MATH 4500 - Homework 8

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Exercise 1. When $X^2 = -det(X)\,\mathbf{1}$, show that

$$e^X = \cos(\sqrt{det(X)})\,\mathbf{1} + \frac{\sin(\sqrt{det(X)})}{\sqrt{det(X)}} X.$$

Answer: Evidently, X^n for even exponents would be the identity times some constant $(-1)^{n/2} det(X)^{n/2}$, and for odd exponents you can look at the previous even term and multiply it times X. Hence,

$$X^n = \begin{cases} (-1)^m det(X)^m X & n = 2m+1 \\ (-1)^m det(X)^m\,\mathbf{1} & n = 2m. \end{cases}$$

We can use the definition of the matrix exponential, and separate even from odd terms:

$$\begin{aligned} e^X &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{X^n}{n!} \\ &= \mathbf{1} + \sum_{n \text{ odd}}^{\infty} \frac{X^n}{n!} + \sum_{n \text{ even}}^{\infty} \frac{X^n}{n!} \\ &= \mathbf{1} + \sum_{m=0}^{\infty} \frac{(-1)^m det(X)^m}{(2m+1)!} X + \sum_{m=1}^{\infty} \frac{(-1)^m det(X)^m}{(2m)!}\,\mathbf{1} \\ &= \mathbf{1}(1 + \sum_{m=1}^{\infty} \frac{(-1)^m \sqrt{det(X)}^{2m}}{(2m)!}) + \frac{X}{\sqrt{det(X)}} \sum_{m=0}^{\infty} \frac{(-1)^m \sqrt{det(X)}^{2m+1}}{(2m+1)!} \\ &= \cos(\sqrt{det(X)})\,\mathbf{1} + \frac{X}{\sqrt{det(X)}} \sin(\sqrt{det(X)}) \end{aligned}$$

Exercise 2. Using Exercise 1, and the fact that $Tr(X) = 0$, show that if

$$e^X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

then $\cos(det(X)) = -1$, in which case $\sin(det(X)) = 0$, and there is a contradiction.

Answer: We suppose, for the sake of contradiction, that there exists some $X \in \mathfrak{sl}(2, \mathbb{C})$ such that

$$e^X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in SL(2, \mathbb{C}).$$

By assumption $Tr(X) = 0$, and hence from Cayley-Hamilton's theorem $X^2 = -det(X)\,\mathbf{1}$. We can then use the conclusion of Exercise 1, which states that

$$e^X = \cos(\sqrt{det(X)})\,\mathbf{1} + \frac{X}{\sqrt{det(X)}}\sin(\sqrt{det(X)}).$$

We then compute the trace of e^X, but the second term of the latter expression has trace 0 and the trace is linear so $Tr(e^X) = 2\cos(\sqrt{det(X)}) = -2$. It follows that $\cos(\sqrt{det(X)}) = -1$, and at the values at which cos is -1 sin is 0, hence $\sin(\sqrt{det(X)}) = 0$, which implies $e^X = -\mathbf{1}$. This is a contradiction.

Exercise 3. Show that $SU(n)$ is a normal subgroup of $U(n)$ by describing it as the kernel of a homomorphism.

Answer: We look at the homomorphism $\varphi : U(n) \to C^{\times}$, defined by

$$\varphi(A) = det(A).$$

We have seen before that this is a homomorphism, since

$$\varphi(AB) = det(AB) = det(A)det(B) = \varphi(A)\varphi(B)$$

Because 1 is the multiplicative identity in $C^{\times}$, it follows that

$$ker(\varphi) = \{A \in U(n) \mid det(A) = 1\} = SU(n).$$

Finally, as we have previously seen, the kernel of a homomorphism is a normal subgroup. To see why this is the case, take any $A \in ker(\phi)$ and $B \in G$ where $\phi : G \to G'$ is a homomorphism. Then

$$\phi(BAB^{-1}) = \phi(B)\phi(A)\phi(B^{-1}) = \phi(B)\phi(B^{-1}) = \phi(BB^{-1}) = \phi(1) = 1$$

which implies that $BAB^{-1} \in ker(\phi)$.

Exercise 4. Show that $T_1(SU(n))$ is an ideal of $T_1(U(n))$ by checking that it has the required closure properties.

Answer: We take arbitrary elements $X \in T_1(SU(n))$ and $Y \in T_1(U(n))$, then we show that $[X, Y] \in T_1(SU(n))$. But recall from Section 5.3 that

$$T_1(U(n)) = \{X \mid X + \overline{X}^T = 0\}$$
$$T_1(SU(n)) = \{Y \mid Y + \overline{Y}^T = 0 \text{ and } Tr(Y) = 0\}$$

Hence, for X, Y as described above,

$$\begin{aligned}(XY - YX) + \overline{(XY - YX)}^T &= (XY - YX) + \overline{Y}^T\overline{X}^T - \overline{X}^T\overline{Y}^T \\ &= (XY - YX) + (-Y)(-X) - (-X)(-Y) \\ &= XY - YX + YX - XY = 0\end{aligned}$$

And using the fact that $Tr(XY) = Tr(YX)$, which we proved in Exercise 6 of Homework 7, and that the trace is linear, then

$$Tr(XY - YX) = Tr(XY) - Tr(YX) = 0.$$

It follows that $[X, Y] = XY - YX \in T_{\mathbf{1}}(SU(n))$.

Exercise 5. Find a 1-dimensional ideal $\mathcal{J}$ in $\mathfrak{u}(n)$, and show that $\mathcal{J}$ is the tangent space of $Z(U(n))$.

Answer: We start from the fact that

$$Z(U(n)) = \{e^{\mathbf{i}\,\theta}\mathbf{1} \mid \theta \in \mathbb{R}\}$$

Indeed, $Z(U(n))$ is a normal subgroup of $U(n)$ since for any $x \in Z(U(n))$ and $y \in U(n)$, we have that

$$\begin{aligned} yxy^{-1} &= yy^{-1}x \quad \text{defn of center} \\ &= x \in Z(U(n)).\end{aligned}$$

We can then compute $T_{\mathbf{1}}(Z(U(n))$ by differentiating any path through the group at $\mathbf{1}$:

$$\begin{aligned}\frac{d}{dt}e^{\mathbf{i}\,\theta t}\Big|_{t=0} &= \mathbf{i}\,\theta e^{\mathbf{i}\,\theta t}\Big|_{t=0} \\ &= \mathbf{i}\,\theta\,\mathbf{1} \quad \theta \in \mathbb{R}\end{aligned}$$

Hence, any $x \in T_{\mathbf{1}}(Z(U(n))$ is of the form above. Furthermore, take any vector $y = \mathbf{i}\,\theta\,\mathbf{1}$, then

$$\begin{aligned} e^y &= e^{\mathbf{i}\,\theta\,\mathbf{1}} \\ &= \sum_{n=0}^{\infty}\frac{\mathbf{i}^n\,\theta^n}{n!}\mathbf{1}^n = \mathbf{1}\sum_{n=0}^{\infty}\frac{\mathbf{i}^n\,\theta^n}{n!} = e^{\mathbf{i}\,\theta}\,\mathbf{1} \in Z(U(n)).\end{aligned}$$

It follows that

$$T_{\mathbf{1}}(Z(U(n))) = \{\mathbf{i}\,\theta\,\mathbf{1} \mid \theta \in \mathbb{R}\}.$$

By the first proof in Section 6.1, this means that $T_{\mathbf{1}}(Z(U(n)))$ is an ideal of $\mathfrak{u}(n)$, since $Z(U(n))$ is a normal subgroup of $U(n)$. Furthermore, $T_{\mathbf{1}}(Z(U(n)))$ is a one dimensional vector space over $\mathbb{R}$, since the set $\{\mathbf{i}\,\mathbf{1}\}$ is clearly linearly independent, spans the vector space, and contains one vector.

Now, to see why $Z(U(n)) = \{e^{\mathbf{i}\theta}\mathbf{1} \mid \theta \in \mathbb{R}\}$, we can appeal to the spectral theorem from linear algebra and the definition of center (commutes with all else). Take any $A \in Z(U(n))$, and the orthogonal matrix P that diagonalizes it. Then,

$$D = PAP^{\text{-}1} = PP^{\text{-}1}A = A$$

Furthermore, suppose A has different eigenvalues. Then if P' is a permutation matrix, we know that $P'AP'^{\text{-}1} \neq A$ so $A \notin Z(U(n))$ - a contradiction. Hence, A is of the form $\lambda\mathbf{1}$. In addition, since $A \in U(n)$, it follows that

$$|det(A)| = |\lambda^n| = 1 \implies \lambda = e^{\mathbf{i}\theta} \quad \forall\theta \in \mathbb{R}$$

.

Note that we could have also pretended that we had the ansatz $\mathcal{J} = \{\mathbf{i}\theta\mathbf{1} \mid \theta \in \mathbb{R}\}$, which is clearly one dimensional as argued above, and can be shown to be an ideal. Then independently found $T_{\mathbf{1}}(Z(U(n)))$ as we did above, or argued that it has dimension 1 and found that $T_{\mathbf{1}}(Z(U(n))) = \mathcal{J}$ by inclusion.

Exercise 6. Also show that the $Z(U(n))$ is the image, under the exponential map, of the ideal $\mathcal{J}$ in Exercise 5.

Answer: We found in Exercise 5 the one dimensional ideal

$$\mathcal{J} = \{\mathbf{i}\theta\mathbf{1} \mid \theta \in \mathbb{R}\},$$

and we have already shown that for any $y \in \mathcal{J}$, $e^y \in Z(U(n))$.