Adding toroidal flow in GEM for adiabatic electron model

汤炜康

Contents

1 Guiding center drift due to toroidal flow **2**

2 Parallel acceleration due to toroidal flow **5**

3 Changes in weight equation **7**

1 Guiding center drift due to toroidal flow

There are three terms needed to be added in the drift velocity of the guiding center V_G, according to Eq. 21 in [1], i. e.

$$\frac{cm_a}{e_a B}\hat{b} \times (\mathbf{U} \cdot \nabla \mathbf{U}), \tag{1}$$

$$\frac{cm_a v_\parallel^{'}}{e_a B}\hat{b} \times (\hat{b} \cdot \nabla \mathbf{U}), \tag{2}$$

$$\frac{cm_a v_\parallel^{'}}{e_a B}\hat{b} \times (\mathbf{U} \cdot \nabla \hat{b}). \tag{3}$$

In the right-handed coordinate system (R,Z,ζ), considering the toroidal flow with the form of $\mathbf{U}(\mathbf{R}) = -\omega_0(\psi_p)R^2\nabla\zeta$,

$$\mathbf{U} \cdot \nabla \mathbf{U} = -\frac{U^2}{R}\hat{R} \tag{4}$$

$$\begin{aligned} \hat{b} \times (\mathbf{U} \cdot \nabla \mathbf{U}) &= \frac{U^2}{R}\hat{R} \times \hat{b} \\ &= \frac{U^2}{RB}\hat{R} \times (\frac{f}{R}\hat{\zeta} + \frac{\psi_p^{'}(r)}{R}(\frac{\partial r}{\partial R}\hat{Z} - \frac{\partial r}{\partial Z}\hat{R})) \\ &= \frac{U^2}{RB}(-\frac{f}{R}\hat{Z} + \frac{\psi_p^{'}(r)}{R}\frac{\partial r}{\partial R}\hat{\zeta}) \end{aligned} \tag{5}$$

Then Eq. 1 becomes

$$\frac{cm_a U^2}{e_a B^2 R^2}(-f\hat{Z} + \psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta}). \tag{6}$$

Eq. 6 $\cdot\nabla x$ is

$$\frac{cm_a U^2}{e_a B^2 R^2} \cdot (-f)\frac{\partial r}{\partial Z}. \tag{7}$$

This term can be coded as `-Up**2*fp*srbzp/bfldp**2/radiusp**2`, where `U(0:nr,0:ntheta)` is a new variable needed to be declared, representing the equlibrium toroidal flow. Actually, we only need to declare a new variable `omega(0:nr)`, `U(0:nr,0:ntheta)=-omega(0:nr)*radius(0:nr,0:ntheta)`.

For Eq. 6 $\cdot\nabla y$,

$$y = \frac{r_0}{q_0}\int_0^\theta \hat{q}(r,\theta^{'})d\theta^{'} - \zeta) = \frac{r_0}{q_0}(q\theta_f - \zeta) \tag{8}$$

$$\nabla y = \frac{\partial y}{\partial r}\nabla r + \frac{r_0}{q_0}\hat{q}\nabla\theta - \frac{r_0}{q_0}\nabla\zeta, \tag{9}$$

where $\hat{q} = q\frac{\partial\theta_f}{\partial\theta}$.

$$\begin{aligned} \text{Eq. } 6\cdot\nabla y &= \frac{cm_aU^2}{e_aB^2R^2}(-f\hat{Z} + \psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta})\cdot\nabla y \\ &= \frac{cm_aU^2}{e_aB^2R^2}(-f\frac{\partial y}{\partial r}\frac{\partial r}{\partial Z} - \frac{r_0}{q_0}\hat{q}f\frac{\partial\theta}{\partial Z} - \frac{r_0}{q_0R}\psi_p^{'}(r)\frac{\partial r}{\partial R}), \end{aligned} \tag{10}$$

i. e. `-Up**2/bfldp**2/radiusp**2 * (fp*dydrp*srbzp + r0/q0*qhatp*fp *thbzp + r0/q0/radiusp*psipp*srbrp).`

So far, all the terms about Eq. 1 is finished. Let's start with Eqs. 2 and 3. Still in the (R, Z, ζ) coordinate, one could find Eq. 2 is identical to Eq. 3 as

$$\mathbf{U}\cdot\nabla\hat{b} = \hat{b}\cdot\nabla\mathbf{U} = -\frac{Uf}{R^2B}\hat{R} - \frac{U}{R^2B}\psi_p^{'}(r)\frac{\partial r}{\partial Z}\hat{\zeta}. \tag{11}$$

To derive Eq. 2 or Eq. 3,

$$\begin{aligned} &\frac{cm_av_\parallel^{'}}{e_aB}\hat{b}\times(\mathbf{U}\cdot\nabla\hat{b}) \\ =&\frac{cm_av_\parallel^{'}}{e_aB}[-\frac{Uf^2}{B^2R^3}\hat{Z} + \frac{Uf\psi_p^{'}(r)}{B^2R^3}\frac{\partial r}{\partial R}\hat{\zeta} \\ &-\frac{U}{B^2R^3}\psi_p^{'2}(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}\hat{R} - \frac{U}{B^2R^3}\psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2\hat{Z}] \\ =&\frac{cm_av_\parallel{'}U}{e_aB^3R^3}[-\psi_p^{'2}(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}\hat{R} - (f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2)\hat{Z} + f\psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta}]. \end{aligned} \tag{12}$$

Then,

$$\begin{aligned} \text{Eq. } 12\cdot\nabla x &= -\frac{cm_av_\parallel{'}U}{e_aB^3R^3}[\psi_p^{'2}(r)(\frac{\partial r}{\partial R})^2\frac{\partial r}{\partial Z} + (f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2)\frac{\partial r}{\partial Z}] \\ &= -\frac{cm_av_\parallel{'}U}{e_aB^3R^3}[\psi_p^{'2}(r)(\frac{\partial r}{\partial R})^2 + f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2]\frac{\partial r}{\partial Z}, \end{aligned} \tag{13}$$

which can be coded as `-Up*vpar/bfldp**3/radiusp**3*(psipp**2*srbrp**2 + fp**2 + psipp**2*srbzp**2)*srbzp`.

Using Eq. 9,

$$
\begin{aligned}
\text{Eq. } 12\cdot\nabla y = & -\frac{cm_a v_\parallel{}' U}{e_a B^3 R^3}[\psi_p'^2(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R}+\frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial R}) \\
& +(f^2+\psi_p'^2(r)(\frac{\partial r}{\partial Z})^2)(\frac{\partial y}{\partial r}\frac{\partial r}{\partial Z}+\frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial Z}) \\
& +\frac{r_0}{q_0 R}f\psi_p'(r)\frac{\partial r}{\partial R}].
\end{aligned}
\tag{14}
$$

That is, `-Up*vpar/bfldp**3/radiusp**3*(psipp**2*srbrp*srbzp*(dydrp*srbrp + r0/q0*qhatp*thbrp) + (fp**2 + psipp**2*srbzp**2)*(dydrp*srbzp + r0/q0*qhatp*thbzp) + r0/q0/radiusp*fp*psipp*srbrp)`.

2 Parallel acceleration due to toroidal flow

An auxiliary guiding center variable $v_\parallel{}''$ is defined according to

$$\varepsilon = \mu\mathbf{B}_0(\mathbf{R}) + \frac{1}{2}mv_\parallel''^2 - \frac{1}{2}mU^2(\mathbf{R}) + q\Phi_1(\mathbf{R}) \tag{15}$$

Notice that $v_\parallel{}''$ depends on $(\mathbf{R}, \varepsilon, \mu)$ but not γ, and $v_\parallel{}'' = v_\parallel{}' + \mathcal{O}(\delta)$. The new parallel velocity is defined with ε^μ. Since $d\varepsilon^\mu/dt = \mathcal{O}(\delta^2)$,

$$\begin{aligned} mv_\parallel''\frac{dv_\parallel''}{dt} &= -\mu\frac{dB(\mathbf{R})}{dt} + m\frac{dU^2}{dt} - q\frac{d\Phi_1}{dt} \\ &= -\mu v_\parallel''\mathbf{b}\cdot\nabla B + mv_\parallel''\mathbf{b}\cdot\nabla U^2 - qv_\parallel''\mathbf{b}\cdot\nabla\Phi_1 + \mathcal{O}(\delta^2) \end{aligned} \tag{16}$$

We will neglect the $\mathcal{O}(\delta^2)$ terms, which include the parallel nonlinearity.

$$\frac{dv_\parallel''}{dt} = -\frac{\mu}{m}\mathbf{b}\cdot\nabla B + \mathbf{b}\cdot\nabla U^2 - \frac{q}{m}\mathbf{b}\cdot\nabla\Phi_1 + \mathcal{O}(\delta^2) \tag{17}$$

For any scalar $s(r, \theta, \zeta)$,

$$\mathbf{b}\cdot\nabla s = \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial s}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta + \frac{f}{BR^2}\frac{\partial s}{\partial\zeta}. \tag{18}$$

$$\begin{aligned} \mathbf{b}\cdot\nabla s &= \frac{1}{B}(\nabla\zeta\times\nabla\psi_p + \frac{f}{R}\hat{\zeta})\cdot(\frac{\partial s}{\partial r}\nabla r + \frac{\partial s}{\partial\theta}\nabla\theta + \frac{\partial s}{\partial\zeta}\nabla\zeta) \\ &= \frac{\psi_p'}{B}(\nabla\zeta\times\nabla r)\cdot(\frac{\partial s}{\partial r}\nabla r + \frac{\partial s}{\partial\theta}\nabla\theta + \frac{\partial s}{\partial\zeta}\nabla\zeta) + \frac{f}{BR^2}\frac{\partial s}{\partial\zeta} \\ &= \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial s}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta + \frac{f}{BR^2}\frac{\partial s}{\partial\zeta} \end{aligned}$$

So,

$$\mathbf{b}\cdot\nabla B = \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial B}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta \tag{19}$$

$$\mathbf{b}\cdot\nabla U^2 = \frac{2U}{B}\frac{\psi_p'}{R}\frac{\partial U}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta, \tag{20}$$

$$\mathbf{b}\cdot\nabla\Phi_1 = \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial\Phi_1}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta \tag{21}$$

with

$$\hat{\zeta} \cdot \nabla r \times \nabla \theta = |\nabla r \times \nabla \theta|. \tag{22}$$

The $\partial U/\partial\theta$ and $\partial\Phi_1/\partial\theta$ will be calculated in the `gem_equil.f90`. Others are all existing variables in GEM. Φ_1 is the electric potential determined by the charge-neutrality, with $\mathbf{E_n} = -\nabla\Phi_1$. For a plasma with a single ion species with ion temperature T_i and electron temperature T_e,

$$e\Phi_1 = \frac{m_i\omega_0^2}{2(1+T_i/T_e)}(R^2 - \langle R^2\rangle). \tag{23}$$

Attention, here the bracket $\langle\cdots\rangle$ stands for the flux surface average.

$$\langle R^2\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} R^2(r,\theta)d\theta \tag{24}$$

Thus,

$$\partial_\theta(R^2 - \langle R^2\rangle) = 2R\partial_\theta R - \frac{R^2}{2\pi} \tag{25}$$

3 Changes in weight equation

Let $f = f_0 + \delta f$, the perturbed part of distribution function

$$\delta f = -q(\phi - \mathbf{U}\cdot\mathbf{A})\frac{f_0}{T} + h \tag{26}$$

Here, h is the non-adiabatic part of δf.

Write $h = h + h_1 + h_2 \cdots$

$$h_2 = \delta f + \frac{q}{T} f_0[(\phi - \mathbf{U}\cdot\mathbf{A}) - \langle(\phi - \mathbf{U}\cdot\mathbf{A})\rangle + \langle\mathbf{v}'\cdot\mathbf{A}\rangle] \tag{27}$$

$$\begin{aligned}
&\frac{\partial h_2}{\partial t} + \langle\dot{\mathbf{R}}\rangle\cdot\nabla h_2 + \langle\dot{\varepsilon}^u\rangle\frac{\partial h_1}{\partial\varepsilon^u} - \langle\dot{\varepsilon}^u\rangle\frac{\partial}{\partial\varepsilon^u}\langle\frac{q}{T}f_0\mathbf{v}'\cdot\mathbf{A}\rangle \\
= &-\left\langle \left.\frac{d\mathbf{R}}{dt}\right|_1 \right\rangle \cdot \frac{\partial f_0}{\partial\mathbf{R}} \\
&- q\mathbf{v}_g\cdot\nabla\langle\Psi\rangle\frac{f_0}{T} \\
&+ q[\mathbf{U}(\mathbf{R})\cdot\nabla\langle\Psi\rangle - \langle\mathbf{U}\cdot\nabla\Psi\rangle]\frac{f_0}{T} \\
&- \frac{q}{\Omega}\left\langle(\nabla\Psi\times\mathbf{b})\cdot\nabla\psi_p\ \omega_0' R\left(\frac{B_t}{B_0}v_\parallel' + U\right)\right\rangle\frac{f_0}{T} \\
&+ S_1 + S_2 + S_3
\end{aligned} \tag{28}$$

There are two terms to be added in the weight equation. For electrostatic model, $\Psi = \phi$.

The first term,

$$q[\mathbf{U}(\mathbf{R})\cdot\nabla\langle\phi\rangle - \langle\mathbf{U}\cdot\nabla\phi\rangle]\frac{f_0}{T} \tag{29}$$

All right, let's proceed a further step. Write

$$\mathbf{U}(\mathbf{x}) = \mathbf{U}(\mathbf{R}) + \boldsymbol{\rho}\cdot\nabla\mathbf{U}, \tag{30}$$

then

$$\mathbf{U}(\mathbf{R})\cdot\nabla\langle\phi\rangle - \langle\mathbf{U}\cdot\nabla\phi\rangle = -\langle\boldsymbol{\rho}\cdot\nabla\mathbf{U}\cdot\nabla\phi\rangle. \tag{31}$$

Remember,

$$\hat{\zeta}\cdot\nabla\hat{R}=\frac{1}{R}\hat{\zeta} \text{ and } \hat{\zeta}\cdot\nabla\hat{\zeta}=-\frac{1}{R}\hat{R}.$$

These two terms results from the reconverting from the Cartesian coordinate to the cylindrical coordinate in deriving the material derivative, see Eq. 7.106 in [2].

$$\boldsymbol{\rho}=\rho(\mathbf{e_1}\mathrm{sin}\gamma+\mathbf{e_2}\mathrm{cos}\gamma),$$

where $\rho=v'_\perp/\Omega=\sqrt{2\mu B/m}/(qB/m), \mathbf{e_1}=\nabla r/|\nabla r|, \mathbf{e_2}=\mathbf{b}\times\mathbf{e_1}$ and γ is the gyro angle.

Let

$$\mathbf{e_1}=\mathrm{e_{1R}}(r,\theta)\hat{R}+\mathrm{e_{1Z}}(r,\theta)\hat{Z}$$
$$\mathbf{e_2}=\mathrm{e_{2R}}(r,\theta)\hat{R}+\mathrm{e_{2Z}}(r,\theta)\hat{Z}+\mathrm{e_{2\zeta}}(r,\theta)\hat{\zeta},$$

then we have

$$\boldsymbol{\rho}=\rho\Big((\mathrm{e_{1R}}\mathrm{sin}\gamma+\mathrm{e_{2R}}\mathrm{cos}\gamma)\hat{R}+(\mathrm{e_{1Z}}\mathrm{sin}\gamma+\mathrm{e_{2Z}}\mathrm{cos}\gamma)\hat{Z}+(\mathrm{e_{2\zeta}}\mathrm{cos}\gamma)\hat{\zeta}\Big)$$

$$\begin{aligned}\boldsymbol{\rho}\cdot\nabla\mathbf{U}&=\boldsymbol{\rho}\cdot\nabla(U\hat{\zeta})\\&=\boldsymbol{\rho}\cdot\nabla U\hat{\zeta}+U\boldsymbol{\rho}\cdot\nabla\hat{\zeta}\\&=\boldsymbol{\rho}\cdot\left(\frac{U}{R}\hat{R}-\omega'R\frac{\partial r}{\partial R}\hat{R}-\omega'R\frac{\partial r}{\partial Z}\hat{Z}\right)\hat{\zeta}-\frac{U\rho_\zeta}{R}\hat{R}\\&=\left(\frac{U\rho_R}{R}-\omega'R\frac{\partial r}{\partial R}\rho_R-\omega'R\frac{\partial r}{\partial Z}\rho_Z\right)\hat{\zeta}-\frac{U\rho_\zeta}{R}\hat{R}\end{aligned}\tag{32}$$

$$\begin{aligned}&\left(\frac{U\rho_R}{R}-\omega'R\frac{\partial r}{\partial R}\rho_R-\omega'R\frac{\partial r}{\partial Z}\rho_Z\right)\hat{\zeta}\cdot\nabla\phi\\=&\left(\frac{U\rho_R}{R}-\omega'R\frac{\partial r}{\partial R}\rho_R-\omega'R\frac{\partial r}{\partial Z}\rho_Z\right)\hat{\zeta}\cdot\frac{\partial\phi}{\partial y}\nabla y\\=&\left(-\frac{U\rho_R}{R}+\omega'R\frac{\partial r}{\partial R}\rho_R+\omega'R\frac{\partial r}{\partial Z}\rho_Z\right)\frac{r_0}{q_0R}\frac{\partial\phi}{\partial y}\end{aligned}\tag{33}$$

$$-\frac{U\rho_\zeta}{R}\hat{R}\cdot\nabla\phi = -\frac{U\rho_\zeta}{R}\hat{R}\cdot\left(\frac{\partial\phi}{\partial x}\nabla x + \frac{\partial\phi}{\partial y}\nabla y + \frac{\partial\phi}{\partial z}\nabla z\right)$$
$$= -\frac{U\rho_\zeta}{R}\left(\frac{\partial\phi}{\partial x}\frac{\partial r}{\partial R} + \frac{\partial\phi}{\partial y}\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R} + \frac{r_0}{q_0}\hat{q}\frac{\partial\theta}{\partial R}\right) + \frac{\partial\phi}{\partial z}q_0R_0\frac{\partial\theta}{\partial R}\right) \tag{34}$$

$$-\langle\boldsymbol{\rho}\cdot\nabla\mathbf{U}\cdot\nabla\phi\rangle = \left(\frac{U}{R}\left\langle\rho_R\frac{\partial\phi}{\partial y}\right\rangle - \omega' R\frac{\partial r}{\partial R}\left\langle\rho_R\frac{\partial\phi}{\partial y}\right\rangle - \omega' R\frac{\partial r}{\partial Z}\left\langle\rho_Z\frac{\partial\phi}{\partial y}\right\rangle\right)\frac{r_0}{q_0R}$$
$$+\frac{U}{R}\left(\left\langle\rho_\zeta\frac{\partial\phi}{\partial x}\right\rangle\frac{\partial r}{\partial R} + \left\langle\rho_\zeta\frac{\partial\phi}{\partial y}\right\rangle\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R} + \frac{r_0}{q_0}\hat{q}\frac{\partial\theta}{\partial R}\right) + \left\langle\rho_\zeta\frac{\partial\phi}{\partial z}\right\rangle q_0R_0\frac{\partial\theta}{\partial R}\right) \tag{35}$$

$$\rho_R = \rho(\mathrm{e_{1R}}\mathrm{sin}\gamma + \mathrm{e_{2R}}\mathrm{cos}\gamma)$$
$$\rho_Z = \rho(\mathrm{e_{1Z}}\mathrm{sin}\gamma + \mathrm{e_{2Z}}\mathrm{cos}\gamma)$$
$$\rho_\zeta = \rho\mathrm{e}_{2\zeta}\mathrm{cos}\gamma$$
$$\mathrm{e_{1R}} = \frac{\partial r}{\partial R}\Big/|\nabla r| = \texttt{srbr/gr}$$
$$\mathrm{e_{1Z}} = \frac{\partial r}{\partial Z}\Big/|\nabla r| = \texttt{srbz/gr}$$
$$\mathrm{e_{2R}} = -\frac{f}{RB}\mathrm{e_{1Z}}$$
$$\mathrm{e_{2Z}} = \frac{f}{RB}\mathrm{e_{1R}}$$
$$\mathrm{e}_{2\zeta} = \frac{\psi_p'}{RB}\left[-\mathrm{e_{1R}}\frac{\partial r}{\partial R} - \mathrm{e_{1Z}}\frac{\partial r}{\partial Z}\right]$$

The gyro average of ρ_R, ρ_Z and ρ_ζ could be obtained by the 4-point averaging method. The 4 points are at $\gamma = 0, \pi/2, \pi$ and $3\pi/2$, respectively.

The second term,

$$\begin{aligned}
&-\frac{q}{\Omega}\left\langle(\nabla\phi\times\mathbf{b})\cdot\nabla\psi_p\ \omega_0' R\left(\frac{B_t}{B_0}v_\parallel'+U\right)\right\rangle\frac{f_0}{T}\\
=&-\frac{q}{\Omega}\left\langle(\nabla\phi\times\mathbf{b})\cdot\nabla\psi_p\right\rangle\omega_0' R\left(\frac{B_t}{B_0}v_\parallel'+U\right)\frac{f_0}{T}
\end{aligned}\tag{36}$$

Considering

$$\begin{aligned}
\mathbf{B}&=\nabla\psi\times\nabla(q\theta_f-\zeta)\\
&=\frac{q_0}{r_0}\frac{d\psi}{dx}\nabla x\times\nabla y\\
&=C(x)\nabla x\times\nabla y
\end{aligned}\tag{37}$$

then

$$\mathbf{b}=\frac{\nabla x\times\nabla y}{|\nabla x\times\nabla y|}.\tag{38}$$

So we have

$$\begin{aligned}
\nabla\phi\times\mathbf{b}\cdot\nabla\psi_p&=\nabla\psi_p\times\nabla\phi\cdot\mathbf{b}\\
&=\psi_p'\nabla x\times\nabla\phi\cdot\frac{\nabla x\times\nabla y}{|\nabla x\times\nabla y|}\\
&=\frac{\psi_p'}{|\nabla x\times\nabla y|}\left(\frac{\partial\phi}{\partial y}\nabla x\times\nabla y+\frac{\partial\phi}{\partial z}\nabla x\times\nabla z\right)\cdot\nabla x\times\nabla y\\
&=\psi_p'|\nabla x\times\nabla y|\frac{\partial\phi}{\partial y}+\frac{\psi_p'}{|\nabla x\times\nabla y|}\frac{\partial\phi}{\partial z}(\nabla x\times\nabla z)\cdot(\nabla x\times\nabla y)
\end{aligned}\tag{39}$$

$$\begin{aligned}
(\nabla x\times\nabla z)\cdot(\nabla x\times\nabla y)&=(\nabla x\times\nabla y)\times\nabla x\cdot\nabla z\\
&=(|\nabla x|^2\nabla y-|\nabla x\cdot\nabla y|\nabla x)\cdot\nabla z\\
&=|\nabla x|^2\nabla y\cdot\nabla z-|\nabla x\cdot\nabla y|\nabla x\cdot\nabla z\\
&=|\nabla x|^2\nabla y\cdot\nabla z-q_0R_0|\nabla x\cdot\nabla y|\ |\nabla r\cdot\nabla\theta|
\end{aligned}\tag{40}$$

$$\begin{aligned}
\nabla y\cdot\nabla z&=\left(\frac{\partial y}{\partial r}\nabla r+\frac{r_0}{q_0}\hat{q}\nabla\theta-\frac{r_0}{q_0}\nabla\zeta\right)\cdot q_0R_0\nabla\theta\\
&=q_0R_0\frac{\partial y}{\partial r}|\nabla r\cdot\nabla\theta|+r_0R_0\hat{q}|\nabla\theta|^2
\end{aligned}\tag{41}$$

According to Eq. 39 - 41, Eq. 36 can be coded using existing variables.

We may need the following operator to simplify the coding process,

$$\hat{R} \cdot \nabla y = \frac{\partial y}{\partial r}\frac{\partial r}{\partial R} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial R}$$

$$\hat{Z} \cdot \nabla y = \frac{\partial y}{\partial r}\frac{\partial r}{\partial Z} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial Z}$$

$$\hat{R} \cdot \nabla z = q_0 R_0 \frac{\partial \theta}{\partial R}$$

$$\hat{Z} \cdot \nabla z = q_0 R_0 \frac{\partial \theta}{\partial Z}$$

$$\hat{\zeta} \cdot \nabla y = -\frac{r_0}{q_0 R}$$

References

[1] H. Sugama and W. Horton. Nonlinear electromagnetic gyrokinetic equation for plasmas with large mean flows. *Physics of Plasmas*, 5(7):2560–2573, 1998.

[2] Bastian E. Rapp. Chapter 7 - vector calculus. In Bastian E. Rapp, editor, *Microfluidics: Modelling, Mechanics and Mathematics*, Micro and Nano Technologies, pages 137 – 188. Elsevier, Oxford, 2017.