Adding toroidal flow in GEM for adiabatic electron model

汤炜康

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Contents

1 Guiding center drift due to toroidal flow

There are three terms needed to be added in the drift velocity of the guiding center V_G, according to Eq. 21 in [1], i. e.

$$\frac{cm_a}{e_a B}\hat{b} \times (\mathbf{U} \cdot \nabla \mathbf{U}), \tag{1}$$

$$\frac{cm_a v_\parallel^{'}}{e_a B}\hat{b} \times (\hat{b} \cdot \nabla \mathbf{U}), \tag{2}$$

$$\frac{cm_a v_\parallel^{'}}{e_a B}\hat{b} \times (\nabla \mathbf{U} \cdot \hat{b}). \tag{3}$$

In the right-handed coordinate system (R,Z,ζ), considering the toroidal flow with the form of $\mathbf{U}(\mathbf{R}) = -\omega_0(\psi_p) R^2 \nabla \zeta$,

$$\mathbf{U} \cdot \nabla \mathbf{U} = -\frac{U^2}{R}\hat{R} \tag{4}$$

$$\begin{aligned}
\hat{b} \times (\mathbf{U} \cdot \nabla \mathbf{U}) &= \frac{U^2}{R}\hat{R} \times \hat{b} \\
&= \frac{U^2}{RB}\hat{R} \times (\frac{f}{R}\hat{\zeta} + \frac{\psi_p^{'}(r)}{R}(\frac{\partial r}{\partial R}\hat{Z} - \frac{\partial r}{\partial Z}\hat{R})) \\
&= \frac{U^2}{RB}(-\frac{f}{R}\hat{Z} + \frac{\psi_p^{'}(r)}{R}\frac{\partial r}{\partial R}\hat{\zeta})
\end{aligned} \tag{5}$$

Then Eq. 1 becomes

$$\frac{cm_a U^2}{e_a B^2 R^2}(-f\hat{Z} + \psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta}). \tag{6}$$

Eq. 6 $\cdot \nabla x$ is

$$\frac{cm_a U^2}{e_a B^2 R^2} \cdot (-f)\frac{\partial r}{\partial Z}. \tag{7}$$

This term can be coded as `-Up**2*fp*srbzp/bfldp**2/radiusp**2`, where `U(0:nr,0:ntheta)` is a new variable needed to be declared, representing the equlibrium toroidal flow. Actually, we only need to declare a new variable `omega(0:nr)`, `U(0:nr,0:ntheta)=-omega(0:nr)*radius(0:nr,0:ntheta)`.

For Eq. 6 $\cdot\nabla y$,

$$y = \frac{r_0}{q_0}\int_0^{\theta} \hat{q}(r,\theta^{'})d\theta^{'} - \zeta) = \frac{r_0}{q_0}(q\theta_f - \zeta) \tag{8}$$

$$\nabla y = \frac{\partial y}{\partial r}\nabla r + \frac{r_0}{q_0}\hat{q}\nabla\theta - \frac{r_0}{q_0}\nabla\zeta, \tag{9}$$

where $\hat{q} = q\frac{\partial\theta_f}{\partial\theta}$.

$$\begin{aligned}\text{Eq. } 6\cdot\nabla y &= \frac{cm_aU^2}{e_aB^2R^2}(-f\hat{Z} + \psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta})\cdot\nabla y \\ &= \frac{cm_aU^2}{e_aB^2R^2}(-f\frac{\partial y}{\partial r}\frac{\partial r}{\partial Z} - \frac{r_0}{q_0}\hat{q}f\frac{\partial\theta}{\partial Z} - \frac{r_0}{q_0R}\psi_p^{'}(r)\frac{\partial r}{\partial R}),\end{aligned} \tag{10}$$

i. e. `-Up**2/bfldp**2/radiusp**2 * (fp*dydrp*srbzp + r0/q0*qhatp*fp *thbzp + r0/q0/radiusp*psipp*srbrp).`

So far, all the terms about Eq. 1 is finished. Let's start with Eqs. 2 and 3. Still in the (R,Z,ζ) coordinate, one could find Eq. 2 is identical to Eq. 3 as

$$\mathbf{U}\cdot\nabla\hat{b} = \hat{b}\cdot\nabla\mathbf{U} = -\frac{Uf}{R^2B}\hat{R} - \frac{U}{R^2B}\psi_p^{'}(r)\frac{\partial r}{\partial Z}\hat{\zeta}. \tag{11}$$

To derive Eq. 2 or Eq. 3,

$$\begin{aligned}&\frac{cm_av_{\parallel}^{'}}{e_aB}\hat{b}\times(\mathbf{U}\cdot\nabla\hat{b}) \\ =&\frac{cm_av_{\parallel}^{'}}{e_aB}[-\frac{Uf^2}{B^2R^3}\hat{Z} + \frac{Uf\psi_p^{'}(r)}{B^2R^3}\frac{\partial r}{\partial R}\hat{\zeta} \\ &-\frac{U}{B^2R^3}\psi_p^{'2}(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}\hat{R} - \frac{U}{B^2R^3}\psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2\hat{Z}] \\ =&\frac{cm_av_{\parallel}{'}U}{e_aB^3R^3}[-\psi_p^{'2}(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}\hat{R} - (f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2)\hat{Z} + f\psi_p^{'}(r)\frac{\partial r}{\partial R}\hat{\zeta}].\end{aligned} \tag{12}$$

Then,

$$\begin{aligned}\text{Eq. } 12\cdot\nabla x &= -\frac{cm_av_{\parallel}{'}U}{e_aB^3R^3}[\psi_p^{'2}(r)(\frac{\partial r}{\partial R})^2\frac{\partial r}{\partial Z} + (f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2)\frac{\partial r}{\partial Z}] \\ &= -\frac{cm_av_{\parallel}{'}U}{e_aB^3R^3}[\psi_p^{'2}(r)(\frac{\partial r}{\partial R})^2 + f^2 + \psi_p^{'2}(r)(\frac{\partial r}{\partial Z})^2]\frac{\partial r}{\partial Z},\end{aligned} \tag{13}$$

which can be coded as `-Up*vpar/bfldp**3/radiusp**3*(psipp**2*srbrp**2 + fp**2 + psipp**2*srbzp**2)*srbzp`.

Using Eq. 9,

$$\begin{aligned}\text{Eq. } 12 \cdot \nabla y = & -\frac{c m_a {v_\parallel}' U}{e_a B^3 R^3}[\psi_p'^2(r)\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial R}) \\ & + (f^2 + \psi_p'^2(r)(\frac{\partial r}{\partial Z})^2)(\frac{\partial y}{\partial r}\frac{\partial r}{\partial Z} + \frac{r_0}{q_0}\hat{q}\frac{\partial \theta}{\partial Z}) \\ & + \frac{r_0}{q_0 R} f \psi_p'(r)\frac{\partial r}{\partial R}].\end{aligned} \tag{14}$$

That is, `-Up*vpar/bfldp**3/radiusp**3*(psipp**2*srbrp*srbzp*(dydrp*srbrp + r0/q0*qhatp*thbrp) + (fp**2 + psipp**2*srbzp**2)*(dydrp*srbzp + r0/q0*qhatp*thbzp) + r0/q0/radiusp*fp*psipp*srbrp)`.

2 Parallel acceleration due to toroidal flow

An auxiliary guiding center variable $v_\parallel{}''$ is defined according to

$$\varepsilon = \mu \mathbf{B}_0(\mathbf{R}) + \frac{1}{2} m v_\parallel''^2 - \frac{1}{2} m U^2(\mathbf{R}) + q\Phi_1(\mathbf{R}) \tag{15}$$

Notice that $v_\parallel{}''$ depends on $(\mathbf{R}, \varepsilon, \mu)$ but not γ, and $v_\parallel{}'' = v_\parallel{}' + \mathcal{O}(\delta)$. The new parallel velocity is defined with ε^μ. Since $d\varepsilon^\mu/dt = \mathcal{O}(\delta^2)$,

$$\begin{aligned} m v_\parallel'' \frac{dv_\parallel''}{dt} &= -\mu \frac{dB(\mathbf{R})}{dt} + m\frac{dU^2}{dt} - q\frac{d\Phi_1}{dt} \\ &= -\mu v_\parallel'' \mathbf{b} \cdot \nabla B + m v_\parallel'' \mathbf{b} \cdot \nabla U^2 - q v_\parallel'' \mathbf{b} \cdot \nabla \Phi_1 + \mathcal{O}(\delta^2) \end{aligned} \tag{16}$$

We will neglect the $\mathcal{O}(\delta^2)$ terms, which include the parallel nonlinearity.

$$\frac{dv_\parallel''}{dt} = -\frac{\mu}{m} \mathbf{b} \cdot \nabla B + \mathbf{b} \cdot \nabla U^2 - \frac{q}{m} \mathbf{b} \cdot \nabla \Phi_1 + \mathcal{O}(\delta^2) \tag{17}$$

For any scalar $s(r, \theta, \zeta)$,

$$\mathbf{b} \cdot \nabla s = \frac{1}{B} \frac{\psi_p'}{R} \frac{\partial s}{\partial \theta} \hat{\zeta} \cdot \nabla r \times \nabla \theta + \frac{f}{BR^2} \frac{\partial s}{\partial \zeta}. \tag{18}$$

$$\begin{aligned} \mathbf{b} \cdot \nabla s &= \frac{1}{B} (\nabla \zeta \times \nabla \psi_p + \frac{f}{R} \hat{\zeta}) \cdot (\frac{\partial s}{\partial r} \nabla r + \frac{\partial s}{\partial \theta} \nabla \theta + \frac{\partial s}{\partial \zeta} \nabla \zeta) \\ &= \frac{\psi_p'}{B} (\nabla \zeta \times \nabla r) \cdot (\frac{\partial s}{\partial r} \nabla r + \frac{\partial s}{\partial \theta} \nabla \theta + \frac{\partial s}{\partial \zeta} \nabla \zeta) + \frac{f}{BR^2} \frac{\partial s}{\partial \zeta} \\ &= \frac{1}{B} \frac{\psi_p'}{R} \frac{\partial s}{\partial \theta} \hat{\zeta} \cdot \nabla r \times \nabla \theta + \frac{f}{BR^2} \frac{\partial s}{\partial \zeta} \end{aligned}$$

So,

$$\mathbf{b} \cdot \nabla B = \frac{1}{B} \frac{\psi_p'}{R} \frac{\partial B}{\partial \theta} \hat{\zeta} \cdot \nabla r \times \nabla \theta \tag{19}$$

$$\mathbf{b} \cdot \nabla U^2 = \frac{2U}{B} \frac{\psi_p'}{R} \frac{\partial U}{\partial \theta} \hat{\zeta} \cdot \nabla r \times \nabla \theta, \tag{20}$$

$$\mathbf{b} \cdot \nabla \Phi_1 = \frac{1}{B} \frac{\psi_p'}{R} \frac{\partial \Phi_1}{\partial \theta} \hat{\zeta} \cdot \nabla r \times \nabla \theta \tag{21}$$

with

$$\hat{\zeta} \cdot \nabla r \times \nabla \theta = |\nabla r \times \nabla \theta|. \tag{22}$$

The $\partial U/\partial \theta$ and $\partial \Phi_1/\partial \theta$ will be calculated in the `gem_equil.f90`. Others are all existing variables in GEM. Φ_1 is the electric potential determined by the charge-neutrality, with $\mathbf{E_n} = -\nabla \Phi_1$. For a plasma with a single ion species with ion temperature T_i and electron temperature T_e,

$$e\Phi_1 = \frac{m_i \omega_0^2}{2(1 + T_i/T_e)} (R^2 - \langle R^2 \rangle). \tag{23}$$

Attention, here the bracket $\langle \cdots \rangle$ stands for the flux surface average.

$$\langle R^2 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} R^2(r, \theta) d\theta \tag{24}$$

Thus,

$$\partial_\theta (R^2 - \langle R^2 \rangle) = 2R\partial_\theta R - \frac{R^2}{2\pi} \tag{25}$$

3 Changes in weight equation

$$\delta f = -q(\phi - \mathbf{U}\cdot\mathbf{A})\frac{f_0}{T} + h \tag{26}$$

Here, h is the non-adiabatic part of δf. Write $h = h + h_1 + h_2 \cdots$

$$h_2 = \delta f + \frac{q}{T} f_0[(\phi - \mathbf{U}\cdot\mathbf{A}) - \langle(\phi - \mathbf{U}\cdot\mathbf{A})\rangle + \langle\mathbf{v}'\cdot\mathbf{A}\rangle] \tag{27}$$

$$\begin{aligned}
&\frac{\partial h_2}{\partial t} + \langle\dot{\mathbf{R}}\rangle\cdot\nabla h_2 + \langle\dot{\varepsilon}^u\rangle\frac{\partial h_1}{\partial\varepsilon^u} - \langle\dot{\varepsilon}^u\rangle\frac{\partial}{\partial\varepsilon^u}\langle\frac{q}{T}f_0\mathbf{v}'\cdot\mathbf{A}\rangle \\
=& -\left\langle\frac{d\mathbf{R}}{dt}\bigg|_1\right\rangle\cdot\frac{\partial f_0}{\partial\mathbf{R}} \\
&- q\mathbf{v}_g\cdot\nabla\langle\Psi\rangle\frac{f_0}{T} \\
&+ q[\mathbf{U}(\mathbf{R})\cdot\nabla\langle\Psi\rangle - \langle\mathbf{U}\cdot\nabla\Psi\rangle]\frac{f_0}{T} \\
&- \frac{q}{\Omega}\left\langle(\nabla\Psi\times\mathbf{b})\cdot\nabla\psi_p\;\omega_0' R\left(\frac{B_t}{B_0}v_\parallel' + U\right)\right\rangle\frac{f_0}{T} \\
&+ S_1 + S_2 + S_3
\end{aligned} \tag{28}$$

There are two terms to be added in the weight equation. For electrostatic model, $\Psi = \phi$.

The first term,

$$q[\mathbf{U}\cdot\nabla\langle\phi\rangle - \langle\mathbf{U}\cdot\nabla\phi\rangle]\frac{f_0}{T} \tag{29}$$

$$\begin{aligned}
\mathbf{U}\cdot\nabla\langle\phi\rangle &= U\hat{\zeta}\cdot\left(\frac{\partial\langle\phi\rangle}{\partial x}\nabla x + \frac{\partial\langle\phi\rangle}{\partial y}\nabla y + \frac{\partial\langle\phi\rangle}{\partial z}\nabla z\right) \\
&= U\frac{\partial\langle\phi\rangle}{\partial y}\hat{\zeta}\cdot\boldsymbol{\nabla} y \\
&= -\frac{r_0 U}{q_0 R}\frac{\partial\langle\phi\rangle}{\partial y}
\end{aligned} \tag{30}$$

$$\langle\mathbf{U}\cdot\nabla\phi\rangle = \left\langle -\frac{r_0 U}{q_0 R}\frac{\partial\phi}{\partial y}\right\rangle = -\frac{r_0 U}{q_0 R}\left\langle\frac{\partial\phi}{\partial y}\right\rangle \tag{31}$$

The second term,

$$
\begin{aligned}
&-\frac{q}{\Omega}\left\langle (\nabla\phi\times\mathbf{b})\cdot\nabla\psi_p\;\omega_0' R\left(\frac{B_t}{B_0}v_\parallel' + U\right)\right\rangle\frac{f_0}{T} \\
=&-\frac{q}{\Omega}\left\langle (\nabla\phi\times\mathbf{b})\cdot\nabla\psi_p\right\rangle\omega_0' R\left(\frac{B_t}{B_0}v_\parallel' + U\right)\frac{f_0}{T}
\end{aligned}
\tag{32}
$$

Considering

$$
\begin{aligned}
\mathbf{B} &= \nabla\psi\times\nabla(q\theta_f-\zeta) \\
&= \frac{q_0}{r_0}\frac{d\psi}{dx}\nabla x\times\nabla y \\
&= C(x)\nabla x\times\nabla y
\end{aligned}
\tag{33}
$$

then

$$
\mathbf{b} = \frac{\nabla x\times\nabla y}{|\nabla x\times\nabla y|}. \tag{34}
$$

So we have

$$
\begin{aligned}
\nabla\phi\times\mathbf{b}\cdot\nabla\psi_p &= \nabla\psi_p\times\nabla\phi\cdot\mathbf{b} \\
&= \psi_p'\nabla x\times\nabla\phi\cdot\frac{\nabla x\times\nabla y}{|\nabla x\times\nabla y|} \\
&= \frac{\psi_p'}{|\nabla x\times\nabla y|}\left(\frac{\partial\phi}{\partial y}\nabla x\times\nabla y + \frac{\partial\phi}{\partial z}\nabla x\times\nabla z\right)\cdot\nabla x\times\nabla y \\
&= \psi_p'|\nabla x\times\nabla y|\frac{\partial\phi}{\partial y} + \frac{\psi_p'}{|\nabla x\times\nabla y|}\frac{\partial\phi}{\partial z}(\nabla x\times\nabla z)\cdot(\nabla x\times\nabla y)
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
(\nabla x\times\nabla z)\cdot(\nabla x\times\nabla y) &= (\nabla x\times\nabla y)\times\nabla x\cdot\nabla z \\
&= (|\nabla x|^2\nabla y - |\nabla x\cdot\nabla y|\nabla x)\cdot\nabla z \\
&= |\nabla x|^2\nabla y\cdot\nabla z - |\nabla x\cdot\nabla y|\nabla x\cdot\nabla z \\
&= |\nabla x|^2\nabla y\cdot\nabla z - q_0R_0|\nabla x\cdot\nabla y|\;|\nabla r\cdot\nabla\theta|
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
\nabla y\cdot\nabla z &= \left(\frac{\partial y}{\partial r}\nabla r + \frac{r_0}{q_0}\hat{q}\nabla\theta - \frac{r_0}{q_0}\nabla\zeta\right)\cdot q_0R_0\nabla\theta \\
&= q_0R_0\frac{\partial y}{\partial r}|\nabla r\cdot\nabla\theta| + r_0R_0\hat{q}|\nabla\theta|^2
\end{aligned}
\tag{37}
$$

According to Eq. 35 - 37, Eq. 32 can be coded using existing variables.

References

[1] H. Sugama and W. Horton. Nonlinear electromagnetic gyrokinetic equation for plasmas with large mean flows. *Physics of Plasmas*, 5(7):2560–2573, 1998.