MATH 4500 - Homework 4

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March 8, 2019

Exercise 1. If $B = \begin{pmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$, show that $JBJ^{-1} = \overline{B}$.

Answer: We show this by direct computation:

$$\begin{aligned} JBJ^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \overline{\beta} & \overline{\alpha} \\ -\alpha & \beta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \overline{\alpha} & -\overline{\beta} \\ \beta & \alpha \end{pmatrix} \\ &= \overline{B} \end{aligned}$$

Exercise 2. Conversely, show that if $JBJ^{-1} = \overline{B}$ and $B = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$ then we have $\overline{c} = f$ and $\overline{d} = -e$, so B has the form $\begin{pmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$.

Answer: First note that $JBJ^{-1} = \overline{B}$ implies $J^{-1}\overline{B}J = B$, and we compute the quantity on the LHS:

$$\begin{aligned} J^{-1}\overline{B}J &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{c} & \overline{d} \\ \overline{e} & \overline{f} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\overline{e} & -\overline{f} \\ \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \overline{f} & -\overline{e} \\ -\overline{d} & \overline{c} \end{pmatrix} \end{aligned}$$

But we can match the coefficients to those of B since $J^{-1}\overline{B}J = B$, which gives $\overline{f} = c$, $-\overline{e} = d$, $-\overline{d} = e$ and $\overline{c} = f$. But since $\overline{\overline{a}} = a$, the first two equations are the same as the last two, and we can conclude $\overline{c} = f$ and $\overline{d} = -e$.

Exercise 3. Use block multiplication, and the results of Exercises 1 and 2, to show that B_{2n} has the form $C(A)$ if and only if $J_{2n}B_{2n}{J_{2n}}^{-1} = \overline{B_{2n}}$.

Answer: Note that J_{2n} and ${J_{2n}}^{-1}$ are both block matrices: we can show that ${J_{2n}}^{-1}$ is also a block matrix by the uniqueness of the inverse, since we can construct a block matrix with n

blocks, each of which is J^{-1}. Also, note that $J^{-1} = -J$, which will be used later to simplify the expressions. Clearly, using block multiplication, the product of these matrices is the identity.

Now we compute $J_{2n}B_{2n}J_{2n}{}^{-1}$:

$$
\begin{aligned}
J_{2n}B_{2n}J_{2n}{}^{-1} &= -\begin{pmatrix} J & 0 & 0 & \dots & 0 \\ 0 & J & 0 & \dots & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & \dots & J \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ .. & .. & .. & .. \\ B_{n1} & B_{n2} & \dots & B_{nn} \end{pmatrix} \begin{pmatrix} J & 0 & 0 & \dots & 0 \\ 0 & J & 0 & \dots & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & \dots & J \end{pmatrix} \\
&= -\begin{pmatrix} JB_{11} & JB_{12} & \dots & JB_{1n} \\ JB_{21} & JB_{22} & \dots & JB_{2n} \\ .. & .. & .. & .. \\ JB_{n1} & JB_{n2} & \dots & JB_{nn} \end{pmatrix} \begin{pmatrix} J & 0 & 0 & \dots & 0 \\ 0 & J & 0 & \dots & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & \dots & J \end{pmatrix} \\
&= -\begin{pmatrix} JB_{11}J & JB_{12}J & \dots & JB_{1n}J \\ JB_{21}J & JB_{22}J & \dots & JB_{2n}J \\ .. & .. & .. & .. \\ JB_{n1}J & JB_{n2}J & \dots & JB_{nn}J \end{pmatrix}
\end{aligned}
$$

Notice for each of these blocks, $JB_{ij}J^{-1} = \overline{B_{ij}}$ if and only if B_{ij} has the form $C(A)$ by Exercises 1 and 2. In order for the matrix B_{2n} to have the form $C(A)$, all of these blocks must have the form $C(A)$, and if they all are of the form $C(A)$ then clearly $JB_{ij}J^{-1} = \overline{B_{ij}}$ for all i, j, and thus $J_{2n}B_{2n}J_{2n}{}^{-1} = \overline{B_{2n}}$ if and only if B_{2n} is of the form $C(A)$.

Exercise 4. By taking det of both sides of the equation in Exercise 3, show that $det(B_{2n})$ is real.

Answer: We take the determinant of both sides, as suggested:

$$
\begin{aligned}
det(B_{2n}) &= det(J_{2n}\overline{B_{2n}}J_{2n}{}^{-1}) \\
&= det(J_{2n})det(\overline{B_{2n}})det(J_{2n}{}^{-1}) \\
&= det(J_{2n})det(J_{2n}{}^{-1})det(\overline{B_{2n}}) \\
&= det(\mathbf{1})det(\overline{B_{2n}}) \\
&= det(\overline{B_{2n}})
\end{aligned}
$$

But note that if $det(A) = a$ for some matrix $A \in \mathbb{C}_{n\times n}$, and some $a \in \mathbb{C}$, then $det(\overline{C}) = \overline{a}$, since the determinant is the sum of products of entries in B_{2n}, and both addition and multiplication commute with conjugation. Hence,

$$
det(B_{2n}) = a = \overline{a} = det(\overline{B_{2n}}) \implies det(B_{2n}) \in \mathbb{R}
$$

Exercise 5. Assuming now that B_{2n} is in the complex form of $Sp(n)$, and hence is unitary, show that $det(B_{2n}) = \pm 1$.

Answer: We are assuming that B_{2n} is unitary, and hence by definition we have:

$$
\begin{aligned}
B_{2n}B_{2n}^{\dagger} &= \mathbf{1} \\
det(B_{2n}B_{2n}^{\dagger}) &= det(\mathbf{1}) \\
det(B_{2n})det(B_{2n}^{\dagger}) &= 1 \\
det(B_{2n})\overline{det(B_{2n})} &= 1 \\
|det(B_{2n})|^2 &= 1
\end{aligned}
$$

But by Exercise 4, $det(B_{2n}) \in \mathbb{R}$, so it follows that $det(B_{2n}) = \pm 1$.