MATH 4500 - Homework 3

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Exercise 1. Check that $u_4\overline{u_3}u_2\overline{u_1} = i$ and $\overline{u_1}u_2\overline{u_3}u_4 = 1$, so the product of the four reections is indeed $q \mapsto \mathbf{i}\,q$.

Answer: From the previous exercises on the book, the values of u_i are as follows:

$$u_1 = \mathbf{i} \quad u_2 = \frac{\mathbf{i}-1}{\sqrt{2}} \quad u_3 = \mathbf{k} \quad u_4 = \frac{\mathbf{k}-\mathbf{j}}{\sqrt{2}}$$
$$\implies \overline{u_1} = -\mathbf{i} \quad \overline{u_3} = -\mathbf{k}$$

So we can compute the quantities above using associativity:

$$\begin{aligned}
u_4\overline{u_3}u_2\overline{u_1} &= u_4\overline{u_3}u_2\overline{u_1} \\
&= (u_4\overline{u_3})(u_2\overline{u_1}) \\
&= (\frac{\mathbf{k}-\mathbf{j}}{\sqrt{2}}\,\mathbf{k})(\frac{\mathbf{i}-1}{\sqrt{2}}\,\mathbf{i}) \\
&= \frac{1}{2}(-1-\mathbf{i})(-1-\mathbf{i}) \\
&= \frac{1}{2}2\,\mathbf{i} = \mathbf{i}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{u_1}u_2\overline{u_3}u_4 &= (\overline{u_1}u_2)(\overline{u_3}u_4) \\
&= (\mathbf{i}\,\frac{\mathbf{i}-1}{\sqrt{2}})(\mathbf{k}\,\frac{\mathbf{k}-\mathbf{j}}{\sqrt{2}}) \\
&= \frac{1}{2}(-1-\mathbf{i})(-1+\mathbf{i}) \\
&= 1
\end{aligned}$$

Exercise 2. Check that $q \mapsto u^{-1}qu$ is an automorphism of $\mathbb{H}$ for any unit quaternion u.

Answer: We check that $\varphi(q) = u^{-1}qu$ indeed defines an automosphism.

- Injectivity. Suppose that $\varphi(q_1) = \varphi(q_2)$. We show that this implies $q_1 = q_2$:

$$\begin{aligned}
\varphi(q_1) &= \varphi(q_2) \\
u^{-1}q_1u &= u^{-1}q_2u \\
u(u^{-1}q_1u)u^{-1} &= u(u^{-1}q_2u)u^{-1} \quad \text{Multiply on the left and right} \\
(uu^{-1})q_1(uu^{-1}) &= (uu^{-1})q_2(uu^{-1}) \quad \text{Associative} \\
q_1 &= q_2
\end{aligned}$$

- Surjectivity. For any $q' \in \mathbb{H}$, there exists $q \in \mathbb{H}$ such that $u^{\text{-}1} qu = q'$. Clearly,

$$\begin{aligned} u^{\text{-}1} qu &= q' \\ u(u^{\text{-}1} qu)u^{\text{-}1} &= uq'u^{\text{-}1} \\ q &= uq'u^{\text{-}1} \end{aligned}$$

  Note that such a $q \in \mathbb{H}$ since $u, u^{\text{-}1} \in \mathbf{S}^3$ so $uq' \in \mathbb{H}$ and $uq'u^{\text{-}1} \in \mathbb{H}$.

- Homomorphism. For all $q_1, q_2 \in \mathbb{H}$, we have that

$$\begin{aligned} \varphi(q_1 q_2) &= u^{\text{-}1} q_1 q_2 u \\ &= u^{\text{-}1} q_1 (1) q_2 u \\ &= u^{\text{-}1} q_1 (uu^{\text{-}1}) q_2 u \\ &= (u^{\text{-}1} q_1 u)(u^{\text{-}1} q_2 u) \\ &= \varphi(q_1)\varphi(q_2) \end{aligned}$$

Exercise 3. Give an example of a matrix in $O(3)$ that is not in $SO(3)$, and interpret it geometrically.

> **Answer:** An example of such a matrix is a single reflection, which is by definition not orientation preserving and does not have determinant 1. For example, a reflection across the plane passing throught $\mathcal{O}$, perpendicular to the x axis:
>
> $$R_x \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
>
> Clearly, this matrix satisfies $R_x^T R_x = \mathbf{1}$ since its a diagonal matrix with diagonals ± 1 and consequently $R_x \in O(3)$, but $det(R_x) = -1$ so $R_x \notin SO(3)$.

Exercise 4. Bearing in mind that matrix multiplication is a continuous operation, show that if there are continuous paths in G from 1 to $A \in G$ and to $B \in G$ then there is a continuous path in G from A to AB.

> **Answer:** First consider the continuous path $\phi : I \to G$, where $i = [0, 1]$ such that $\phi(0) = 1$ and $\phi(1) = B$ such as the one given in the problem. Now we construct another continuous map $\tilde{\phi} : I \to G$ by composition of continuous maps, defined by $\tilde{\phi} = A \circ \phi$ (since matrix multiplication is a continuous operation). Clearly, $\tilde{\phi}(0) = A1 = A$, while $\tilde{\phi}(1) = AB$. This shows that, if there is a continuous map from 1 to B in G, then there is also a continuous map from A to AB.

Exercise 5. Similarly, show that if there is a continuous path in G from 1 to A, then there is also a continuous path from $A^{\text{-}1}$ to 1.

> **Answer:** We construct a $\tilde{\phi} : I \to G$ as in the last problem but using $A^{\text{-}1}$ instead of A. That is, $\tilde{\phi} = A^{\text{-}1} \circ \phi$ where $\phi(0) = 1$ and $\phi(1) = A$. Clearly, this is also a composition of continuous maps, which is itself continuous, and $\tilde{\phi}(0) = A^{\text{-}1} 1 = A^{\text{-}1}$ while $\tilde{\phi}(1) = A^{\text{-}1} A = 1$ as desired. Notice that $A^{\text{-}1} \in G$ since G is a group.