MATH 4500 - Homework 9

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Exercise 1. Recall that a linear map between two abstract Lie algebras $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$ and that two Lie algebras are isomorphic if there is a bijective Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$.

(a) Prove that $\mathfrak{sl}_2(\mathbb{C})$ admits a $\mathbb{C}$-linear basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $[X, Y] = H$, $[H, X] = 2X$, and $[H, Y] = -2Y$.

Answer: We first show that $\{X, Y, H\}$ spans $\mathfrak{sl}_2(\mathbb{C})$ by showing that any matrix that satisfies

$$A \in M_2(\mathbb{C}) \quad Tr(A) = 0.$$

All matrices satisfying this are of the form:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \beta X + \gamma Y + \alpha H$$

These matrices are also clearly linearly independent (only their trivial linear combination can equal 0), and hence $\{X, Y, H\}$ form a basis.

(b) Bearing in mind that for any vector space V, $\mathfrak{gl}(V) := \{A \in End(V)\}$ with the bracket $[A, B] = AB - BA$ is an abstract Lie algebra, prove that we have a Lie algebra homomorphism

$$\begin{aligned} ad : \mathfrak{sl}_2(\mathbb{C}) &\to \mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C})) \\ x &\mapsto [x, -] \end{aligned}$$

which is injective, but not surjective.

Answer: First we show that this is indeed a homomorphism:

$$\begin{aligned} ad([x, y])(z) &= [[x, y], z] \\ &= -[[y, z], x] - [[z, x], y] \quad \text{Jacobi id} \\ &= [x, [y, z]] - [y, [x, z]] \\ &= (ad(x) \circ ad(y))(z) - (ad(y) \circ ad(x))(z) \\ &= ([ad(x), ad(y)])(z) \end{aligned}$$

Now we show that the map is injective by showing that $\ker ad = 0$. Take any $v = aX + bY + cH \in \ker ad$. Then by assumption, we have

$$\begin{aligned} ad(v)(X) &= ad(aX + bY + cH) \\ &= a * ad(X)(X) + b * ad(Y)(X) + c * ad(H)(X) \\ &= -2bH + 2cX \\ &= 0 \\ &\implies b = c = 0 \quad \text{since } H, X \text{ linearly ind.} \end{aligned}$$

In addition, since $v \in \ker(ad)$ then $ad(v)(Y) = a * ad(X)(Y) = a * H = 0$ and this also implies that $a = 0$.

Finally, $\mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C})$ is the set of all linear maps from $\mathfrak{sl}_2(\mathbb{C})$ to $\mathfrak{sl}_2(\mathbb{C})$, and since $dim(\mathfrak{sl}_2(\mathbb{C})) = 3$, then $dim(\mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C})) = 9$, so ad cannot be surjective.

(c) Find the eigenvalues of the linear map $ad(H) : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$.

Answer: Since $\{X, Y, H\}$ from a basis, any vector is a linear combination of them. But we know that

$$\begin{aligned} ad(H)(X) &= [H, X] = 2X \\ ad(H)(Y) &= [H, Y] = -2Y \\ ad(H)(H) &= [H, H] = 0 \end{aligned}$$

So X, Y, H are the only possible eigenvectors, and their eigenvalues are $2, -2$ and 0.

Exercise 2 (6.5.4)**.** Prove that each 4×4 skew-symmetric matrix is uniquely decomposable as a sum

$$\begin{pmatrix} 0 & -a & -b & -c \\ a & 0 & -c & b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{pmatrix} + \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & z & -y \\ y & -z & 0 & x \\ z & y & -x & 0 \end{pmatrix}$$

Answer: We prove that the following system has a unique solution:

$$\begin{pmatrix} 0 & -a-x & -b-y & -c-z \\ a+x & 0 & -c+z & b-y \\ b+y & c-z & 0 & -a+x \\ c+z & -b+y & a-x & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{pmatrix}$$

This gives us 6 equations, and six unknowns. Looking at these equations closely, only a_1, a_6 involve the variables a, x. The case is similar for the other variables, so that there really are 3 systems of 2 equations and two variables which can be solved as follows:

$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ x \end{pmatrix} = \begin{pmatrix} a_1 \\ a_6 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b \\ y \end{pmatrix} = \begin{pmatrix} a_2 \\ a_5 \end{pmatrix} \quad \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c \\ z \end{pmatrix} = \begin{pmatrix} a_3 \\ a_4 \end{pmatrix}$$

But this concludes the proof since the matrices above have determinants -2,2, and -2 respectively, and are thus invertible and hence the system has a unique solution.

Exercise 3 (6.5.5)**.** Setting $I = -E_{12} - E_{34}$, $J = -E_{13} + E_{24}$, and $K = -E_{14} - E_{23}$, show that $[I, J] = 2K$, $[J, K] = 2I$, and $[K, I] = 2J$.

Answer: We start by noting the identity:

$$E_{ji} = e_{ji} - e_{ij} = -(e_{ij} - e_{ji}) = -E_{ij}.$$

Using this identity and the bilinearity of the Lie bracket we complete the proof:

$$\begin{aligned}
[I, J] &= [-E_{12} - E_{34}, -E_{13} + E_{24}] \\
&= [E_{12}, E_{13}] - [E_{12}, E_{24}] + [E_{34}, E_{13}] - [E_{34}, E_{24}] \\
&= -[E_{12}, E_{31}] - [E_{12}, E_{24}] + [E_{34}, E_{13}] + [E_{43}, E_{24}] \\
&= -E_{23} - E_{14} + E_{41} + E_{32} \\
&= -2E_{23} - 2E_{14} = 2K
\end{aligned}$$

$$\begin{aligned}
[J, K] &= [-E_{13} + E_{24}, -E_{14} - E_{23}] \\
&= [E_{13}, E_{14}] + [E_{13}, E_{23}] - [E_{24}, E_{14}] - [E_{24}, E_{23}] \\
&= -E_{34} - E_{12} + E_{21} + E_{43} \\
&= -2E_{34} - 2E_{12} = 2I
\end{aligned}$$

$$\begin{aligned}
[K, I] &= [-E_{14} - E_{23}, -E_{12} - E_{34}] \\
&= [E_{14}, E_{12}] + [E_{14}, E_{34}] + [E_{23}, E_{12}] + [E_{23}, E_{34}] \\
&= -E_{42} - E_{13} + E_{31} + E_{24} \\
&= -2E_{13} + 2E_{24} = 2J
\end{aligned}$$

Exercise 4 (6.5.6)**.** Deduce from Exercises 2 and 3 that $\mathfrak{so}(4)$ is isomorphic to the direct product $\mathfrak{so}(3) \times \mathfrak{so}(3)$ (also known as the direct sum and commonly written $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$).

Answer: We construct a basis as done in the textbook for $\mathfrak{so}(n)$ and denote it by $E_{ij}^{(n)}$. For $\mathfrak{so}(4)$ it is easier to consider the elements $\{I, J, K, I', J', K'\}$ where I, J, K are defined in the previous exercise and

$$I' = -E_{12}^{(4)} + E_{34}^{(4)} \quad J = -E_{13}^{(4)} - E_{24}^{(4)} \quad K' = -E_{14}^{(4)} + E_{23}^{(4)}.$$

It is easy to see, by the computations from Exercise 3, that

$$[I', J'] = -2K' \quad [J', K'] = -2I' \quad [K', I'] = -2J'.$$

In addition,

$$[I, I'] = [J, J'] = [K, K'] = 0$$

because all of these commutators are in the form $[E_{ij}, E_{kl}]$ for distinct i, j, k, l (Exercise 6.5.2), and we can also compute some other cross commutators:

$$\begin{aligned}
[I, K'] &= [-E_{12}^{(4)} - E_{34}^{(4)}, -E_{13}^{(4)} - E_{24}^{(4)}] \\
&= -E_{23}^{(4)} + E_{14}^{(4)} + E_{41}^{(4)} - E_{32}^{(4)} = 0
\end{aligned}$$

$$\begin{aligned}[J, I'] &= [-E^{(4)}_{13} + E^{(4)}_{24}, -E^{(4)}_{12} + E^{(4)}_{34}] \\ &= -E^{(4)}_{32} - E^{(4)}_{14} - E^{(4)}_{41} - E^{(4)}_{23} = 0\end{aligned}$$

and from these, we can use the Jacobi identity to show that $[B, B'] = 0$ for all $B \in \{I, J, K\}$. To make computations easier, we match B_1, B_2, B_3 with I, J, K respectively, and likewise for the primed elements. For the basis elements of $\mathfrak{so}(3)$, we will denote $E^{(3)}_{12}, E^{(3)}_{13}, E^{(3)}_{23}$ as A_1, A_2 and A_3 respectively. We define a map $\varphi : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathfrak{so}(4)$ by

$$\varphi((\sum_{i=1}^{3} a_i A_i, x_i A_i)) = 2\sum_{i=1}^{3} a_i B_i - 2\sum_{i=1}^{3} x_i B'_i$$

for some $a, x \in \mathbb{R}^3$. By Exercise 2, this map is one-to-one and onto. Hence, we only need to show that this is a homomorphism:

$$\begin{aligned}\varphi([(a, x), (b, y)]) &= \varphi(([a, b], [x, y])) \\ &= \varphi((\sum_{i,j} a_i b_i [A_i, A_j], \sum_{i,j} x_i y_i [A'_i, A'_j])) \\ &= \varphi((\sum_{i,j} a_i b_j A_k \epsilon_{ijk}, \sum_{i,j} x_i y_j A'_k \epsilon_{ijk})) \\ &= 2\sum_{i,j} a_i b_j B_k \epsilon_{ijk} - 2\sum_{i,j} x_i y_j B'_k \epsilon_{ijk} \\ &= \sum_{i,j} a_i b_j [B_i, B_j] + \sum_{i,j} x_i y_j [B'_i, B'_j] \\ &= \sum_{i,j} [a_i B_i + x_i B'_i, b_j B_j + y_j B'_j] \\ &= [\sum_i a_i B_i + x_i B'_i, \sum_j b_j B_j + y_j B'_j] \\ &= [\varphi((a, x)), \varphi((b, y))]\end{aligned}$$

Notice that the penultimate step above holds because of the bilinearity of the lie bracket, and because $[B_i, B'_j] = 0$ for all i, j. This shows that the map is a bijective homomorphism, and hence $\mathfrak{so}(4) \cong \mathfrak{so}(3) \times \mathfrak{so}(3)$.

Exercise 5 (8.1.3)**.** Give an example of an infinite intersection of open sets that is not open.

Answer: We can consider the intercept of open ϵ-balls around some point P:

$$\bigcap_{\epsilon \neq 0} N_\epsilon(P) = \{P\}$$

Clearly, each $N_\epsilon(P)$ is open, but the intercept of all of them is the set which only contains P and which is closed since $\mathbb{R}^k \backslash P$ is open.

Exercise 6 (8.2.1)**.** Prove that $U(n)$, $SU(n)$, and $Sp(n)$ are closed subsets of the appropriate matrix spaces.

Answer:

U(n): We can consider the continuos function $\phi(A) = A\overline{A}^T$ which maps $GL(n, \mathbb{C})$ onto itself. Note that ϕ is continuous since it is componentwise continuous because addition, multiplication, and conjugation are all continuous. Clearly, $\phi(\mathbf{1})^{-1} = U(n)$, and $\{\mathbf{1}\}$ is closed, so its preimage $U(n)$ must also be closed.

SU(n): We can look at the map $det : GL(n, \mathbb{C}) \to \mathbb{C}^\times$. Since $\{1\}$ is closed (in $\mathbb{C}^\times$), then its preimage $\{A \in GL(n, \mathbb{C}) \mid det(A) = 1\}$ must also be closed since det is a continuous function. But $SU(n) = U(n) \cap det(1)^{-1}$, and we showed above that $U(n)$ is closed so $SU(n)$ must be closed since it is the intersection of two closed sets.

Sp(n): We do the same thing as for $U(n)$, but instead we consider $GL(n, \mathbb{H})$. We define a map $\phi : GL(n, \mathbb{H}) \to GL(n, \mathbb{H})$ given by $\phi(A) = A\overline{A}^T$ where the conjugation is in $\mathbb{H}$. Again, this map is continuous since it is elementwise continuous because (quaternion) conjugation, addition, and multiplication are continuous. Clearly, $\phi(\mathbf{1})^{-1} = Sp(n)$, and $\{\mathbf{1}\}$ is closed in $GL(n, \mathbb{H})$. Hence, its preimage $Sp(n)$ must also be closed in $GL(n, \mathbb{H})$.