Adding toroidal flow in GEM for adiabatic electron model

汤炜康

Contents

1 Guiding center drift due to toroidal flow

There are three terms needed to be added in the drift velocity of the guiding center V_G, according to Eq. 21 in [1], i. e.

$$\frac{cm_a}{e_a B}\hat{b} \times (\mathbf{U} \cdot \nabla \mathbf{U}), \tag{1}$$

$$\frac{cm_a v_\parallel^{'}}{e_a B}\hat{b} \times (\hat{b} \cdot \nabla \mathbf{U}), \tag{2}$$

$$\frac{cm_a v_\parallel^{'}}{e_a B}\hat{b} \times (\mathbf{U} \cdot \nabla \hat{b}). \tag{3}$$

In the right-handed coordinate system (R,Z,ζ), considering the toroidal flow with the form of $\mathbf{U}(\mathbf{R}) = -\omega_0(\psi_p)R^2\nabla\zeta$,

$$\mathbf{U} \cdot \nabla \mathbf{U} = -\frac{U^2}{R}\hat{R} \tag{4}$$

$$\begin{aligned} \hat{b} \times (\mathbf{U} \cdot \nabla \mathbf{U}) &= \frac{U^2}{R}\hat{R} \times \hat{b} \\ &= \frac{U^2}{RB}\hat{R} \times (\frac{f}{R}\hat{\zeta} + \frac{\psi_p^{'}(r)}{R}(\frac{\partial r}{\partial R}\hat{Z} - \frac{\partial r}{\partial Z}\hat{R})) \\ &= \frac{U^2}{RB}(-\frac{f}{R}\hat{Z} + \frac{\psi_p^{'}(r)}{R}\frac{\partial r}{\partial R}\hat{\zeta}) \end{aligned} \tag{5}$$

Then Eq. 1 becomes

$$\frac{cm_a U^2}{e_a B^2 R^2}(-f\hat{Z} + \psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta}). \tag{6}$$

Eq. 6 $\cdot\nabla x$ is

$$\frac{cm_a U^2}{e_a B^2 R^2} \cdot (-f)\frac{\partial r}{\partial Z}. \tag{7}$$

To implement this term, a new variable `ut` needed to be declared, representing the equilibrium toroidal flow. Actually, we only need to declare a new variable `omg`, then `ut=-omg*radius`, see **Appendix**.

For Eq. 6 $\cdot\nabla y$,

$$y = \frac{r_0}{q_0}\int_0^\theta \hat{q}(r,\theta^{'})d\theta^{'} - \zeta) = \frac{r_0}{q_0}(q\theta_f - \zeta) \tag{8}$$

$$\nabla y = \frac{\partial y}{\partial r}\nabla r + \frac{r_0}{q_0}\hat{q}\nabla\theta - \frac{r_0}{q_0}\nabla\zeta, \tag{9}$$

where $\hat{q} = q\frac{\partial\theta_f}{\partial\theta}$.

$$\begin{aligned}\text{Eq. 6}\cdot\nabla y &= \frac{cm_aU^2}{e_aB^2R^2}(-f\hat{Z} + \psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta})\cdot\nabla y \\ &= \frac{cm_aU^2}{e_aB^2R^2}(-f\frac{\partial y}{\partial r}\frac{\partial r}{\partial Z} - \frac{r_0}{q_0}\hat{q}f\frac{\partial\theta}{\partial Z} - \frac{r_0}{q_0R}\psi_p^{'}(r)\frac{\partial r}{\partial R}),\end{aligned} \tag{10}$$

$$\frac{cm_aU^2}{e_aB^2R^2}(-f\hat{Z}\cdot\nabla y + \psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta}\cdot\nabla y)$$

So far, all the terms about Eq. 1 is finished. Let's start with Eqs. 2 and 3. Still in the (R,Z,ζ) coordinate, one could find Eq. 2 is identical to Eq. 3 as

$$\mathbf{U}\cdot\nabla\hat{b} = \hat{b}\cdot\nabla\mathbf{U} = -\frac{Uf}{R^2B}\hat{R} - \frac{U}{R^2B}\psi_p^{'}(r)\frac{\partial r}{\partial Z}\hat{\zeta}. \tag{11}$$

To derive Eq. 2 or Eq. 3,

$$\begin{aligned}&\frac{cm_av_\parallel^{'}}{e_aB}\hat{b}\times(\mathbf{U}\cdot\nabla\hat{b}) \\ =&\frac{cm_av_\parallel^{'}}{e_aB}[-\frac{Uf^2}{B^2R^3}\hat{Z} + \frac{Uf\psi_p^{'}(r)}{B^2R^3}\frac{\partial r}{\partial R}\hat{\zeta} \\ &-\frac{U}{B^2R^3}\psi_p^{'2}(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}\hat{R} - \frac{U}{B^2R^3}\psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2\hat{Z}] \\ =&\frac{cm_av_\parallel{'}U}{e_aB^3R^3}[-\psi_p^{'2}(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}\hat{R} - (f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2)\hat{Z} + f\psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta}].\end{aligned} \tag{12}$$

Then,

$$\begin{aligned}\text{Eq. 12}\cdot\nabla x &= -\frac{cm_av_\parallel{'}U}{e_aB^3R^3}[\psi_p^{'2}(r)(\frac{\partial r}{\partial R})^2\frac{\partial r}{\partial Z} + (f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2)\frac{\partial r}{\partial Z}] \\ &= -\frac{cm_av_\parallel{'}U}{e_aB^3R^3}[\psi_p^{'2}(r)(\frac{\partial r}{\partial R})^2 + f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2]\frac{\partial r}{\partial Z}.\end{aligned} \tag{13}$$

Using Eq. 9,

$$
\begin{aligned}
\text{Eq. } 12 \cdot \nabla y = &- \frac{c m_a {v_\parallel}' U}{e_a B^3 R^3}[\psi_p'^2(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial R}) \\
&+ (f^2 + \psi_p'^2(r)(\frac{\partial r}{\partial Z})^2)(\frac{\partial y}{\partial r}\frac{\partial r}{\partial Z} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial Z}) \\
&+ \frac{r_0}{q_0 R} f \psi_p'(r)\frac{\partial r}{\partial R}].
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\frac{c m_a {v_\parallel}' U}{e_a B^3 R^3}\Bigg[&- \psi_p'^2(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}\hat{R}\cdot\nabla y \\
&- \left(f^2 + \psi_p'^2(r)(\frac{\partial r}{\partial Z})^2\right)\hat{Z}\cdot\nabla y \\
&+ f\psi_p'(r)\frac{\partial r}{\partial R}\hat{\zeta}\cdot\nabla y \Bigg]
\end{aligned}
$$

2 Parallel acceleration due to toroidal flow

An auxiliary guiding center variable $v_\parallel''$ is defined according to

$$\varepsilon = \mu \mathbf{B}_0(\mathbf{R}) + \frac{1}{2} m v_\parallel''^2 - \frac{1}{2} m U^2(\mathbf{R}) + q\Phi_1(\mathbf{R}) \tag{15}$$

Notice that $v_\parallel''$ depends on $(\mathbf{R}, \varepsilon, \mu)$ but not γ, and $v_\parallel'' = v_\parallel' + \mathcal{O}(\delta)$. The new parallel velocity is defined with ε^μ. Since $d\varepsilon^\mu/dt = \mathcal{O}(\delta^2)$,

$$\begin{aligned} m v_\parallel'' \frac{dv_\parallel''}{dt} &= -\mu \frac{dB(\mathbf{R})}{dt} + m\frac{dU^2}{dt} - q\frac{d\Phi_1}{dt} \\ &= -\mu v_\parallel'' \mathbf{b}\cdot\nabla B + m v_\parallel'' \mathbf{b}\cdot\nabla U^2 - q v_\parallel'' \mathbf{b}\cdot\nabla\Phi_1 + \mathcal{O}(\delta^2) \end{aligned} \tag{16}$$

We will neglect the $\mathcal{O}(\delta^2)$ terms, which include the parallel nonlinearity.

$$\frac{dv_\parallel''}{dt} = -\frac{\mu}{m}\mathbf{b}\cdot\nabla B + \mathbf{b}\cdot\nabla U^2 - \frac{q}{m}\mathbf{b}\cdot\nabla\Phi_1 + \mathcal{O}(\delta^2) \tag{17}$$

For any scalar $s(r, \theta, \zeta)$,

$$\mathbf{b}\cdot\nabla s = \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial s}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta + \frac{f}{BR^2}\frac{\partial s}{\partial\zeta}. \tag{18}$$

$$\begin{aligned} \mathbf{b}\cdot\nabla s &= \frac{1}{B}(\nabla\zeta\times\nabla\psi_p + \frac{f}{R}\hat{\zeta})\cdot(\frac{\partial s}{\partial r}\nabla r + \frac{\partial s}{\partial\theta}\nabla\theta + \frac{\partial s}{\partial\zeta}\nabla\zeta) \\ &= \frac{\psi_p'}{B}(\nabla\zeta\times\nabla r)\cdot(\frac{\partial s}{\partial r}\nabla r + \frac{\partial s}{\partial\theta}\nabla\theta + \frac{\partial s}{\partial\zeta}\nabla\zeta) + \frac{f}{BR^2}\frac{\partial s}{\partial\zeta} \\ &= \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial s}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta + \frac{f}{BR^2}\frac{\partial s}{\partial\zeta} \end{aligned}$$

So,

$$\mathbf{b}\cdot\nabla B = \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial B}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta \tag{19}$$

$$\mathbf{b}\cdot\nabla U^2 = \frac{2U}{B}\frac{\psi_p'}{R}\frac{\partial U}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta, \tag{20}$$

$$\mathbf{b}\cdot\nabla\Phi_1 = \frac{1}{B}\frac{\psi_p'}{R}\frac{\partial\Phi_1}{\partial\theta}\hat{\zeta}\cdot\nabla r\times\nabla\theta \tag{21}$$

with

$$\hat{\zeta} \cdot \nabla r \times \nabla \theta = |\nabla r \times \nabla \theta|. \tag{22}$$

The $\partial U/\partial \theta$ and $\partial \Phi_1/\partial \theta$ will be calculated in the `gem_equil.f90`. Others are all existing variables in GEM. Φ_1 is the electric potential determined by the charge-neutrality, with $\mathbf{E_n} = -\nabla \Phi_1$. For a plasma with a single ion species with ion temperature T_i and electron temperature T_e,

$$e\Phi_1 = \frac{m_i \omega_0^2}{2(1 + T_i/T_e)}(R^2 - \langle R^2 \rangle). \tag{23}$$

Attention, here the bracket $\langle \cdots \rangle$ stands for the flux surface average.

$$\langle R^2 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} R^2(r, \theta) d\theta \tag{24}$$

Thus,

$$\partial_\theta (R^2 - \langle R^2 \rangle) = 2R\partial_\theta R - \frac{R^2}{2\pi} \tag{25}$$

3 Changes in weight equation

Let $f = f_0 + \delta f$, the perturbed part of distribution function

$$\delta f = -q(\phi - \mathbf{U} \cdot \mathbf{A})\frac{f_0}{T} + h \tag{26}$$

Here, h is the non-adiabatic part of δf.

Write $h = h + h_1 + h_2 \cdots$

$$h_2 = \delta f + \frac{q}{T} f_0[(\phi - \mathbf{U} \cdot \mathbf{A}) - \langle(\phi - \mathbf{U} \cdot \mathbf{A})\rangle + \langle \mathbf{v}' \cdot \mathbf{A} \rangle] \tag{27}$$

$$\begin{aligned}
&\frac{\partial h_2}{\partial t} + \langle \dot{\mathbf{R}} \rangle \cdot \nabla h_2 + \langle \dot{\varepsilon}^u \rangle \frac{\partial h_1}{\partial \varepsilon^u} - \langle \dot{\varepsilon}^u \rangle \frac{\partial}{\partial \varepsilon^u} \langle \frac{q}{T} f_0 \mathbf{v}' \cdot \mathbf{A} \rangle \\
= &- \left\langle \frac{d\mathbf{R}}{dt}\bigg|_1 \right\rangle \cdot \frac{\partial f_0}{\partial \mathbf{R}} \\
&- q\mathbf{v}_g \cdot \nabla \langle \Psi \rangle \frac{f_0}{T} \\
&+ q[\mathbf{U}(\mathbf{R}) \cdot \nabla \langle \Psi \rangle - \langle \mathbf{U} \cdot \nabla \Psi \rangle] \frac{f_0}{T} \\
&- \frac{q}{\Omega} \left\langle (\nabla \Psi \times \mathbf{b}) \cdot \nabla \psi_p \; \omega_0' R \left(\frac{B_t}{B_0} v_\parallel' + U \right) \right\rangle \frac{f_0}{T} \\
&+ S_1 + S_2 + S_3
\end{aligned} \tag{28}$$

There are two terms to be added in the weight equation. For electrostatic model, $\Psi = \phi$.

The first term,

$$q[\mathbf{U}(\mathbf{R}) \cdot \nabla \langle \phi \rangle - \langle \mathbf{U} \cdot \nabla \phi \rangle] \frac{f_0}{T} \tag{29}$$

All right, let's proceed a further step. Write

$$\mathbf{U}(\mathbf{x}) = \mathbf{U}(\mathbf{R}) + \boldsymbol{\rho} \cdot \nabla \mathbf{U}, \tag{30}$$

then

$$\mathbf{U}(\mathbf{R}) \cdot \nabla \langle \phi \rangle - \langle \mathbf{U} \cdot \nabla \phi \rangle = -\langle \boldsymbol{\rho} \cdot \nabla \mathbf{U} \cdot \nabla \phi \rangle. \tag{31}$$

Remember,

$$\hat{\zeta}\cdot\nabla\hat{R}=\frac{1}{R}\hat{\zeta} \text{ and } \hat{\zeta}\cdot\nabla\hat{\zeta}=-\frac{1}{R}\hat{R}.$$

These two terms results from the reconverting from the Cartesian coordinate to the cylindrical coordinate in deriving the material derivative, see Eq. 7.106 in [2].

$$\boldsymbol{\rho}=\rho(\mathbf{e_1}\mathrm{sin}\gamma+\mathbf{e_2}\mathrm{cos}\gamma),$$

where $\rho=v'_{\perp}/\Omega=\sqrt{2\mu B/m}/(qB/m), \mathbf{e_1}=\nabla r/|\nabla r|, \mathbf{e_2}=\mathbf{b}\times\mathbf{e_1}$ and γ is the gyro angle.

Let

$$\mathbf{e_1}=\mathrm{e_{1R}}(r,\theta)\hat{R}+\mathrm{e_{1Z}}(r,\theta)\hat{Z}$$

$$\mathbf{e_2}=\mathrm{e_{2R}}(r,\theta)\hat{R}+\mathrm{e_{2Z}}(r,\theta)\hat{Z}+\mathrm{e}_{2\zeta}(r,\theta)\hat{\zeta},$$

then we have

$$\boldsymbol{\rho}=\rho\Big((\mathrm{e_{1R}}\mathrm{sin}\gamma+\mathrm{e_{2R}}\mathrm{cos}\gamma)\hat{R}+(\mathrm{e_{1Z}}\mathrm{sin}\gamma+\mathrm{e_{2Z}}\mathrm{cos}\gamma)\hat{Z}+(\mathrm{e}_{2\zeta}\mathrm{cos}\gamma)\hat{\zeta}\Big)$$

$$\begin{aligned}\boldsymbol{\rho}\cdot\nabla\mathbf{U}&=\boldsymbol{\rho}\cdot\nabla(U\hat{\zeta})\\&=\boldsymbol{\rho}\cdot\nabla U\hat{\zeta}+U\boldsymbol{\rho}\cdot\nabla\hat{\zeta}\\&=\boldsymbol{\rho}\cdot\left(\frac{U}{R}\hat{R}-\omega' R\frac{\partial r}{\partial R}\hat{R}-\omega' R\frac{\partial r}{\partial Z}\hat{Z}\right)\hat{\zeta}-\frac{U\rho_\zeta}{R}\hat{R}\\&=\left(\frac{U\rho_R}{R}-\omega' R\frac{\partial r}{\partial R}\rho_R-\omega' R\frac{\partial r}{\partial Z}\rho_Z\right)\hat{\zeta}-\frac{U\rho_\zeta}{R}\hat{R}\end{aligned}\tag{32}$$

$$\begin{aligned}&\left(\frac{U\rho_R}{R}-\omega' R\frac{\partial r}{\partial R}\rho_R-\omega' R\frac{\partial r}{\partial Z}\rho_Z\right)\hat{\zeta}\cdot\nabla\phi\\=&\left(\frac{U\rho_R}{R}-\omega' R\frac{\partial r}{\partial R}\rho_R-\omega' R\frac{\partial r}{\partial Z}\rho_Z\right)\hat{\zeta}\cdot\frac{\partial\phi}{\partial y}\nabla y\\=&\left(-\frac{U\rho_R}{R}+\omega' R\frac{\partial r}{\partial R}\rho_R+\omega' R\frac{\partial r}{\partial Z}\rho_Z\right)\frac{r_0}{q_0R}\frac{\partial\phi}{\partial y}\end{aligned}\tag{33}$$

$$
\begin{aligned}
-\frac{U\rho_\zeta}{R}\hat{R}\cdot\nabla\phi &= -\frac{U\rho_\zeta}{R}\hat{R}\cdot\left(\frac{\partial\phi}{\partial x}\nabla x+\frac{\partial\phi}{\partial y}\nabla y+\frac{\partial\phi}{\partial z}\nabla z\right)\\
&= -\frac{U\rho_\zeta}{R}\left(\frac{\partial\phi}{\partial x}\frac{\partial r}{\partial R}+\frac{\partial\phi}{\partial y}\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R}+\frac{r_0}{q_0}\hat{q}\frac{\partial\theta}{\partial R}\right)+\frac{\partial\phi}{\partial z}q_0R_0\frac{\partial\theta}{\partial R}\right)
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
-\left\langle\boldsymbol{\rho}\cdot\nabla\mathbf{U}\cdot\nabla\phi\right\rangle &= \left(\frac{U}{R}\left\langle\rho_R\frac{\partial\phi}{\partial y}\right\rangle-\omega' R\frac{\partial r}{\partial R}\left\langle\rho_R\frac{\partial\phi}{\partial y}\right\rangle-\omega' R\frac{\partial r}{\partial Z}\left\langle\rho_Z\frac{\partial\phi}{\partial y}\right\rangle\right)\frac{r_0}{q_0R}\\
&+\frac{U}{R}\left(\left\langle\rho_\zeta\frac{\partial\phi}{\partial x}\right\rangle\frac{\partial r}{\partial R}+\left\langle\rho_\zeta\frac{\partial\phi}{\partial y}\right\rangle\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R}+\frac{r_0}{q_0}\hat{q}\frac{\partial\theta}{\partial R}\right)+\left\langle\rho_\zeta\frac{\partial\phi}{\partial z}\right\rangle q_0R_0\frac{\partial\theta}{\partial R}\right)
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
&\left[\left(\omega' R\frac{\partial r}{\partial R}-\frac{U}{R}\right)\left\langle\rho_R\frac{\partial\phi}{\partial y}\right\rangle+\omega' R\frac{\partial r}{\partial Z}\left\langle\rho_Z\frac{\partial\phi}{\partial y}\right\rangle\right]\hat{\zeta}\cdot\nabla y\\
&+\frac{U}{R}\left(\left\langle\rho_\zeta\frac{\partial\phi}{\partial x}\right\rangle\frac{\partial r}{\partial R}+\left\langle\rho_\zeta\frac{\partial\phi}{\partial y}\right\rangle\hat{R}\cdot\nabla y+\left\langle\rho_\zeta\frac{\partial\phi}{\partial z}\right\rangle\hat{R}\cdot\nabla z\right)
\end{aligned}
$$

$$
\begin{aligned}
\rho_R &= \rho(\mathrm{e_{1R}}\sin\gamma+\mathrm{e_{2R}}\cos\gamma)\\
\rho_Z &= \rho(\mathrm{e_{1Z}}\sin\gamma+\mathrm{e_{2Z}}\cos\gamma)\\
\rho_\zeta &= \rho\mathrm{e_{2\zeta}}\cos\gamma\\
\mathrm{e_{1R}} &= \frac{\partial r}{\partial R}\Big/|\nabla r| = \mathtt{srbr/gr}\\
\mathrm{e_{1Z}} &= \frac{\partial r}{\partial Z}\Big/|\nabla r| = \mathtt{srbz/gr}\\
\mathrm{e_{2R}} &= -\frac{f}{RB}\mathrm{e_{1Z}}\\
\mathrm{e_{2Z}} &= \frac{f}{RB}\mathrm{e_{1R}}
\end{aligned}
$$

$$e_{2\zeta} = \frac{\psi_p'}{RB}\left[- e_{1R}\frac{\partial r}{\partial R} - e_{1Z}\frac{\partial r}{\partial Z}\right]$$

The gyro average of ρ_R, ρ_Z and ρ_ζ could be obtained by the 4-point averaging method. The 4 points are at $\gamma = \pi/2, 3\pi/2, 0$ and π, respectively.

The second term,

$$\begin{aligned} & -\frac{q}{\Omega}\left\langle (\nabla\phi \times \mathbf{b}) \cdot \nabla\psi_p \; \omega_0' R\left(\frac{B_t}{B_0}v_\parallel' + U\right)\right\rangle \frac{f_0}{T} \\ = & -\frac{q}{\Omega}\left\langle (\nabla\phi \times \mathbf{b}) \cdot \nabla\psi_p \right\rangle \omega_0' R\left(\frac{B_t}{B_0}v_\parallel' + U\right)\frac{f_0}{T} \end{aligned} \tag{36}$$

Considering

$$\begin{aligned} \mathbf{B} &= \nabla\psi \times \nabla(q\theta_f - \zeta) \\ &= \frac{q_0}{r_0}\frac{d\psi}{dx}\nabla x \times \nabla y \\ &= C(x)\nabla x \times \nabla y \end{aligned} \tag{37}$$

then

$$\mathbf{b} = \frac{\nabla x \times \nabla y}{|\nabla x \times \nabla y|}. \tag{38}$$

So we have

$$\begin{aligned} \nabla\phi \times \mathbf{b} \cdot \nabla\psi_p &= \nabla\psi_p \times \nabla\phi \cdot \mathbf{b} \\ &= \psi_p' \nabla x \times \nabla\phi \cdot \frac{\nabla x \times \nabla y}{|\nabla x \times \nabla y|} \\ &= \frac{\psi_p'}{|\nabla x \times \nabla y|}\left(\frac{\partial\phi}{\partial y}\nabla x \times \nabla y + \frac{\partial\phi}{\partial z}\nabla x \times \nabla z\right) \cdot \nabla x \times \nabla y \\ &= \psi_p' |\nabla x \times \nabla y| \frac{\partial\phi}{\partial y} + \frac{\psi_p'}{|\nabla x \times \nabla y|}\frac{\partial\phi}{\partial z}(\nabla x \times \nabla z) \cdot (\nabla x \times \nabla y) \end{aligned} \tag{39}$$

$$\begin{aligned} (\nabla x \times \nabla z) \cdot (\nabla x \times \nabla y) &= (\nabla x \times \nabla y) \times \nabla x \cdot \nabla z \\ &= (|\nabla x|^2 \nabla y - |\nabla x \cdot \nabla y| \nabla x) \cdot \nabla z \\ &= |\nabla x|^2 \nabla y \cdot \nabla z - |\nabla x \cdot \nabla y| \nabla x \cdot \nabla z \\ &= |\nabla x|^2 \nabla y \cdot \nabla z - q_0 R_0 |\nabla x \cdot \nabla y| \; |\nabla r \cdot \nabla\theta| \end{aligned} \tag{40}$$

$$
\begin{aligned}
\nabla y \cdot \nabla z &= \left(\frac{\partial y}{\partial r} \nabla r + \frac{r_0}{q_0} \hat{q} \nabla \theta - \frac{r_0}{q_0} \nabla \zeta \right) \cdot q_0 R_0 \nabla \theta \\
&= q_0 R_0 \frac{\partial y}{\partial r} |\nabla r \cdot \nabla \theta| + r_0 R_0 \hat{q} |\nabla \theta|^2
\end{aligned}
\tag{41}
$$

According to Eq. 39 - 41, Eq. 36 can be coded using existing variables. The blue parts above are small ordering terms that are NOT implemented for now.

4 Verification of the model

Above are all the changes for adiabatic electron model. Let's start to test the code.

Appendix

We may need the following operator to simplify the coding process.

$$\hat{R}\cdot\nabla y = \frac{\partial y}{\partial r}\frac{\partial r}{\partial R} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial R}$$

$$\hat{Z}\cdot\nabla y = \frac{\partial y}{\partial r}\frac{\partial r}{\partial Z} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial Z}$$

$$\hat{R}\cdot\nabla z = q_0 R_0\frac{\partial \theta}{\partial R}$$

$$\hat{Z}\cdot\nabla z = q_0 R_0\frac{\partial \theta}{\partial Z}$$

$$\hat{\zeta}\cdot\nabla y = -\frac{r_0}{q_0 R}$$

New variables declared in GEM:
omg(nr)—ω, angular frequency of toroidal flow
domg(nr)—dω/dr
ut(nr,ntheta)—U, toroidal flow
phi1(nr,ntheta)—Φ_1
bdgut,bdgbfld,bdgphi1::(nr,ntheta)—$\mathbf{b}\cdot\nabla$U, $\mathbf{b}\cdot\nabla$B, $\mathbf{b}\cdot\nabla\Phi_1$
hrdgy(nr,ntheta)—$\hat{R}\cdot\nabla y$
hzdgy(nr,ntheta)—$\hat{Z}\cdot\nabla y$
hrdgz(nr,ntheta)—$\hat{R}\cdot\nabla z$
hzdgz(nr,ntheta)—$\hat{Z}\cdot\nabla z$
hztdgy(nr,ntheta)—$\hat{\zeta}\cdot\nabla y$
radius2(nr,ntheta)—R^2
rhoreyp—$\left\langle \rho_R \frac{\partial \phi}{\partial y}\right\rangle$
rhozeyp—$\left\langle \rho_Z \frac{\partial \phi}{\partial y}\right\rangle$
rhoztexp—$\left\langle \rho_\zeta \frac{\partial \phi}{\partial x}\right\rangle$
rhozteyp—$\left\langle \rho_\zeta \frac{\partial \phi}{\partial y}\right\rangle$
rhoztezp—$\left\langle \rho_\zeta \frac{\partial \phi}{\partial z}\right\rangle$

References

[1] H. Sugama and W. Horton. Nonlinear electromagnetic gyrokinetic equation for plasmas with large mean flows. *Physics of Plasmas*, 5(7):2560–2573, 1998.

[2] Bastian E. Rapp. Chapter 7 - vector calculus. In Bastian E. Rapp, editor, *Microfluidics: Modelling, Mechanics and Mathematics*, Micro and Nano Technologies, pages 137 – 188. Elsevier, Oxford, 2017.