Chear 5.6

$$\cancel{V(t,y) = Wt} \quad V(y,t) = W(F,t)$$

$$\cancel{V_t = W} \quad V_t + V_y \frac{dy}{dt} = W_t \quad \cancel{(1)}$$

$$\frac{dy}{dt} = \cancel{\frac{y}{Fa}} \frac{1}{2} a$$

$$\frac{dy}{dF} = \frac{1}{Fa}$$

$$(1) \quad V_t = W_t - V_y \frac{1}{2} a.$$

$$V_y \frac{dy}{dF} = W_F$$

$$(2) \quad V_y = aF W_F \quad \Rightarrow V_{yy} \frac{1}{Fa} = aF W_{FF} + a W_F$$

$$(1)' \quad V_t = W_t - \frac{a^2}{2} F W_F$$

$$\cancel{(3) \; V_{yy} = a^2 F^2 V_{FF}} \quad (3) \; V_{yy} = a^2 F^2 W_{FF} + a^2 F W_F$$

$$V_t + \frac{1}{2} V_{yy} = W_t - \frac{a^2}{2} F W_F + \frac{1}{2} a^2 F^2 W_{FF} + \frac{1}{2} a^2 F V_F$$

$$= W_t + \frac{1}{2} a^2 F^2 W_{FF}$$

$$e^{-\frac{(y-x)^2}{2t}} \; e^{-\frac{(z-y)^2}{2s}}$$

$$= e^{-\frac{1}{2\sigma^2}(y-\mu)^2 + C}$$

$$\frac{(y-x)^2}{t} + \frac{(z-y)^2}{s} = \frac{(y-\mu)^2}{\sigma^2} + C$$

$$\left(\frac{1}{t}+\frac{1}{s}\right)y^2 - 2\left[\frac{x}{t}+\frac{z}{s}\right]y + \frac{x^2}{t} + \frac{z^2}{s} = \frac{y^2}{\sigma^2} - \frac{2\mu}{\sigma^2}y + \frac{\mu^2}{\sigma^2} + C$$

i) $\sigma^2 = \frac{ts}{s+t}$ ✓

ii) $\cancel{\mu = x+z}$ $\qquad \frac{x}{t}+\frac{z}{s} = \frac{\mu}{\sigma^2}$

iii) $\cancel{\frac{x^2}{t}+\frac{z^2}{s}}$ $\qquad \mu = \frac{ts}{s+t}\left[\frac{x}{t}+\frac{z}{s}\right]$

ii) $\mu = \frac{1}{s+t}\left[sx+tz\right]$ ✓ $= \left[\frac{s}{s+t}x + \frac{t}{s+t}z\right]$

iii) $C = \frac{x^2}{t}+\frac{z^2}{s} - \frac{\frac{1}{(s+t)^2}\left[sx+tz\right]^2}{\frac{ts}{s+t}}$

$C = \frac{x^2}{t}+\frac{z^2}{s} - \frac{1}{(s+t)}\frac{1}{st}(sx+tz)^2$ ✓

$$= \frac{x^2}{t} - \frac{s x^2}{t(t+s)} + \frac{z^2}{s} - \frac{t z^2}{s(t+s)} - \frac{2xz}{(s+t)}$$

$$= \frac{x^2}{t+s} + \frac{z^2}{t+s} - \frac{2xz}{t+s}$$

$$\Rightarrow G(y-x, t)\sqrt{2\pi t} \quad G(z-y, s)\sqrt{2\pi s}$$

$$= G\left(y - \left[\frac{s}{s+t}x + \frac{t}{s+t}z\right], \frac{ts}{s+t}\right)\sqrt{2\pi \frac{ts}{s+t}}$$

$$\cdot \underbrace{e^{\left[-\frac{x^2}{t+s} - \frac{z^2}{t+s} + \frac{2xz}{t+s}\right]\frac{1}{2}}}$$

$$= e^{-\frac{1}{2}\frac{(x-z)^2}{t+s}} = G(z-x, t+s)\sqrt{2\pi(t+s)}$$

$$\therefore$$

$$G(y-x, t) \cdot G(z-y, s) = G\left(y - \left[\frac{s}{s+t}x + \frac{t}{s+t}z\right], \frac{ts}{s+t}\right) G(z-x, t+s)$$

$$\Rightarrow \int_{-\infty}^{\infty} y\, G(y,s)\, G(y-x,t)\,dy = \frac{s}{s+t} \times G(x, t+s)$$

$$\Rightarrow \int_{-\infty}^{\infty} y\, G(y,s) \frac{d}{dx} N\left(\frac{x-y}{\sqrt{t}}\right) dy = \cancel{\frac{s}{s+t} \times G(x, t+s)}$$

$$= \frac{s}{s+t} \times \frac{d}{dx} N\left(\frac{x}{t+s}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} y\, G(y,s) \frac{d^k}{dx^k} N\left(\frac{x-y}{\sqrt{t}}\right) dy = \frac{s}{s+t} \frac{d^{k-1}}{dx^{k-1}} \left(x \frac{d}{dx} N\left(\frac{x}{t+s}\right)\right)$$

$$\parallel$$

$$\int_{-\infty}^{\infty} y\, G(y,s)\, N^{(k)}\left(\frac{x-y}{\sqrt{t}}\right) \frac{1}{t^{k/2}}\, dy$$

$$\omega = x-y$$
$$y = x-\omega$$

$$\parallel$$

$$\int_{-\infty}^{\infty} (x-\omega)\, G(x-\omega,s)\, N^{(k)}\left(\frac{\omega}{\sqrt{t}}\right) \frac{1}{t^{k/2}}\, d\omega$$

$$\parallel$$

$$\int_{-\infty}^{\infty} (x-\omega)\, G(x-\omega,s) \frac{d^k}{dx^k} N\left(\frac{\omega}{\sqrt{t}}\right) d\omega$$

$$\parallel$$

$$x \int_{-\infty}^{\infty} G(x-\omega,s) \frac{d^k}{dx^k} N\left(\frac{\omega}{\sqrt{t}}\right) d\omega - \int_{-\infty}^{\infty} \omega \frac{d^k}{dx^k} N\left(\frac{\omega}{\sqrt{t}}\right) G(\omega-x,s)\, d\omega$$

$$\therefore \int_{-\infty}^{\infty} \omega \frac{d^k}{dx^k} N\left(\frac{\omega}{\sqrt{t}}\right) G(\omega - x, s)\, d\omega$$

$$= x \frac{d^k}{dx^k} N\left(\frac{x}{\sqrt{s+t}}\right) - \frac{s}{s+t} \frac{d^{k-1}}{dx^k}\left(x \frac{d}{dx} N\left(\frac{x}{t+s}\right)\right)$$