Source and velocity recovery from one time measurements in the advection diffusion equation

Inferencia de parámetros en PDE's

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Advection diffusion equation

We can model the propagation of contaminants in a medium with the advection diffusion equation.

$$\begin{aligned} u_t(x,t) &= \Delta u(x,t) + \psi \cdot \nabla u(x,t) + f(x) \\ u(x,0) &= 0\,, \end{aligned} \tag{1}$$

where

- $u(x,t)$: contaminant density
- ψ: constant velocity of the medium
- $f(x)$: contaminant source intensity in the position x

QUESTION: How many time measurements $u(\cdot, T_i)$ do we need to recover f and ψ uniquely?

Uniqueness result

If we only allow sources f with compact support, then one time measurement $u(x, T)$ is enough. The main argument is the following:

1. If we have solutions u_k of the equation (1) with parameters ψ_k, f_k such that $\psi_1 \neq \psi_2$ and $u_1(\cdot, T) = u_2(\cdot, T)$, then

$$\widehat{f}_1(\xi) = \widehat{f}_2(\xi) \frac{a_1}{a_2} \frac{(e^{a_2 T} - 1)}{(e^{a_1 T} - 1)}, \quad a_k = (i\psi_k - 2\pi\xi)(2\pi\xi), \quad \xi \neq 0 \, .$$

2. $\widehat{f}_k$ is an entire function.
3. $\widehat{f}_k$ has at most $O(r)$ zeros inside a ball of radius r.
4. $(e^{a_1 T} - 1)$ has $O(r^2)$ zeros inside a ball of radius r, all of them different from the zeros of $(e^{a_2 T} - 1)$.

Adjoint equation

Given the data $u_d(\cdot) = u(\cdot, T)$, we want minimize (over ψ and f)

$$F(u) = \frac{1}{2} \int_{\mathbb{R}} (u(x, T) - u_d(x))^2 dx \; ,$$

where u satisfies the restriction

$$0 = g(u, \psi, f) = \begin{pmatrix} u_t - u_{xx} - \psi u_x - f(x) \\ u(x, 0) \end{pmatrix} . \qquad (2)$$

We calculate

$$F_u \dot{u} = \int_{\mathbb{R}} (u(x, T) - u_d(x)) \dot{u}(x, T) dx, \quad g_u \dot{u} = \begin{pmatrix} \dot{u}_t - \dot{u}_{xx} - \psi \dot{u}_x \\ \dot{u}(x, 0) \end{pmatrix}$$

$$g_\psi \dot{\psi} = \begin{pmatrix} -u_x \dot{\psi} \\ 0 \end{pmatrix}, \quad g_f \dot{f} = \begin{pmatrix} -\dot{f} \\ 0 \end{pmatrix}$$

Adjoint equation

The adjoint vector λ is the solution of the adjoint equation

$$F_u \dot{u} = -\left\langle \begin{pmatrix} \lambda(x,t) \\ \lambda_0(x) \end{pmatrix}, g_u \dot{u} \right\rangle_{L^2} \tag{3}$$

The chain rule and the adjoint equation give

$$d_f F \dot{f} = F_u d_f u \dot{f} = -\left\langle \begin{pmatrix} \lambda(x,t) \\ \lambda_0(x) \end{pmatrix}, g_u d_f u \dot{f} \right\rangle_{L^2},$$

and taking the derivative with respect to f of the restriction we obtain

$$g_u d_f u \dot{f} + g_f \dot{f} = 0 \,.$$

Therefore

$$d_f F \dot{f} = \left\langle \begin{pmatrix} \lambda(x,t) \\ \lambda_0(x) \end{pmatrix}, g_f \dot{f} \right\rangle_{L^2} = \left\langle \begin{pmatrix} \lambda(x,t) \\ \lambda_0(x) \end{pmatrix}, \begin{pmatrix} -\dot{f} \\ 0 \end{pmatrix} \right\rangle_{L^2}$$

$$d_\psi F \dot{\psi} = \left\langle \begin{pmatrix} \lambda(x,t) \\ \lambda_0(x) \end{pmatrix}, \begin{pmatrix} -u_x \dot{\psi} \\ 0 \end{pmatrix} \right\rangle_{L^2}$$

Adjoint equation

Integration by parts shows that the adjoint equation can be written as

$$
\begin{aligned}
\lambda_t(x,t) &= -\lambda_{xx}(x,t) + \psi\lambda_x(x,t) \\
\lambda(x,T) &= u_d(x) - u(x,T) \\
\lambda(x,0) &= \lambda_0(x)
\end{aligned}
\tag{4}
$$

Summary: Given any ψ and f

1. Solve for $u(x,t)$ in the advection diffusion equation.
2. Use $u(x,T)$ and the data $u_d(x)$ to find the solution λ of the adjoint equation.
3. Use u and λ to evaluate

$$
d_f F = -\int_0^T \lambda(x,t)dt, \quad d_\psi F = -\int_0^T \int_{\mathbb{R}} \lambda(x,t)u_x(x,t)dxdt
$$

Numerical implementation

Now we can minimize F over ψ and f with a simple gradient descent:

$$f_{n+1} = f_n - \alpha d_f F, \quad \psi_{n+1} = \psi_n - \beta d_\psi F$$