co-domains should be clear from the context.

We can extend an element $g \in C_{00}([A,B])$ to an element of $C(\mathbb{R})$ or $C_{00}([A_1, B_1])$ if $[A, B] \subset [A_1, B_1]$ by defining $g$ as $0$ outside $[A, B]$; we will not distinguish $g$ from its extensions.

The operator $W_k$ is closely related to convolutions. Indeed, we can write

$$
\begin{aligned}
W_k(g) &= \int_{-\infty}^{+\infty} g(s)\, f_U\left(\left(\left(s\left(\frac{k}{k-1}\right)\right)^a - x\right)\ sign(s)\right) ds \\
&= \int_{-\infty}^{0} \left(\frac{k-1}{k}\right)^a g(\left(\frac{k}{k-1}\right)^a s)\, f_U((x-s))\, ds \\
&\quad + \int_{0}^{\infty} \left(\frac{k-1}{k}\right)^a g(\left(\frac{k}{k-1}\right)^a s)\, f_U((s-x))\, ds
\end{aligned}
$$

Thus, we can write:

$$
W_k(g) = A_k(g) * f_U + B_k(g) * \mathcal{O}(f_U) \tag{25}
$$

where

$$
\begin{aligned}
A_k(g)(s) &= \left(\frac{k-1}{k}\right)^a g(\left(\frac{k}{k-1}\right)^a s) I(s \le 0) \\
B_k(g)(s) &= \left(\frac{k-1}{k}\right)^a g(\left(\frac{k}{k-1}\right)^a s) I(s \ge 0) \\
\mathcal{O}(f_U)(s) &= f_U(-s)
\end{aligned}
$$

Note that (25) can be used to define $W_k$ in $L^p$ spaces.

## 3.1   $W$ restricted to $C_{00}([-M,M])$

For any $r > 0$, let $B(0, r)$ be the open disc in the complex plane centered at $0$ and of radius $r$.

**Lemma.** *$f_U$ satisfies the assumptions in section 1, $W$ as in (24), $M \ge 1$. Then,*

    *i. $W(C([-M, M])) \subseteq C_{00}([-M, M])$, in particular, $W$ is an endomorphism of $C_{00}([-M, M])$*

ii. $W : C([-M, M]) \longmapsto C_{00}([-M, M])$ *is compact*

iii. $W$ *is order preserving: if* $g \in C(\mathbb{R})$ *satisfies* $g(x) \geq 0$ *for all* $x$, *then* $W(g)(x) \geq 0$ *for all* $x$

iv. $\int_{-\infty}^{+\infty} W(g)(x) \, dx = \int_{-\infty}^{+\infty} g(s) \, ds$ *for all* $g \in L^{\infty}([-M, M])$ *(g extended as 0 outside* $[-M.M]$*)*

v. *Let* $\varphi \in C_{00}([-M, M])$ , $\varphi \in Ker(\mathbb{I} - W)$, *then*

    a. $|\varphi| \in Ker(\mathbb{I} - W)$

    b. $\varphi(-x) \in Ker(\mathbb{I} - W)$

    c. *if* $\varphi(0) = 0$ *then* $\varphi = 0$

vi. *The spectrum of* $W$ *satisfies*

$$1 \in spec \left( W \Big|_{C_{00}([-M,M])} \right) \subset \overline{B(0,1)}$$

vii. *(Perron) There is an eigenvector* $\phi_1$ *of* $W$ *corresponding to the eigenvalue* 1 *such that*

    a. $\phi_1(x) \geq 0$

    b. $\int \phi_1 = 1$

    c. 1 *is a simple eigenvalue (i.e. of multiplicity 1)*

    *Moreover,* $\phi_1$ *is supported on* $[-1, 1]$.

viii. *If* $\lambda \in spec \left( W \Big|_{C_{00}([-M,M])} \right)$, *and* $|\lambda| = 1$ *then* $\lambda = 1$

ix. *If* $1 \leq M_1 < M_2$ *then*

$$spec(W \Big|_{C_{00}([-M_1,M_1])}) \subseteq spec(W \Big|_{C_{00}([-M_2,M_2])})$$

**Remark** Some comments before going through the proof. Part vii.c) is a consequence of the assumption that $f_U$ doesn't have too many zeros on $[0, \epsilon]$ for some $\epsilon > 0$, as per the assumption (3) about $f_U$. This assumption is needed to ensure simplicity of 1, and the spectral gap, and it is standard in the context of Perron's theorem. Conceivably, the assumptions on $f_U$ can be relaxed, but without a strong way of identifying the weak limit of $\{Z(k)\}_{k \geq 1}$, one would expect the proof of weak convergence will be more delicate.

*Proof.* Most of these are basic facts of integral operators with a stochastic kernel, and we will skip some details.

If $g \in C([-M, M]$, then, by part a) of the lemma in the introduction, $supp(W(g)) \subseteq [-M, M]$. Since $W(g)$ is continuous everywhere, and vanishes outside [-M.M], by continuity, $W(g)$ vanishes at $\pm M$, i.e. $W(g) \in C_{00}([-M, M])$. This shows i).

Part ii) is a standard result of integral operators.

Part iii) is immediate, since, in case $g$ is non-negative, the integrand in (24) is non-negative. We encountered W in the context of writing the kernel mapping a probability density to another probability density. Part iv) is basically a re-statement of that fact.

Part iv) can be re-stated as follows. $C_{00}([-M, M])'$, the dual of $C_{00}([-M, M])$ consists of finitely additive measures. By abuse of notation, let $dx$ be the Lebesgue measure on $[-M, M]$, as element of $C_{00}([-M, M])'$ in the canonical way. Then iv) says that $W'(dx) = dx$.

For v.a), let $\varphi = W(\varphi)$.

$$
\begin{aligned}
|\varphi(x)| &= \left| \int_{-\infty}^{+\infty} \varphi(s) \ f_U((s-x) \ sign(s)) \ ds \right| \\
&\leq \int_{-\infty}^{+\infty} |\varphi(s)| \ f_U((s-x) \ sign(s)) \ ds \\
&= W(|\varphi|)(x)
\end{aligned}
$$

Integrating over $x$ and using iv), be get

$$
\int |\varphi| \ dx \leq \int W(|\varphi|) \ dx = \int |\varphi| \ dx
$$

so the first inequality is an equality, and $|\varphi(x)| = W(|\varphi|)(x)$ a.e., but since both are continuous functions, then theay are equal everywhere.

v.b) follows from a simple change of variables in the integral.

For part v.c), let $\varphi \in Ker(\mathbb{I} - W)$ be such that $\varphi(0) = 0$. Then $\varphi_1(x) = \frac{|\varphi(x)| + |\varphi(-x)|}{2}$ is also in $Ker(\mathbb{I} - W)$, it is non-negative, symmetric, and vanishes at 0. Using the definition of $W$ and these properties of $\varphi_1$

$$
0 = \varphi_1(0) = \int_0^1 \varphi_1(s) \ 2 \ f_U(s) \ ds.
$$

but $f_U(s) > 0$ for almost all points (by the assumption *essential support* $f_U = [0, 1]$), therefore $\varphi_1(s) = 0$ for $s \in [0, 1]$.

Let $A = \sup\{x \in [0, M] \mid \varphi_1(s) = 0 \, for \, s \in [0, x]\}$. So far we know $1 \le A \le M$. If $A < M$ then

$$0 = \varphi_1(A) = \int_A^M \{f_U(x + A) + f_U(x - A)\} \; \varphi_1(x) \; dx,$$

but $f_U(x + A) = 0$ for $x > 0$, since $A \ge 1$, so

$$0 = \varphi_1(A) = \int_A^M f_U(x - A) \; \varphi_1(x) \; dx,$$

then $f_U(x - A) \; \varphi_1(x) = $ for all $x \in [A, \min\{1, M\}]$, where $f_U(x - A) > 0$ almost everywhere, so $\varphi_1(x) = $ for all $x \in [A, \min\{1, M\}]$, which contradicts the definition of $A$. It follows that $A = M$, and $\varphi_1 = 0$ and so $\varphi = 0$.

For part vi), notice that from iv), $W'(dx) = dx$, so 1 is an eigenvalue of $W'$ with eigenvector $dx$, but $spec(W') = spec(W)$, so $1 \in spec(W)$.

If $\lambda \in spec(W)$, $\lambda \ne 0$, then, since $W$ is compact, there is an eigenvector $\psi$ corresponding to $\lambda$, normalized so that $\int |\psi| \, dx = 1$. Then

$$
\begin{aligned}
|\lambda| &= \left| \lambda \int \psi \; dx \right| = \left| \int W(\psi) \; dx \right| \\
&\le \int |W(\psi)| \; dx \; \le \; \int W(|\psi|) \; dx = \int |\psi| \; dx = 1
\end{aligned}
$$

For vii), since 1 is an eigenvalue, it has an eigenvector $\phi$, which we can normalize so that $\int |\phi| \, dx = 1$. By part v), $\phi_1 = |\phi|$ is also an eigenvector corresponding to the eigenvalue 1.

If $\phi_2$ is another eigenvector corresponding to the eigenvalue 1, then $\phi_2 - \frac{\phi_2(0)}{\phi_1(0)}\phi_1 \in Ker(\mathbb{I} - W)$ and vanishes at 0. By part v.c), $\phi_2 = \frac{\phi_2(0)}{\phi_1(0)}\phi_1$, so $Ker(\mathbb{I} - W) = < \phi_1 >$, i.e.: 1 has geometric multiplicity 1.

If $(\mathbb{I} - W)^2 H = 0$, then $(\mathbb{I} - W)H \in Ker(\mathbb{I} - W)$, so $(\mathbb{I} - W)H = \alpha\phi_1$ for some $\alpha$. But $0 = \int (\mathbb{I} - W)H \; dx = \alpha \int \phi_1 \; dx = \alpha$, so $(\mathbb{I} - W)H = 0$. This shows 1 has algebraic multiplicity 1.

For part viii), let $\lambda \in spec(W)$, $|\lambda| = 1$, and let $\phi$ be an eigenvector normalized as $\int |\phi| \, dx = 1$. We have

$$|\psi(x)| = |W(\psi)(x)| \le W(|\psi|)(x)$$

and since $\int |\psi| = 1 = \int W(|\psi|)$, then the inequality is an equality a.e., and $|\psi(x)| \in Ker(\mathbb{I} - W)$, so $|\psi(x)| = \alpha\varphi_1$ for some $\alpha$, and om account of the $L^1$ normalization, $\alpha = 1$. We can write $\psi(x) = \epsilon(x)\varphi_1(x)$ with $\epsilon$ continuous where $\varphi_1(x) > 0$, and $|\epsilon(x)| = 1$ for all $x$. Then, for any $x$:

$$
\begin{aligned}
\varphi_1(x) &= \bar{\lambda}\, \overline{\epsilon(x)} \int_{-M}^{M} \epsilon(s)\, \varphi_1(s)\, f_U((s-x)sign(s))\, ds \\
&= \int_{-M}^{M} \bar{\lambda}\, \overline{\epsilon(x)}\, \epsilon(s)\, \varphi_1(s)\, f_U((s-x)sign(s))\, ds \\
&\leq \int_{-M}^{M} \Re\left\{\bar{\lambda}\, \overline{\epsilon(x)}\, \epsilon(s)\right\} \varphi_1(s)\, f_U((s-x)sign(s))\, ds \\
&\leq \int_{-M}^{M} \varphi_1(s)\, f_U((s-x)sign(s))\, ds = \varphi_1(x)
\end{aligned}
$$

so
$$
\Re\left\{\bar{\lambda}\, \overline{\epsilon(x)}\, \epsilon(s)\right\} = 1 \text{ where } \varphi_1(s)\, f_U((s-x)sign(s)) > 0
$$

Taking $x > 0$ small, $s = -x$ , $\varphi_1(s)\, f_U((s-x)sign(s)) = \varphi_1(-x)\, f_U(2x) > 0$ for a.e. $x$ near 0, so, for a.e. small $x > 0$

$$
\Re\left\{\bar{\lambda}\, \overline{\epsilon(x)}\, \epsilon(-x)\right\} = 1
$$
$$
\bar{\lambda}\, \overline{\epsilon(x)}\, \epsilon(-x) = 1
$$
$$
\lambda = \epsilon(x)\overline{\epsilon(-x)}.
$$

But $\epsilon$ is continuous at 0, so letting $x \to +0$, we get $\lambda = |\epsilon(0)|^2 = 1$

For ix), if $\lambda \neq 0$ is in $spec(W\big|_{C_{00}([-M_1,M_1])})$, and $\phi$ is a corresponding eigenvalue, then, extending $\phi$ as 0 outside $[-M_1, M_1]$ we have:

$$
\begin{aligned}
W\big|_{C_{00}([-M_2,M_2])}(\phi)(x) &= \int_{-M_2}^{M_2} \phi(s)f_U((s-x)sign(s))\, ds \\
&= \int_{-M_1}^{M_1} \phi(s)f_U((s-x)sign(s))\, ds
\end{aligned}
$$

because $\phi(s) = 0$ for $|s| \geq M_1$.

If $|x| \leq M_1$ then

$$
\int_{-M_1}^{M_1} \phi(s)f_U((s-x)sign(s))\, ds = W\big|_{C_{00}([-M_1,M_1])}(\phi)(x) = \lambda\, \phi(x)
$$

If $M_1 < x \leq M_2$ then

$$\int_{-M_1}^{M_1} \phi(s) f_U((s-x)sign(s)) \ ds = \int_0^{M_1} \phi(-s) f_U(s+x) + \phi(s) f_U(s-x) \ ds = 0$$

because under the integral, $s+x > 0+M_1 \geq 1$, so $f_U(s+x) = 0$, and $s-x \leq M_1-x < 0$, so $f_U(s-x) = 0$. Similarly, the integral is 0 when $-M_2 \leq x < -M_1$. But the extension of $\phi$ is also 0 at $x$ such that $M_1 < |x| \leq M_1$, so

$$W\Big|_{C_{00}([-M_2,M_2])}(\phi)(x) = \lambda \ \phi(x)$$

for all $x \in [-M_2, M_2]$, hence $\lambda \in spec(W\Big|_{C_{00}([-M_2,M_2])})$   $\square$

**Corollary 3.1.** *(Spectral gap) There is $r$, $0 < r < 1$ such that*

$$spec(W) \subset \overline{B(0,r)} \cup \{1\}.$$

*Proof.* 1 is the only point of $spec(W)$ at the boundary of $B(0,1)$, and 0 is the only accumulation point of $spec(W)$.   $\square$

## 3.2   $W$ restricted to $L^1[-M, M]$

$W$ can be extended to several spaces; in particular, if $M \geq 1$, considering that the integral

$$\int_{-M}^{M} \int_{-M}^{M} |g(s)| \ f_U((s-x)sign(s)) \ ds \ dx$$

is finite for all $g \in L^1[-M, M]$, we can extend $W$ to $L^1[-M, M]$ as

$$W(g)(x) = \int_{-M}^{M} g(s) \ f_U((s-x)sign(s)) \ ds \tag{26}$$

for those $x$ for which the integrand is in $L^1$, i.e.: for a.e. $x$. This defines a map from $L^1[-M, M]$ to itself.

We explore the spectrum of the resulting operator. When working in $C_{00}$ we have access to the pointwise definition of $W(g)(x)$. In $L^1$, we will make use of duality.

**Lemma.** *Let $M \geq 1$; let $W$ be defined by (26); let $\phi_1$ be the eigenvector of $W\big|_{C_{00}([-1,1])}$ corresponding to the eigenvalue 1, normalized as $\phi_1 \geq 0$, $\int \phi_1 = 1$. Then*

   *i.* $W\big|_{L^1[-M,M]}$ *is compact.*

   *ii. There is $r$, $0 < r < 1$ such that*

$$spec(W\big|_{L^1[-M,M]}) \subset \overline{B(0,r)} \cup \{1\}.$$

   *iii.* $Ker(W\big|_{L^1[-M,M]} - 1) = <\phi_1> .$

Before going into the proof, let's mention that ii) is the statement of existence of the spectral gap for the Markov operator $W$, and iii) is equivalent to existence and uniqueness of the invariant measure (invariant for the Markov chain defined by $W$). One would expect, given the form of kernel of $W$, that it would be easy to show that if $h \in L^1[-M, M]$ satisfies $W(h) = h$, then $h$ is continuous, and reduce the problem of identifying $h$ to the case of continuous eigenvectors, however, it

*Proof.* For i), replace $f_U$ by $H \in L^1[0,1]$, $H$ continuous. The operator with $f_U$ replaced by $H$ is compact in $L^1$, and converges (in the operator norm for endomorphisms of $L^1[-M, M]$) to $W$ as $H \longrightarrow f_U$ in $L^1$. So $W$ is compact.

For ii), note that

$$spec(W\big|_{L^1[-M,M]}) = spec(W^*\big|_{L^\infty[-M,M]}) \subseteq spec(W^*\big|_{C_{00}([-M,M])^*})$$
$$= spec(W\big|_{C_{00}([-M,M])}) \subset \overline{B(0,r)} \cup \{1\}$$

because, respectivly: Schauder's theorem, $L^\infty[-M, M] \subseteq C_{00}([-M, M])^*$, Schauder's theorem again, and the Lemma in the previous section.

Let $\lambda \in spec(W\big|_{L^1[-M,M]})$, $\lambda \neq 0$. From Fredholm theory, $dim(Ker(W - \lambda)) = co-dim(Im(W-\lambda))$, so we can find $h_1, \ldots, h_d$ that form a basis of $Ker\left(W\big|_{L^1[-M,M]} - \lambda\right)$, and:

$$
\begin{aligned}
L^1[-M, M] &= <h_1, \ldots, h_d> \oplus V \text{ for some subspace } V \\
L^1[-M, M] &= <\omega_1, \ldots, \omega_d> \oplus Im(W - \lambda) \text{ for some } \omega_1, \ldots, \omega_d \\
&\omega_1, \ldots, \omega_d \text{ are linearly independent} \\
&(W - \lambda) : V \longrightarrow Im(W - \lambda) \text{ is an isomorphism}
\end{aligned}
$$

Applying Ritz lemma, we can find $\omega_1, \ldots, \omega_d$ in $C_{00}([-M, M])$.

Let $\zeta_1, \ldots, \zeta_d \in L^1[-M, M]^* = L^\infty[-M, M]$ be defined as

$$\zeta_i(\omega_j) = \delta_{i,j} = \quad 1 \text{ if } i = j, 0 \text{ otherwise}$$
$$\zeta_i = 0 \text{ on } Im(W - \lambda).$$

The fact that $\zeta_1, \ldots, \zeta_d$ are continuous on $L^1[-M, M]$ follows from the decomposition of $L^1$. Note that

$$
\begin{aligned}
0 &= \ <\zeta_i, h> \ \text{ for all } h \in Im(W - \lambda) \\
\text{iff } 0 &= \ <\zeta_i, (W - \lambda)(g)> \ \text{ for all } g \in L^1[-M, M] \\
\text{iff } 0 &= \ <(W^* - \lambda)\zeta_i, g> \ \text{ for all } g \in L^1[-M, M] \\
\text{iff } 0 &= (W^* - \lambda)\zeta_i \\
\text{iff } \zeta_i &\in Ker(W^*\big|_{L^\infty} - \lambda)
\end{aligned}
$$

Note that the inclusion map $\iota : C_{00}([-1, 1]) \longrightarrow L^1[-M, M]$ implies a dual inclusion map $\iota^* : L^1[-M, M]^* \longrightarrow C_{00}([-1, 1])^*$, so (identifying elemnts of $L^1[-M, M]^*$ with their images under $\iota^*$), we can consider $\zeta_1, \ldots, \zeta_d$ as elements of $C_{00}([-1, 1])^*$. Also, (ignoring $\iota^*$ in the notation) $\zeta_1, \ldots, \zeta_d \in Ker\left(W^*\big|_{C_{00}([-M,M])^*} - \lambda\right)$.

$\zeta_1, \ldots, \zeta_d$ are linearly independent in $C_{00}([-M, M])^*$, because, if for some scalars $\alpha_1, \ldots$ we have

$$0 = \sum_i \alpha_i \ \zeta_i \text{ in } C_{00}([-M, M])^*$$

then evaluating $0$ and $\sum_i \alpha_i \ \zeta_i$ at $\omega_j \in C_{00}([-M, M])^*$, we get $0 = \alpha_j$. In particular:

$$
\begin{aligned}
d = dim\left(Ker\left(W\big|_{L^1[-M,M]} - \lambda\right)\right) &= dim\left(Ker\left(W^*\big|_{L^\infty[-M,M]} - \lambda\right)\right) \\
\leq dim\left(Ker\left(W^*\big|_{C_{00}([-M,M])^*} - \lambda\right)\right) &= dim\left(Ker\left(W\big|_{C_{00}([-M,M])} - \lambda\right)\right).
\end{aligned}
$$

We apply this inequality to the case $\lambda = 1$, which is in $spec\left(W\big|_{L^1[-M,M]} - 1\right)$, and for which we know simplicity of eigenvalue 1 to get

$$
\begin{aligned}
1 \leq dim\left(Ker\left(W\big|_{L^1[-M,M]} - 1\right)\right) \\
\leq dim\left(Ker\left(W\big|_{C_{00}([-M,M])} - 1\right)\right) = 1
\end{aligned}
$$

Therefore, since $\phi_1 \in Ker\left(W\big|_{L^1[-M,M]} - 1\right)$, iii) follows. $\qquad\square$

## 3.3   $W$ on $C_{00}(\mathbb{R})$ and $L^1(\mathbb{R})$

There are some un-refereed sources that seem to suggest that, because $W$ is an integral operator involving a compactly supported function, then it is automatically compact. $W$ is close to a convolution operator, and so it cannot be compact when acting on the entire line. It turns out that part of the range of the Fourier transform of $f_U$ is contained in the spectrum of $W$. To settle any doubt, we include the result in this section.

Since $f_U$ has compact support, the Fourier transform of $f_U$ given by $\hat{f}_U(z) = \int_0^1 e^{-2\pi izs} f_u(s)\, ds$, is an entire function of $z \in \mathbb{C}$. For $\xi \in \mathbb{R}$, small:

$$\hat{f}_U(\xi) = \int_0^1 (1 - 2\pi i\xi s - 2\pi^2 \xi^2 s^2 + \ldots) f_u(s)\, ds$$
$$= 1 - 2\pi i\xi\mu - 2\pi^2 \xi^2 (\mu^2 + \sigma^2) + \ldots$$

so the parabola $x = 1 - \frac{1}{2}\left(\frac{\mu^2 + \sigma^2}{\mu^2}\right) y^2$ osculates the curve $\hat{f}_U(\mathbb{R})$ at $x = 1$, $y = 0$. In particular, the range of $\hat{f}_U$ has lots of points (it is not discrete).

The fact that $supp(f_U) \subseteq [0, \infty)$ implies that $|\hat{f}_U(z)| \leq 1$ if the imaginary part of $z$ satisfies $\Im(z) \leq 0$.

**Lemma.** *Let $W$ as in (24), i.e.:*

$$W(g)(x) = \int_{-\infty}^{+\infty} g(s)\, f_U((s-x)\ sign(s))\ ds.$$

*where the we take $g \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$. Let $\phi_1$ be the eigenvector of $W$ in $C_{00}([-1,1])$ corresponding to the eigenvalue 1, normalized so that $\phi_1 \geq 0$, $\int \phi_1 = 1$. Let $\mathbb{C}^- = \{z \in \mathbb{C} | \Im(z) \leq 0\}$ be the closed lower half complex plane. Then:*

*i. $W : C(\mathbb{R}) \longmapsto C(\mathbb{R})$ is not compact*

*ii. $\hat{f}_U(\mathbb{C}^-) \subseteq spec\left(W\big|_{C(\mathbb{R})}\right)$*

*iii. $W : C_{00}(\mathbb{R}) \longmapsto C_{00}(\mathbb{R})$ is not compact*

*iv. $W : L^1(\mathbb{R}) \longmapsto L^1(\mathbb{R})$ is not compact*

*v. $\hat{f}_U(\mathbb{C}^-) \subseteq spec(W\big|_{L^1(\mathbb{R})})$*

*vi. $Ker\left(W\big|_{L^1(\mathbb{R})} - 1\right) = <\phi_1>$, and 1 has algebraic multiplicity 1 as eigenvalue of $W\big|_{L^1(\mathbb{R})}$.*

*Proof.* Let $M \geq 2$, $z \in \mathbb{C}^-$. Let

$$A_M(x) = \begin{cases} 1 & \text{if } |x| \leq M \\ 0 & \text{if } |x| \geq M+1 \\ piecewise\ linear & \text{in between} \end{cases}$$

and $g_M(x) = e^{-2\pi i |x| z} A_M(x)$. We have

$$W(g_M)(x) = \int_{-M-1}^{M+1} e^{-2\pi i |s| z} A_M(s)\ f_U((s-x)\ sign(s))\ ds$$

$$= \int_0^{M+1} e^{-2\pi i s z} A_M(s) \Big( f_U(s-x) + f_U(s+x) \Big)\ ds = W(g_M)(-x),$$

i.e.: $W(g_M)$ is even.

Since $g_M$ is supported on $[-M-1, M+1]$, then $W(g_M)(x) = 0 = \hat{f}_U(z) g_M(x)$ when $|x| \geq M+1$.

If $1 \leq x \leq M-1$, then $f_U(s+x) = 0$ for any $s \geq 0$, and so

$$W(g_M)(x) = \int_0^{M+1} e^{-2\pi i s z} A_M(s) f_U(s-x)\ ds$$

$$= \int_{-x}^{M+1-x} e^{-2\pi i (s+x) z} A_M(s+x) f_U(s)\ ds$$

$$= e^{-2\pi i x z} \int_0^1 e^{-2\pi i s z} A_M(s+x) f_U(s)\ ds,$$

but, in the integrand, $s + x \leq 1 + M - 1 = M$, so $A_M(s+x) = 1$, and

$$W(g_M)(x) = e^{-2\pi i x z} \int_0^1 e^{-2\pi i s z} f_U(s)\ ds = e^{-2\pi i x z} \hat{f}_U(z) = \hat{f}_U(z) g_M(x).$$

Therefore

$$W(g_M)(x) - \hat{f}_U(z) g_M(x) = 0$$

if $1 \leq x \leq M-1$. Since $W(g_M)$ and $g_M$ are even, it follows that $W(g_M)(x) - \hat{f}_U(z) g_M(x) = 0$ if $1 \leq |x| \leq M-1$ or $M+1 \leq |x|$. In any case, $|W(g_M)(x) - \hat{f}_U(z) g_M(x)| \leq 2$ for all $x$.

$i$ follows from $ii$, since compact operators have discrete spectra, and $\hat{f}_U(\mathbb{C}^-)$ is not discrete.

To show $ii$, let $z \in \mathbb{C}^-$.

If $\hat{f}_U(z) \in spec\left( W\Big|_{C_{00}([-1,1])} \right)$ then there is $\rho \in C_{00}([-1,1]) \subset C(\mathbb{R})$, $\rho \neq 0$, such that $(W - \hat{f}_U(z))(\rho) = 0$. Thus, $\hat{f}_U(z) \in spec\left( W\Big|_{C(\mathbb{R})} \right)$.

If $\hat{f}_U(z) \notin spec\left(W\big|_{C_{c0}([-1,1])}\right)$. Taking $g(x) = e^{-2\pi i|x|z}$, which is in $C(\mathbb{R})$, and applying the previous computation letting $M \leftarrow \infty$, we get $\Delta_z = W(g) - \hat{f}_U(z)g$ is in $C_{c0}([-1,1])$, and since $\hat{f}_U(z) \notin spec\left(W\big|_{C_{c0}([-1,1])}\right)$, we can write $\Delta_z = (W - \hat{f}_U(z))(\rho_z)$ for some $\rho_z \in C_{c0}([-1,1])$, then

$$(W - \hat{f}_U(z))(g - \rho_z) = 0$$

and $g - \rho_z \neq 0$, since $\rho_z$ has compact support and $g$ does not. This shows $\hat{f}_U(z) \in spec\left(W\big|_{C(\mathbb{R})}\right)$.

If $z_o \in \mathbb{C}^-$ and $\hat{f}_U(z_o) \neq 0$ is in $spec\left(W\big|_{C_{c0}([-1,1])}\right)$, using the analyticity of $\hat{f}_U$ and the discreteness of $spec\left(W\big|_{C_{c0}([-1,1])}\right)$ away from $0$ , we can find a sequence $z_j$ in $\mathbb{C}^-$ such that $z_j \leftarrow z_o$, $\hat{f}_U(z_j) \notin spec\left(W\big|_{C_{c0}([-1,1])}\right)$, hence, from our previous result, $\hat{f}_U(z_j) \in spec\left(W\big|_{C(\mathbb{R})}\right)$, so passing to the limit, $\hat{f}_U(z_o) \in spec\left(W\big|_{C(\mathbb{R})}\right)$. $0$ is an accumulation point of $\hat{f}_U\big|_{\mathbb{R}}$, so it is in $spec\left(W\big|_{C_{c0}([-1,1])}\right)$, which cover the case $\hat{f}_U(z_o) = 0$.

To show $iii$, assume $W : C_{c0}(\mathbb{R}) \longmapsto C_{c0}(\mathbb{R})$ is compact. Pick any $z$ such that $\hat{f}_U(z) \neq 0$. Since the sequence $\{g_M | M = 2, 3, ...\}$ is contained in $C_{c0}(\mathbb{R})$ and it is bounded, the sequence $\{W(g_M) | M = 2, 3, ...\}$ would contained a convergent subsequence. Any accumulation point $G$ of $\{W(g_M)\}$ is of the form $G(x) = e^{-2\pi i|x|\xi} \hat{f}_U(z)$ for $1 \leq |x|$, hence not in $C_{c0}(\mathbb{R})$, a contradiction.

$iv$ follows from $v$, because compact operators have discrete spectrum, and $\hat{f}_U(\mathbb{C}^-)$ is not discrete.

To show $v$, let $\Im(z) < 0$, and such that $\hat{f}_U(z) \neq 0$. In part $ii$ we showed that there is an eigenvector of $W$ in $C(\mathbb{R})$ of the form $V(x) = e^{-2\pi i|x|z} - \rho_z(x)$ with $\rho_z \in C_{c0}([-1,1])$, therefore $V$ is integrable, so $\hat{f}_U(z) \in spec(W\big|_{L^1(\mathbb{R})})$. By continuity we get $\hat{f}_U(\mathbb{C}^-) \subseteq spec(W\big|_{L^1(\mathbb{R})})$.

To show $vi$, recall $\phi_1$ is continuous and is supported on $[-1,1]$, so $\phi_1 \in L^1(\mathbb{R})$, which shows the inclusion $< \phi_1 > \subseteq Ker\left(W\big|_{L^1(\mathbb{R})} - 1\right)$.

For the other inclusion, let $g \in Ker\left(W\big|_{L^1(\mathbb{R})} - 1\right)$, $g \neq 0$. We will show that

$g$ has compact support, then, applying the result of section (3.2) we conclude $g \in < \phi_1 >$. If

$$g(x) = \int_{-\infty}^{+\infty} g(s) \, f_U\Big((s-x)sign(s)\Big) \, ds \tag{27}$$

(in $L^1(\mathbb{R})$) then

$$|g(x)| \le \int_{-\infty}^{+\infty} |g(s)| \, f_U\Big((s-x)sign(s)\Big) \, ds \tag{28}$$

and since both sides have the same integral over $\mathbb{R}$ we conclude that equality holds in (28) a.e., so $|g| \in Ker\left(W\Big|_{L^1(\mathbb{R})} - 1\right)$. Let $G(x) = |g(x)| + |g(-x)|$, then $G \in Ker\left(W\Big|_{L^1(\mathbb{R})} - 1\right)$, $G \ge 0$, $G \ne 0$, and $G$ is even. Moreover, $G$ has compact support iff $g$ has compact support, so we can work with $G$.

Let $V(a) = \int_a^{+\infty} G(x) \, dx$. Since $G$ is non-negative and integrable, $V$ is continuous and non-increasing. We claim that $V(a) = V(1)$ for all $a \ge 1$. Indeed, given $a \ge 1$,

$$\int_a^{+\infty} G(x) \, dx = \int_a^{+\infty} \int_0^{+\infty} G(s) \left[f_U(s-x) + f_U(s+x)\right] \, ds \, dx$$

But if $x \ge a \ge 1$ and $u \ge 0$, then $f_U(s+x) = 0$, so

$$\int_a^{+\infty} G(x) \, dx = \int_a^{+\infty} \int_0^{+\infty} G(s) \, f_U(s-x) \, ds \, dx$$

$$= \int_a^{+\infty} \int_{-x}^{+\infty} G(h+x) \, f_U(h) \, dh \, dx$$

$$= \int_a^{+\infty} \int_0^1 G(h+x) \, f_U(h) \, dh \, dx$$

$$= \int_0^1 f_U(h) \int_a^{+\infty} G(h+x) \, dx \, dh = \int_0^1 f_U(h) \int_{a+h}^{+\infty} G(x) \, dx \, dh$$

$$\le \int_0^1 f_U(h) \int_a^{+\infty} G(x) \, dx \, dh = \int_a^{+\infty} G(x) \, dx$$

therefore, the above inequality is an equality, and we get

$$\int_{a+h}^{+\infty} G(x) \, dx = \int_a^{+\infty} G(x) \, dx$$

a.e. $h \in [0,1]$ with respect to the measure $f_U \, dh$, in particular,

$$V(a+h) = V(a)$$

for some $h > 0$. Applying this result to $a = 1$ and using the monotonicity of $V$, we conclude that $V(1) = V(1+h)$ for all $h \in [0, \epsilon)$ for some $\epsilon > 0$.

If $V(a) < V(1)$ for some $a > 1$, take $a_0 = \inf\{a > 1 | V(a) < V(1)\}$ and apply the result to $a_0$. We have $V(1) = V(a_0)$, and so $V(1) = V(a_0 + h)$ for all small $h > 0$, but also, $V(a) < V(1)$ for some points $a > a_0$. A contradiction.

Since $\lim_{a\to\infty} V(a) = 0$, we conclude $V(1) = 0$, and $G$ is supported on $[-1, 1]$, which completes the proof of $Ker \left( W\Big|_{L^1(\mathbb{R})} - 1 \right) = < \phi_1 >$.

We can show that 1 is an algebraically simple eigenvalue of $W\Big|_{L^1(\mathbb{R})}$ as before, because is $h \in L^1(\mathbb{R})$ satisfies $(W-1)^2(h) = 0$, then $g = (W-1)(h) \in Ker \left( W\Big|_{L^1(\mathbb{R})} - 1 \right) = < \phi_1 >$, so $(W - 1)(h) = g = \lambda\phi_1$ for some constant $\lambda$. Integrating over $\mathbb{R}$ we get $0 = \int(W - 1)(h) = \lambda \int \phi_1 = \lambda$, hence $(W - 1)(h) = 0$. This completes the proof of $vi$. $\qquad\square$

## 3.4   Iterates of $W$ on $C_{00}([-M, M])$ and $L^1[-M, M]$

We want to make parallel statements about $W$ acting on $C_{00}([-M, M])$ and on $L^1[-M, M]$. We describe decompositions of these spaces that are akin to an invariant manifold description of the dynamics generated by iterates of $W$.

**Lemma 3.2.** *Let $\varphi_1$ be the eigenfunction of $C_{00}([-1, 1])$ that is non-negative, normalized so that $\int \varphi_1 \, ds = 1$. Let $M > 1$. For $\mathbb{B}$ either of $C_{00}([-M, M])$ or $L^1[-M, M]$, let $S_1 = < \varphi_1 >$ be the one-dimensional space spanned by $\varphi_1$ in $\mathbb{B}$ and let $S_2 = Ker(dx) = \{g \in \mathbb{B} \mid \int_{-M}^{M} g \, dx = 0\}$. Then*

*i.* $S_1$, $S_2$, $C_{00}([-M_0, M_0]$ with $1 \leq M_0 < M$ are invariant subspaces of $W$

*ii.* $\mathbb{B}$ is isomorphic to the direct sum $< \varphi_1 > \oplus Ker(dx)$

*iii. There is $r$ , $0 < r < 1$ such that*

$$spec(W\Big|_{S_2}) \subset \overline{B(0, r)}$$

*iv. There is a norm $\mathcal{N}$ on $\mathbb{B}$, equivalent to the standard norm in $\mathbb{B}$, such that $W\Big|_{S_2}$ is a contraction*

*Proof.* Part i) follows from the facts that $\phi_1$ is an eigenvalue, $W$ preserves integrals, and $W$ does not increase the support of elements in $C_{00}([-M_0, M_0]$.

Let

$$\pi_1 : \mathbb{B} \longmapsto S_1$$

$$\pi_1(g) = \left( \int g\varphi_1 \right) \varphi_1$$

and

$$\pi_2 : \mathbb{B} \longmapsto S_2$$

$$\pi_2(g) = g - \left(\int g\varphi_1\right) \varphi_1.$$

The map $\pi_1 \oplus \pi_2$ is (well defined), linear, continuous (giving $S_1$, $S_2$ the metric induced by the norm in $\mathbb{B}$), invertible. This shows ii).

For iii, $spec(W\big|_{S_2})$ is compact, so, with the possible exception of 0, the spectrum coincides with the point spectrum, eigenvectors of $W\big|_{S_2}$ are eigenvectors of $W$, so $spec(W\big|_{S_2}) \subseteq spec(W) \subseteq \{1\} \cup \overline{B(0,r)}$; but $1 \notin spec(W\big|_{S_2})$, else "the" eigenvector of $W$ corresponding to 1 would be in $S_2$, hence

$$spec(W\big|_{S_2}) \subseteq \overline{B(0,r)}.$$

For part iv), let $\mathcal{W} = W|_{S_2}$, which is an endomorphism of $S_2$. By part iii), The map $(\mathbb{I} - z\mathcal{W})$ is invertible for $|z| < 1/r$, so $(\mathbb{I} - z\mathcal{W})|_{S_2}^{-1} = \sum_{j\geq0} z^j \mathcal{W}^j$ converges for $|z| < 1/r$. In particular:

$$\overline{\lim}_{n\to\infty} \|\mathcal{W}^n\|^{1/n} \leq r$$

where $\|\ \|$ is the operator norm for automorphisms of $\mathbb{B}$ restricted and co-restricted to $S_2$. Therefore, if $r_1$ satisfies $r < r_1 < 1$, there is $B$ such that $\|\mathcal{W}^n\| \leq B\, r_1^n$, and there is $z$ such that $r_1 < z < 1$.

Let if $\phi \in \mathbb{B}$. Write $\phi = \alpha\varphi_1 + g$ with $g \in S_2$. Consider

$$\mathcal{N}(\phi) =: |\alpha|\|\varphi_1\| + \sum_{j\geq0} z^{-j}\|\mathcal{W}^j g\|.$$

Since

$$\sum_{j\geq0} z^{-j}\|\mathcal{W}^j g\| \ \leq \ \sum_{j\geq0} z^{-j}\|\mathcal{W}^j\|\,\|g\| \ \leq \ \sum_{j\geq0} z^{-j} B r_1^j\,\|g\| = \frac{B}{1 - \frac{r_1}{z}}\|g\| \ < \ \infty,$$

the series in the definition of $\mathcal{N}(\phi)$ is convergent.

$\|g\| = z^{-j}\|\mathcal{W}^j\, g\|\big|_{j=0} \leq \mathcal{N}(g)$, form where it follows that $\mathcal{N}$ is equivalent to the norm $\|\ \|$.

$\mathcal{N}$ satisfies the properties of a norm.

Finally,

$$\mathcal{N}(\mathcal{W}(g)) \ = \ \sum_{j\geq0} z^{-j}\|\mathcal{W}^{j+1}g\| \ = \ z \sum_{j\geq1} z^{-j}\|\mathcal{W}^j g\| \ \leq \ z\mathcal{N}(g).$$

Since $z < 1$, we get that $\mathcal{W}$ restricted to $S_2$ is a contraction. $\qquad\square$

# 4   Weak convergence of $\{Z(j)\}_{j \geq 1}$

In this section we complete the proof of the Theorem stated in the Section 1. Parts
$a)$, $b)$ are dealt with in Lemma 2.3. Part $c)$ is proved in Section 3.3.

**Lemma 4.1.** *Part d of the Theorem stated in the Section 1 holds.*

*Proof.* To show $d)$, i.e.: $f_k \to f_Z$ in $L^1(\mathbb{R})$, first choose $M_0 > 2$ as in Lemma 2.3,
so that the tail probabilities of $Z_j$ are suitably bounded for $j$ sufficiently large. By
modifying $\gamma_2$ and $\gamma_3$ we may have that there are constants $C$, $\gamma_2$ that depend only on
$\mu$ and $a$ such that:

$$P(|Z(j)| \geq M) \leq C \ exp(-\gamma_2 M^{(1-a)/a} j^{1-a})$$

for $M \geq M_0$ and $j \geq 1$.

For such value $M_0$, by the result in section 3.4, there is a norm $\mathcal{N}$ on $L^1[-M_0, M_0]$
that is equivalent to the standard $L^1$ norm under which the operator $W$ restricted to
a suitable subspace is a contraction. Explicitly, there is $\lambda \in (0, 1)$ such that for any
$g \in L^1[-M_0, M_0]$, if $\int g(x)dx = 0$, then $\mathcal{N}(W(g)) \leq \lambda \mathcal{N}(g)$.

Let us consider the following modification of the operator $W_k$ defined in (24). To
lighten the notation, let $K_k(x, s) = f_U\big((s \left(\frac{k}{k-1}\right)^a - z) \ sign(s)\big)$. For $g \in L^1[-M_0, M_0]$,
define $V_k(g)$ as:

$$V_k(g)(z) = \begin{cases} \int_{-\infty}^{+\infty} g(s)K_k(x, s)ds & \text{if } |z| \leq M_0 \\ 0 & \text{if } |z| > M_0 \end{cases}$$

We claim that $V_k \to W$ as operators on $L^1[-M_0, M_0]$. Indeed, if $g \in L^1[-M_0, M_0]$,
then for $z \in [-M_0, M_0]$,

$$V_k(g)(z) - W(g)(z) = \int_{|x| \leq M_0} g(x)\big[K_k(x, s) - K_\infty(x, s)\big]dx$$

so

$$\int_{|z| \leq M_0} |V_k(g) - W(g)|dz$$

$$\leq \int_{|x| \leq M_0} |g(x)| \int_{|z| \leq M_0} \big||K_k(x, s) - K_\infty(x, s)\big|dzdx$$

$$\leq \int_{|x| \leq M_0} |g(x)|dx \sup_{|\xi| \leq M_0((\frac{k}{k-1})^a - 1)} \int \big|f_U(\xi + z) - f_U(z)\big|dz$$

thus,

$$\|V_k - W\|_{Hom(L^1)} \leq \sup_{|\xi| \leq M_0((\frac{k}{k-1})^a - 1)} \int |f_U(\xi + z) - f_U(z)| dz.$$

Therefore, there is $\lambda_1 \in (0, 1)$ and $k_0$ such that for any $g \in L^1[-M_0, M_0]$, if $\int g(x) dx = 0$, and $k \geq k_0$ then $\mathcal{N}(V_k(g)) \leq \lambda_1 \mathcal{N}(g)$.

Let $\Delta_k = f_k - \phi_1 - (\int_{|x| \leq M_0} f_k - 1) = f_k - \phi_1 + (\int_{|x| > M_0} f_k) = f_k - \phi_1 + P(|Z_k| > M_0)$. We seek an inductive bound of $\Delta_k$. We are interest in the a bound for $f_k - \phi_1$, but in order to use the contraction property of $V_k$, we subtract a constant from $f_k - \phi_1$.

We have

$$\Delta_{k+1} = f_{k+1}(z) - \phi_1(z) + \int_{|x| > M_0} f_{k+1}$$

$$= \int f_k(x) K_k(x, s) dx - \int \phi_1(x) K_\infty(x, s) dx + \int_{|x| > M_0} f_{k+1}$$

$$= \int_{|x| \leq M_0} \big(f_k(x) - \phi_1(x)\big) K_k(x, s) dx + \int_{|x| \leq M_0} \phi_1(x) K_k(x, s) dx$$

$$+ \int_{|x| > M_0} f_k(x) K_k(x, s) dx$$

$$- \int \phi_1(x) K_\infty(x, s) dx + \int_{|x| > M_0} f_{k+1}$$

$$= \int_{|x| \leq M_0} \big(f_k(x) - \phi_1(x)\big) K_k(x, s) dx$$

$$+ \int_{|x| > M_0} f_k(x) K_k(x, s) dx$$

$$+ \int \phi_1(x) \big(K_k(x, s) - K_\infty(x, s)\big) dx$$

$$+ \int_{|x| > M_0} f_{k+1}$$

$$= \int_{|x|\leq M_0} \Delta_k(x) K_k(x,s) dx$$

$$+ \left(\int_{|x|>M_0} f_k\right)\left(\int_{|x|\leq M_0} K_k(x,s) dx\right)$$

$$+ \int_{|x|>M_0} f_k(x) K_k(x,s) dx$$

$$+ \int \phi_1(x)\big(K_k(x,s) - K_\infty(x,s)\big) dx$$

$$+ \int_{|x|>M_0} f_{k+1} = I + II + III + IV + V$$

We bound each term separately.

$I = \int_{|x|\leq M_0} \Delta_k(x) K_k(x,s) dx = V_k(\Delta_k)$, so $\mathcal{N}(I) \leq \lambda_1 \mathcal{N}(\Delta_k)$

$II = \left(\int_{|x|>M_0} f_k\right)\left(\int_{|x|\leq M_0} K_k(x,s) dx\right) = P(|Z(k)| \geq M_0) W_k(I_{|x|\leq M_0})$, so

$$\int_{|z|\leq M_0} II(z) dz \leq P(|Z(k)| \geq M_0) \int W_k(I_{|x|\leq M_0})$$

$$= P(|Z(k)| \geq M_0) \int I_{|x|\leq M_0} = 2\, M_0 P(|Z(k)| \geq M_0)$$

$$\leq C\, M_0 exp(-\gamma_2 M_0^{(1-a)/a} k^{1-a})$$

$III = \int_{|x|>M_0} f_k(x) K_k(x,s) dx = W_k(f_k I_{(|x|>M_0)})$, therefore

$$\int_{|z|\leq M_0} III(z) dz \leq \int f_k I_{(|x|>M_0)} = P(|Z_k| \geq M_0) \leq C\, exp(-\gamma_2 M_0^{(1-a)/a} k^{1-a}).$$

$IV = \int \phi_1(x)\big(K_k(x,s) - K_\infty(x,s)\big) dx$, terefore

$$\int_{|z|\leq M_0} IV(z) dz \leq \int \phi_1(x) \int_{|z|\leq M_0} \big|K_k(x,s) - K_\infty(x,s)\big| ds dx$$

$$\leq \sup_{|\xi|\leq M_0((\frac{k}{k-1})^a-1)} \int \big|f_U(\xi+s) - f_U(s)\big| ds.$$

$V = \int_{|x|>M_0} f_{k+1} = P(|Z_{k+1}| \geq M_0) \leq C \; exp(-\gamma_2 M_0^{(1-a)/a}(k+1)^{1-a})$, therefore

$$\int_{|z|\leq M_0} V(z)dz \leq C \; M_0 exp(-\gamma_2 M_0^{(1-a)/a}(k+1)^{1-a})$$

Putting all these five inequalities together, using the equivalence of $\mathcal{N}$ and the usual $L^1$ norm, we get ($C$ represents a constant, not allways the same from occurrence to occurrence)

$\mathcal{N}(\Delta_{k+1}) \leq \lambda_1 \mathcal{N}(\Delta_k) + C \; M_0 exp(-\gamma_2 M_0^{(1-a)/a}k^{1-a})$

$+ C \sup_{|\xi|\leq M_0((\frac{k}{k-1})^a-1)} \int \left| f_U(\xi+s) - f_U(s) \right| ds + C \; M_0 exp(-\gamma_2 M_0^{(1-a)/a}(k+1)^{1-a})$

$\leq \lambda_1 \mathcal{N}(\Delta_k) + C \; M_0 exp(-\gamma_2 M_0^{(1-a)/a}k^{1-a})$

$+ C \sup_{|\xi|\leq M_0((\frac{k}{k-1})^a-1)} \int \left| f_U(\xi+s) - f_U(s) \right| ds$

$= \lambda_1 \mathcal{N}(\Delta_k) + \epsilon_k$

where

$$\epsilon_k = C \; M_0 exp(-\gamma_2 M_0^{(1-a)/a}k^{1-a}) + C \sup_{|\xi|\leq M_0((\frac{k}{k-1})^a-1)} \int \left| f_U(\xi+s) - f_U(s) \right| ds \to 0$$

Thus, we have obtained an inequality of the form $A_{k+1} \leq \lambda_1 A_k + \epsilon_k$, with $0 < \lambda_1 < 1$, and $\epsilon_k \to 0$. This implies that $A_k \to 0$. Indeed, consider that,

$$\frac{A_{k+1}}{\lambda_1^{k+1}} \leq \frac{A_k}{\lambda_1^k} + \frac{\epsilon_k}{\lambda_1^{k+1}}$$

so, for $q < p$:

$$\frac{A_p}{\lambda_1^p} \leq \frac{A_q}{\lambda_1^q} + \sum_{j=q}^{p-1} \frac{\epsilon_j}{\lambda_1^{j+1}}$$

then

$$A_p \leq \lambda_1^p \frac{A_q}{\lambda_1^q} + \sum_{j=q}^{p-1} \epsilon_j \lambda_1^{p-j-1}$$

$$\leq \lambda_1^p \frac{A_q}{\lambda_1^q} + \sup_{j\geq q} \epsilon_j \sum_{j=q}^{p-1} \lambda_1^{p-j-1}$$

$$\leq \lambda_1^p \frac{A_q}{\lambda_1^q} + \sup_{j\geq q} \epsilon_j \sum_{j=0}^{\infty} \lambda_1^j$$

$$= \lambda_1^p \frac{A_q}{\lambda_1^q} + \sup_{j\geq q} \epsilon_j \frac{1}{1-\lambda_1}.$$

so $\int_{|z|\leq M_0}|\Delta_p(z)|dz \to 0$ as $p \to \infty$.

Now part $d)$ is trivial:

$$\int |f_k - \phi_1| = \int_{|z|\leq M_0}|f_k - \phi_1| + \int_{|z|>M_0}|f_k - \phi_1|$$

but $\phi_1$ is supported on $[-1,1]$, so $\int_{|z|>M_0}|f_k - \phi_1| = \int_{|z|>M_0} f_k = P(|Z_k| > M_0) \to 0$ as $k \to \infty$, and

$$\int_{|z|\leq M_0}|f_k - \phi_1| = \int_{|z|\leq M_0}|\Delta_k - P(|Z_k| > M_0)|$$
$$\leq \int_{|z|\leq M_0}|\Delta_k| + 2M_0\ P(|Z_k| > M_0) \to 0$$

$\square$

**Lemma 4.2.** *Part $f)$ of the Theorem stated in the Section 1 holds.*

*Proof.* Part $f)$ is a simple consequence of $d)$. For any Borel set $B \subset \mathbb{R}$, $P(Z_k \in B) = \int_B f_k \to \int_B \phi_1 = P(Z \in B)$. $\square$

**Lemma 4.3.** *Part $e)$ of the Theorem stated in the Section 1 holds.*

*Proof.* Let $f_U \in L^p([0,1])$ for some $p > 1$. Choose $N$ minimal positive integer such that $p \geq 1 + \frac{1}{N-1}$. Since $f_U \in L^1([0,1])$, then $f_U \in L^{p_1}([0,1])$ where $\frac{1}{p_1} = 1 - \frac{1}{N}$.

We claim that for $j = 1, \ldots, N$, $f_j$, the probability density function of $Z_j$, is in $L^{p_j}$ where $\frac{1}{p_j} = 1 - \frac{j}{N}$. Indeed, $Z_1 = -U_1$, so $f_1(s) = f_U(-s) \in L^{p_1}$ by assumption. Now we proceed inductively. If $f_j \in L^{p_j}$ for some $1 \leq j < N$, then $f_{j+1} = W_{j+1}(f_j)$. Recall $W_{j+1}$ can be written as in (25) in terms of convolutions, so $f_{j+1} = A_{j+1}(f_j) * f_U + B_{j+1}(f_j) * \mathcal{O}(f_U)$. We apply Young's inequality to conclude $\|f_{j+1}\|_r \leq C\|f_j\|_{p_j}\|f_U\|_{p_1}$ where

$$\frac{1}{r} = \frac{1}{p_j} + \frac{1}{p_1} - 1 = 1 - \frac{j}{N} + 1 - \frac{1}{N} - 1 = 1 - \frac{j+1}{N}$$

which completes the induction.

In particular, when $j = N$ we get $\|f_j\|_\infty \leq C\|f_U\|_{p_1}^j$ for a constant $C$ that only depends on $p_1$.

For $j = N+1, \ldots,$ $\|f_j\|_{infty} = \|W_j(f_{j-1})\|_{infty} \leq \|f_{j-1}\|_{infty}$, thus, the family $\{f_j\}_{j>N}$ is uniformly bounded. Moreover, the family is equicontinuous on compact subsets of $\mathbb{R}$, the continuity modulus dominated by the $L^1$ continuity modulus of $f_U$. Therefore the family is relatively compact on intervals, and since it converges in $L^1$ to $\phi_1$, it converges to $\phi_1$ uniformly on any baunded interval. To show convergence over the entire line we need only uniform bounds of the tails.

Let $M_0$ as in Lemma 2.3. For $j > N$, and $|z| > 3M_0$ we have

$$0 \leq f_j(z) = \int f_{j-1}(x) f_U\big((x\rho_j - z)sign(x)\big) dx$$

$$= \int_{|x-z|\leq 1} \rho_j^{-1} f_{j-1}(\rho_j^{-1}x) f_U\big((x-z)sign(x)\big) dx$$

$$\leq \left(\int_{|x-z|\leq 1} [\rho_j^{-1} f_{j-1}(\rho_j^{-1}x)]^{q_1} dx\right)^{\frac{1}{q_1}} \left(\int f_U(t)^{p_1} dt\right)^{\frac{1}{p_1}}$$

where we wrote $\rho_j$ for $\left(\frac{j}{j-1}\right)^a$ to simplify the notation, and $\frac{1}{q_1} = 1 - \frac{1}{p_1} = \frac{1}{N}$, so $q_1 = N$.

In the integral, $|x - z| \leq 1$, so $|x| \geq |z| - 1 \geq 3M_0 - 1 > 2M_0$, then

$$\left(\int_{|x-z|\leq 1} [\rho_j^{-1} f_{j-1}(\rho_j^{-1}x)]^{q_1} dx\right)^{\frac{1}{q_1}}$$

$$\leq \left(\rho_j^{-1}\|f_{j-1}\|_\infty\right)^{\frac{N-1}{N}} \left(\int_{|x|\geq \rho_j^{-1}2M_0} f_{j-1}(x)dx\right)^{\frac{1}{N}}$$

$$\leq \|f_N\|_\infty^{\frac{N-1}{N}} P(|Z_{j-1}| \geq \rho_j^{-1}2M_0)^{\frac{1}{N}}$$

$$\leq C\|f_N\|_\infty^{\frac{N-1}{N}} exp\left(-\gamma_5 M_0^{\frac{1-a}{a}} j^{1-a}\right)$$

where we made use of $\rho_j^{-1} < 1$, $\|f_j\|_\infty \leq \|f_N\|_\infty$ for $j \geq N$, the bound for the tail probability of $Z_{j-1}$ given in Lemma 2.3, $C$ and $\gamma_5$ are constants that depend only on $p_1, a$ and $\mu = E(U_1)$.

There $f_j \to 0$ uniformly in the complement of $[-3M_0, 3M_0]$

$\square$

REFERENCES
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