

# Adding toroidal flow in GEM for adiabatic electron model

汤炜康

## Contents

1	Guiding center drift due to toroidal flow	2
2	Parallel acceleration due to toroidal flow	5
3	Changes in weight equation	7

# 1 Guiding center drift due to toroidal flow

There are three terms needed to be added in the drift velocity of the guiding center  $V_G$ , according to Eq. 21 in [1], i. e.

$$\frac{cm_a}{e_a B} \hat{b} \times (\mathbf{U} \cdot \nabla \mathbf{U}), \quad (1)$$

$$\frac{cm_a v_{\parallel}'}{e_a B} \hat{b} \times (\hat{b} \cdot \nabla \mathbf{U}), \quad (2)$$

$$\frac{cm_a v_{\parallel}'}{e_a B} \hat{b} \times (\nabla \mathbf{U} \cdot \hat{b}). \quad (3)$$

In the right-handed coordinate system  $(R, Z, \zeta)$ , considering the toroidal flow with the form of  $\mathbf{U}(\mathbf{R}) = -\omega_0(\psi_p) R^2 \nabla \zeta$ ,

$$\mathbf{U} \cdot \nabla \mathbf{U} = -\frac{U^2}{R} \hat{R} \quad (4)$$

$$\begin{aligned} \hat{b} \times (\mathbf{U} \cdot \nabla \mathbf{U}) &= \frac{U^2}{R} \hat{R} \times \hat{b} \\ &= \frac{U^2}{RB} \hat{R} \times \left( \frac{f}{R} \hat{\zeta} + \frac{\psi_p'(r)}{R} \left( \frac{\partial r}{\partial R} \hat{Z} - \frac{\partial r}{\partial Z} \hat{R} \right) \right) \\ &= \frac{U^2}{RB} \left( -\frac{f}{R} \hat{Z} + \frac{\psi_p'(r)}{R} \frac{\partial r}{\partial R} \hat{\zeta} \right) \end{aligned} \quad (5)$$

Then Eq. 1 becomes

$$\frac{cm_a U^2}{e_a B^2 R^2} \left( -f \hat{Z} + \psi_p'(r) \frac{\partial r}{\partial R} \hat{\zeta} \right). \quad (6)$$

Eq. 6  $\cdot \nabla x$  is

$$\frac{cm_a U^2}{e_a B^2 R^2} \cdot (-f) \frac{\partial r}{\partial Z}. \quad (7)$$

This term can be coded as `-Up**2*fp*srbzp/bfldp**2/radiusp**2`, where `U(0:nr,0:ntheta)` is a new variable needed to be declared, representing the equilibrium toroidal flow. Actually, we only need to declare a new variable `omega(0:nr)`, `U(0:nr,0:ntheta)=-omega(0:nr)*radius(0:nr,0:ntheta)`.

For Eq. 6  $\cdot \nabla y$ ,

$$y = \frac{r_0}{q_0} \int_0^\theta \hat{q}(r, \theta') d\theta' - \zeta = \frac{r_0}{q_0} (q\theta_f - \zeta) \quad (8)$$

$$\nabla y = \frac{\partial y}{\partial r} \nabla r + \frac{r_0}{q_0} \hat{q} \nabla \theta - \frac{r_0}{q_0} \nabla \zeta, \quad (9)$$

where  $\hat{q} = q \frac{\partial \theta_f}{\partial \theta}$ .

$$\begin{aligned} \text{Eq. 6} \cdot \nabla y &= \frac{cm_a U^2}{e_a B^2 R^2} (-f \hat{Z} + \psi'_p(r) \frac{\partial r}{\partial R} \hat{\zeta}) \cdot \nabla y \\ &= \frac{cm_a U^2}{e_a B^2 R^2} (-f \frac{\partial y}{\partial r} \frac{\partial r}{\partial Z} - \frac{r_0}{q_0} \hat{q} f \frac{\partial \theta}{\partial Z} - \frac{r_0}{q_0 R} \psi'_p(r) \frac{\partial r}{\partial R}), \end{aligned} \quad (10)$$

i. e.  $-\text{Up}^{**2}/\text{bfldp}^{**2}/\text{radiusp}^{**2} * (\text{fp} * \text{dydrp} * \text{srbzp} + \text{r0}/\text{q0} * \text{qhatp} * \text{fp} * \text{thbzb} + \text{r0}/\text{q0}/\text{radiusp} * \text{psipp} * \text{srbpr})$ .

So far, all the terms about Eq. 1 is finished. Let's start with Eqs. 2 and 3. Still in the  $(R, Z, \zeta)$  coordinate, one could find Eq. 2 is identical to Eq. 3 as

$$\mathbf{U} \cdot \nabla \hat{b} = \hat{b} \cdot \nabla \mathbf{U} = -\frac{Uf}{R^2 B} \hat{R} - \frac{U}{R^2 B} \psi'_p(r) \frac{\partial r}{\partial Z} \hat{\zeta}. \quad (11)$$

To derive Eq. 2 or Eq. 3,

$$\begin{aligned} &\frac{cm_a v'_\parallel}{e_a B} \hat{b} \times (\mathbf{U} \cdot \nabla \hat{b}) \\ &= \frac{cm_a v'_\parallel}{e_a B} \left[ -\frac{Uf^2}{B^2 R^3} \hat{Z} + \frac{Uf\psi'_p(r)}{B^2 R^3} \frac{\partial r}{\partial R} \hat{\zeta} \right. \\ &\quad \left. - \frac{U}{B^2 R^3} \psi_p'^2(r) \frac{\partial r}{\partial R} \frac{\partial r}{\partial Z} \hat{R} - \frac{U}{B^2 R^3} \psi_p'^2(r) \left( \frac{\partial r}{\partial Z} \right)^2 \hat{Z} \right] \\ &= \frac{cm_a v'_\parallel U}{e_a B^3 R^3} \left[ -\psi_p'^2(r) \frac{\partial r}{\partial R} \frac{\partial r}{\partial Z} \hat{R} - (f^2 + \psi_p'^2(r) \left( \frac{\partial r}{\partial Z} \right)^2) \hat{Z} + f\psi'_p(r) \frac{\partial r}{\partial R} \hat{\zeta} \right]. \end{aligned} \quad (12)$$

Then,

$$\begin{aligned} \text{Eq. 12} \cdot \nabla x &= -\frac{cm_a v'_\parallel U}{e_a B^3 R^3} \left[ \psi_p'^2(r) \left( \frac{\partial r}{\partial R} \right)^2 \frac{\partial r}{\partial Z} + (f^2 + \psi_p'^2(r) \left( \frac{\partial r}{\partial Z} \right)^2) \frac{\partial r}{\partial Z} \right] \\ &= -\frac{cm_a v'_\parallel U}{e_a B^3 R^3} \left[ \psi_p'^2(r) \left( \frac{\partial r}{\partial R} \right)^2 + f^2 + \psi_p'^2(r) \left( \frac{\partial r}{\partial Z} \right)^2 \right] \frac{\partial r}{\partial Z}, \end{aligned} \quad (13)$$

which can be coded as  $-U_p v_{\text{par}} / b_{\text{fldp}}^3 / \text{radiusp}^3 * (\text{psipp}^2 * \text{srbrp}^2 + f_p^2 + \text{psipp}^2 * \text{srbzp}^2) * \text{srbzp}$ .

Using Eq. 9,

$$\begin{aligned} \text{Eq. 12} \cdot \nabla y = & - \frac{cm_a v_{\parallel}' U}{e_a B^3 R^3} [\psi_p'^2(r) \frac{\partial r}{\partial R} \frac{\partial r}{\partial Z} (\frac{\partial y}{\partial r} \frac{\partial r}{\partial R} + \frac{r_0}{q_0} \hat{q} \frac{\partial \theta}{\partial R}) \\ & + (f^2 + \psi_p'^2(r) (\frac{\partial r}{\partial Z})^2) (\frac{\partial y}{\partial r} \frac{\partial r}{\partial Z} + \frac{r_0}{q_0} \hat{q} \frac{\partial \theta}{\partial Z}) \\ & + \frac{r_0}{q_0 R} f \psi_p'(r) \frac{\partial r}{\partial R}]. \end{aligned} \quad (14)$$

That is,  $-U_p v_{\text{par}} / b_{\text{fldp}}^3 / \text{radiusp}^3 * (\text{psipp}^2 * \text{srbrp} * \text{srbzp} * (\text{dydrp} * \text{srbrp} + r_0 / q_0 * \text{qhatp} * \text{thbrp}) + (f_p^2 + \text{psipp}^2 * \text{srbzp}^2) * (\text{dydrp} * \text{srbzp} + r_0 / q_0 * \text{qhatp} * \text{thbzp}) + r_0 / q_0 / \text{radiusp} * f_p * \text{psipp} * \text{srbrp})$ .

## 2 Parallel acceleration due to toroidal flow

An auxiliary guiding center variable  $v_{\parallel}''$  is defined according to

$$\varepsilon = \mu \mathbf{B}_0(\mathbf{R}) + \frac{1}{2} m v_{\parallel}''^2 - \frac{1}{2} m U^2(\mathbf{R}) + q \Phi_1(\mathbf{R}) \quad (15)$$

Notice that  $v_{\parallel}''$  depends on  $(\mathbf{R}, \varepsilon, \mu)$  but not  $\gamma$ , and  $v_{\parallel}'' = v_{\parallel}' + \mathcal{O}(\delta)$ . The new parallel velocity is defined with  $\varepsilon^{\mu}$ . Since  $d\varepsilon^{\mu}/dt = \mathcal{O}(\delta^2)$ ,

$$\begin{aligned} m v_{\parallel}'' \frac{dv_{\parallel}''}{dt} &= -\mu \frac{dB(\mathbf{R})}{dt} + m \frac{dU^2}{dt} - q \frac{d\Phi_1}{dt} \\ &= -\mu v_{\parallel}'' \mathbf{b} \cdot \nabla B + m v_{\parallel}'' \mathbf{b} \cdot \nabla U^2 - q v_{\parallel}'' \mathbf{b} \cdot \nabla \Phi_1 + \mathcal{O}(\delta^2) \end{aligned} \quad (16)$$

We will neglect the  $\mathcal{O}(\delta^2)$  terms, which include the parallel nonlinearity.

$$\frac{dv_{\parallel}''}{dt} = -\frac{\mu}{m} \mathbf{b} \cdot \nabla B + \mathbf{b} \cdot \nabla U^2 - \frac{q}{m} \mathbf{b} \cdot \nabla \Phi_1 + \mathcal{O}(\delta^2) \quad (17)$$

For any scalar  $s(r, \theta, \zeta)$ ,

$$\mathbf{b} \cdot \nabla s = \frac{1}{B} \frac{\psi_p'}{R} \frac{\partial s}{\partial \theta} \hat{\zeta} \cdot \nabla r \times \nabla \theta + \frac{f}{BR^2} \frac{\partial s}{\partial \zeta}. \quad (18)$$

$$\begin{aligned} \mathbf{b} \cdot \nabla s &= \frac{1}{B} (\nabla \zeta \times \nabla \psi_p + \frac{f}{R} \hat{\zeta}) \cdot (\frac{\partial s}{\partial r} \nabla r + \frac{\partial s}{\partial \theta} \nabla \theta + \frac{\partial s}{\partial \zeta} \nabla \zeta) \\ &= \frac{\psi_p'}{B} (\nabla \zeta \times \nabla r) \cdot (\frac{\partial s}{\partial r} \nabla r + \frac{\partial s}{\partial \theta} \nabla \theta + \frac{\partial s}{\partial \zeta} \nabla \zeta) + \frac{f}{BR^2} \frac{\partial s}{\partial \zeta} \\ &= \frac{1}{B} \frac{\psi_p'}{R} \frac{\partial s}{\partial \theta} \hat{\zeta} \cdot \nabla r \times \nabla \theta + \frac{f}{BR^2} \frac{\partial s}{\partial \zeta} \end{aligned}$$

So,

$$\mathbf{b} \cdot \nabla B = \frac{1}{B} \frac{\psi_p'}{R} \frac{\partial B}{\partial \theta} \hat{\zeta} \cdot \nabla r \times \nabla \theta \quad (19)$$

$$\mathbf{b} \cdot \nabla U^2 = \frac{2U}{B} \frac{\psi_p'}{R} \frac{\partial U}{\partial \theta} \hat{\zeta} \cdot \nabla r \times \nabla \theta, \quad (20)$$

$$\mathbf{b} \cdot \nabla \Phi_1 = \frac{1}{B} \frac{\psi_p'}{R} \frac{\partial \Phi_1}{\partial \theta} \hat{\zeta} \cdot \nabla r \times \nabla \theta \quad (21)$$

with

$$\hat{\zeta} \cdot \nabla r \times \nabla \theta = |\nabla r \times \nabla \theta|. \quad (22)$$

The  $\partial U/\partial \theta$  and  $\partial \Phi_1/\partial \theta$  will be calculated in the `gem_equil.f90`. Others are all existing variables in GEM.  $\Phi_1$  is the electric potential determined by the charge-neutrality, with  $\mathbf{E}_n = -\nabla \Phi_1$ . For a plasma with a single ion species with ion temperature  $T_i$  and electron temperature  $T_e$ ,

$$e\Phi_1 = \frac{m_i \omega_0^2}{2(1 + T_i/T_e)} (R^2 - \langle R^2 \rangle). \quad (23)$$

Attention, here the bracket  $\langle \cdots \rangle$  stands for the flux surface average.

$$\langle R^2 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} R^2(r, \theta) d\theta \quad (24)$$

Thus,

$$\partial_\theta (R^2 - \langle R^2 \rangle) = 2R \partial_\theta R - \frac{R^2}{2\pi} \quad (25)$$

### 3 Changes in weight equation

Let  $f = f_0 + \delta f$ , the perturbed part of distribution function

$$\delta f = -q(\phi - \mathbf{U} \cdot \mathbf{A}) \frac{f_0}{T} + h \quad (26)$$

Here,  $h$  is the non-adiabatic part of  $\delta f$ .

Write  $h = h + h_1 + h_2 \dots$

$$h_2 = \delta f + \frac{q}{T} f_0 [(\phi - \mathbf{U} \cdot \mathbf{A}) - \langle (\phi - \mathbf{U} \cdot \mathbf{A}) \rangle + \langle \mathbf{v}' \cdot \mathbf{A} \rangle] \quad (27)$$

$$\begin{aligned} & \frac{\partial h_2}{\partial t} + \langle \dot{\mathbf{R}} \rangle \cdot \nabla h_2 + \langle \varepsilon^u \rangle \frac{\partial h_1}{\partial \varepsilon^u} - \langle \varepsilon^u \rangle \frac{\partial}{\partial \varepsilon^u} \langle \frac{q}{T} f_0 \mathbf{v}' \cdot \mathbf{A} \rangle \\ &= - \left\langle \frac{d\mathbf{R}}{dt} \right|_1 \cdot \frac{\partial f_0}{\partial \mathbf{R}} \\ & \quad - q \mathbf{v}_g \cdot \nabla \langle \Psi \rangle \frac{f_0}{T} \\ & \quad + q [\mathbf{U}(\mathbf{R}) \cdot \nabla \langle \Psi \rangle - \langle \mathbf{U} \cdot \nabla \Psi \rangle] \frac{f_0}{T} \\ & \quad - \frac{q}{\Omega} \left\langle (\nabla \Psi \times \mathbf{b}) \cdot \nabla \psi_p \omega'_0 R \left( \frac{B_t}{B_0} v'_\parallel + U \right) \right\rangle \frac{f_0}{T} \\ & \quad + S_1 + S_2 + S_3 \end{aligned} \quad (28)$$

There are two terms to be added in the weight equation. For electrostatic model,  $\Psi = \phi$ .

The first term,

$$q [\mathbf{U}(\mathbf{R}) \cdot \nabla \langle \phi \rangle - \langle \mathbf{U} \cdot \nabla \phi \rangle] \frac{f_0}{T} \quad (29)$$

All right, let's proceed a further step. Write

$$\mathbf{U}(\mathbf{x}) = \mathbf{U}(\mathbf{R}) + \boldsymbol{\rho} \cdot \nabla \mathbf{U}, \quad (30)$$

then

$$\mathbf{U}(\mathbf{R}) \cdot \nabla \langle \phi \rangle - \langle \mathbf{U} \cdot \nabla \phi \rangle = - \langle \boldsymbol{\rho} \cdot \nabla \mathbf{U} \cdot \nabla \phi \rangle. \quad (31)$$

Remember,

$$\hat{\zeta} \cdot \nabla \hat{R} = \frac{1}{R} \hat{\zeta} \text{ and } \hat{\zeta} \cdot \nabla \hat{\zeta} = -\frac{1}{R} \hat{R}.$$

These two terms results from the reconverting from the Cartesian coordinate to the cylindrical coordinate in deriving the material derivative, see Eq. 7.106 in [2].

$$\boldsymbol{\rho} = \rho(\mathbf{e}_1 \sin \gamma + \mathbf{e}_2 \cos \gamma),$$

where  $\rho = v'_\perp / \Omega = \sqrt{2\mu B / m} / (qB / m)$ ,  $\mathbf{e}_1 = \nabla r / |\nabla r|$ ,  $\mathbf{e}_2 = \mathbf{b} \times \mathbf{e}_1$  and  $\gamma$  is the gyro angle.

Let

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{e}_{1R}(r, \theta) \hat{R} + \mathbf{e}_{1Z}(r, \theta) \hat{Z} \\ \mathbf{e}_2 &= \mathbf{e}_{2R}(r, \theta) \hat{R} + \mathbf{e}_{2Z}(r, \theta) \hat{Z} + \mathbf{e}_{2\zeta}(r, \theta) \hat{\zeta}, \end{aligned}$$

then we have

$$\boldsymbol{\rho} = \rho \left( (\mathbf{e}_{1R} \sin \gamma + \mathbf{e}_{2R} \cos \gamma) \hat{R} + (\mathbf{e}_{1Z} \sin \gamma + \mathbf{e}_{2Z} \cos \gamma) \hat{Z} + (\mathbf{e}_{2\zeta} \cos \gamma) \hat{\zeta} \right)$$

$$\begin{aligned} \boldsymbol{\rho} \cdot \nabla \mathbf{U} &= \boldsymbol{\rho} \cdot \nabla (U \hat{\zeta}) \\ &= \boldsymbol{\rho} \cdot \nabla U \hat{\zeta} + U \boldsymbol{\rho} \cdot \nabla \hat{\zeta} \\ &= \boldsymbol{\rho} \cdot \left( \frac{U}{R} \hat{R} - \omega' R \frac{\partial r}{\partial R} \hat{R} - \omega' R \frac{\partial r}{\partial Z} \hat{Z} \right) \hat{\zeta} - \frac{U \rho_\zeta}{R} \hat{R} \\ &= \left( \frac{U \rho_R}{R} - \omega' R \frac{\partial r}{\partial R} \rho_R - \omega' R \frac{\partial r}{\partial Z} \rho_Z \right) \hat{\zeta} - \frac{U \rho_\zeta}{R} \hat{R} \end{aligned} \quad (32)$$

$$\begin{aligned} &\left( \frac{U \rho_R}{R} - \omega' R \frac{\partial r}{\partial R} \rho_R - \omega' R \frac{\partial r}{\partial Z} \rho_Z \right) \hat{\zeta} \cdot \nabla \phi \\ &= \left( \frac{U \rho_R}{R} - \omega' R \frac{\partial r}{\partial R} \rho_R - \omega' R \frac{\partial r}{\partial Z} \rho_Z \right) \hat{\zeta} \cdot \frac{\partial \phi}{\partial y} \nabla y \\ &= \left( -\frac{U \rho_R}{R} + \omega' R \frac{\partial r}{\partial R} \rho_R + \omega' R \frac{\partial r}{\partial Z} \rho_Z \right) \frac{r_0}{q_0 R} \frac{\partial \phi}{\partial y} \end{aligned} \quad (33)$$



$$\begin{aligned}
-\frac{U\rho_\zeta}{R}\hat{R}\cdot\nabla\phi &= -\frac{U\rho_\zeta}{R}\hat{R}\cdot\left(\frac{\partial\phi}{\partial x}\nabla x + \frac{\partial\phi}{\partial y}\nabla y + \frac{\partial\phi}{\partial z}\nabla z\right) \\
&= -\frac{U\rho_\zeta}{R}\left(\frac{\partial\phi}{\partial x}\frac{\partial r}{\partial R} + \frac{\partial\phi}{\partial y}\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R} + \frac{r_0}{q_0}\hat{q}\frac{\partial\theta}{\partial R}\right) + \frac{\partial\phi}{\partial z}q_0R_0\frac{\partial\theta}{\partial R}\right)
\end{aligned} \tag{34}$$

$$\begin{aligned}
-\langle\boldsymbol{\rho}\cdot\nabla\mathbf{U}\cdot\nabla\phi\rangle &= \left(\frac{U\langle\rho_R\rangle}{R} - \omega'R\frac{\partial r}{\partial R}\langle\rho_R\rangle - \omega'R\frac{\partial r}{\partial Z}\langle\rho_Z\rangle\right)\frac{r_0}{q_0R}\left\langle\frac{\partial\phi}{\partial y}\right\rangle \\
&+ \frac{U\langle\rho_\zeta\rangle}{R}\left(\left\langle\frac{\partial\phi}{\partial x}\right\rangle\frac{\partial r}{\partial R} + \left\langle\frac{\partial\phi}{\partial y}\right\rangle\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial R} + \frac{r_0}{q_0}\hat{q}\frac{\partial\theta}{\partial R}\right) + \left\langle\frac{\partial\phi}{\partial z}\right\rangle q_0R_0\frac{\partial\theta}{\partial R}\right)
\end{aligned} \tag{35}$$

$$\rho_R = \rho(\mathbf{e}_{1R}\sin\gamma + \mathbf{e}_{2R}\cos\gamma)$$

$$\rho_Z = \rho(\mathbf{e}_{1Z}\sin\gamma + \mathbf{e}_{2Z}\cos\gamma)$$

$$\rho_\zeta = \rho\mathbf{e}_{2\zeta}\cos\gamma$$

$$\mathbf{e}_{1R} = \frac{\partial r}{\partial R}\bigg/\left|\nabla r\right| = \mathbf{srbr}/\mathbf{gr}$$

$$\mathbf{e}_{1Z} = \frac{\partial r}{\partial Z}\bigg/\left|\nabla r\right| = \mathbf{thbr}/\mathbf{gr}$$

$$\mathbf{e}_{2R} = -\frac{f}{RB}\mathbf{e}_{1Z}$$

$$\mathbf{e}_{2Z} = \frac{f}{RB}\mathbf{e}_{1R}$$

$$\mathbf{e}_{2\zeta} = \frac{\psi'_p}{RB}\left[-\mathbf{e}_{1R}\frac{\partial r}{\partial R} - \mathbf{e}_{1Z}\frac{\partial r}{\partial Z}\right]$$

The gyro average of  $\rho_R, \rho_Z$  and  $\rho_\zeta$  could be obtained by the 4-point average method. The 4 points are at  $\gamma = 0, \pi/2, \pi$  and  $3\pi/2$ , respectively.

The second term,

$$\begin{aligned}
& -\frac{q}{\Omega} \left\langle (\nabla\phi \times \mathbf{b}) \cdot \nabla\psi_p \omega'_0 R \left( \frac{B_t}{B_0} v'_\parallel + U \right) \right\rangle \frac{f_0}{T} \\
& = -\frac{q}{\Omega} \left\langle (\nabla\phi \times \mathbf{b}) \cdot \nabla\psi_p \right\rangle \omega'_0 R \left( \frac{B_t}{B_0} v'_\parallel + U \right) \frac{f_0}{T}
\end{aligned} \tag{36}$$

Considering

$$\begin{aligned}
\mathbf{B} &= \nabla\psi \times \nabla(q\theta_f - \zeta) \\
&= \frac{q_0}{r_0} \frac{d\psi}{dx} \nabla x \times \nabla y \\
&= C(x) \nabla x \times \nabla y
\end{aligned} \tag{37}$$

then

$$\mathbf{b} = \frac{\nabla x \times \nabla y}{|\nabla x \times \nabla y|}. \tag{38}$$

So we have

$$\begin{aligned}
\nabla\phi \times \mathbf{b} \cdot \nabla\psi_p &= \nabla\psi_p \times \nabla\phi \cdot \mathbf{b} \\
&= \psi'_p \nabla x \times \nabla\phi \cdot \frac{\nabla x \times \nabla y}{|\nabla x \times \nabla y|} \\
&= \frac{\psi'_p}{|\nabla x \times \nabla y|} \left( \frac{\partial\phi}{\partial y} \nabla x \times \nabla y + \frac{\partial\phi}{\partial z} \nabla x \times \nabla z \right) \cdot \nabla x \times \nabla y \\
&= \psi'_p |\nabla x \times \nabla y| \frac{\partial\phi}{\partial y} + \frac{\psi'_p}{|\nabla x \times \nabla y|} \frac{\partial\phi}{\partial z} (\nabla x \times \nabla z) \cdot (\nabla x \times \nabla y)
\end{aligned} \tag{39}$$

$$\begin{aligned}
(\nabla x \times \nabla z) \cdot (\nabla x \times \nabla y) &= (\nabla x \times \nabla y) \times \nabla x \cdot \nabla z \\
&= (|\nabla x|^2 \nabla y - |\nabla x \cdot \nabla y| \nabla x) \cdot \nabla z \\
&= |\nabla x|^2 \nabla y \cdot \nabla z - |\nabla x \cdot \nabla y| \nabla x \cdot \nabla z \\
&= |\nabla x|^2 \nabla y \cdot \nabla z - q_0 R_0 |\nabla x \cdot \nabla y| |\nabla r \cdot \nabla \theta|
\end{aligned} \tag{40}$$

$$\begin{aligned}
\nabla y \cdot \nabla z &= \left( \frac{\partial y}{\partial r} \nabla r + \frac{r_0}{q_0} \hat{q} \nabla \theta - \frac{r_0}{q_0} \nabla \zeta \right) \cdot q_0 R_0 \nabla \theta \\
&= q_0 R_0 \frac{\partial y}{\partial r} |\nabla r \cdot \nabla \theta| + r_0 R_0 \hat{q} |\nabla \theta|^2
\end{aligned} \tag{41}$$

According to Eq. 39 - 41, Eq. 36 can be coded using existing variables.

## References

- [1] H. Sugama and W. Horton. Nonlinear electromagnetic gyrokinetic equation for plasmas with large mean flows. *Physics of Plasmas*, 5(7):2560–2573, 1998.
- [2] Bastian E. Rapp. Chapter 7 - vector calculus. In Bastian E. Rapp, editor, *Microfluidics: Modelling, Mechanics and Mathematics*, Micro and Nano Technologies, pages 137 – 188. Elsevier, Oxford, 2017.