MATH 4500 - Homework 5

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Definition 1. Suppose V is a finite dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$), and suppose B is a function $V \times V \mathbb{R}$ (or $V \times V \to \mathbb{C}$). We say B is a bilinear form if for any $u, v, w \in V$ and $a \in \mathbb{R}$ (or $\mathbb{C}$)

$$B(u + v, w) = B(u, w) + B(v, w) \tag{1}$$
$$B(u, v + w) = B(u, v) + B(u, w) \tag{2}$$
$$B(au, v) = aB(u, v) \tag{3}$$
$$B(u, av) = \bar{a}B(u, v) \tag{4}$$

Exercise 1. Show that $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined as $B(u, v) = u \cdot v = uv^T$ is a symmetric bilinear form over $\mathbb{R}^n$.

Answer: We first show that it is a bilinear form by direct computation:

1. $B(u + v, w) = (u + v)w^T = uw^T + vw^T = B(u, w) + B(v, w)$
2. $B(u, v + w) = u(v + w)^T = u(v^T + w^T) = uv^T + uw^T = B(u, w) + B(u, w)$
3. $B(au, v) = (au)v^T = auv^T = a(uv^T) = aB(u, v)$
4. $B(u, av) = u(av)^T = u(av^T) = auv^T = aB(u, v)$

Note that $a \in \mathbb{R}$ so the fourth condition is satisfied. B is clearly also symmetric:

$$B(v, u) = vu^T = (vu^T)^T = uv^T = B(u, v) \quad \text{since } vu^T \in \mathbb{R}$$

Exercise 2. Show that $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, defined as $H(u, v) = uv^\dagger$ is a bilinear form over $\mathbb{C}^n$ but it is not symmetric. Here, $\dagger$ means conjugate transpose.

Answer: We also show this by direct computation:

1. $B(u + v, w) = (u + v)w^\dagger = uw^\dagger + vw^\dagger = B(u, w) + B(v, w)$
2. $B(u, v + w) = u(v + w)^\dagger = u(v^\dagger + w^\dagger) = uv^\dagger + uw^\dagger = B(u, w) + B(u, w)$
3. $B(au, v) = (au)v^\dagger = auv^\dagger = a(uv^\dagger) = aB(u, v)$
4. $B(u, av) = u(av)^\dagger = u(\bar{a}v^\dagger) = \bar{a}uv^\dagger = \bar{a}B(u, v)$

However, this form is not symmetric

$$B(v, u)^\dagger = (vu^\dagger)^\dagger = uv^\dagger = B(u, v)$$

as $B(v, u)^\dagger \neq B(v, u)$ since it might be the case that $B(v, u) \neq \overline{B(v, u)}$

Exercise 3. Suppose $\omega : \mathbb{R}^4 \times \mathbb{R}^4$ is a bilinear form. On a basis e_1, e_2, e_3, e_4 we have

$$\omega(e_1, e_4) = \omega(e_2, e_3) = -\omega(e_4, e_1) = -\omega(e_3, e_2)$$

$$\omega(e_i, e_j) = 0 \quad \text{on all other basis vectors}$$

Show that this bilinear form ω is antisymmetric. We define

$$G := \{A \in GL_4(\mathbb{R}) \mid \omega(Ax, Ay) = \omega(x, y) \text{ for all } x, y \in \mathbb{R}^4\}$$

Prove that G is a group. This group is denoted by $Sp(2, \mathbb{R})$ and called the symplectic group.

Answer: To make computations easier, I will denote $\omega(e_i, e_j) = w_{ij}$ in this problem. First note that we can write any vector x as $x = \sum_{i=1}^{4} a_i e_i$ since $\{e_1, e_2, e_3, e_4\}$ are a basis for $\mathbb{R}^4$. Then, we can check that this form is antisymmetric by direct computation - letting $u = \sum_{i=1}^{4} u_i e_i$ and $v = \sum_{i=1}^{4} v_i e_i$. Then:

$$\begin{aligned}
\omega(u, v) &= \omega(\sum_i u_i e_i, \sum_j v_j e_j) \\
&= \sum_i \omega(u_i e_i, \sum_j v_j e_j) \quad \text{by (1)} \\
&= \sum_i u_i \omega(e_i, \sum_j v_j e_j) \quad \text{by (3)} \\
&= \sum_i u_i \sum_j v_j \omega(e_i, e_j) \quad \text{by (2) then (4)} \\
&= u_1 v_4 w_{14} + u_2 v_3 w_{23} + u_3 v_2 w_{32} + u_4 v_1 w_{41} \\
&= -u_1 v_4 w_{41} - u_2 v_3 w_{32} - u_3 v_2 w_{23} - u_4 v_1 w_{14}
\end{aligned}$$

But following the exact same process until the penultimate step, (or by replacing any $u_i \leftrightarrow v_i$), clearly

$$\begin{aligned}
\omega(v, u) &= v_1 u_4 w_{14} + v_3 u_2 w_{23} + v_3 u_2 w_{23} + v_4 u_1 w_{14} \\
&= -\omega(u, v) \quad \text{by computation above}
\end{aligned}$$

Now we show that G is a group, by first noting that we can compute $\omega(u, v)$ by

$$\omega(u, v) = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & w \\ 0 & 0 & w & 0 \\ 0 & -w & 0 & 0 \\ -w & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix},$$

where $\omega(e_1, e_4) = w$. Hence, for any $A \in G$, if we denote the matrix in the equation above as W we must have:

$$\begin{aligned}
&\omega(u, v) = \omega(Au, Av) \quad \forall u, v \in \mathbb{R}^4 \\
&\Leftrightarrow u^T W v = (Au)^T W (Av) = u^T A^T W A v \quad \forall u, v \in \mathbb{R}^4 \\
&\Leftrightarrow A^T W A = W
\end{aligned}$$

We can use this to show that G is a group.

1. Clearly, $\mathbf{1} \in G$, since $\mathbf{1}^T W \mathbf{1} = W$.
2. Take any $A \in G$, and note that $0 \neq w^4 = det(W) = det(W)det(A^T W A) = det(A)^2 det(W)$, hence $det(A) \neq 0$ and A is invertible. Furthermore,

$$\begin{aligned} A^T W A &= W \\ \Rightarrow A^{-1^T} A^T W A A^{-1} &= A^{-1^T} W A^{-1} \\ W &= A^{-1^T} W A^{-1}, \end{aligned}$$

   and clearly $A^{-1} \in G$.
3. Take $A_1, A_2 \in G$, then

$$(A_1 A_2)^T W A_1 A_2 = A_2^T A_1^T W A_1 A_2 = A_2^T W A_2 = W.$$

   Hence, $A_1 A_2 \in G$
4. Matrix multiplication is associative.

Exercise 4. Show that for $u, v \in \mathbb{R}^4$

$$B_{3,1}(u,v) = u_1 v_1 + u_2 v_2 + u_3 v_3 - u_4 v_4$$

is a symmetric bilinear form. Is $B_{3,1}(u,u)$ an inner product? We define

$$G := \{A \in GL_4(\mathbb{R}) \mid B_{3,1}(Ax, Ay) = B_{3,1}(x,y) \text{ for all } x, y \in \mathbb{R}^4\}$$

Prove that G is a group. It is denoted by $SO(3,1)$ and is called the Lorentz group and it plays an important role in physics.

Answer: We first show that $B_{3,1}$ is a bilinear from:

1. $$\begin{aligned} B_{3,1}(u+v,w) &= \sum_{i=1}^{3}(u+v)_i w_i - (u+v)_4 w_4 = \sum_{i=1}^{3}(u_i+v_i)w_i - (u_4+v_4)w_4 \\ &= \sum_{i=1}^{3} u_i w_i - u_4 w_4 + \sum_{i=1}^{3} v_i w_i - v_4 w_4 = B_{3,1}(u,w) + B_{3,1}(v,w) \end{aligned}$$

2. $$\begin{aligned} B_{3,1}(u,v+w) &= \sum_{i=1}^{3} u_i(v+w)_i - u_4(v+w)_4 = \sum_{i=1}^{3} u_i v_i - u_4 v_4 + \sum_{i=1}^{3} u_i w_i - u_4 w_4 \\ &= B_{3,1}(u,v+w) \quad \text{same properties as above used} \end{aligned}$$

3. $$B_{3,1}(au,v) = \sum_{i=1}^{3}(au)_i v_i - au_4 v_4 = a\sum_{i=1}^{3} u_i v_i - au_4 v_4 = a(\sum_{i=1}^{3} u_i v_i - u_4 v_4) = aB_{3,1}(u,v)$$

4. The following hold, additionally, because we are in $\mathbb{R}^4$:

$$B_{3,1}(u,av) = \sum_{i=1}^{3} u_i(av)_i - au_4 v_4 = a\sum_{i=1}^{3} u_i v_i - au_4 v_4 = a(\sum_{i=1}^{3} u_i v_i - u_4 v_4) = aB_{3,1}(u,v)$$

Similarly, the form is obviously symmetric under the exchange of $u \leftrightarrow v$ since multiplication is commutative, so the form is actually symmetric under the exchange of $u_i \leftrightarrow v_i$ for any i.

$B_{3,1}$ is not an inner product, nonetheless, since an inner product $\langle\cdot\rangle$ must satisfy for any $v \in \mathbb{R}^n$, $\langle v, v\rangle = 0 \Leftrightarrow v = 0$, and we can pick $0 \neq v = (1,0,0,1) \in \mathbb{R}^4$, for which $B_{3,1}(v,v) = 1\cdot 1 + 0\cdot 0 + 0\cdot 0 - 1\cdot 1 = 0$.

Finally, it is easy to see that we can compute $B_{3,1}(u,v)$ for any $u, v \in \mathbb{R}^4$ using the matrix η below:

$$B_{3,1}(u,v) = u^T \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -1 \end{pmatrix} = u^T \eta v \quad \mathbf{1} \in GL(3,\mathbb{R}).$$

Hence, any $A \in G$ must by definition satisfy:

$$\begin{aligned} &B_{3,1}(u,v) = B_{3,1}(Au, Av) \quad \forall u, v \in \mathbb{R}^4 \\ &\Leftrightarrow u^T \eta v = (Au)^T \eta (Av) = u^T A^T \eta A v \quad \forall u, v \in \mathbb{R}^4 \\ &\Leftrightarrow \eta = A^T \eta A \end{aligned}$$

Notice that $det(\eta) = -1 \neq 0$ and hence, all the arguments from Exercise 3 hold, which shows that G is indeed a group as well.

1. Clearly, $\mathbf{1} \in G$ since $\mathbf{1}^T \eta \mathbf{1} = \eta$.
2. $\mathbf{1}^T \eta \mathbf{1} = \eta$ implies $det(A)^2 det(\eta) = \det(\eta)1 \neq 0$ so A is invertible. Furthermore,
$$A^T \eta A = \eta \Rightarrow A^{\text{-}1^T} \eta A^{\text{-}1} = \eta \Rightarrow A^{\text{-}1} \in G.$$
3. For any $A_1, A_2 \in G$, we have
$$(A_1A_2)^T \eta A_1 A_2 = A_2^T (A_1^T \eta A_1) A_2 = A_2^T \eta A_2 = \eta \Rightarrow A_1 A_2 \in G.$$
4. Matrix multiplication is associative.

Exercise 5. Show, directly from the definition of matrix exponentiation, that

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \implies e^A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Answer: First note that

$$J^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\mathbf{1} \text{ where } J \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So clearly, the sequence $J, J^2, J^3, J^4, J^5, \ldots$ evaluates to $J, -\mathbf{1}, -J, \mathbf{1}, J, -\mathbf{1}, -J, \mathbf{1}, \ldots$ and

$$\theta^n J^n = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}^n = \begin{cases} (-1)^k \theta^{2k}\, \mathbf{1}, n = 2k \text{ for some } k \in \mathbb{N} \\ (-1)^k \theta^{2k} K, n = 2k+1 \text{ for some } k \in \mathbb{N} \end{cases}$$

By the definition of the exponential of a matrix, we have

$$\begin{aligned} e^A &= \mathbf{1} + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \ldots \\ &= \mathbf{1} + \frac{\theta}{1!} J - \frac{\theta^2}{2!} \mathbf{1} - \frac{\theta^3}{3!} J + \frac{\theta^4}{4!} \mathbf{1} + \ldots \\ &= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots)\, \mathbf{1} + (\frac{\theta}{1!} - \frac{\theta^3}{3!} + \ldots) J \\ &= \cos\theta\, \mathbf{1} + \sin\theta J \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{aligned}$$

Exercise 6. Suppose that D is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, ..., \lambda_k$. By computing the powers D^n show that e^D is a diagonal matrix with diagonal entries $e^{\lambda_1}, e^{\lambda_2}, ..., e^{\lambda_k}$.

Answer: Since a diagonal matrix is a block matrix, we know that the entries of D^n are $\lambda_1^n, \lambda_2^n, ..., \lambda_k^n$. Hence, the entries of the matrix $\frac{1}{n!}D^n$ are just the diagonal entries $\frac{1}{n!}\lambda_1^n, \frac{1}{n!}\lambda_2^n, ..., \frac{1}{n!}\lambda_k^n$. Using the definition of matrix exponentials, we have

$$\begin{aligned}
e^D &= \mathbf{1} + \sum_{i=1}^{\infty} \frac{1}{i!} D^i \\
&= \mathbf{1} + \sum_{i=1}^{n} diag(\frac{1}{i!}\lambda_1^i, \frac{1}{i!}\lambda_2^i, ..., \frac{1}{i!}\lambda_k^i) \\
&= \sum_{i=0}^{\infty} diag(\frac{1}{i!}\lambda_1^i, \frac{1}{i!}\lambda_2^i, ..., \frac{1}{i!}\lambda_k^i) \\
&= diag(\sum_{i=0}^{\infty} \frac{1}{i!}\lambda_1^i, \sum_{i=0}^{\infty} \frac{1}{i!}\lambda_2^i, ..., \sum_{i=0}^{\infty} \frac{1}{i!}\lambda_k^i) \\
&= diag(e^{\lambda_1}, e^{\lambda_2}, ..., e^{\lambda_k})
\end{aligned}$$