Example 1.15 You know that a certain letter is equally likely to be in any one of three different folders. Let α_i be the probability that you will find your letter upon making a quick examination of folder i if the letter is, in fact, in folder i, $i = 1, 2, 3$. (We may have $\alpha_i < 1$.) Suppose you look in folder 1 and do not find the letter. What is the probability that the letter is in folder 1?

Solution: Let F_i, $i = 1, 2, 3$ be the event that the letter is in folder i; and let E be the event that a search of folder 1 does not come up with the letter. We desire $P(F_1|E)$. From Bayes' formula we obtain

$$P(F_1|E) = \frac{P(E|F_1)P(F_1)}{\sum_{i=1}^{3} P(E|F_i)P(F_i)}$$
$$= \frac{(1-\alpha_1)\frac{1}{3}}{(1-\alpha_1)\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \frac{1-\alpha_1}{3-\alpha_1} \quad \blacksquare$$

Exercises

1. A box contains three marbles: one red, one green, and one blue. Consider an experiment that consists of taking one marble from the box then replacing it in the box and drawing a second marble from the box. What is the sample space? If, at all times, each marble in the box is equally likely to be selected, what is the probability of each point in the sample space?

***2.** Repeat Exercise 1 when the second marble is drawn without replacing the first marble.

3. A coin is to be tossed until a head appears twice in a row. What is the sample space for this experiment? If the coin is fair, what is the probability that it will be tossed exactly four times?

4. Let E, F, G be three events. Find expressions for the events that of E, F, G

(a) only F occurs,
(b) both E and F but not G occur,
(c) at least one event occurs,
(d) at least two events occur,
(e) all three events occur,
(f) none occurs,
(g) at most one occurs,
(h) at most two occur.

***5.** An individual uses the following gambling system at Las Vegas. He bets \$1 that the roulette wheel will come up red. If he wins, he quits. If he loses then he makes the same bet a second time only this time he bets \$2; and then regardless of the outcome, quits. Assuming that he has a probability of $\frac{1}{2}$ of winning each bet, what is the probability that he goes home a winner? Why is this system not used by everyone?

6. Show that $E(F \cup G) = EF \cup EG$.

7. Show that $(E \cup F)^c = E^c F^c$.

8. If $P(E) = 0.9$ and $P(F) = 0.8$, show that $P(EF) \geqslant 0.7$. In general, show that

$$P(EF) \geqslant P(E) + P(F) - 1$$

This is known as Bonferroni's inequality.

***9.** We say that $E \subset F$ if every point in E is also in F. Show that if $E \subset F$, then

$$P(F) = P(E) + P(FE^c) \geqslant P(E)$$

10. Show that

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leqslant \sum_{i=1}^{n} P(E_i)$$

This is known as Boole's inequality.

Hint: Either use Equation (1.2) and mathematical induction, or else show that $\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i$, where $F_1 = E_1$, $F_i = E_i \bigcap_{j=1}^{i-1} E_j^c$, and use property (iii) of a probability.

11. If two fair dice are tossed, what is the probability that the sum is i, $i = 2, 3, \ldots, 12$?

12. Let E and F be mutually exclusive events in the sample space of an experiment. Suppose that the experiment is repeated until either event E or event F occurs. What does the sample space of this new super experiment look like? Show that the probability that event E occurs before event F is $P(E)/[P(E) + P(F)]$.

Hint: Argue that the probability that the original experiment is performed n times and E appears on the nth time is $P(E) \times (1-p)^{n-1}$, $n = 1, 2, \ldots$, where $p = P(E) + P(F)$. Add these probabilities to get the desired answer.

13. The dice game craps is played as follows. The player throws two dice, and if the sum is seven or eleven, then she wins. If the sum is two, three, or twelve, then she loses. If the sum is anything else, then she continues throwing until she either throws that number again (in which case she wins) or she throws a seven (in which case she loses). Calculate the probability that the player wins.

14. The probability of winning on a single toss of the dice is p. A starts, and if he fails, he passes the dice to B, who then attempts to win on her toss. They continue tossing the dice back and forth until one of them wins. What are their respective probabilities of winning?

15. Argue that $E = EF \cup EF^c$, $E \cup F = E \cup FE^c$.

16. Use Exercise 15 to show that $P(E \cup F) = P(E) + P(F) - P(EF)$.

***17.** Suppose each of three persons tosses a coin. If the outcome of one of the tosses differs from the other outcomes, then the game ends. If not, then the persons start over and retoss their coins. Assuming fair coins, what is the probability that the game will end with the first round of tosses? If all three coins are biased and have probability $\frac{1}{4}$ of landing heads, what is the probability that the game will end at the first round?

18. Assume that each child who is born is equally likely to be a boy or a girl. If a family has two children, what is the probability that both are girls given that (a) the eldest is a girl, (b) at least one is a girl?

***19.** Two dice are rolled. What is the probability that at least one is a six? If the two faces are different, what is the probability that at least one is a six?

20. Three dice are thrown. What is the probability the same number appears on exactly two of the three dice?

21. Suppose that 5 percent of men and 0.25 percent of women are color-blind. A color-blind person is chosen at random. What is the probability of this person being male? Assume that there are an equal number of males and females.

22. A and B play until one has 2 more points than the other. Assuming that each point is independently won by A with probability p, what is the probability they will play a total of $2n$ points? What is the probability that A will win?

23. For events $E_1, E_2, \ldots, E_n$ show that

$$P(E_1 E_2 \cdots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2) \cdots P(E_n|E_1 \cdots E_{n-1})$$

24. In an election, candidate A receives n votes and candidate B receives m votes, where $n > m$. Assume that in the count of the votes all possible orderings of the $n+m$ votes are equally likely. Let $P_{n,m}$ denote the probability that from the first vote on A is always in the lead. Find

(a) $P_{2,1}$ (b) $P_{3,1}$ (c) $P_{n,1}$ (d) $P_{3,2}$ (e) $P_{4,2}$
(f) $P_{n,2}$ (g) $P_{4,3}$ (h) $P_{5,3}$ (i) $P_{5,4}$
(j) Make a conjecture as to the value of $P_{n,m}$.

***25.** Two cards are randomly selected from a deck of 52 playing cards.

(a) What is the probability they constitute a pair (that is, that they are of the same denomination)?
(b) What is the conditional probability they constitute a pair given that they are of different suits?

26. A deck of 52 playing cards, containing all 4 aces, is randomly divided into 4 piles of 13 cards each. Define events E_1, E_2, E_3, and E_4 as follows:

$$\begin{aligned} E_1 &= \{\text{the first pile has exactly 1 ace}\}, \\ E_2 &= \{\text{the second pile has exactly 1 ace}\}, \\ E_3 &= \{\text{the third pile has exactly 1 ace}\}, \\ E_4 &= \{\text{the fourth pile has exactly 1 ace}\} \end{aligned}$$

Use Exercise 23 to find $P(E_1E_2E_3E_4)$, the probability that each pile has an ace.

***27.** Suppose in Exercise 26 we had defined the events E_i, $i = 1, 2, 3, 4$, by

$$\begin{aligned} E_1 &= \{\text{one of the piles contains the ace of spades}\}, \\ E_2 &= \{\text{the ace of spades and the ace of hearts are in different piles}\}, \\ E_3 &= \{\text{the ace of spades, the ace of hearts, and the} \\ &\qquad \text{ace of diamonds are in different piles}\}, \\ E_4 &= \{\text{all 4 aces are in different piles}\} \end{aligned}$$

Now use Exercise 23 to find $P(E_1E_2E_3E_4)$, the probability that each pile has an ace. Compare your answer with the one you obtained in Exercise 26.

28. If the occurrence of B makes A more likely, does the occurrence of A make B more likely?

29. Suppose that $P(E) = 0.6$. What can you say about $P(E|F)$ when

(a) E and F are mutually exclusive?
(b) $E \subset F$?
(c) $F \subset E$?

***30.** Bill and George go target shooting together. Both shoot at a target at the same time. Suppose Bill hits the target with probability 0.7, whereas George, independently, hits the target with probability 0.4.

(a) Given that exactly one shot hit the target, what is the probability that it was George's shot?
(b) Given that the target is hit, what is the probability that George hit it?

31. What is the conditional probability that the first die is six given that the sum of the dice is seven?

***32.** Suppose all n men at a party throw their hats in the center of the room. Each man then randomly selects a hat. Show that the probability that none of the n men selects his own hat is

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - + \cdots \frac{(-1)^n}{n!}$$

Note that as $n \to \infty$ this converges to e^{-1}. Is this surprising?

33. In a class there are four freshman boys, six freshman girls, and six sophomore boys. How many sophomore girls must be present if sex and class are to be independent when a student is selected at random?

34. Mr. Jones has devised a gambling system for winning at roulette. When he bets, he bets on red, and places a bet only when the ten previous spins of the roulette have landed on a black number. He reasons that his chance of winning is quite large since the probability of eleven consecutive spins resulting in black is quite small. What do you think of this system?

35. A fair coin is continually flipped. What is the probability that the first four flips are

(a) H, H, H, H?
(b) T, H, H, H?
(c) What is the probability that the pattern T, H, H, H occurs before the pattern H, H, H, H?

36. Consider two boxes, one containing one black and one white marble, the other, two black and one white marble. A box is selected at random and a marble is drawn at random from the selected box. What is the probability that the marble is black?

37. In Exercise 36, what is the probability that the first box was the one selected given that the marble is white?

38. Urn 1 contains two white balls and one black ball, while urn 2 contains one white ball and five black balls. One ball is drawn at random from urn 1 and placed in urn 2. A ball is then drawn from urn 2. It happens to be white. What is the probability that the transferred ball was white?

39. Stores A, B, and C have 50, 75, and 100 employees, and, respectively, 50, 60, and 70 percent of these are women. Resignations are equally likely among all employees, regardless of sex. One employee resigns and this is a woman. What is the probability that she works in store C?

***40.** (a) A gambler has in his pocket a fair coin and a two-headed coin. He selects one of the coins at random, and when he flips it, it shows heads. What is the probability that it is the fair coin? (b) Suppose that he flips the same coin a second time and again it shows heads. Now what is the probability that it is the air coin? (c) Suppose that he flips the same coin a third time and it shows tails. Now what is the probability that it is the fair coin?

41. In a certain species of rats, black dominates over brown. Suppose that a black rat with two black parents has a brown sibling.

(a) What is the probability that this rat is a pure black rat (as opposed to being a hybrid with one black and one brown gene)?
(b) Suppose that when the black rat is mated with a brown rat, all five of their offspring are black. Now, what is the probability that the rat is a pure black rat?

42. There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the three coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

43. Suppose we have ten coins which are such that if the ith one is flipped then heads will appear with probability $i/10$, $i = 1, 2, \ldots, 10$. When one of the coins is randomly selected and flipped, it shows heads. What is the conditional probability that it was the fifth coin?

44. Urn 1 has five white and seven black balls. Urn 2 has three white and twelve black balls. We flip a fair coin. If the outcome is heads, then a ball from urn 1 is selected, while if the outcome is tails, then a ball from urn 2 is selected. Suppose that a white ball is selected. What is the probability that the coin landed tails?

*45. An urn contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the urn c additional balls of the same color are put in with it. Now suppose that we draw another ball. Show that the probability that the first ball drawn was black given that the second ball drawn was red is $b/(b+r+c)$.

46. Three prisoners are informed by their jailer that one of them has been chosen at random to be executed, and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information, since he already knows that at least one will go free. The jailer refuses to answer this question, pointing out that if A knew which of his fellows were to be set free, then his own probability of being executed would rise from $\frac{1}{3}$ to $\frac{1}{2}$, since he would then be one of two prisoners. What do you think of the jailer's reasoning?

47. For a fixed event B, show that the collection $P(A|B)$, defined for all events A, satisfies the three conditions for a probability. Conclude from this that

$$P(A|B) = P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B)$$

Then directly verify the preceding equation.

*48. Sixty percent of the families in a certain community own their own car, thirty percent own their own home, and twenty percent own both their own car and their own home. If a family is randomly chosen, what is the probability that this family owns a car or a house but not both?

References

Reference [2] provides a colorful introduction to some of the earliest developments in probability theory. References [3], [4], and [7] are all excellent introductory texts in modern probability theory. Reference [5] is the definitive work which established the axiomatic foundation of modern mathematical probability theory. Reference [6] is a nonmathematical introduction to probability theory and its applications, written by one of the greatest mathematicians of the eighteenth century.

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