Stochastic Volatility

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Abstract

1 Hedging in continuous time

A *hedging strategy* for an instrument with value V is a portfolio Π composed as follows.

- A short position on the instrument V.
- x_i shares of the instrument S_i.
- y shares of a risk free bond with value A.

The portfolio value satisfies

$$\Pi = \sum_i x_i S_i + yA - V \,, \qquad t \geq 0, \tag{1.1}$$

and the initial condition

$$\Pi = 0, \qquad t = 0\,.$$

The objective is to find x_i and y such that the portfolio is *self-financing*; that is,

$$\sum_i x_i dS_i + ydA - dV = \sum_i dx_i S_i + dyA\,, \qquad t \geq 0 \tag{1.2}$$

and the portfolio value does not change; that is,

$$d\Pi = 0, \qquad t \geq 0\,. \tag{1.3}$$

We need models for dS_i and dV in order to decide if the previous equations hold.

In the Black Scholes model we assume that

$$dS = \mu S dt + \sigma S dW\,, \tag{1.4}$$

and use the *no arbitrage* hypothesis to derive the value of V and dV. Namely, V must be a function of S and t; that is, $V = V(S,t)$. A portfolio R with x shares of S and y shares of A is *replicating* if and only if $\Pi = R - V$ is a hedging strategy. It follows that equations (1.2) and (1.3) hold. We calculate, using Ito's formula,

$$dV = \left(V_t + \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS}\right)dt + \left(\sigma S V_S\right)dW\,, \tag{1.5}$$

and

$$d\Pi = xdS + ydA - dV + dxS + dyA\,. \tag{1.6}$$

Now substitute (1.2) and (1.5) in (1.6).

$$\begin{aligned} d\Pi = 2&\Big(x\mu S - \mu S V_S - V_t - \frac{1}{2}\sigma^2 S^2 V_{SS} + r(V - xS)\Big)dt \\ &+ 2\Big(x\sigma S - \sigma S V_S\Big)dW\,. \end{aligned} \tag{1.7}$$

In order to make both terms in equation (1.7) equal to zero we must pick $x = V_S$ and V will be the solution of the Black-Scholes equation

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0\,. \tag{1.8}$$

In the case of a European call option with strike price K we obtain the Black-Scholes formula,

$$V = SN(d_1) - e^{-r(T-t)}KN(d_2)\,,$$

with

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + \big(r + \frac{1}{2}\sigma^2\big)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}\,. \end{aligned}$$

These equations can be rewritten in terms of the *forward price* $F(t,T)$ as

$$\begin{aligned} V &= e^{-r(T-t)}(e^{r(T-t)}SN(d_1) - KN(d_2)) \\ &= e^{-r(T-t)}(F(t,T)N(d_1) - KN(d_2))\,, \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} d_1 &= \frac{\ln(e^{r(T-t)}S/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_1 &= \frac{\ln(F(t,T)/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}\,. \end{aligned} \tag{1.10}$$

The formulae in terms of the forward price is known as *Black's formula*.

2 The SABR stochastic volatility model ($\beta = 1$)

A stochastic volatility models has an extra source of randomness that impacts the stock price volatility. The SABR model for the stock price is defined by equation (1.4) and the additional equation for the annualized volatility

$$d\sigma = \gamma dt + \nu\sigma dZ\,, \tag{2.1}$$

where Z is a Brownian motion such that $dWdZ = \rho dt$. This approach has two crucial properties:

1. The option price V is now a function of S, σ and t.

2. We need more than one risky asset in the hedging strategy. The usual choice is a *liquid* option on the same underlying with value $U = U(S, \sigma, t)$ and the equation for the hedging portfolio, after we add z units of U, is

$$\Pi = xS + zU + yA - V \ .$$

We can write down an equation for the dynamics of Π using Ito's lemma:

$$dV = \left(V_t + \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\nu^2\sigma^2 V_{\sigma\sigma} + \rho\sigma^2\nu S V_{S\sigma}\right)dt + \sigma S V_S dW + (\gamma + \nu\sigma)V_\sigma dZ \ , \tag{2.2}$$

$$dU = \left(U_t + \mu S U_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \frac{1}{2}\nu^2\sigma^2 U_{\sigma\sigma} + \rho\sigma^2\nu S U_{S\sigma}\right)dt + \sigma S U_S dW + (\gamma + \nu\sigma)U_\sigma dZ \ , \tag{2.3}$$

and

$$d\Pi = xdS + zdU + ydA - dV + dxS + dzU + dyA \ . \tag{2.4}$$

Now substitute (1.2), (2.2), and (2.3) in (2.4).

$$\begin{aligned} d\Pi = 2\Bigg(& x\mu S + z\left[U_t + \mu S U_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \frac{1}{2}\nu^2\sigma^2 U_{\sigma\sigma} + \rho\sigma^2\nu S U_{S\sigma}\right] \\ &- \left[V_t + \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\nu^2\sigma^2 V_{\sigma\sigma} + \rho\sigma^2\nu S V_{S\sigma}\right] \\ &+ r(V - xS - zU)\Bigg)dt \\ &+ 2\Big(x\sigma S + z\sigma S U_S - \sigma S V_S\Big)dW + 2(\gamma + \nu\sigma)\Big(zU_\sigma - V_\sigma\Big)dZ \ . \end{aligned} \tag{2.5}$$

The only way the third term in equation (2.5) vanishes is with the choice

$$z = \frac{V_\sigma}{U_\sigma} \ ; \tag{2.6}$$

thus, we can not hedge the option V with only the underlying S. In order to make the last two terms in equation (1.7) equal to zero we must choose

$$x = V_S - zU_S \ .$$

With this choice of x and z we obtain

$$\begin{aligned} d\Pi = 2\Bigg(& \frac{V_\sigma}{U_\sigma}\left[U_t + rSU_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \frac{1}{2}\nu^2\sigma^2 U_{\sigma\sigma} + \rho\sigma^2\nu S U_{S\sigma} - rU\right] \\ &- \left[V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\nu^2\sigma^2 V_{\sigma\sigma} + \rho\sigma^2\nu S V_{S\sigma} - rV\right]\Bigg)dt \ . \end{aligned}$$

The RHS of the last equation vanishes iff there is a function $\phi(S, \sigma, t)$ such that

$$\begin{aligned} U_t + rSU_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \frac{1}{2}\nu^2\sigma^2 U_{\sigma\sigma} + \rho\sigma^2\nu S U_{S\sigma} - rU &= \phi U_\sigma \\ V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\nu^2\sigma^2 V_{\sigma\sigma} + \rho\sigma^2\nu S V_{S\sigma} - rV &= \phi V_\sigma \ . \end{aligned}$$

The function ϕ is written in terms of the *market price of volatility risk* $\lambda(S, \sigma, t)$ as

$$\phi = -(\gamma - \lambda\nu\sigma) \ .$$

and the final PDE for V (and U) becomes

$$V_t + rSV_S + (\gamma - \lambda\nu\sigma)V_\sigma + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\nu^2\sigma^2 V_{\sigma\sigma} + \rho\sigma^2\nu S V_{S\sigma} - rV = 0$$

This is the same PDE satisfied by

$$V(S,\sigma,t) = E_{S,\sigma,t}[e^{-r(T-t)} f(S_T,\sigma_T,T)] ,$$

assuming the *risk neutral dynamics* for the variables S and σ:

$$\begin{aligned} dS &= rSdt + \sigma S dW^* \\ d\sigma &= (\gamma - \lambda\nu\sigma)dt + \nu\sigma dZ^* . \end{aligned} \tag{2.7}$$

Each choice of the function $\lambda(S,\sigma,t)$ gives a rule to calculate the values of all options. Moreover, assuming all prices are calculated with the same λ, it is possible to hedge an option using the underlying and another option on the same underlying.

We will assume

- $\gamma = \overline{\gamma}\sigma$.
- λ is a constant.

Under this model, a *forward contract* has present value equal to

$$e^{-r(T-t)} E_*[S_T - F(t,T)|\mathcal{F}_t] = e^{-r(T-t)}[e^{r(T-t)}S_t - F(t,T)] ,$$

and therefore $F(t,T) = e^{r(T-t)}S_t$. In the same way, an imaginary forward contract with payoff $\sigma_T - G(t,T)$ has present value

$$e^{-r(T-t)} E_*[\sigma_T - G(t,T)|\mathcal{F}_t] = e^{-r(T-t)}[E_*[\sigma_T|\mathcal{F}_t] - G(t,T)] ,$$

and therefore $G(t,T) = e^{(\overline{\gamma}-\lambda\nu)(T-t)}\sigma_t$.

Finally, we can write down equations in terms in terms of the *forward variables*

$$\begin{aligned} \widehat{F}_t &= e^{r(T-t)}S_t \\ \widehat{\alpha}_t &= e^{(\overline{\gamma}-\lambda\nu)(T-t)}\sigma_t . \end{aligned}$$

The result is

$$\begin{aligned} d\widehat{F} &= e^{-(\overline{\gamma}-\lambda\nu)(T-t)}\widehat{\alpha}\widehat{F}dW^* \\ d\widehat{\alpha} &= \nu\widehat{\alpha}dZ^* . \end{aligned} \tag{2.8}$$

Hagan considers the case $\overline{\gamma} - \lambda\nu = 0$ and the PDE in the forward variables is

$$V_t + \frac{1}{2}a^2F^2V_{FF} + \frac{1}{2}\nu^2a^2V_{aa} + \rho a^2\nu F V_{Fa} - rV = 0 \tag{2.9}$$

Notice that at maturity the forward variables match the the original variables; that is, $F_T = S_T$ and $\widehat{\alpha}_T = \sigma_T$. Therefore the call value satisfies

$$C_T = (S_T - K)_+ = (\widehat{F}_T - K)_+$$

If the underlying is fairly liquid we will have quotes for $\widehat{F}_t$ using the current *futures price* of the underlying. The *forward volatility* $\widehat{\alpha}$ is harder to quote and the standard practice is to add its current value as the parameter α of the model.

3 Hagan's SABR approximation

European option market prices are quoted with the implied volatility Σ. The relationship between V and Σ is summarized in the equations

$$V = e^{-r(T-t)}(FN(d_1) - KN(d_2)) \,, \tag{3.1}$$

and

$$\begin{aligned} d_1 &= \frac{\ln(F/K) + \frac{1}{2}\Sigma^2(T-t)}{\Sigma\sqrt{T-t}} \\ d_2 &= d_1 - \Sigma\sqrt{T-t} \,. \end{aligned} \tag{3.2}$$

Since $V = V(F,a,t)$ we have $\Sigma = \Sigma(F,a,t)$. Moreover, we can get a PDE for Σ using the PDE (2.9). The result is

$$\begin{aligned} \Sigma_t &+ \frac{1}{2}a^2F^2\Sigma_{FF} + \frac{1}{2}\nu^2a^2\Sigma_{aa} + \rho a^2\nu F\Sigma_{Fa} \\ &+ \frac{d_2}{\Sigma}\Big\{\frac{1}{2}a^2F\big[Fd_1(\Sigma_F)^2 - 2\frac{\Sigma_F}{\sqrt{T-t}}\big] \\ &+ \frac{1}{2}v^2a^2d_1(\Sigma_a)^2 \\ &+ \rho a^2\nu\big[Fd_1\Sigma_F\Sigma_a - \frac{\Sigma_a}{\sqrt{T-t}}\big]\Big\} \,. \end{aligned}$$

This is a nonlinear PDE and we will use it soon to find an approximation of Σ, but we need to find the boundary condition at the final time for Σ.

4 Implied volatility close to maturity.

We start with the function $W = E_*[(F-K)_+]$ which satisfies the PDE

$$W_t + \frac{1}{2}a^2F^2W_{FF} + \frac{1}{2}\nu^2a^2W_{aa} + \rho a^2\nu FW_{Fa} = 0 \,, \tag{4.1}$$

with the boundary condition

$$W(F,a,T) = (F-K)_+ \tag{4.2}$$

Taking a time derivative on both sides of (4.1), we can see that W_t also satisfies the PDE (4.1). Now substitute (4.2) in (4.1). The result is the boundary condition $W_t(F,K,T) + \frac{1}{2}a^2K^2\delta_{(F-K)} = 0$. We can *approximate* W_t close to maturity as a multiple of a Gaussian function.

5 Asymptotic expansion over ν

The PDE we use to calculate option prices is

$$\begin{aligned} W_t + \frac{1}{2}a^2F^2W_{FF} + \frac{1}{2}\nu^2a^2W_{aa} + \rho a^2\nu FW_{Fa} = 0 \\ W(F,a,T) = (F-K)_+ \,, \end{aligned} \tag{5.1}$$

where $W = E_*[(\widehat{F}_T - K)_+]$. We will try to approximate the solution with an asymptotic expansion

$$W(F,a,t,\nu) = W^0(F,a,t) + W^1(F,a,t)\nu + W^2(F,a,t)\nu^2 + \ldots \tag{5.2}$$

Substitute (5.2) in (5.1) and collect $O(1)$ terms. The result is

$$\begin{aligned} W_t^0 + \frac{1}{2}a^2F^2W_{FF}^0 &= 0 \\ W^0(F,a,T) &= (F-K)_+ \ , \end{aligned} \tag{5.3}$$

There are no derivatives w.r.t a in the previous equation, therefore the solution is given by Black's formula with a playing the role of the volatility; this is,

$$\begin{aligned} W^0(F,a,t) &= FN(d_1) - KN(d_2) \\ d_1 &= \frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2}a^2(T-t)}{a\sqrt{T-t}} \ . \end{aligned}$$

For convenience we will denote $\tau = T-t$. Now we collect $O(\nu)$ terms in equation (5.1). The result is

$$\begin{aligned} W_t^1 + \frac{1}{2}a^2F^2W_{FF}^1 + \rho a^2 F W_{Fa}^0 &= 0 \\ W^1(F,a,T) &= 0 \ . \end{aligned} \tag{5.4}$$

We calculate W_{Fa}^0 using the financial greeks and, after rearranging terms, the previous equation becomes

$$\begin{aligned} W_t^1 + \frac{1}{2}a^2F^2W_{FF}^1 &= \rho a F d_2 \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} \\ W^1(F,a,T) &= 0 \ . \end{aligned} \tag{5.5}$$

Finally we define $y = d_2\sqrt{\tau} = \frac{\ln\left(\frac{F}{K}\right) - \frac{1}{2}a^2(T-t)}{a}$ and the PDE in the new variables reads

$$\begin{aligned} W_t^1 + \frac{1}{2}W_{yy}^1 &= \rho a K y \frac{e^{-\frac{y^2}{2\tau}}}{\sqrt{2\pi\tau}} \\ &= -\rho a K \tau \frac{d^2}{dy^2} N(y/\sqrt{\tau}) \\ W^1(y,a,T) &= 0 \ . \end{aligned} \tag{5.6}$$

Now we can use the heat kernel to calculate the solution of the previous *non-homogeneous* heat equation. The key relationships we will use in the next calculations are

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{e^{-\frac{(y-x)^2}{2t}}}{\sqrt{2\pi t}} \frac{d^k}{dy^k} N(y/\sqrt{s})dy = \frac{d^k}{dx^k} N(x/\sqrt{s+t}) \ , \qquad k > 1 \\ &\int_{-\infty}^{\infty} \frac{e^{-\frac{(y-x)^2}{2t}}}{\sqrt{2\pi t}} \ y \ \frac{d^k}{dy^k} N(y/\sqrt{s})dy = x\frac{d^k}{dx^k} N(x/\sqrt{s+t}) - \frac{s}{s+t}\frac{d^{k-1}}{dx^{k-1}}\left[x\frac{d}{dx}N\left(\frac{x}{\sqrt{s+t}}\right)\right] \ , \qquad k > 1 \end{aligned} \tag{5.7}$$

Duhamel's principle gives

$$\begin{aligned} W^1(y,a,t_0) &= -\int_{t_0}^{T}\int_{-\infty}^{\infty} \rho a K \tau \frac{d^2}{dy^2} N(y/\sqrt{\tau}) \frac{e^{-\frac{(x-y)^2}{2(t-t_0)}}}{\sqrt{2\pi(t-t_0)}} dx dt \\ &= -\int_{t_0}^{T} \rho a K \tau \frac{d^2}{dy^2} N(y/\sqrt{T-t_0}) dt \\ &= -\rho a K \frac{d^2}{dy^2} N(y/\sqrt{T-t_0}) \int_{t_0}^{T} \tau dt \\ &= -\rho a K \frac{d^2}{dy^2} N(y/\sqrt{T-t_0}) \frac{1}{2}(T-t_0)^2 \ . \end{aligned}$$

In other words,

$$W^1(y,a,t) = -\frac{1}{2}\rho a K \tau^2 \frac{d^2}{dy^2} N(y/\sqrt{\tau})\,.$$

Going back to the original variables, we have found the first correction; this is,

$$W = FN(d_1) - KN(d_2) - \nu \frac{\rho a K \tau^{3/2}}{2} d_2 \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} + O(\nu^2)\,.$$

We continue the approximation by collecting $O(\nu^2)$ terms in equation (5.1). The result is

$$W_t^2 + \frac{1}{2}a^2F^2W_{FF}^2 + \frac{1}{2}a^2W_{aa}^0 + \rho a^2 F W_{Fa}^1 = 0\,.$$

Using the explicit formulae for W^0 and W^1 and the finance greeks we calculate

$$\begin{aligned}
W_a^0 &= K\tau \frac{d}{dy} N(y/\sqrt{\tau}) \\
W_{aa}^0 &= K\tau\left(\tau - \frac{y}{a}\right)\frac{d^2}{dy^2}N(y/\sqrt{\tau}) \\
W_F^1 &= -\frac{1}{2}\rho\frac{K}{F}\tau^2\frac{d^3}{dy^3}N(y/\sqrt{\tau}) \\
W_{Fa}^1 &= -\frac{1}{2}\rho\frac{K}{F}\tau^2\left(\tau - \frac{y}{a}\right)\frac{d^4}{dy^4}N(y/\sqrt{\tau})
\end{aligned}$$

In terms of the variable y we obtain

$$W_t^2 + \frac{1}{2}W_{yy}^2 = \frac{1}{2}aK\tau(a\tau - y)\frac{d^2}{dy^2}N(y/\sqrt{\tau}) - \frac{1}{2}\rho^2 aK\tau^2(a\tau - y)\frac{d^4}{dy^4}N(y/\sqrt{\tau})\,.$$

Duhamel's principle and equation (5.7) give

$$\begin{aligned}
W^2(y,a,t_0) &= \int_{t_0}^{T}\left[\frac{1}{2}aK\tau(a\tau - x)\frac{d^2}{dx^2}N(x/\sqrt{T-t_0}) - \frac{1}{2}\rho^2 aK\tau^2(a\tau - x)\frac{d^4}{dx^4}N(x/\sqrt{T-t_0})\right. \\
&+ \frac{1}{2}aK\frac{\tau^2}{T-t_0}\frac{d}{dx}\left(x\frac{d}{dx}N\left(\frac{x}{\sqrt{T-t_0}}\right)\right) \\
&\left.- \frac{1}{2}\rho^2 aK\frac{\tau^3}{T-t_0}\frac{d^3}{dx^3}\left(x\frac{d}{dx}N\left(\frac{x}{\sqrt{T-t_0}}\right)\right)\right] dt \\
&= \int_0^{T-t_0}\left[\frac{1}{2}aK\tau(a\tau - x)\frac{d^2}{dx^2}N(x/\sqrt{T-t_0}) - \frac{1}{2}\rho^2 aK\tau^2(a\tau - x)\frac{d^4}{dx^4}N(x/\sqrt{T-t_0})\right. \\
&+ \frac{1}{2}aK\frac{\tau^2}{T-t_0}\frac{d}{dx}\left(x\frac{d}{dx}N\left(\frac{x}{\sqrt{T-t_0}}\right)\right) \\
&\left.- \frac{1}{2}\rho^2 aK\frac{\tau^3}{T-t_0}\frac{d^3}{dx^3}\left(x\frac{d}{dx}N\left(\frac{x}{\sqrt{T-t_0}}\right)\right)\right] d\tau \\
&= \left[\frac{1}{2}aK\tau^2(a\tau/3 - x/2)\frac{d^2}{dx^2}N(x/\sqrt{T-t_0}) - \frac{1}{2}\rho^2 aK\tau^3(a\tau/4 - x/3)\frac{d^4}{dx^4}N(x/\sqrt{T-t_0})\right. \\
&+ \frac{1}{6}aK\frac{\tau^3}{T-t_0}\frac{d}{dx}\left(x\frac{d}{dx}N\left(\frac{x}{\sqrt{T-t_0}}\right)\right) \\
&\left.- \frac{1}{8}\rho^2 aK\frac{\tau^4}{T-t_0}\frac{d^3}{dx^3}\left(x\frac{d}{dx}N\left(\frac{x}{\sqrt{T-t_0}}\right)\right)\right]_{\tau=0}^{\tau=T-t_0}\,.
\end{aligned}$$

In other words

$$
\begin{aligned}
W^2(y,a,t) = \Bigg[& \frac{1}{2} aK\tau^2 (a\tau/3 - x/2) \frac{d^2}{dx^2} N(x/\sqrt{\tau}) - \frac{1}{2}\rho^2 aK\tau^3 (a\tau/4 - x/3) \frac{d^4}{dx^4} N(x/\sqrt{\tau}) \\
& + \frac{1}{6} aK\tau^2 \frac{d}{dx}\left(x \frac{d}{dx} N\left(\frac{x}{\sqrt{\tau}}\right)\right) \\
& - \frac{1}{8}\rho^2 aK\tau^3 \frac{d^3}{dx^3}\left(x \frac{d}{dx} N\left(\frac{x}{\sqrt{\tau}}\right)\right) \Bigg] .
\end{aligned}
$$