Adding toroidal flow in GEM for adiabatic electron model

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“You don’t have to be the chosen one. The secret is to build the resolve and spirit to enjoy the plateaus, the times when it doesn’t feel like you’re improving and you question why you’re doing this. If you’re patient, the plateaus will become springboards.”

— Steve Nash

1 Guiding center drift due to toroidal flow

In the right-handed coordinate system (R,Z,ζ), considering the toroidal flow with the form of $\mathbf{U}(\mathbf{R}) = -\omega(r)R^2\nabla\zeta$, where r is a flux label. We have

$$Centrifugal\ force\ :\ F_{cf} = m_a\omega^2 R\hat{R} \tag{1}$$

$$Coriolis\ force\ :\ F_{co} = 2m_a v_\parallel \hat{b}\times\hat{\omega} \tag{2}$$

The corresponding F_{cf} drift is

$$\frac{F_{cf}\times\hat{b}}{e_aB} = \frac{m_a\omega^2 R}{e_aB}\hat{R}\times\hat{b} = \frac{m_aU^2}{e_aBR}\hat{R}\times\hat{b} \tag{3}$$

and the F_{co} drift is

$$\frac{F_{co}\times\hat{b}}{e_aB} = \frac{2m_a v_\parallel}{e_aB}\hat{b}\times\hat{\omega}\times\hat{b} \tag{4}$$

$$\begin{aligned}\hat{b}\times\hat{\omega}\times\hat{b} = \hat{b}\times(\hat{\omega}\times\hat{b}) &= \hat{\omega} - (\hat{b}\cdot\hat{\omega})\hat{b}\\ &= -\frac{U}{R}\hat{\zeta} + (\hat{b}\cdot\frac{U}{R}\hat{\zeta})\hat{b}\\ &= \frac{Uf}{R^2}\hat{b} - \frac{U}{R}\hat{\zeta}\end{aligned}$$

There are three drift terms needed to be added in the velocity of guiding center V_G, according to Eq. 21 in [1], i. e.

$$\frac{cm_a}{e_aB}\hat{b}\times(\mathbf{U}\cdot\nabla\mathbf{U}), \tag{5}$$

$$\frac{cm_a v_\parallel^{'}}{e_aB}\hat{b}\times(\hat{b}\cdot\nabla\mathbf{U}), \tag{6}$$

$$\frac{cm_a v_\parallel^{'}}{e_aB}\hat{b}\times(\mathbf{U}\cdot\nabla\hat{b}). \tag{7}$$

$$\mathbf{U}\cdot\nabla\mathbf{U} = -\frac{U^2}{R}\hat{R} \tag{8}$$

Eq. 5 is consistent with the F_{cf} drift.

$$\begin{aligned}\hat{b} \times (\mathbf{U} \cdot \nabla \mathbf{U}) &= \frac{U^2}{R} \hat{R} \times \hat{b} \\ &= \frac{U^2}{RB} \hat{R} \times (\frac{f}{R}\hat{\zeta} + \frac{\psi_p^{'}(r)}{R}(\frac{\partial r}{\partial R}\hat{Z} - \frac{\partial r}{\partial Z}\hat{R})) \\ &= \frac{U^2}{RB}(-\frac{f}{R}\hat{Z} + \frac{\psi_p^{'}(r)}{R}\frac{\partial r}{\partial R}\hat{\zeta})\end{aligned} \tag{9}$$

Then Eq. 5 becomes

$$\frac{cm_a U^2}{e_a B^2 R^2}(-f\hat{Z} + \psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta}). \tag{10}$$

Eq. 10 $\cdot \nabla x$ is

$$\frac{cm_a U^2}{e_a B^2 R^2} \cdot (-f)\frac{\partial r}{\partial Z}. \tag{11}$$

To implement this term, a new variable `ut` needed to be declared, representing the equilibrium toroidal flow. Actually, we only need to declare a new variable `omg`, then `ut=-omg*radius`, see **Appendix**.

For Eq. 10 $\cdot \nabla y$,

$$y = \frac{r_0}{q_0}\int_0^\theta \hat{q}(r,\theta^{'})d\theta^{'} - \zeta) = \frac{r_0}{q_0}(q\theta_f - \zeta) \tag{12}$$

$$\nabla y = \frac{\partial y}{\partial r}\nabla r + \frac{r_0}{q_0}\hat{q}\nabla\theta - \frac{r_0}{q_0}\nabla\zeta, \tag{13}$$

where $\hat{q} = q\frac{\partial \theta_f}{\partial \theta}$.

$$\begin{aligned}\text{Eq. 10} \cdot \nabla y &= \frac{cm_a U^2}{e_a B^2 R^2}(-f\hat{Z} + \psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta}) \cdot \nabla y \\ &= \frac{cm_a U^2}{e_a B^2 R^2}(-f\frac{\partial y}{\partial r}\frac{\partial r}{\partial Z} - \frac{r_0}{q_0}\hat{q}f\frac{\partial \theta}{\partial Z} - \frac{r_0}{q_0 R}\psi_p^{'}(r)\frac{\partial r}{\partial R}),\end{aligned} \tag{14}$$

$$\frac{cm_a U^2}{e_a B^2 R^2}(-f\hat{Z} \cdot \nabla y + \psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta} \cdot \nabla y)$$

So far, all the terms about Eq. 5 is finished. Let's start with Eqs. 6 and 7. Still in the (R, Z, ζ) coordinate, one could find Eq. 6 is identical to Eq. 7 as

$$\mathbf{U} \cdot \nabla \hat{b} = \hat{b} \cdot \nabla \mathbf{U} = -\frac{Uf}{R^2 B}\hat{R} - \frac{U}{R^2 B}\psi_p^{'}(r)\frac{\partial r}{\partial Z}\hat{\zeta}. \tag{15}$$

To derive Eq. 6 or Eq. 7,

$$\begin{aligned}
&\frac{cm_a v_{\parallel}^{'}}{e_a B}\hat{b} \times (\mathbf{U} \cdot \nabla \hat{b}) \\
=&\frac{cm_a v_{\parallel}^{'}}{e_a B}[-\frac{Uf^2}{B^2 R^3}\hat{Z} + \frac{Uf\psi_p^{'}(r)}{B^2 R^3}\frac{\partial r}{\partial R}\hat{\zeta} \\
&-\frac{U}{B^2 R^3}\psi_p^{'2}(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}\hat{R} - \frac{U}{B^2 R^3}\psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2\hat{Z}] \\
=&\frac{cm_a v_{\parallel}{'}U}{e_a B^3 R^3}[-\psi_p^{'2}(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}\hat{R} - (f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2)\hat{Z} + f\psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta}].
\end{aligned} \tag{16}$$

Then,

$$\begin{aligned}
\text{Eq. 16} \cdot \nabla x &= -\frac{cm_a v_{\parallel}{'}U}{e_a B^3 R^3}[\psi_p^{'2}(r)(\frac{\partial r}{\partial R})^2\frac{\partial r}{\partial Z} + (f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2)\frac{\partial r}{\partial Z}] \\
&= -\frac{cm_a v_{\parallel}{'}U}{e_a B^3 R^3}[\psi_p^{'2}(r)(\frac{\partial r}{\partial R})^2 + f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2]\frac{\partial r}{\partial Z}.
\end{aligned} \tag{17}$$

Using Eq. 13,

$$\begin{aligned}
\text{Eq. 16} \cdot \nabla y =& -\frac{cm_a v_{\parallel}{'}U}{e_a B^3 R^3}[\psi_p^{'2}(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial R}) \\
&+ (f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2)(\frac{\partial y}{\partial r}\frac{\partial r}{\partial Z} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial Z}) \\
&+ \frac{r_0}{q_0 R}f\psi_p^{'}(r)\frac{\partial r}{\partial R}].
\end{aligned} \tag{18}$$

$$\begin{aligned}
\frac{cm_a v_{\parallel}{'}U}{e_a B^3 R^3}\Bigg[&-\psi_p^{'2}(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}\hat{R} \cdot \nabla y \\
&-\left(f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2\right)\hat{Z} \cdot \nabla y \\
&+ f\psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta} \cdot \nabla y \qquad \Bigg]
\end{aligned}$$

The remainder is $\mathbf{U}$ itself. As $\mathbf{U}$ is perpendicular to the (x, z) plane, we only need to add $\mathbf{U} \cdot \nabla y = U\hat{\zeta} \cdot \nabla y$.

2 Parallel acceleration due to toroidal flow

An auxiliary guiding center variable $v_\parallel{}''$ is defined according to

$$\varepsilon = \mu \mathbf{B}_0(\mathbf{R}) + \frac{1}{2} m v_\parallel''^2 - \frac{1}{2} m U^2(\mathbf{R}) + q\Phi_1(\mathbf{R}) \tag{19}$$

Notice that $v_\parallel{}''$ depends on $(\mathbf{R}, \varepsilon, \mu)$ but not γ, and $v_\parallel{}'' = v_\parallel{}' + \mathcal{O}(\delta)$. The new parallel velocity is defined with ε^μ. Since $d\varepsilon^\mu/dt = \mathcal{O}(\delta^2)$,

$$\begin{aligned} m v_\parallel'' \frac{dv_\parallel''}{dt} &= -\mu \frac{dB(\mathbf{R})}{dt} + m\frac{dU^2}{dt} - q\frac{d\Phi_1}{dt} \\ &= -\mu v_\parallel'' \mathbf{b}\cdot\nabla B + m v_\parallel'' \mathbf{b}\cdot\nabla U^2 - q v_\parallel'' \mathbf{b}\cdot\nabla\Phi_1 + \mathcal{O}(\delta^2) \end{aligned} \tag{20}$$

We will neglect the $\mathcal{O}(\delta^2)$ terms, which include the parallel nonlinearity.

$$\frac{dv_\parallel''}{dt} = -\frac{\mu}{m}\mathbf{b}\cdot\nabla B + \mathbf{b}\cdot\nabla U^2 - \frac{q}{m}\mathbf{b}\cdot\nabla\Phi_1 + \mathcal{O}(\delta^2) \tag{21}$$

Term 1 of the r.h.s. of the above equation is the mirror force, which has been implemented in GEM.

For any scalar $s(r,\theta,\zeta)$,

$$\mathbf{b}\cdot\nabla s = \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial s}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta + \frac{f}{BR^2}\frac{\partial s}{\partial\zeta}. \tag{22}$$

$$\begin{aligned} \mathbf{b}\cdot\nabla s &= \frac{1}{B}(\nabla\zeta\times\nabla\psi_p + \frac{f}{R}\hat{\zeta})\cdot(\frac{\partial s}{\partial r}\nabla r + \frac{\partial s}{\partial\theta}\nabla\theta + \frac{\partial s}{\partial\zeta}\nabla\zeta) \\ &= \frac{\psi_p'}{B}(\nabla\zeta\times\nabla r)\cdot(\frac{\partial s}{\partial r}\nabla r + \frac{\partial s}{\partial\theta}\nabla\theta + \frac{\partial s}{\partial\zeta}\nabla\zeta) + \frac{f}{BR^2}\frac{\partial s}{\partial\zeta} \\ &= \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial s}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta + \frac{f}{BR^2}\frac{\partial s}{\partial\zeta} \end{aligned}$$

So,

$$\mathbf{b}\cdot\nabla B = \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial B}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta \tag{23}$$

$$\mathbf{b}\cdot\nabla U^2 = \frac{2U}{B}\frac{\psi_p'}{R}\frac{\partial U}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta, \tag{24}$$

$$\mathbf{b}\cdot\nabla\Phi_1 = \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial\Phi_1}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta \tag{25}$$

with

$$\hat{\zeta}\cdot\nabla r\times\nabla\theta = |\nabla r\times\nabla\theta|. \tag{26}$$

The $\partial U/\partial\theta$ and $\partial\Phi_1/\partial\theta$ will be calculated in the `gem_equil.f90`. Others are all existing variables in GEM. Φ_1 is the electric potential determined by the charge-neutrality, with $\mathbf{E_n} = -\nabla\Phi_1$. For a plasma with a single ion species with ion temperature T_i and electron temperature T_e,

$$e\Phi_1 = \frac{m_i\omega^2(r)}{2(1+T_i/T_e)}(R^2 - \langle R^2\rangle). \tag{27}$$

Attention, here the bracket $\langle\cdots\rangle$ stands for the flux surface average.

$$\langle R^2\rangle = \frac{\int_0^{2\pi}\int_{-\pi}^{\pi} R^2\mathcal{J}\,d\theta d\zeta}{\int_0^{2\pi}\int_{-\pi}^{\pi}\mathcal{J}\,d\theta d\zeta} \tag{28}$$

$$\langle R^2\rangle = \frac{\int_{-\pi}^{\pi} R^2\mathcal{J}\,d\theta}{\int_{-\pi}^{\pi}\mathcal{J}\,d\theta} \tag{29}$$

3 Changes in weight equation

Let $f = f_0 + \delta f$, the perturbed part of distribution function

$$\delta f = -q(\phi - \mathbf{U} \cdot \mathbf{A})\frac{f_0}{T} + h \tag{30}$$

Here, h is the non-adiabatic part of δf.

Write $h = h + h_1 + h_2 \cdots$

$$h_2 = \delta f + \frac{q}{T} f_0[(\phi - \mathbf{U} \cdot \mathbf{A}) - \langle(\phi - \mathbf{U} \cdot \mathbf{A})\rangle + \langle \mathbf{v}' \cdot \mathbf{A}\rangle] \tag{31}$$

$$\begin{aligned}
&\frac{\partial h_2}{\partial t} + \langle \dot{\mathbf{R}} \rangle \cdot \nabla h_2 + \langle \dot{\varepsilon}^u \rangle \frac{\partial h_1}{\partial \varepsilon^u} - \langle \dot{\varepsilon}^u \rangle \frac{\partial}{\partial \varepsilon^u} \langle \frac{q}{T} f_0 \mathbf{v}' \cdot \mathbf{A} \rangle \\
= &- \left\langle \frac{d\mathbf{R}}{dt}\bigg|_1 \right\rangle \cdot \frac{\partial f_0}{\partial \mathbf{R}} \\
&- q\mathbf{v}_g \cdot \nabla \langle \Psi \rangle \frac{f_0}{T} \\
&+ q[\mathbf{U}(\mathbf{R}) \cdot \nabla \langle \Psi \rangle - \langle \mathbf{U} \cdot \nabla \Psi \rangle] \frac{f_0}{T} \\
&- \frac{q}{\Omega} \left\langle (\nabla \Psi \times \mathbf{b}) \cdot \nabla \psi_p \; \omega' R \left(\frac{B_t}{B_0} v'_\parallel + U \right) \right\rangle \frac{f_0}{T} \\
&+ S_1 + S_2 + S_3
\end{aligned} \tag{32}$$

There are two terms to be added in the weight equation. For electrostatic model, $\Psi = \phi$.

The first term,

$$q[\mathbf{U}(\mathbf{R}) \cdot \nabla \langle \phi \rangle - \langle \mathbf{U} \cdot \nabla \phi \rangle] \frac{f_0}{T} \tag{33}$$

All right, let's proceed a further step. Write

$$\mathbf{U}(\mathbf{x}) = \mathbf{U}(\mathbf{R}) + \boldsymbol{\rho} \cdot \nabla \mathbf{U}, \tag{34}$$

then

$$\mathbf{U}(\mathbf{R}) \cdot \nabla \langle \phi \rangle - \langle \mathbf{U} \cdot \nabla \phi \rangle = -\langle \boldsymbol{\rho} \cdot \nabla \mathbf{U} \cdot \nabla \phi \rangle. \tag{35}$$

Remember,

$$\hat{\zeta}\cdot\nabla\hat{R} = \frac{1}{R}\hat{\zeta} \text{ and } \hat{\zeta}\cdot\nabla\hat{\zeta} = -\frac{1}{R}\hat{R}.$$

These two terms results from the reconverting from the Cartesian coordinate to the cylindrical coordinate in deriving the material derivative, see Eq. 7.106 in [2].

$$\boldsymbol{\rho} = \rho(\mathbf{e_1}\mathrm{sin}\gamma + \mathbf{e_2}\mathrm{cos}\gamma),$$

where $\rho = v'_{\perp}/\Omega = \sqrt{2\mu B/m}/(qB/m), \mathbf{e_1} = \nabla r/|\nabla r|, \mathbf{e_2} = \mathbf{b}\times\mathbf{e_1}$ and γ is the gyro angle.

Let

$$\mathbf{e_1} = \mathrm{e_{1R}}(r,\theta)\hat{R} + \mathrm{e_{1Z}}(r,\theta)\hat{Z}$$
$$\mathbf{e_2} = \mathrm{e_{2R}}(r,\theta)\hat{R} + \mathrm{e_{2Z}}(r,\theta)\hat{Z} + \mathrm{e_{2\zeta}}(r,\theta)\hat{\zeta},$$

then we have

$$\boldsymbol{\rho} = \rho\Big((\mathrm{e_{1R}}\mathrm{sin}\gamma + \mathrm{e_{2R}}\mathrm{cos}\gamma)\hat{R} + (\mathrm{e_{1Z}}\mathrm{sin}\gamma + \mathrm{e_{2Z}}\mathrm{cos}\gamma)\hat{Z} + (\mathrm{e_{2\zeta}}\mathrm{cos}\gamma)\hat{\zeta}\Big)$$

$$\begin{aligned}\boldsymbol{\rho}\cdot\nabla\mathbf{U} &= \boldsymbol{\rho}\cdot\nabla(U\hat{\zeta})\\ &= \boldsymbol{\rho}\cdot\nabla U\hat{\zeta} + U\boldsymbol{\rho}\cdot\nabla\hat{\zeta}\\ &= \boldsymbol{\rho}\cdot\left(\frac{U}{R}\hat{R} - \omega' R\frac{\partial r}{\partial R}\hat{R} - \omega' R\frac{\partial r}{\partial Z}\hat{Z}\right)\hat{\zeta} - \frac{U\rho_\zeta}{R}\hat{R}\\ &= \left(\frac{U\rho_R}{R} - \omega' R\frac{\partial r}{\partial R}\rho_R - \omega' R\frac{\partial r}{\partial Z}\rho_Z\right)\hat{\zeta} - \frac{U\rho_\zeta}{R}\hat{R}\end{aligned} \tag{36}$$

$$\begin{aligned}&\left(\frac{U\rho_R}{R} - \omega' R\frac{\partial r}{\partial R}\rho_R - \omega' R\frac{\partial r}{\partial Z}\rho_Z\right)\hat{\zeta}\cdot\nabla\phi\\ =&\left(\frac{U\rho_R}{R} - \omega' R\frac{\partial r}{\partial R}\rho_R - \omega' R\frac{\partial r}{\partial Z}\rho_Z\right)\hat{\zeta}\cdot\frac{\partial\phi}{\partial y}\nabla y\\ =&\left(-\frac{U\rho_R}{R} + \omega' R\frac{\partial r}{\partial R}\rho_R + \omega' R\frac{\partial r}{\partial Z}\rho_Z\right)\frac{r_0}{q_0 R}\frac{\partial\phi}{\partial y}\end{aligned} \tag{37}$$

$$
\begin{aligned}
-\frac{U\rho_\zeta}{R}\hat{R}\cdot\nabla\phi &= -\frac{U\rho_\zeta}{R}\hat{R}\cdot\left(\frac{\partial\phi}{\partial x}\nabla x+\frac{\partial\phi}{\partial y}\nabla y+\frac{\partial\phi}{\partial z}\nabla z\right)\\
&= -\frac{U\rho_\zeta}{R}\left(\frac{\partial\phi}{\partial x}\frac{\partial r}{\partial R}+\frac{\partial\phi}{\partial y}\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R}+\frac{r_0}{q_0}\hat{q}\frac{\partial\theta}{\partial R}\right)+\frac{\partial\phi}{\partial z}q_0R_0\frac{\partial\theta}{\partial R}\right)
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
-\left\langle\boldsymbol{\rho}\cdot\nabla\mathbf{U}\cdot\nabla\phi\right\rangle &= \left(\frac{U}{R}\left\langle\rho_R\frac{\partial\phi}{\partial y}\right\rangle-\omega' R\frac{\partial r}{\partial R}\left\langle\rho_R\frac{\partial\phi}{\partial y}\right\rangle-\omega' R\frac{\partial r}{\partial Z}\left\langle\rho_Z\frac{\partial\phi}{\partial y}\right\rangle\right)\frac{r_0}{q_0R}\\
&+\frac{U}{R}\left(\left\langle\rho_\zeta\frac{\partial\phi}{\partial x}\right\rangle\frac{\partial r}{\partial R}+\left\langle\rho_\zeta\frac{\partial\phi}{\partial y}\right\rangle\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R}+\frac{r_0}{q_0}\hat{q}\frac{\partial\theta}{\partial R}\right)+\left\langle\rho_\zeta\frac{\partial\phi}{\partial z}\right\rangle q_0R_0\frac{\partial\theta}{\partial R}\right)
\end{aligned}
\tag{39}
$$

$$
\begin{gathered}
\left[\left(\omega' R\frac{\partial r}{\partial R}-\frac{U}{R}\right)\left\langle\rho_R\frac{\partial\phi}{\partial y}\right\rangle+\omega' R\frac{\partial r}{\partial Z}\left\langle\rho_Z\frac{\partial\phi}{\partial y}\right\rangle\right]\hat{\zeta}\cdot\nabla y\\
+\frac{U}{R}\left(\left\langle\rho_\zeta\frac{\partial\phi}{\partial x}\right\rangle\frac{\partial r}{\partial R}+\left\langle\rho_\zeta\frac{\partial\phi}{\partial y}\right\rangle\hat{R}\cdot\nabla y+\left\langle\rho_\zeta\frac{\partial\phi}{\partial z}\right\rangle\hat{R}\cdot\nabla z\right)
\end{gathered}
$$

$$
\begin{aligned}
\rho_R &= \rho(\mathrm{e_{1R}}\sin\gamma+\mathrm{e_{2R}}\cos\gamma)\\
\rho_Z &= \rho(\mathrm{e_{1Z}}\sin\gamma+\mathrm{e_{2Z}}\cos\gamma)\\
\rho_\zeta &= \rho\mathrm{e_{2\zeta}}\cos\gamma\\
\mathrm{e_{1R}} &= \frac{\partial r}{\partial R}\Big/|\nabla r| = \texttt{srbr/gr}\\
\mathrm{e_{1Z}} &= \frac{\partial r}{\partial Z}\Big/|\nabla r| = \texttt{srbz/gr}\\
\mathrm{e_{2R}} &= -\frac{f}{RB}\mathrm{e_{1Z}}\\
\mathrm{e_{2Z}} &= \frac{f}{RB}\mathrm{e_{1R}}
\end{aligned}
$$

$$e_{2\zeta} = \frac{\psi_p'}{RB}\Big[- e_{1R}\frac{\partial r}{\partial R} - e_{1Z}\frac{\partial r}{\partial Z}\Big]$$

The gyro average of ρ_R, ρ_Z and ρ_ζ could be obtained by the 4-point averaging method. The 4 points are at $\gamma = \pi/2, 3\pi/2, 0$ and π, respectively.

The second term,

$$\begin{aligned} & -\frac{q}{\Omega}\Big\langle (\nabla\phi \times \mathbf{b})\cdot\nabla\psi_p \; \omega' R\Big(\frac{B_t}{B_0}v_\parallel' + U\Big)\Big\rangle \frac{f_0}{T} \\ = & -\frac{q}{\Omega}\Big\langle (\nabla\phi \times \mathbf{b})\cdot\nabla\psi_p \Big\rangle \omega' R\Big(\frac{B_t}{B_0}v_\parallel' + U\Big)\frac{f_0}{T} \end{aligned} \tag{40}$$

Considering

$$\begin{aligned} \mathbf{B} &= \nabla\psi \times \nabla(q\theta_f - \zeta) \\ &= \frac{q_0}{r_0}\frac{d\psi}{dx}\nabla x \times \nabla y \\ &= C(x)\nabla x \times \nabla y \end{aligned} \tag{41}$$

then

$$\mathbf{b} = \frac{\nabla x \times \nabla y}{|\nabla x \times \nabla y|}. \tag{42}$$

So we have

$$\begin{aligned} \nabla\phi \times \mathbf{b}\cdot\nabla\psi_p &= \nabla\psi_p \times \nabla\phi \cdot \mathbf{b} \\ &= \psi_p'\nabla x \times \nabla\phi \cdot \frac{\nabla x \times \nabla y}{|\nabla x \times \nabla y|} \\ &= \frac{\psi_p'}{|\nabla x \times \nabla y|}\Big(\frac{\partial\phi}{\partial y}\nabla x \times \nabla y + \frac{\partial\phi}{\partial z}\nabla x \times \nabla z\Big)\cdot\nabla x \times \nabla y \\ &= \psi_p'|\nabla x \times \nabla y|\frac{\partial\phi}{\partial y} + \frac{\psi_p'}{|\nabla x \times \nabla y|}\frac{\partial\phi}{\partial z}(\nabla x \times \nabla z)\cdot(\nabla x \times \nabla y) \end{aligned} \tag{43}$$

$$\begin{aligned} (\nabla x \times \nabla z)\cdot(\nabla x \times \nabla y) &= (\nabla x \times \nabla y) \times \nabla x \cdot \nabla z \\ &= (|\nabla x|^2\nabla y - |\nabla x \cdot \nabla y|\nabla x)\cdot\nabla z \\ &= |\nabla x|^2\nabla y \cdot \nabla z - |\nabla x \cdot \nabla y|\nabla x \cdot \nabla z \\ &= |\nabla x|^2\nabla y \cdot \nabla z - q_0 R_0|\nabla x \cdot \nabla y| \; |\nabla r \cdot \nabla\theta| \end{aligned} \tag{44}$$

$$
\begin{aligned}
\nabla y \cdot \nabla z &= \left(\frac{\partial y}{\partial r} \nabla r + \frac{r_0}{q_0} \hat{q} \nabla \theta - \frac{r_0}{q_0} \nabla \zeta \right) \cdot q_0 R_0 \nabla \theta \\
&= q_0 R_0 \frac{\partial y}{\partial r} |\nabla r \cdot \nabla \theta| + r_0 R_0 \hat{q} |\nabla \theta|^2
\end{aligned}
\tag{45}
$$

According to Eq. 43 - 45, Eq. 40 can be coded using existing variables. The blue parts above are small ordering terms that are NOT implemented for now.

4 Verification of the model

Above are all the changes for adiabatic electron model. Let's start to test the code. We'll start with the well-known Cyclone Base Case (CBC), those parameters could be found in Ref. [3].

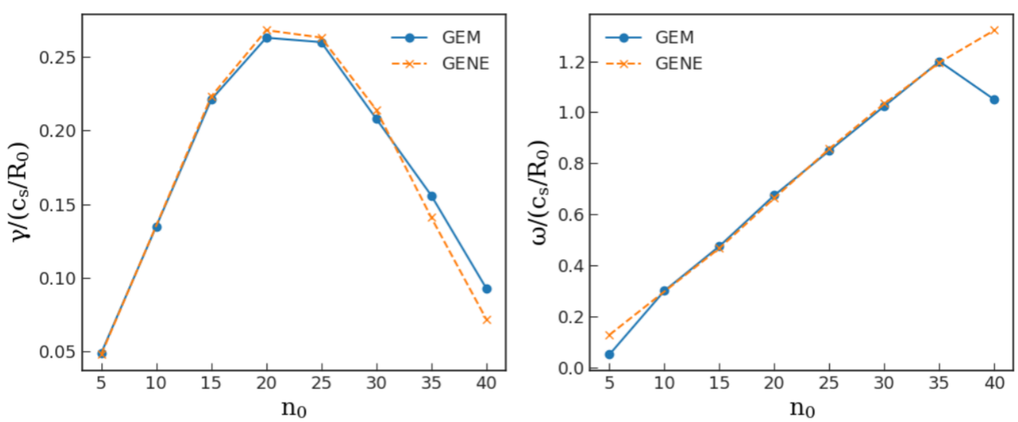

Figure 1: CBC benchmark

First of all, results of ion temperature gradient (ITG) mode with adiabatic electron response are compared as a baseline case. As illustrated in figure 1, results obtained by the two codes (GENE and GEM) show a good consistency and conformity for both ITG growth rate and frequency, from a scanning over toroidal mode number over $n_0 \in [5,40]$. The GENE results correspond to table III and figure 3 in Ref. [3].

Considering a rigid flow profile $\omega(r) = \omega_0$, the corresponding growth rate and mode frequency are shown in figure 2. Strong suppression effect on ITG can be found for sufficiently large toroidal flow, $v_t \sim 0.1 v_{th}$.

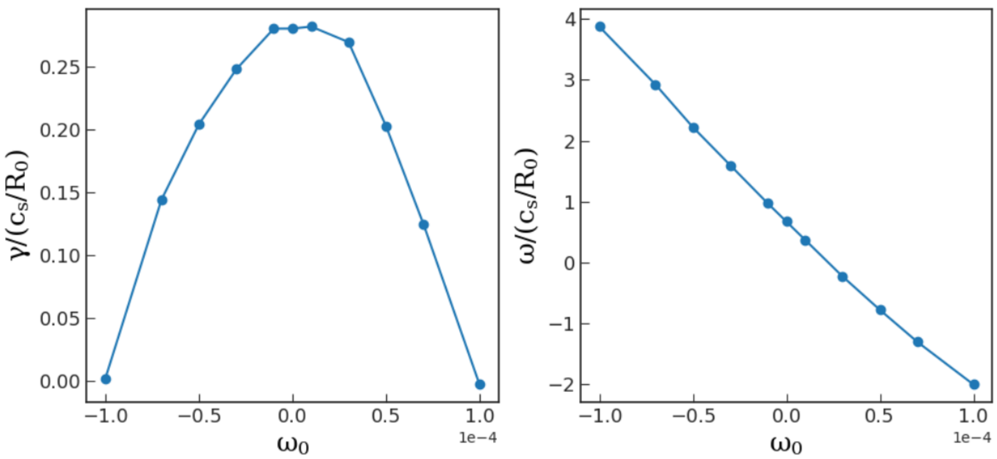

Figure 2: ITG growth rate and frequency with rigid flow.

Appendix

We may need the following operator to simplify the coding process.

$$\hat{R}\cdot\nabla y = \frac{\partial y}{\partial r}\frac{\partial r}{\partial R} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial R}$$

$$\hat{Z}\cdot\nabla y = \frac{\partial y}{\partial r}\frac{\partial r}{\partial Z} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial Z}$$

$$\hat{R}\cdot\nabla z = q_0 R_0\frac{\partial \theta}{\partial R}$$

$$\hat{Z}\cdot\nabla z = q_0 R_0\frac{\partial \theta}{\partial Z}$$

$$\hat{\zeta}\cdot\nabla y = -\frac{r_0}{q_0 R}$$

New variables declared in GEM:
omg(nr)—ω, angular frequency of toroidal flow
domg(nr)—dω/dr
ut(nr,ntheta)—U, toroidal flow
phi1(nr,ntheta)—Φ_1
bdgut,bdgbfld,bdgphi1::(nr,ntheta)—$\mathbf{b}\cdot\nabla\mathrm{U},\ \mathbf{b}\cdot\nabla\mathrm{B},\ \mathbf{b}\cdot\nabla\Phi_1$
hrdgy(nr,ntheta)—$\hat{R}\cdot\nabla y$

hzdgy(nr,ntheta)—$\hat{Z} \cdot \nabla y$
hrdgz(nr,ntheta)—$\hat{R} \cdot \nabla z$
hzdgz(nr,ntheta)—$\hat{Z} \cdot \nabla z$
hztdgy(nr,ntheta)—$\hat{\zeta} \cdot \nabla y$
radius2(nr,ntheta)—R^2
rhoreyp—$\left\langle \rho_R \frac{\partial \phi}{\partial y} \right\rangle$
rhozeyp—$\left\langle \rho_Z \frac{\partial \phi}{\partial y} \right\rangle$
rhoztexp—$\left\langle \rho_\zeta \frac{\partial \phi}{\partial x} \right\rangle$
rhozteyp—$\left\langle \rho_\zeta \frac{\partial \phi}{\partial y} \right\rangle$
rhoztezp—$\left\langle \rho_\zeta \frac{\partial \phi}{\partial z} \right\rangle$

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