MATH 4500 - Homework 7

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Exercise 1. Using bilinearity, or otherwise, show that $U, V \in \mathbb{R}\,\mathbf{i} + \mathbb{R}\,\mathbf{j} + \mathbb{R}\,\mathbf{k}$ implies $[U, V] \in \mathbb{R}\,\mathbf{i} + \mathbb{R}\,\mathbf{j} + \mathbb{R}\,\mathbf{k}$.

Answer: We show this using the definition of the Lie bracket. First, we compute the values of several Lie brackets that will be useful:

$$\begin{aligned}
[i, j] &= ij - ji = 2k \\
[j, k] &= jk - kj = 2i \\
[k, i] &= ik - ki = 2j
\end{aligned}$$

Notice that, just like with the cross product in $\mathbb{R}^3$, we can write this more succintly using the antisymemtric levi-civita tensor, if we let $e_1 = \mathbf{i}, e_2 = \mathbf{j}, e_3 = \mathbf{k}$:

$$[e_i, e_j] = 2e_k\epsilon_{ijk}$$

Now we let $U = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k} = \sum u_l e_l$ and $V = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k} = \sum v_l e_l$, and compute $[U, V]$:

$$\begin{aligned}
[\sum_l u_l e_l, \sum_m v_m e_m] &= \sum_l \sum_m u_l v_m [e_l, e_m] && \text{bilinear} \\
&= \sum_j \sum_m u_l v_m e_k \epsilon_{lmk} && \text{from above}
\end{aligned}$$

But every term in the sum is in $\mathbb{R}\,\mathbf{i} + \mathbb{R}\,\mathbf{j} + \mathbb{R}\,\mathbf{k}$, and hence, so is $[U, V]$.

Exercise 2. Prove the Jacobi identity by using the denition $[X, Y] = XY - YX$ to expand $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$. Assume only that the product is associative and that the usual laws for plus and minus apply.

Answer: We use the definition of the Lie bracket. First we look at the term $[X, [Y, Z]]$:

$$\begin{aligned}
[X, [Y, Z]] &= [X, YZ - ZY] && \text{by defn} \\
&= [X, YZ] - [X, ZY] && \text{bilinear} \\
&= XYZ - YZX - XZY + ZYX
\end{aligned}$$

Now we plug this in to the complete expression, noting that each term is a cyclic permutation of the previous one:

$$\begin{aligned}[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] &= XYZ-YZX-XZY+ZYX\\ &+YZX-ZXY-ZYX+YXZ\\ &+ZXY-XYZ-YXZ+XZY\\ &=0\end{aligned}$$

Exercise 3. We explore the correspondence between $SO(3)$ and skew-symmetric matrices.

(a) Find the exponential of the matrix $B=\begin{pmatrix}0&-\theta&0\\ \theta&0&0\\ 0&0&0\end{pmatrix}=\begin{pmatrix}\theta J&0\\ 0&0\end{pmatrix}$

Answer: Using block multiplication, this will converge to

$$e^B=\begin{pmatrix}e^{\theta J}&0\\ 0&e^0\end{pmatrix}$$

with J defined as in Homework 5, Exercise 5. But as we found in that same exercise

$$e^{\theta J}=\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}.$$

It follows that

$$e^B=\begin{pmatrix}\cos\theta&-\sin\theta&0\\ \sin\theta&\cos\theta&0\\ 0&0&1\end{pmatrix}$$

(b) Show that $Ae^BA^T=e^{ABA^T}$ for any orthogonal matrix A.

Answer: First we prove by induction on n that for any matrix $A\in SO(3)$, $(ABA^T)^n=AB^nA^T$. The base case $n=1$ trivially holds, and we assume it holds up to n and show that its true for $n+1$:

$$\begin{aligned}(ABA^T)^{n+1} &= (ABA^T)(ABA^T)^n\\ &=(ABA^T)(AB^nA^T) \quad \text{induction}\\ &=AB(A^TA)B^nA^T\\ &=AB^{n+1}A^T\end{aligned}$$

Hence, we can use the definition of the matrix exponential to conclude the exercise:

$$\begin{aligned}e^{ABA^T} &= \mathbf{1}+\sum_{n=1}(ABA^T)^n\\ &=\mathbf{1}+\sum_{n=1}AB^nA^T\\ &=A(A^T+\sum_{n=1}B^nA^T)\\ &=A(\mathbf{1}+\sum_{n=1}B^n)A^T=Ae^BA^T\end{aligned}$$

(c) Deduce that each matrix in $SO(3)$ equals e^X for some skew-symmetric X.

Answer: First note that any matrix in $Y \in SO(3)$, since it is a rotation of $\mathbb{R}^3$, can be written as $AR_z(\theta)A^T$ for some orthogonal matrix A, with $R_z(\theta) \in SO(3)$ being the rotation on the z-axis (this is just a change of basis). But from part (a), we found that $e^B = R_z(\theta)$, and hence $Ae^BA^T = Y$. Using part (b), we find that $Ae^BA^T = e^{ABA^T} = Y$ and hence any matrix $Y \in SO(3)$ equals e^{ABA^T} for some $A \in O(3)$.

Finally, we show that ABA^T is skew-symmetric:

$$\begin{aligned} ABA^T + (ABA^T)^T &= ABA^T + AB^TA^T \\ &= A(B + B^T)A^T \\ &= A(B - B)A^T = 0 \end{aligned}$$

Exercise 4. Show that the skew-Hermitian matrices in the tangent space of $SU(2)$ can be written in the form $b\,\mathbf{i}+c\,\mathbf{j}+d\,\mathbf{k}$, where $b, c, d \in \mathbb{R}$ and $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are matrices with the same multiplication table as the quaternions $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$.

Answer: By the proof about the tangent space of $SU(n)$ in Section 5.3 , any $X \in SU(n)$ must satisfy

$$X + \overline{X}^T = 0 \quad \text{and} \quad Tr(X) = 0$$

Clearly, any such matrix X must be of the form:

$$\begin{aligned} X &= \begin{pmatrix} d\,\mathbf{i} & -b + -c\,\mathbf{i} \\ b + -c\,\mathbf{i} & -d\,\mathbf{i} \end{pmatrix} \\ &= b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & -\mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix} + d\begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \end{aligned}$$

for $b, c, d \in \mathbb{R}$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$. But these matrices are the ones we saw in page 9 of the textbook, where we saw that the indeed satisfy the multiplication table of the quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

Exercise 5. Also find the tangent space of $Sp(1)$ (which should be the same)

Answer: Once again, we have the equation for any q in the Tangent space of $Sp(1)$.

$$q + \overline{q}^T = 0$$

But we can decompose this in terms of the usual $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\begin{aligned} (a + b\,\mathbf{i}+c\,\mathbf{j}+d\,\mathbf{k}) + \overline{a + b\,\mathbf{i}+c\,\mathbf{j}+d\,\mathbf{k}} &= 0 \\ (a + b\,\mathbf{i}+c\,\mathbf{j}+d\,\mathbf{k}) + (a - b\,\mathbf{i}-c\,\mathbf{j}-d\,\mathbf{k}) &= 0 \\ \implies 2a &= 0 \end{aligned}$$

Hence, the tangent space any linear combination of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. In addition these $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{H}$ trivially satisfy the multiplication table of quaternions.

Exercise 6. Interpret the following sum as $Tr(XY)$ and $Tr(YX)$:

$$\begin{aligned} x_{11}y_{11} + x_{12}y_{21} + ... + x_{1n}y_{n1} \\ +x_{21}y_{12} + x_{22}y_{22} + ... + x_{2n}y_{n2} \\ \vdots \\ +x_{n1}y_{1n} + x_{n2}y_{2n} + ... + x_{2n}y_{n2} \end{aligned}$$

Answer: We have

$$\begin{aligned} Tr(XY) &= Tr(\sum_j X_{ij}Y_{jk}) \\ &= \sum_{k=1}^{n} \delta_{ik} \sum_{j=1}^{n} X_{ij}Y_{jk} \\ &= \sum_{k=1}^{n}\sum_{j=1}^{n} X_{kj}Y_{jk} \end{aligned}$$

Clearly, for any k, we are summing over a row in the expression above. Similarly,

$$\begin{aligned} Tr(YX) &= Tr(\sum_j Y_{ij}X_{jk}) \\ &= \sum_{k=1}^{n} \delta_{ik} \sum_{j=1}^{n} Y_{ij}X_{jk} \\ &= \sum_{k=1}^{n}\sum_{j=1}^{n} Y_{kj}X_{jk} \end{aligned}$$

and so for any k, we are summing over a column in the expression above. It follows that $Tr(XY) = Tr(YX)$, as desired.