nnegative integers; while $\{X(t), t \geqslant 0\}$ is a continuous-time stochastic process dexed by the nonnegative real numbers.

The *state space* of a stochastic process is defined as the set of all possible values at the random variables $X(t)$ can assume.

Thus, a stochastic process is a family of random variables that describes the olution through time of some (physical) process. We shall see much of stochastic ocesses in the following chapters of this text.

**ample 2.53** Consider a particle that moves along a set of $m+1$ nodes, beled $0, 1, \ldots, m$, that are arranged around a circle (see Figure 2.3). At each ep the particle is equally likely to move one position in either the clockwise or unterclockwise direction. That is, if $X_n$ is the position of the particle after its h step then

$$P\{X_{n+1} = i+1 | X_n = i\} = P\{X_{n+1} = i-1 | X_n = i\} = \tfrac{1}{2}$$

here $i+1 \equiv 0$ when $i = m$, and $i - 1 \equiv m$ when $i = 0$. Suppose now that the article starts at 0 and continues to move around according to the preceding rules til all the nodes $1, 2, \ldots, m$ have been visited. What is the probability that node $i = 1, \ldots, m$, is the last one visited?

**Solution:** Surprisingly enough, the probability that node $i$ is the last node visited can be determined without any computations. To do so, consider the first time that the particle is at one of the two neighbors of node $i$, that is, the first time that the particle is at one of the nodes $i-1$ or $i+1$ (with $m+1 \equiv 0$). Suppose it is at node $i-1$ (the argument in the alternative situation is identical). Since neither node $i$ nor $i+1$ has yet been visited, it follows that $i$ will be the last node visited if and only if $i+1$ is visited before $i$. This is so because in

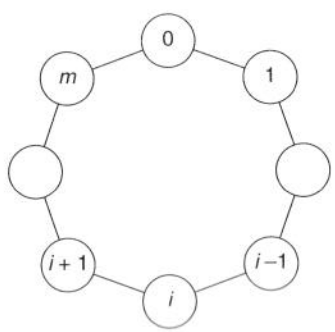


**Figure 2.3.** Particle moving around a circle.



order to visit $i+1$ before $i$ the particle will have to visit all the nodes on the counterclockwise path from $i-1$ to $i+1$ before it visits $i$. But the probability that a particle at node $i-1$ will visit $i+1$ before $i$ is just the probability that a particle will progress $m-1$ steps in a specified direction before progressing one step in the other direction. That is, it is equal to the probability that a gambler who starts with one unit, and wins one when a fair coin turns up heads and loses one when it turns up tails, will have his fortune go up by $m-1$ before he goes broke. Hence, because the preceding implies that the probability that node $i$ is the last node visited is the same for all $i$, and because these probabilities must sum to 1, we obtain

$$P\{i \text{ is the last node visited}\} = 1/m, \qquad i = 1, \ldots, m \quad \blacksquare$$

**Remark** The argument used in Example 2.53 also shows that a gambler who is equally likely to either win or lose one unit on each gamble will be down $n$ before being up 1 with probability $1/(n+1)$; or equivalently

$$P\{\text{gambler is up 1 before being down } n\} = \frac{n}{n+1}$$

Suppose now we want the probability that the gambler is up 2 before being down $n$. Upon conditioning on whether he reaches up 1 before down $n$, we obtain that

$$\begin{aligned} &P\{\text{gambler is up 2 before being down } n\} \\ &\quad = P\{\text{up 2 before down } n | \text{up 1 before down } n\}\frac{n}{n+1} \\ &\quad = P\{\text{up 1 before down } n+1\}\frac{n}{n+1} \\ &\quad = \frac{n+1}{n+2}\frac{n}{n+1} = \frac{n}{n+2} \end{aligned}$$

Repeating this argument yields that

$$P\{\text{gambler is up } k \text{ before being down } n\} = \frac{n}{n+k}$$

## Exercises

**1.** An urn contains five red, three orange, and two blue balls. Two balls are randomly selected. What is the sample space of this experiment? Let $X$ represent the number of orange balls selected. What are the possible values of $X$? Calculate $P\{X = 0\}$.

Let $X$ represent the difference between the number of heads and the number tails obtained when a coin is tossed $n$ times. What are the possible values of $X$?

. In Exercise 2, if the coin is assumed fair, then, for $n = 2$, what are the obabilities associated with the values that $X$ can take on?

4. Suppose a die is rolled twice. What are the possible values that the following ndom variables can take on?

(i) The maximum value to appear in the two rolls.
(ii) The minimum value to appear in the two rolls.
(iii) The sum of the two rolls.
(iv) The value of the first roll minus the value of the second roll.

. If the die in Exercise 4 is assumed fair, calculate the probabilities associated ith the random variables in (i)–(iv).

. Suppose five fair coins are tossed. Let $E$ be the event that all coins land heads. efine the random variable $I_E$

$$I_E = \begin{cases} 1, & \text{if } E \text{ occurs} \\ 0, & \text{if } E^c \text{ occurs} \end{cases}$$

or what outcomes in the original sample space does $I_E$ equal 1? What is $\{I_E = 1\}$?

. Suppose a coin having probability 0.7 of coming up heads is tossed three mes. Let $X$ denote the number of heads that appear in the three tosses. Determine e probability mass function of $X$.

. Suppose the distribution function of $X$ is given by

$$F(b) = \begin{cases} 0, & b < 0 \\ \frac{1}{2}, & 0 \leqslant b < 1 \\ 1, & 1 \leqslant b < \infty \end{cases}$$

Vhat is the probability mass function of $X$?

. If the distribution function of $F$ is given by

$$F(b) = \begin{cases} 0, & b < 0 \\ \frac{1}{2}, & 0 \leqslant b < 1 \\ \frac{3}{5}, & 1 \leqslant b < 2 \\ \frac{4}{5}, & 2 \leqslant b < 3 \\ \frac{9}{10}, & 3 \leqslant b < 3.5 \\ 1, & b \geqslant 3.5 \end{cases}$$

alculate the probability mass function of $X$.



**10.** Suppose three fair dice are rolled. What is the probability at most one six appears?

***11.** A ball is drawn from an urn containing three white and three black balls. After the ball is drawn, it is then replaced and another ball is drawn. This goes on indefinitely. What is the probability that of the first four balls drawn, exactly two are white?

**12.** On a multiple-choice exam with three possible answers for each of the five questions, what is the probability that a student would get four or more correct answers just by guessing?

**13.** An individual claims to have extrasensory perception (ESP). As a test, a fair coin is flipped ten times, and he is asked to predict in advance the outcome. Our individual gets seven out of ten correct. What is the probability he would have done at least this well if he had no ESP? (Explain why the relevant probability is $P\{X \geqslant 7\}$ and not $P\{X = 7\}$.)

**14.** Suppose $X$ has a binomial distribution with parameters 6 and $\frac{1}{2}$. Show that $X = 3$ is the most likely outcome.

**15.** Let $X$ be binomially distributed with parameters $n$ and $p$. Show that as $k$ goes from 0 to $n$, $P(X = k)$ increases monotonically, then decreases monotonically reaching its largest value

(a) in the case that $(n+1)p$ is an integer, when $k$ equals either $(n+1)p - 1$ or $(n+1)p$,
(b) in the case that $(n+1)p$ is not an integer, when $k$ satisfies $(n+1)p - 1 < k < (n+1)p$.

**Hint:** Consider $P\{X = k\}/P\{X = k-1\}$ and see for what values of $k$ it is greater or less than 1.

***16.** An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

**17.** Suppose that an experiment can result in one of $r$ possible outcomes, the $i$th outcome having probability $p_i$, $i = 1, \ldots, r$, $\sum_{i=1}^{r} p_i = 1$. If $n$ of these experiments are performed, and if the outcome of any one of the $n$ does not affect the outcome of the other $n - 1$ experiments, then show that the probability that the first outcome appears $x_1$ times, the second $x_2$ times, and the $r$th $x_r$ times is

$$\frac{n!}{x_1!\,x_2! \ldots x_r!}\, p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r} \qquad \text{when } x_1 + x_2 + \cdots + x_r = n$$

This is known as the *multinomial* distribution.

. Show that when $r = 2$ the multinomial reduces to the binomial.

. In Exercise 17, let $X_i$ denote the number of times the $i$th outcome appears, $= 1, \ldots, r$. What is the probability mass function of $X_1 + X_2 + \ldots + X_k$?

. A television store owner figures that 50 percent of the customers entering s store will purchase an ordinary television set, 20 percent will purchase a color evision set, and 30 percent will just be browsing. If five customers enter his store a certain day, what is the probability that two customers purchase color sets, e customer purchases an ordinary set, and two customers purchase nothing?

**1.** In Exercise 20, what is the probability that our store owner sells three or ore televisions on that day?

**2.** If a fair coin is successively flipped, find the probability that a head first pears on the fifth trial.

**23.** A coin having probability $p$ of coming up heads is successively flipped ntil the $r$th head appears. Argue that $X$, the number of flips required, will be $n$, $\geqslant r$, with probability

$$P\{X = n\} = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \qquad n \geqslant r$$

his is known as the negative binomial distribution.

**Hint:** How many successes must there be in the first $n - 1$ trials?

**4.** The probability mass function of $X$ is given by

$$p(k) = \binom{r+k-1}{r-1} p^r (1-p)^k, \qquad k = 0, 1, \ldots$$

ive a possible interpretation of the random variable $X$.

**Hint:** See Exercise 23.

In Exercises 25 and 26, suppose that two teams are playing a series of games, ach of which is independently won by team $A$ with probability $p$ and by team $B$ ith probability $1 - p$. The winner of the series is the first team to win $i$ games.

**5.** If $i = 4$, find the probability that a total of 7 games are played. Also show hat this probability is maximized when $p = 1/2$.

**6.** Find the expected number of games that are played when

(a) $i = 2$.
(b) $i = 3$.

n both cases, show that this number is maximized when $p = 1/2$.



***27.** A fair coin is independently flipped $n$ times, $k$ times by $A$ and $n - k$ times by $B$. Show that the probability that $A$ and $B$ flip the same number of heads is equal to the probability that there are a total of $k$ heads.

**28.** Suppose that we want to generate a random variable $X$ that is equally likely to be either 0 or 1, and that all we have at our disposal is a biased coin that, when flipped, lands on heads with some (unknown) probability $p$. Consider the following procedure:

1. Flip the coin, and let $0_1$, either heads or tails, be the result.
2. Flip the coin again, and let $0_2$ be the result.
3. If $0_1$ and $0_2$ are the same, return to step 1.
4. If $0_2$ is heads, set $X = 0$, otherwise set $X = 1$.

(a) Show that the random variable $X$ generated by this procedure is equally likely to be either 0 or 1.
(b) Could we use a simpler procedure that continues to flip the coin until the last two flips are different, and then sets $X = 0$ if the final flip is a head, and sets $X = 1$ if it is a tail?

**29.** Consider $n$ independent flips of a coin having probability $p$ of landing heads. Say a changeover occurs whenever an outcome differs from the one preceding it. For instance, if the results of the flips are $H\ H\ T\ H\ T\ H\ H\ T$, then there are a total of five changeovers. If $p = 1/2$, what is the probability there are $k$ changeovers?

**30.** Let $X$ be a Poisson random variable with parameter $\lambda$. Show that $P\{X = i\}$ increases monotonically and then decreases monotonically as $i$ increases, reaching its maximum when $i$ is the largest integer not exceeding $\lambda$.

**Hint:** Consider $P\{X = i\}/P\{X = i - 1\}$.

**31.** Compare the Poisson approximation with the correct binomial probability for the following cases:

(i) $P\{X = 2\}$ when $n = 8,\ p = 0.1$.
(ii) $P\{X = 9\}$ when $n = 10,\ p = 0.95$.
(iii) $P\{X = 0\}$ when $n = 10,\ p = 0.1$.
(iv) $P\{X = 4\}$ when $n = 9,\ p = 0.2$.

**32.** If you buy a lottery ticket in 50 lotteries, in each of which your chance of winning a prize is $\frac{1}{100}$, what is the (approximate) probability that you will win a prize (a) at least once, (b) exactly once, (c) at least twice?

3. Let $X$ be a random variable with probability density

$$f(x) = \begin{cases} c(1-x^2), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the value of $c$?
(b) What is the cumulative distribution function of $X$?

4. Let the probability density of $X$ be given by

$$f(x) = \begin{cases} c(4x-2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the value of $c$?
(b) $P\left\{\frac{1}{2} < X < \frac{3}{2}\right\} = ?$

5. The density of $X$ is given by

$$f(x) = \begin{cases} 10/x^2, & \text{for } x > 10 \\ 0, & \text{for } x \leqslant 10 \end{cases}$$

What is the distribution of $X$? Find $P\{X > 20\}$.

36. A point is uniformly distributed within the disk of radius 1. That is, its density is

$$f(x, y) = C, \qquad 0 \leqslant x^2 + y^2 \leqslant 1$$

Find the probability that its distance from the origin is less than $x$, $0 \leqslant x \leqslant 1$.

37. Let $X_1, X_2, \ldots, X_n$ be independent random variables, each having a uniform distribution over (0,1). Let $M$ = maximum $(X_1, X_2, \ldots, X_n)$. Show that the distribution function of $M$, $F_M(\cdot)$, is given by

$$F_M(x) = x^n, \qquad 0 \leqslant x \leqslant 1$$

What is the probability density function of $M$?

*38. If the density function of $X$ equals

$$f(x) = \begin{cases} ce^{-2x}, & 0 < x < \infty \\ 0, & x < 0 \end{cases}$$

find $c$. What is $P\{X > 2\}$?

39. The random variable $X$ has the following probability mass function

$$p(1) = \tfrac{1}{2}, \qquad p(2) = \tfrac{1}{3}, \qquad p(24) = \tfrac{1}{6}$$

Calculate $E[X]$.



40. Suppose that two teams are playing a series of games, each of which is independently won by team $A$ with probability $p$ and by team $B$ with probability $1-p$. The winner of the series is the first team to win four games. Find the expected number of games that are played, and evaluate this quantity when $p = 1/2$.

41. Consider the case of arbitrary $p$ in Exercise 29. Compute the expected number of changeovers.

42. Suppose that each coupon obtained is, independent of what has been previously obtained, equally likely to be any of $m$ different types. Find the expected number of coupons one needs to obtain in order to have at least one of each type.

**Hint:** Let $X$ be the number needed. It is useful to represent $X$ by

$$X = \sum_{i=1}^{m} X_i$$

where each $X_i$ is a geometric random variable.

43. An urn contains $n + m$ balls, of which $n$ are red and $m$ are black. They are withdrawn from the urn, one at a time and without replacement. Let $X$ be the number of red balls removed before the first black ball is chosen. We are interested in determining $E[X]$. To obtain this quantity, number the red balls from 1 to $n$. Now define the random variables $X_i$, $i = 1, \ldots, n$, by

$$X_i = \begin{cases} 1, & \text{if red ball } i \text{ is taken before any black ball is chosen} \\ 0, & \text{otherwise} \end{cases}$$

(a) Express $X$ in terms of the $X_i$.
(b) Find $E[X]$.

44. In Exercise 43, let $Y$ denote the number of red balls chosen after the first but before the second black ball has been chosen.

(a) Express $Y$ as the sum of $n$ random variables, each of which is equal to either 0 or 1.
(b) Find $E[Y]$.
(c) Compare $E[Y]$ to $E[X]$ obtained in Exercise 43.
(d) Can you explain the result obtained in part (c)?

45. A total of $r$ keys are to be put, one at a time, in $k$ boxes, with each key independently being put in box $i$ with probability $p_i$, $\sum_{i=1}^{k} p_i = 1$. Each time a key is put in a nonempty box, we say that a collision occurs. Find the expected number of collisions.

. If $X$ is a nonnegative integer valued random variable, show that

$$E[X] = \sum_{n=1}^{\infty} P\{X \geqslant n\} = \sum_{n=0}^{\infty} P\{X > n\}$$

**Hint:** Define the sequence of random variables $I_n, n \geqslant 1$, by

$$I_n = \begin{cases} 1, & \text{if } n \leqslant X \\ 0, & \text{if } n > X \end{cases}$$

Now express $X$ in terms of the $I_n$.

7. Consider three trials, each of which is either a success or not. Let $X$ denote e number of successes. Suppose that $E[X] = 1.8$.

(a) What is the largest possible value of $P\{X = 3\}$?
(b) What is the smallest possible value of $P\{X = 3\}$?

both cases, construct a probability scenario that results in $P\{X = 3\}$ having the esired value.

8. If $X$ is uniformly distributed over (0,1), calculate $E[X^2]$.

49. Prove that $E[X^2] \geqslant (E[X])^2$. When do we have equality?

0. Let $c$ be a constant. Show that

(i) $\text{Var}(cX) = c^2\text{Var}(X)$.
(ii) $\text{Var}(c + X) = \text{Var}(X)$.

1. A coin, having probability $p$ of landing heads, is flipped until head appears or the $r$th time. Let $N$ denote the number of flips required. Calculate $E[N]$.

**Hint:** There is an easy way of doing this. It involves writing $N$ as the sum of $r$ geometric random variables.

2. (a) Calculate $E[X]$ for the maximum random variable of Exercise 37.
(b) Calculate $E[X]$ for $X$ as in Exercise 33.
(c) Calculate $E[X]$ for $X$ as in Exercise 34.

3. If $X$ is uniform over (0, 1), calculate $E[X^n]$ and $\text{Var}(X^n)$.

4. Let $X$ and $Y$ each take on either the value 1 or −1. Let

$$\begin{aligned} p(1, 1) &= P\{X = 1, Y = 1\}, \\ p(1, -1) &= P\{X = 1, Y = -1\}, \\ p(-1, 1) &= P\{X = -1, Y = 1\}, \\ p(-1, -1) &= P\{X = -1, Y = -1\} \end{aligned}$$



Suppose that $E[X] = E[Y] = 0$. Show that

(a) $p(1, 1) = p(-1, -1)$
(b) $p(1, -1) = p(-1, 1)$

Let $p = 2p(1, 1)$. Find

(c) $\text{Var}(X)$
(d) $\text{Var}(Y)$
(e) $\text{Cov}(X, Y)$

**55.** Let $X$ be a positive random variable having density function $f(x)$. If $f(x) \leqslant c$ for all $x$, show that, for $a > 0$,

$$P\{X > a\} \geqslant 1 - ac$$

**56.** There are $n$ types of coupons. Each newly obtained coupon is, independently, type $i$ with probability $p_i, i = 1, \ldots, n$. Find the expected number and the variance of the number of distinct types obtained in a collection of $k$ coupons.

**57.** Suppose that $X$ and $Y$ are independent binomial random variables with parameters $(n, p)$ and $(m, p)$. Argue probabilistically (no computations necessary) that $X + Y$ is binomial with parameters $(n + m, p)$.

**58.** Suppose that $X$ and $Y$ are independent continuous random variables. Show that

$$P\{X \leqslant Y\} = \int_{-\infty}^{\infty} F_X(y) f_Y(y)\, dy$$

**59.** Let $X_1$, $X_2$, $X_3$, and $X_4$ be independent continuous random variables with a common distribution function $F$ and let

$$p = P\{X_1 < X_2 > X_3 < X_4\}$$

(a) Argue that the value of $p$ is the same for all continuous distribution functions $F$.
(b) Find $p$ by integrating the joint density function over the appropriate region.
(c) Find $p$ by using the fact that all 4! possible orderings of $X_1, \ldots, X_4$ are equally likely.

**60.** Calculate the moment generating function of the uniform distribution on (0, 1). Obtain $E[X]$ and $\text{Var}[X]$ by differentiating.

**61.** Suppose that $X$ takes on each of the values 1, 2, 3 with probability $\frac{1}{3}$. What is the moment generating function? Derive $E[X]$, $E[X^2]$, and $E[X^3]$ by fferentiating the moment generating function and then compare the obtained sult with a direct derivation of these moments.

2. In deciding upon the appropriate premium to charge, insurance companies metimes use the exponential principle, defined as follows. With $X$ as the random nount that it will have to pay in claims, the premium charged by the insurance mpany is

$$P = \frac{1}{a} \ln\left(E[e^{aX}]\right)$$

here $a$ is some specified positive constant. Find $P$ when $X$ is an exponential ndom variable with parameter $\lambda$, and $a = \alpha\lambda$, where $0 < \alpha < 1$.

3. Calculate the moment generating function of a geometric random variable.

54. Show that the sum of independent identically distributed exponential ndom variables has a gamma distribution.

5. Consider Example 2.48. Find $\mathrm{Cov}(X_i, X_j)$ in terms of the $a_{rs}$.

6. Use Chebyshev's inequality to prove the *weak law of large numbers*. Namely, $X_1, X_2, \ldots$ are independent and identically distributed with mean $\mu$ and variance $^2$ then, for any $\varepsilon > 0$,

$$P\left\{\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| > \varepsilon\right\} \to 0 \qquad \text{as } n \to \infty$$

7. Suppose that $X$ is a random variable with mean 10 and variance 15. What an we say about $P\{5 < X < 15\}$?

8. Let $X_1, X_2, \ldots, X_{10}$ be independent Poisson random variables with mean 1.

(i) Use the Markov inequality to get a bound on $P\{X_1 + \cdots + X_{10} \geqslant 15\}$.
(ii) Use the central limit theorem to approximate $P\{X_1 + \cdots + X_{10} \geqslant 15\}$.

9. If $X$ is normally distributed with mean 1 and variance 4, use the tables to nd $P\{2 < X < 3\}$.

70. Show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$$

**Hint:** Let $X_n$ be Poisson with mean $n$. Use the central limit theorem to show that $P\{X_n \leqslant n\} \to \frac{1}{2}$.

1. Let $X$ denote the number of white balls selected when $k$ balls are chosen at andom from an urn containing $n$ white and $m$ black balls.



(i) Compute $P\{X = i\}$.
(ii) Let, for $i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n$,

$$X_i = \begin{cases} 1, & \text{if the } i\text{th ball selected is white} \\ 0, & \text{otherwise} \end{cases}$$

$$Y_j = \begin{cases} 1, & \text{if white ball } j \text{ is selected} \\ 0, & \text{otherwise} \end{cases}$$

Compute $E[X]$ in two ways by expressing $X$ first as a function of the $X_i$s and then of the $Y_j$s.

*72. Show that $\mathrm{Var}(X) = 1$ when $X$ is the number of men who select their own hats in Example 2.31.

73. For the multinomial distribution (Exercise 17), let $N_i$ denote the number of times outcome $i$ occurs. Find

(i) $E[N_i]$
(ii) $\mathrm{Var}(N_i)$
(iii) $\mathrm{Cov}(N_i, N_j)$
(iv) Compute the expected number of outcomes that do not occur.

74. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed continuous random variables. We say that a record occurs at time $n$ if $X_n > \max(X_1, \ldots, X_{n-1})$. That is, $X_n$ is a record if it is larger than each of $X_1, \ldots, X_{n-1}$. Show

(i) $P\{\text{a record occurs at time } n\} = 1/n$
(ii) $E[\text{number of records by time } n] = \sum_{i=1}^{n} 1/i$
(iii) $\mathrm{Var}(\text{number of records by time } n) = \sum_{i=1}^{n} (i-1)/i^2$
(iv) Let $N = \min\{n : n > 1 \text{ and a record occurs at time } n\}$. Show $E[N] = \infty$.

**Hint:** For (ii) and (iii) represent the number of records as the sum of indicator (that is, Bernoulli) random variables.

75. Let $a_1 < a_2 < \cdots < a_n$ denote a set of $n$ numbers, and consider any permutation of these numbers. We say that there is an inversion of $a_i$ and $a_j$ in the permutation if $i < j$ and $a_j$ precedes $a_i$. For instance the permutation 4, 2, 1, 5, 3 has 5 inversions—(4, 2), (4, 1), (4, 3), (2, 1), (5, 3). Consider now a random permutation of $a_1, a_2, \ldots, a_n$—in the sense that each of the $n!$ permutations is equally likely to be chosen—and let $N$ denote the number of inversions in this permutation. Also, let

$$N_i = \text{number of } k : k < i,\ a_i \text{ precedes } a_k \text{ in the permutation}$$

and note that $N = \sum_{i=1}^{n} N_i$.

(i) Show that $N_1, \ldots, N_n$ are independent random variables.
(ii) What is the distribution of $N_i$?
(iii) Compute $E[N]$ and $\mathrm{Var}(N)$.

**6.** Let $X$ and $Y$ be independent random variables with means $\mu_x$ and $\mu_y$ and ariances $\sigma_x^2$ and $\sigma_y^2$. Show that

$$\mathrm{Var}(XY) = \sigma_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + \mu_x^2\sigma_y^2$$

**7.** Let $X$ and $Y$ be independent normal random variables, each having arameters $\mu$ and $\sigma^2$. Show that $X + Y$ is independent of $X - Y$.

**Hint:** Find their joint moment generating function.

**'8.** Let $\phi(t_1, \ldots, t_n)$ denote the joint moment generating function of $X_1, \ldots, X_n$.

(a) Explain how the moment generating function of $X_i$, $\phi_{X_i}(t_i)$, can be obtained from $\phi(t_1, \ldots, t_n)$.
(b) Show that $X_1, \ldots, X_n$ are independent if and only if

$$\phi(t_1, \ldots, t_n) = \phi_{x_1}(t_1) \ldots \phi_{X_n}(t_n)$$

## References

1. W. Feller, "An Introduction to Probability Theory and Its Applications," Vol. I, John Wiley, New York, 1957.
2. M. Fisz, "Probability Theory and Mathematical Statistics," John Wiley, New York, 1963.
3. E. Parzen, "Modern Probability Theory and Its Applications," John Wiley, New York, 1960.
4. S. Ross, "A First Course in Probability," Sixth Edition, Prentice Hall, New Jersey, 2002.

# 3

# Conditional Probability and Conditional Expectation

## 3.1. Introduction

One of the most useful concepts in probability theory is that of conditional probability and conditional expectation. The reason is twofold. First, in practice, we are often interested in calculating probabilities and expectations when some partial information is available; hence, the desired probabilities and expectations are conditional ones. Secondly, in calculating a desired probability or expectation it is often extremely useful to first "condition" on some appropriate random variable.

## 3.2. The Discrete Case

Recall that for any two events $E$ and $F$, the conditional probability of $E$ given $F$ is defined, as long as $P(F) > 0$, by

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Hence, if $X$ and $Y$ are discrete random variables, then it is natural to define the *conditional probability mass function* of $X$ given that $Y = y$, by

$$\begin{aligned} p_{X|Y}(x|y) &= P\{X = x | Y = y\} \\ &= \frac{P\{X = x, Y = y\}}{P\{Y = y\}} \\ &= \frac{p(x, y)}{p_Y(y)} \end{aligned}$$