Stochastic Volatility

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Abstract

1 Hedging in continuous time

A *hedging strategy* for an instrument with value V is a portfolio Π composed as follows.

- A short position on the instrument V.
- x_i shares of the instrument S_i.
- y shares of a risk free bond with value A.

The portfolio value satisfies

$$\Pi = \sum_i x_i S_i + yA - V \;, \qquad t \geq 0, \tag{1.1}$$

and the initial condition

$$\Pi = 0, \qquad t = 0\;.$$

The objective is to find x_i and y such that the portfolio is *self-financing*; that is,

$$\sum_i x_i dS_i + ydA - dV = \sum_i dx_i S_i + dyA\;, \qquad t \geq 0 \tag{1.2}$$

and the portfolio value does not change; that is,

$$d\Pi = 0, \qquad t \geq 0\;. \tag{1.3}$$

We need models for dS_i and dV in order to decide if the previous equations hold.

In the Black Scholes model we assume that

$$dS = \mu S dt + \sigma S dW\;, \tag{1.4}$$

and use the *no arbitrage* hypothesis to derive the value of V and dV. Namely, V must be a function of S and t; that is, $V = V(S,t)$. A portfolio R with x shares of S and y shares of A is *replicating* if and only if $\Pi = R - V$ is a hedging strategy. It follows that equations (1.2) and (1.3) hold. We calculate, using Ito's formula,

$$dV = \left(V_t + \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} \right) dt + \left(\sigma S V_S \right) dW\;, \tag{1.5}$$

and

$$d\Pi = xdS + ydA - dV + dxS + dyA\,. \tag{1.6}$$

Now substitute (1.2) and (1.5) in (1.6).

$$\begin{aligned} d\Pi = 2\Big(&x\mu S - \mu S V_S - V_t - \frac{1}{2}\sigma^2 S^2 V_{SS} + r(V - xS)\Big)dt \\ &+ 2\Big(x\sigma S - \sigma S V_S\Big)dW\,. \end{aligned} \tag{1.7}$$

In order to make both terms in equation (1.7) equal to zero we must pick $x = V_S$ and V will be the solution of the Black-Scholes equation

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0\,. \tag{1.8}$$

In the case of a European call option with strike price K we obtain the Black-Scholes formula,

$$V = SN(d_1) - e^{-r(T-t)}KN(d_2)\,,$$

with

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + \big(r + \frac{1}{2}\sigma^2\big)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}\,. \end{aligned}$$

These equations can be rewritten in terms of the *forward price* $F(t,T)$ as

$$\begin{aligned} V &= e^{-r(T-t)}(e^{r(T-t)}SN(d_1) - KN(d_2)) \\ &= e^{-r(T-t)}(F(t,T)N(d_1) - KN(d_2))\,, \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} d_1 &= \frac{\ln(e^{r(T-t)}S/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_1 &= \frac{\ln(F(t,T)/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}\,. \end{aligned} \tag{1.10}$$

The formulae in terms of the forward price is known as *Black's formula*.

2 The SABR stochastic volatility model ($\beta = 1$)

A stochastic volatility models has an extra source of randomness that impacts the stock price volatility. The SABR model for the stock price is defined by equation (1.4) and the additional equation for the annualized volatility

$$d\sigma = \nu\sigma dZ\,, \tag{2.1}$$

where Z is a Brownian motion such that $dWdZ = \rho dt$. This approach has two crucial properties:

1. The option price V is now a function of S, σ and t.

2. We need more than one risky asset in the hedging strategy. The usual choice is a *liquid* option on the same underlying with value $U = U(S, \sigma, t)$ and the equation for the hedging portfolio, after we add z units of U, is

$$\Pi = xS + zU + yA - V\ .$$

We can write down an equation for the dynamics of Π using Ito's lemma:

$$dV = \left(V_t + \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\nu^2\sigma^2 V_{\sigma\sigma} + \rho\sigma^2\nu S V_{S\sigma}\right)dt + \sigma S V_S dW + \nu\sigma V_\sigma dZ\ , \tag{2.2}$$

$$dU = \left(U_t + \mu S U_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \frac{1}{2}\nu^2\sigma^2 U_{\sigma\sigma} + \rho\sigma^2\nu S U_{S\sigma}\right)dt + \sigma S U_S dW + \nu\sigma U_\sigma dZ\ , \tag{2.3}$$

and

$$d\Pi = xdS + zdU + ydA - dV + dxS + dzU + dyA\ . \tag{2.4}$$

Now substitute (1.2), (2.2), and (2.3) in (2.4).

$$\begin{aligned} d\Pi = 2\Big(& x\mu S + z\left[U_t + \mu S U_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \frac{1}{2}\nu^2\sigma^2 U_{\sigma\sigma} + \rho\sigma^2\nu S U_{S\sigma}\right] \\ & - \left[V_t + \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\nu^2\sigma^2 V_{\sigma\sigma} + \rho\sigma^2\nu S V_{S\sigma}\right] \\ & + r(V - xS - zU)\Big)dt \\ & + 2\Big(x\sigma S + z\sigma S U_S - \sigma S V_S\Big)dW + 2\Big(z\nu\sigma U_\sigma - \nu\sigma V_\sigma\Big)dZ\ . \end{aligned} \tag{2.5}$$

The only way the third term in equation (2.5) vanishes is with the choice

$$z = \frac{V_\sigma}{U_\sigma}\ ; \tag{2.6}$$

thus, we can not hedge the option V with only the underlying S. In order to make the last two terms in equation (1.7) equal to zero we must choose

$$x = V_S - zU_S\ .$$

With this choice of x and z we obtain

$$\begin{aligned} d\Pi = 2\Bigg(& \frac{V_\sigma}{U_\sigma}\left[U_t + rSU_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \frac{1}{2}\nu^2\sigma^2 U_{\sigma\sigma} + \rho\sigma^2\nu S U_{S\sigma} - rU\right] \\ & - \left[V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\nu^2\sigma^2 V_{\sigma\sigma} + \rho\sigma^2\nu S V_{S\sigma} - rV\right]\Bigg)dt\ . \end{aligned}$$

The RHS of the last equation vanishes iff there is a function $\phi(S, \sigma, t)$ such that

$$\begin{aligned} U_t + rSU_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \frac{1}{2}\nu^2\sigma^2 U_{\sigma\sigma} + \rho\sigma^2\nu S U_{S\sigma} - rU &= \phi U_\sigma \\ V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\nu^2\sigma^2 V_{\sigma\sigma} + \rho\sigma^2\nu S V_{S\sigma} - rV &= \phi V_\sigma\ . \end{aligned}$$

The function ϕ is written in terms of the *market price of risk* λ as

$$\phi = \lambda\nu\sigma\ .$$

and the final PDE for V (and U) becomes

$$V_t + rSV_S - \lambda\nu\sigma V_\sigma + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\nu^2\sigma^2 V_{\sigma\sigma} + \rho\sigma^2\nu S V_{S\sigma} - rV = 0$$

This is the same PDE satisfied by

$$V(S,\sigma,t) = E_{S,\sigma,t}[e^{-r(T-t)} f(S_T, \sigma_T, T)] ,$$

assuming the *risk neutral dynamics* for the variables S and σ:

$$\begin{aligned} dS &= rSdt + \sigma S dW^* \\ d\sigma &= -\lambda\nu\sigma dt + \nu\sigma dZ^* . \end{aligned} \tag{2.7}$$

Finally, we can write down equations in terms in terms of the *forward variables*

$$\begin{aligned} \widehat{F} &= e^{r(T-t)} S \\ \widehat{\sigma} &= e^{-\lambda\nu(T-t)}\sigma . \end{aligned}$$

The result is

$$\begin{aligned} d\widehat{F} &= \sigma \widehat{F} dW^* \\ d\widehat{\sigma} &= \nu\widehat{\sigma} dZ^* . \end{aligned}$$

and the PDE in the forward variables is

$$V_t + \frac{1}{2}\sigma^2 F^2 V_{FF} + \frac{1}{2}\nu^2\sigma^2 V_{\sigma\sigma} + \rho\sigma^2\nu F V_{F\sigma} - rV = 0$$