MATH 4500 - Homework 2

Daniel Alonso Vicuna

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Exercise 1. Show that there is exactly one n-element subgroup of $SO(2)$, for each natural number n, and list its members.

Answer:

First we show that if G is a finite group of order n, then for any $g \in G$ it must be the case that $g^n = e$. Suppose, for the sake of contradiction, that this is not the case. First, it cannot be the case that $g^m \neq e$ for $0 \leq m \leq m$, otherwise $\{e, g, g^2, ..., g_n\}$ are all distinct and the order of G must be greater than n - a contradiction. Hence, suppose that $g^m = e$ for some $0 \leq m \leq n-1$. We can define the group generated by g, that is $\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\}$. By Lagranges theorem, $|\langle g \rangle|$ divides $|G|$ and hence, we can write

$$g^n = g^{|\langle g \rangle| l} = (g^{|\langle g \rangle|})^l = e^l = e \quad l \in \mathbb{Z}$$

Next we show that $\mathbf{S}^1 \cong SO(2)$ by using the obvious isomorphism $\varphi : \mathbf{S}^1 \to SO(2)$:

$$\varphi(e^{i\phi}) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$$

. We can check that the group structure is preserved since

$$\begin{aligned} \varphi(e^{i\phi_1} e^{i\phi_2}) = \varphi(e^{i(\phi_1+\phi_2)}) &= \begin{pmatrix} \cos(\phi_1+\phi_2) & -\sin(\phi_1+\phi_2) \\ \sin(\phi_1+\phi_2) & \cos(\phi_1+\phi_2) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi_1) & -\sin(\phi_1) \\ \sin(\phi_1) & \cos(\phi_1) \end{pmatrix} \begin{pmatrix} \cos(\phi_2) & -\sin(\phi_2) \\ \sin(\phi_2) & \cos(\phi_2) \end{pmatrix} \\ &= \varphi(e^{i\phi_1})\varphi(e^{i\phi_2}) \end{aligned}$$

So now we look at elements in $\mathbf{S}^1$, and let H be a subgroup of $SO(2)$ with n elements. From the proof above, it follows that $g^n = e$ for all $g \in H$. Notice that the only members of $\mathbf{S}^1$ that satisfy this are

$$S = \{e^{\frac{2i\pi k}{n}} \mid k = 0, ..., n-1\}$$

Since $|S| = n$ and $|H| = n$, then it follows that H is uniquely $H = \{\varphi(x) \mid x \in S\}$.

Exercise 2. Show that $\pm\mathbf{1}$ is a normal subgroup of $\mathbf{S}^3$.

Answer: Take any $g \in \mathbf{S}^3$. Then we can show that $g^{-1}\{\pm\mathbf{1}\}g = \{\pm\mathbf{1}\}$. This holds because for any $g = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbf{S}^3$ and $x \in \mathbb{R}$, we have that $xg = x(a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = xa\mathbf{1} + xb\mathbf{i} + xc\mathbf{j} + xd\mathbf{k} = a\mathbf{1}x + b\mathbf{i}x + c\mathbf{j}x + d\mathbf{k}x = gx$, and thus

$$g^{-1}\{\pm\mathbf{1}\}g = \{\pm\mathbf{1}\}g^{-1}g = \{\pm\mathbf{1}\}$$

Thus, we conclude that $\{\pm\mathbf{1}\}$ is normal subgroup of $\mathbf{S}^3$

Exercise 3. Show that $\mathbf{S}^1$ is not a normal subgroup of $\mathbf{S}^3$.

Answer:

In order to show that $\mathbf{S}^1$ is not a normal subgroup of $\mathbf{S}^3$, we can find a $g \in S^3$ such that $g^{-1} S^1 g \neq S^1$. One such g is the quaternion

$$g^{-1} = \frac{1}{\sqrt{2}}(2\,\mathbf{i}+2\,\mathbf{k}) \quad g = \frac{1}{\sqrt{2}}(-2\,\mathbf{i}-2\,\mathbf{k})$$

and we can compute $g^{-1}\,\mathbf{S}^1\,g$:

$$\begin{aligned}
g^{-1}\,\mathbf{S}^1\,g &= \{\frac{1}{2}(-\mathbf{i}-\mathbf{k})(\cos(\theta)+\mathbf{i}\sin(\theta))(\mathbf{i}+\mathbf{k}) \mid \theta \in \mathbb{R}\} \\
&= \{\frac{1}{2}(-\mathbf{i}\cos(\theta)+\sin(\theta)-\mathbf{k}\cos(\theta)-\mathbf{j}\sin(\theta))(\mathbf{i}+\mathbf{k}) \mid \theta \in \mathbb{R}\} \\
&= \{\frac{1}{2}(2\cos(\theta)+2\,\mathbf{k}\sin(\theta)) \mid \theta \in \mathbb{R}\} \\
&= \{\cos(\theta)+\mathbf{k}\sin(\theta) \mid \theta \in \mathbb{R}\} \neq \mathbf{S}^1
\end{aligned}$$

This shows that $\mathbf{S}^1$ is indeed not a normal subgroup of $\mathbf{S}^3$.

Exercise 4. Show that reflection in the hyper-plane orthogonal to a coordinate axis has determinant -1, and generalize this result to any reflection.

Answer: Let $e_1, ..., e_n$ be some basis vectors for $\mathbb{R}^n$. Consider, without loss of generality, a reflection in the hyperplane H orthogonal to e_1. The reflection is a linear map that fixes all points in H and reverses the first coordinate, and can thus be represented by the matrix

$$R_1 = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

where $\mathbf{1} \in SO(n-1)$. Clearly, this matrix has determinant -1. We can generalize this to reflections across any hyperplane by considering the unit vector $\vec{u}_1$ orthogonal to the hyperplane, and extending that to an orthonormal basis $u_2, ..., u_n$, then composing the change of basis matrix

$$U = \begin{pmatrix} u_1 & .... & u_n \end{pmatrix}.$$

Since all of $u_1, ..., u_n$ have norm 1 and are orthogonal, $U^T U = \mathbf{1}$ and so U has determinant ± 1, and the reflection map is actually given by $U^T R_1 U$. Clearly,

$$det(U^T R_1 U) = det(U^T)det(U)det(R_1) = det(U^T U)det(R_1) = 1 * -1 = -1$$

Exercise 5. Observe that the rotations in Exercise 2.6.1 form an S^1, as do the rotations in Exercise 2.6.2, and deduce that $SO(4)$ contains a subgroup isomorphic to T^2.

Answer:

The matrices of Exercise 2.6.1 have the form

$$\begin{pmatrix} R_\theta & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

while matrices in Exercise 2.6.2 have the form

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & R_\phi \end{pmatrix}$$

where $\mathbf{1}, R_\phi, R_\theta \in SO(2)$. For this reason, we can write all such matrices, in general, in the form

$$\begin{pmatrix} R_\theta & 0 \\ 0 & R_\phi \end{pmatrix}$$

I claim that the union of these matrices, which we denote by G, is a subgroup of $SO(4)$ (under matrix multiplication). We check all the necessary properties:

- Clearly, since $\mathbf{1}_{2\times 2} \in SO(2)$, it follows that $\mathbf{1} \in G$.
- Any such matrix has an inverse and is contained in the set, and given by $-\theta$ and $-\phi$ respectively.
- The set is closed under multiplication, since these are block matrices and $SO(2)$ is itself a group.
- Matrix multiplication is associative.

The set of these matrices is isomorphic to the group $SO(2)\times SO(2)$ by page 41 of the textbook, and $SO(2)$ we recall from question 1 that $\mathbf{S}^1 \cong SO(2)$ so $\mathbf{S}^1 \times \mathbf{S}^1 \cong SO(2) \times SO(2)$ and hence $G \subset SO(4)$ is a subgroup and it is isomorphic to $\mathbf{S}^1 \times \mathbf{S}^1 = \mathbf{T}^2$.

Exercise 6. Explain why $S^3 = SU(2)$ is not the same group as $S^1 \times S^1 \times S^1$.

Answer: The notation is misleading: $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ is the 3-fold direct product of $\mathbf{S}$ with itself, while $\mathbf{S}^3 = SU(2)$ is the group of unit quaternions. One way to check that $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \ncong SU(2)$ is to check for invariants in both groups. For example, the number of holes in $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$, also or T^3 is 3, while $SU(2)$ has 0 holes. Hence, $SU(2) \ncong T^3$.