MATH 4500 - Homework 5

Daniel Alonso Vicuna

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Exercise 1. Assuming that $AB = BA$, show that

$$(A+B)^m = A^m + \binom{m}{1}A^{m-1}B + \binom{m}{2}A^{m-2}B^2 + ... + \binom{m}{m-1}AB^{m-1} + B^m,$$

where $\binom{m}{l}$ denotes the number of ways of choosing l things from a set of m things.

Answer: We first prove that if $AB = BA$, then for all $l.m \in \mathbb{Z}$ $BA^lB^m = A^lB^{m+1}$. We show this by induction on l, and this clearly holds for $l = 0$ since $BA^0B^k = BB^k = B^{k+1}$. Now assume it holds up to $l = k$. Then

$$BA^{k+1}B^m = ABA^kB^m = AA^kB^{m+1} = A^{k+1}B^{m+1}.$$

In addition, we show that for $k \geq l \geq 1$,

$$\binom{k}{l} + \binom{k}{l-1} = \binom{k+1}{l}$$

. We use the formula which is proven independently on Exercise 2:

$$\begin{aligned}\binom{k}{l} + \binom{k}{l-1} &= \frac{k!}{l!(k-l)!} + \frac{k!}{(l-1)!(k-l+1)!} \\ &= \frac{k!(k+1-l)}{l!(k-l+1)!} + \frac{k!l}{l!(k-l+1)!} \\ &= \frac{k!(k+1)}{l!(k-l)!} \\ &= \frac{(k+1)!}{l!(k+1-l)!} = \binom{k+1}{l}\end{aligned}$$

Now, we prove the problem statement by induction on m, also using the fact that $1 = \binom{k}{0} = \binom{k}{k}$.

- **Base case:** for $m = 1$, it clearly holds since

$$(A+B)^1 = A^1 + B^1$$

- **Inductive step:** Now we assume that this holds up to k and prove that it must then hold for $k+1$.

$$\begin{aligned}
(A+B)^{k+1} &= (A+B)(A+B)^k \\
&= (A+B)(A^k + \binom{k}{1}A^{k-1}B + \binom{k}{2}A^{k-2}B^2 + ... + \binom{k}{k-1}AB^{k-1} + B^k) \\
&= (A^{k+1} + \binom{k}{1}A^kB + \binom{k}{2}A^{k-1}B^2 + ... + \binom{k}{k-1}A^2B^{k-1} + AB^k) \\
&+ (BA^k + \binom{k}{1}BA^{k-1}B + \binom{k}{2}BA^{k-2}B^2 + ... + \binom{k}{k-1}BAB^{k-1} + B^{k+1}) \\
&= (A^{k+1} + \binom{k}{1}A^kB + \binom{k}{2}A^{k-1}B^2 + ... + \binom{k}{k-1}A^2B^{k-1} + AB^k) \\
&+ (A^kB + \binom{k}{1}A^{k-1}B^2 + \binom{k}{2}A^{k-2}B^3 + ... + \binom{k}{k-1}AB^k + B^{k+1}) \\
&= A^{k+1} + (\binom{k}{1} + \binom{k}{0})A^kB + (\binom{k}{2} + \binom{k}{1})A^{k-2}B^2 + ... \\
&+ \quad (\binom{k}{k} + \binom{k}{k-1})AB^k + B^{k+1} \\
&= A^{k+1} + \binom{k+1}{1}A^kB + \binom{k+1}{2}A^{k-2}B^2 + ... + \binom{k+1}{k}AB^k + B^{k+1}
\end{aligned}$$

This concludes the proof.

Exercise 2. Show that

$$\binom{m}{l} = \frac{m!}{l!(m-l)!}$$

Answer:

We can begin by noticing that we can sample an arrangement(or ordered sequence) from some set S such that $|S| = m$. For the first element, there are m choices, for the second $m-1$ and so on until $m-l+1$, so from independence there are $m!/(m-l)!$ possible arrangements. But the order is irrelevant for drawing l things from S, so every permutation of these l entries of the arrangement is equivalent and there's $k!$ of them, so there is at least a $k-1$ correspondence between arrangements of l elements from a set of m elements, and choices of l things from a set of m things.

Notice that for any two sequences that are not permutations of each other cannot represent an equivalent "choice of l things from a set of m things". Suppose, that sequences x contains an element x_i that is not in y. Then this means there is a "thing" in the choice represented by x that is not in y, so the choices are not the same. This shows that two ordered sequences are the same choice of l items from a set of m items if and only if the two sequences are permutations of each other, and thus the correspondance is exactly $k!-1$ so the number of choices is

$$\binom{m}{l} = \frac{m!/(m-l)!}{k!} = \frac{m!}{(m-l)!l!}$$

Exercise 3. Deduce from Exercises 1 and 2 that the coefficient of $A^{m-l}B^l$ in

$$e^{A+B} = 1 + \frac{A+B}{1!} + \frac{(A+B)^2}{2!} + \frac{(A+B)^3}{3!} +$$

is $\frac{1}{l!(m-l)!}$ when $AB = BA$.

Answer: From Exercise 1, we can see that

$$(A+B)^m = \sum_{i=0} \binom{m}{i} A^{m-i}B^i,$$

so we use this and the expression for $\binom{m}{l}$ that we proved in Exercise 2:

$$\begin{aligned}
e^{A+B} &= \sum_{m=0}^{\infty} \frac{(A+B)^m}{m!} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{l=0}^{m} \binom{m}{l} A^{m-l}B^l \quad \text{Exercise 1} \\
&= \sum_{m=0}^{\infty} \sum_{l=0}^{m} \frac{1}{m!} \frac{m!}{(m-l)!l!} A^{m-l}B^l \quad \text{Exercise 2} \\
&= \sum_{m=0}^{\infty} \sum_{l=0}^{m} \frac{1}{(m-l)!l!} A^{m-l}B^l
\end{aligned}$$

Exercise 4. Show that the coefficient of $A^{m-l}B^l$ in

$$(\mathbf{1} + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + ...)(\mathbf{1} + \frac{B}{1!} + \frac{B^2}{2!} + \frac{B^3}{3!} + ...)$$

is also $\frac{1}{l!(ml)!}$,and hence that $e^{A+B} = e^A e^B$ when $AB = BA$.

Answer: Any term in the product of these two series is of the form $A^{m-l}B^l$. Any such term must have a factor of A^{m-l}, for which the only possibility is $\frac{A^{m-l}}{(m-l)!}$, and a factor B^l, and the only term in the second series of this order is $B^l/l!$. Hence, the term in the product of these two series of the form $A^{m-l}B^l$ has coefficient $\frac{1}{(m-l)!l!}$.

Since the series on the left is e^A and the one on the right is e^B, and their product $e^A e^B$ is equal to e^{A+B} above (we just proved the coefficients of every term are equal), then $e^A e^B = e^{A+B}$. Another way of seeing this is that $AB - BA = 0$, so since the first series converges absolutely to e^A and the second to e^B then their Cauchy product converges to $e^A e^B$ (this would be an "extension" of the Cauchy product).