5. fejezet, Trigonometrikus feladatok

$0, \pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, 2\pi$

$0 \quad 180° \quad 90° \quad 60° \quad 45° \quad 30° \quad 360°$

sin / cos

$\sin(\alpha \pm \beta), \cos(\alpha \pm \beta)$

1) $$\operatorname{tg}\left(\frac{\pi}{12}\right) = \frac{\sin\left(\frac{\pi}{12}\right)}{\cos\left(\frac{\pi}{12}\right)} = \underline{\underline{\frac{\sqrt{3}-1}{\sqrt{3}+1}}}$$

$$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) =$$

$$= \sin\left(\frac{\pi}{3}\right)\cdot\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cdot\cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$$

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{3}\right)\cdot\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\cdot\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}\cdot(\sqrt{3}+1)}{4}$$

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{3-1}{3+2\sqrt{3}+1} = \frac{2}{4+2\sqrt{3}} = \frac{1}{2+\sqrt{3}} \quad \left(= 2-\sqrt{3}\right)$$

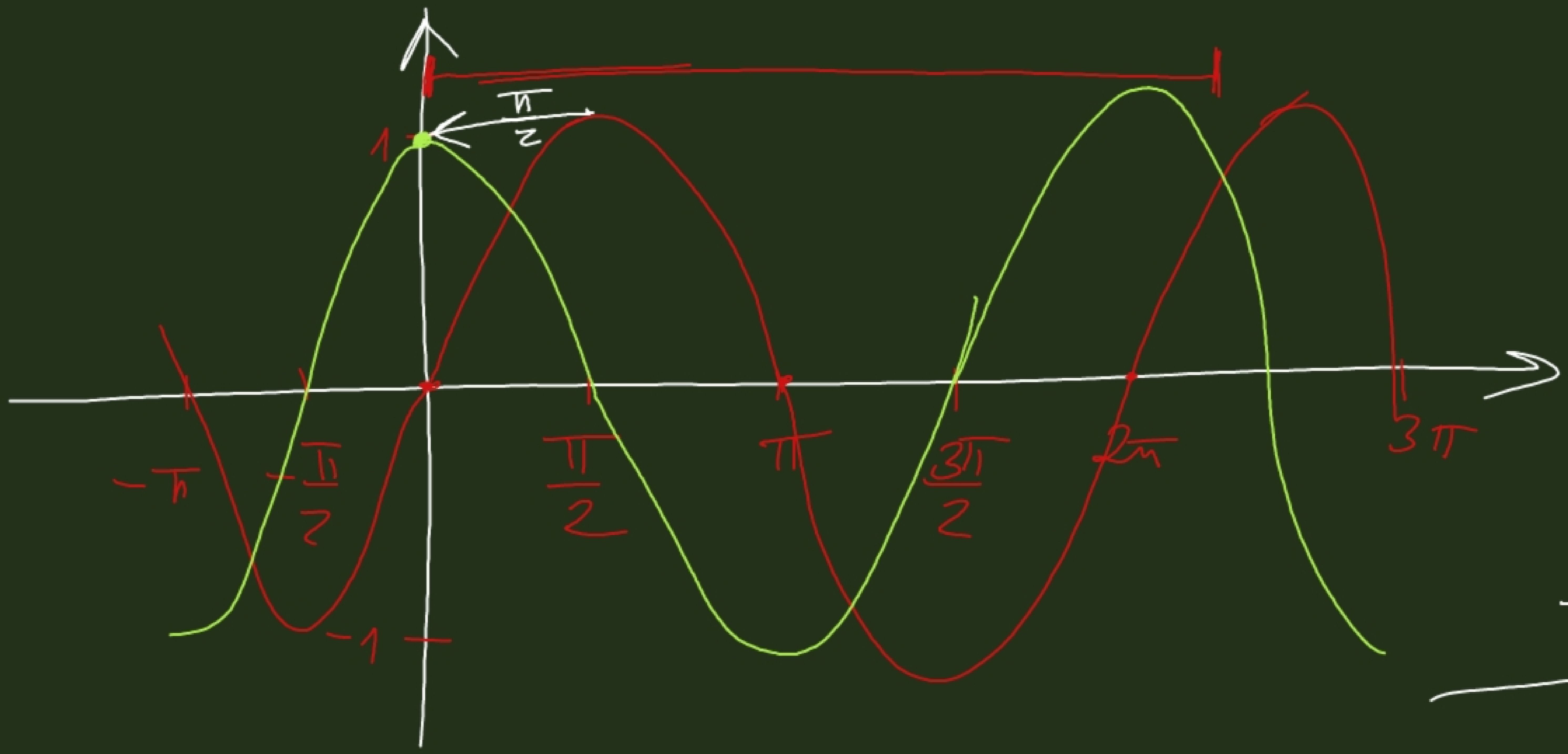
$$\sin(\alpha \pm \beta) = \sin(\alpha) \cdot \cos(\beta) \pm \sin(\beta)\cos(\alpha)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cdot \cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$\sin(2\alpha) = 2 \cdot \sin(\alpha) \cdot \cos(\alpha)$$

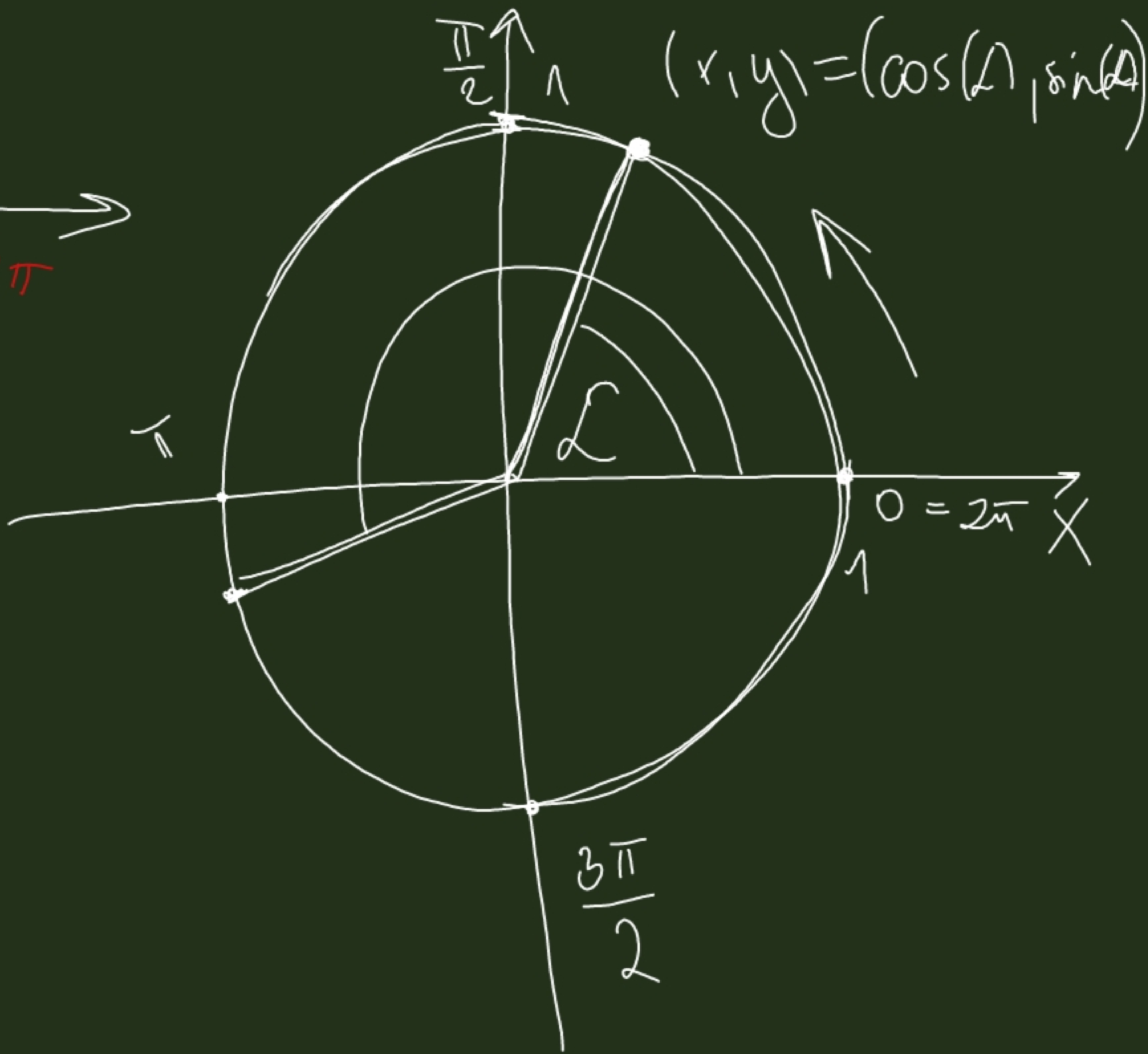
$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$



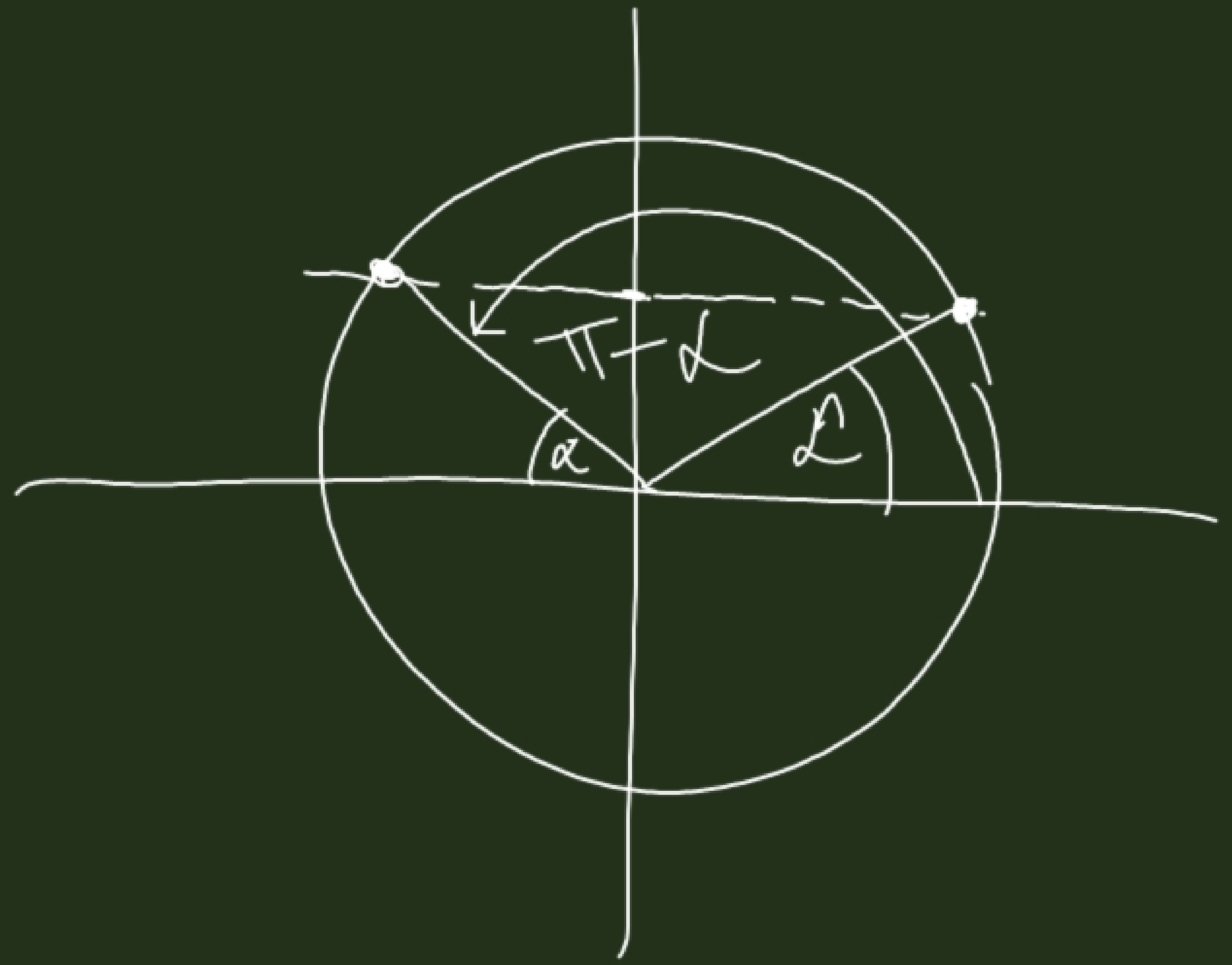
$\frac{\pi}{2}$
1
-1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
3π

$\sin(x)$
$\cos(x)$



$(x, y) = (\cos(\alpha), \sin(\alpha))$
$\frac{\pi}{2}$
1
π
α
$0 = 2\pi$
x
1
$\frac{3\pi}{2}$

4/a $\sin(4x)=\sin(x)$



$$\sin(\alpha)=\sin(\beta)$$

- $\alpha=\beta+k\cdot 2\pi \quad (k\in\mathbb{Z})$
- $\alpha=\pi-\beta+k\cdot 2\pi \quad (k\in\mathbb{Z})$

$$\sin(4x)=\sin(x)$$

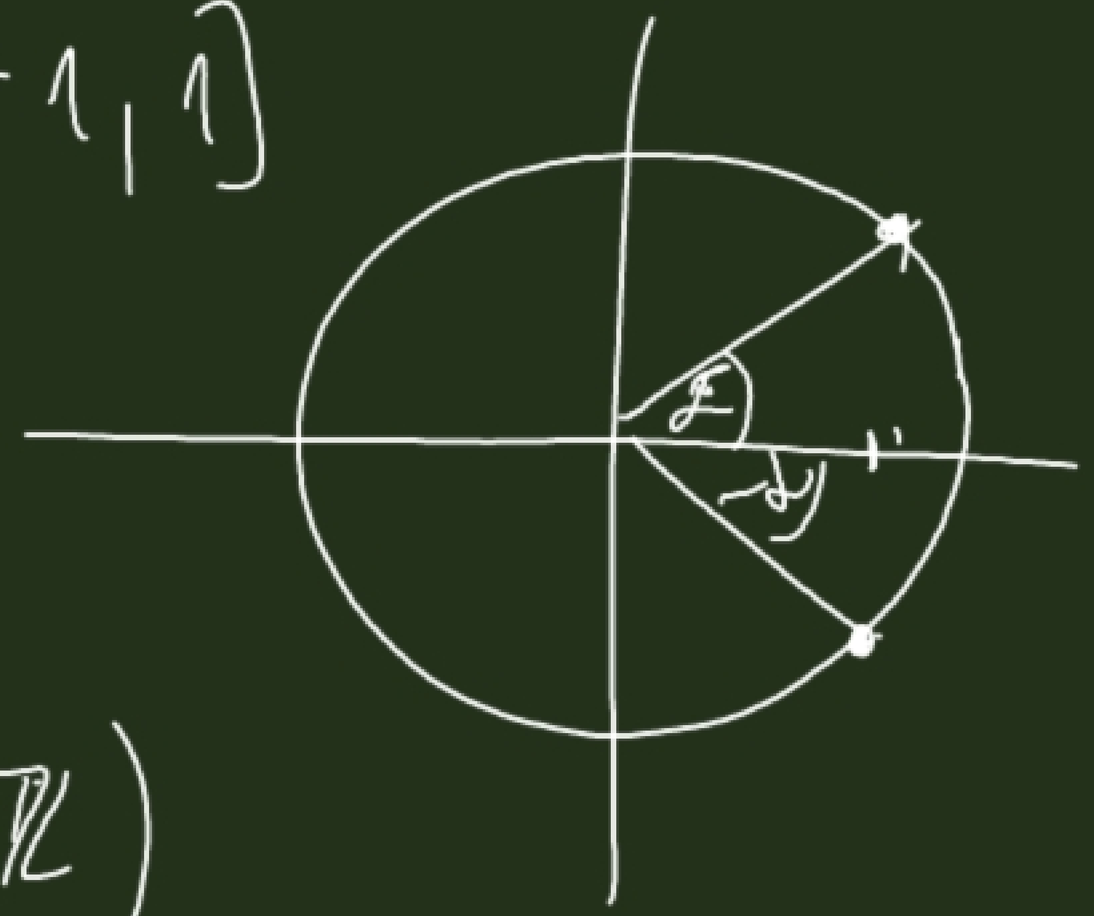
- $4x=x+k\cdot 2\pi \iff \boxed{x=k\cdot\frac{2\pi}{3} \quad (k\in\mathbb{Z})}$
- $4x=\pi-x+k\cdot 2\pi \iff \boxed{x=\frac{\pi}{5}+k\cdot\frac{2\pi}{5} \quad (k\in\mathbb{Z})}$

4|d| $\cos(2x) - 3\cdot\cos(x) + 2 = 0$

$y = \cos(x)$

$$\cos(2x) = \cos^2(x) - \underbrace{\sin^2(x)}_{1-\cos^2(x)} = 2\cos^2(x) - 1$$

$$2\cos^2(x) - 3\cos(x) + 1 = 0 \qquad y = \cos(x), \quad y \in [-1, 1]$$



$$2y^2 - 3y + 1 = 0$$

$$y_1 = 1 \longrightarrow \cos(x) = 1 \iff x = k\cdot 2\pi \quad (k \in \mathbb{Z})$$

$$y_2 = \frac{1}{2} \qquad \cos(x) = \frac{1}{2} \iff x = \pm\frac{\pi}{3} + k\cdot 2\pi \quad (k \in \mathbb{Z})$$

$\boxed{4/e}$ $ctg(x) - tg(x) = 2\sqrt{3}$

$x \neq \frac{\pi}{2} + k \cdot \pi \qquad x \neq k \cdot \pi$

(kik.: $\cos(x) \neq 0$, $\sin(x) \neq 0$

$x \neq k \cdot \frac{\pi}{2} \quad (k \in \mathbb{Z})$)

1. mo.: $\frac{1}{tg(x)} - tg(x) = 2\sqrt{3}$ $\qquad y = tg(x)$

$$-y^2 - 2\sqrt{3}\,y + 1 = 0$$

$$y_{1,2} = \frac{2\sqrt{3} \pm \sqrt{12+4}}{-2} = \frac{2\sqrt{3} \pm 4}{-2} = -(\sqrt{3} \pm 2)$$

$$tg(x) = -\sqrt{3} \pm 2 \quad ?$$

$$\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)} = 2\sqrt{3} \quad / k.n. + \cdot \sin(x)\cdot\cos(x)$$

$$\cos^2(x) - \sin^2(x) = 2\sqrt{3}\,\sin(x)\cdot\cos(x)$$

$$\cos(2x) = \sqrt{3}\cdot\sin(2x)$$

$$\operatorname{ctg}(2x) = \sqrt{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \longrightarrow 2x = \frac{\pi}{6} + k\cdot\pi \quad (k \in \mathbb{Z})$$

$$x = \frac{\pi}{12} + k\cdot\frac{\pi}{2} \quad (k \in \mathbb{Z})$$

4/i) $\sqrt{2}\cdot\sin(x)\cdot\cos\left(\frac{x}{2}\right)=\sqrt{\underbrace{1+\cos(x)}_{\geq 0}}$ $\left(y:=\frac{x}{2}\right.$

$$\cancel{\sqrt{2}}\cdot\sin(2y)\cdot\cos(y)=\sqrt{1+\cos(2y)}=\cancel{\sqrt{2}}\cdot|\cos(y)| \begin{cases}\cos(y)\\ -\cos(y)\end{cases}$$

HF: négyzetreemelés

HF: $z=\sin(y)$

$$\sqrt{1+\cos(2y)}=\sqrt{1+\cos^2(y)-\cancel{\sin^2(y)}}=\sqrt{2\cdot\cos^2(y)}=\sqrt{2}\cdot|\cos(y)|$$

$$1=\cos^2(y)+\cancel{\sin^2(y)}$$

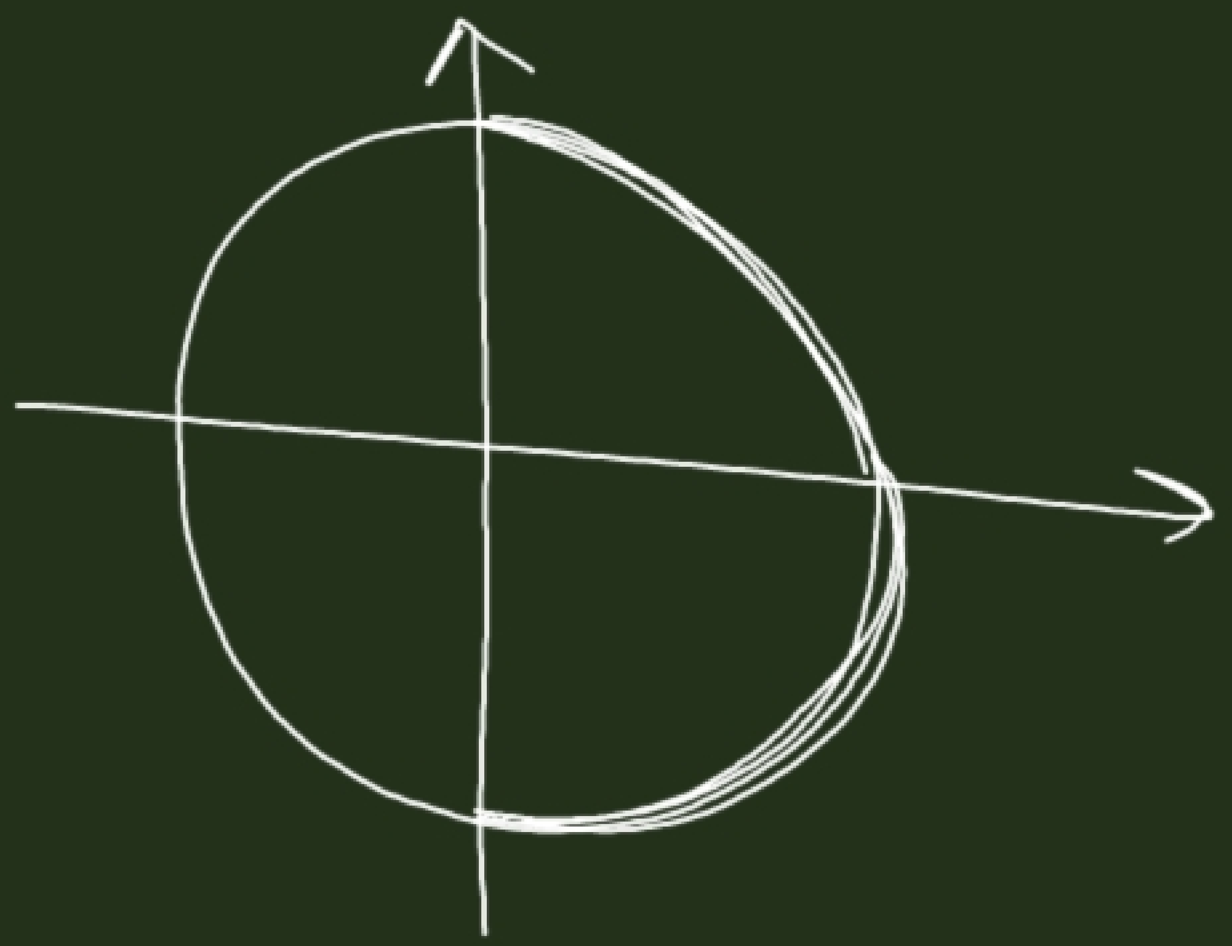
$$\sin(2y)\cdot\cos(y) = |\cos(y)|$$

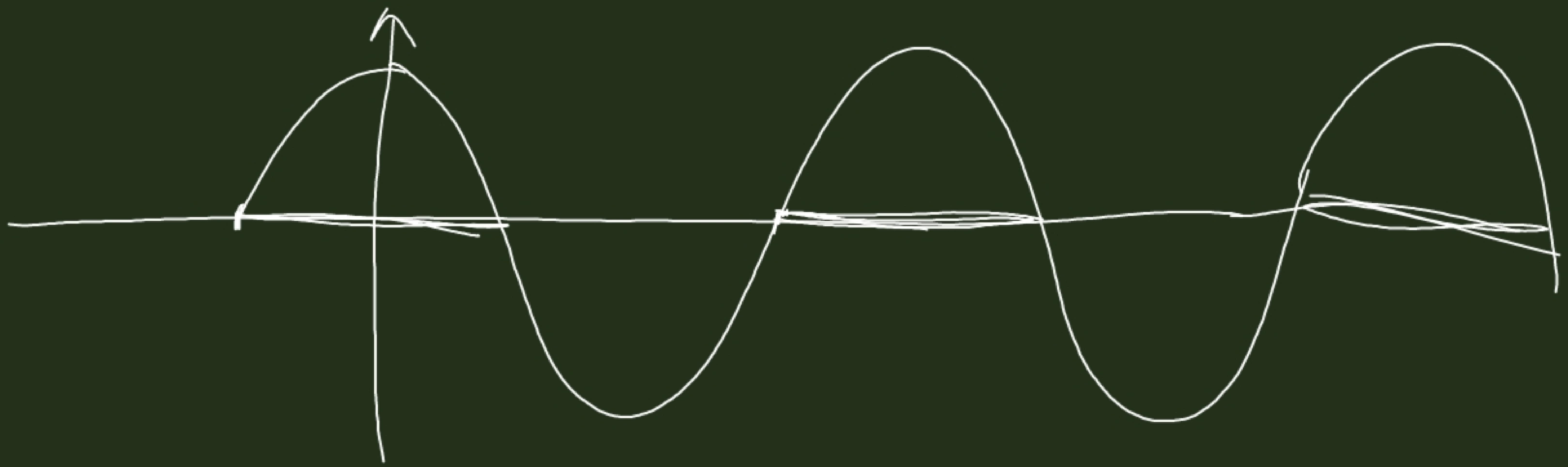
I. $\cos(y) = 0 \iff y = \frac{\pi}{2} + k\cdot\pi \iff \frac{x}{2} = \frac{\pi}{2} + k\cdot\pi \iff x = \pi + k\cdot 2\pi$

$(k \in \mathbb{Z})$

II. $\cos(y) > 0$

$$\boxed{-\frac{\pi}{2} + k\cdot 2\pi < y < \frac{\pi}{2} + k\cdot 2\pi \quad (k \in \mathbb{Z})}$$

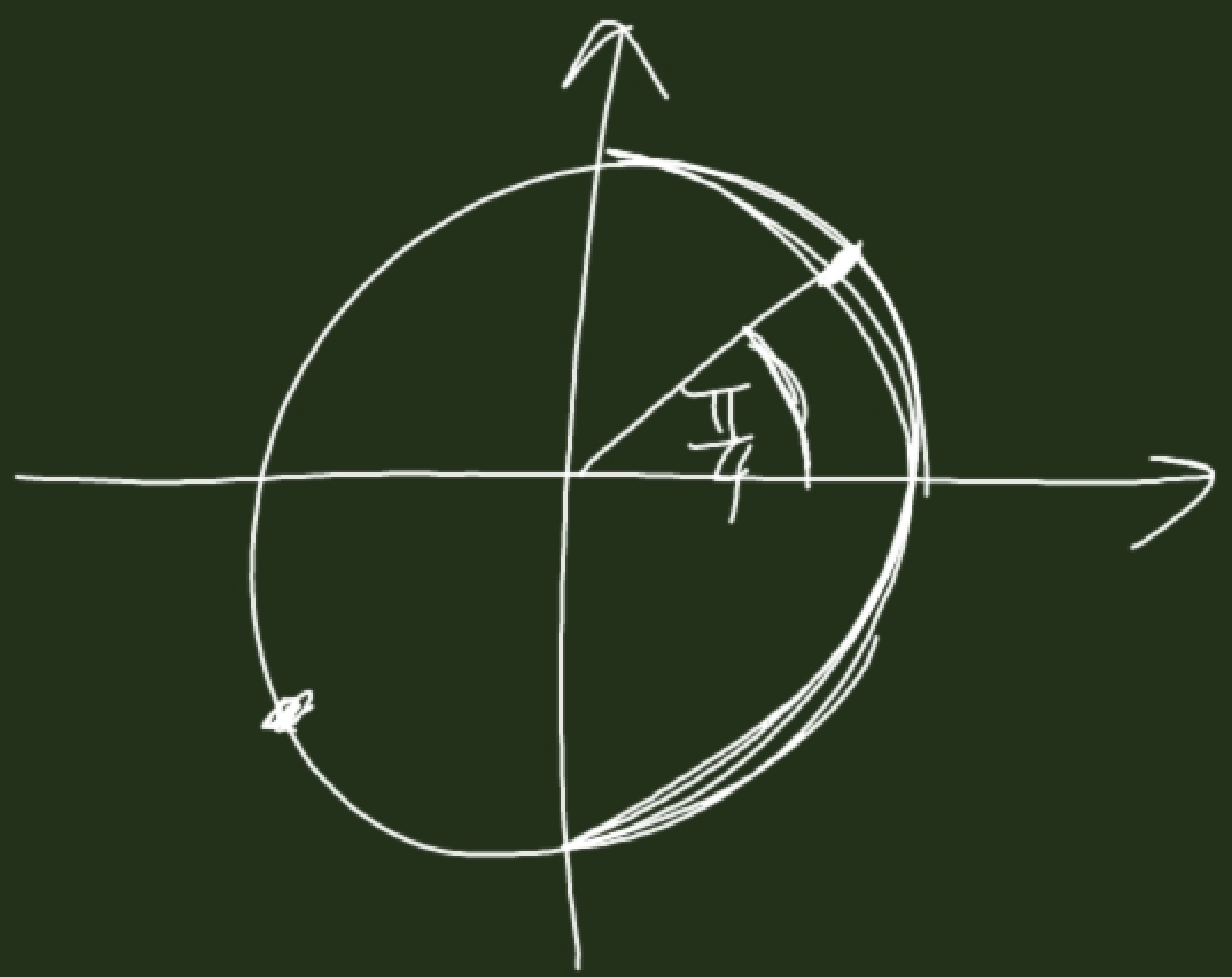




$$\sin(2y)\cdot\cos(y)=\cos(y) \qquad /:\cos(y)\neq 0$$

$$\sin(2y)=1$$

$$2y=\frac{\pi}{2}+k\cdot 2\pi \iff y=\frac{\pi}{4}+k\cdot\pi$$



megoldás: $y=\frac{\pi}{4}+k\cdot 2\pi \quad (k\in\mathbb{Z})$

$$\frac{x}{2}=\frac{\pi}{4}+k\cdot 2\pi$$

$$x=\frac{\pi}{2}+k\cdot 4\pi \quad (k\in\mathbb{Z})$$

$\text{III}.\ \cos(y) < 0 \quad \text{HF.}$

$\boxed{4/m}\ \cos(2x) = \cos(x) - \sin(x)$

$$\cos^2(x) - \sin^2(x) = \cos(x) - \sin(x)$$

$$\big(\cos(x) - \sin(x)\big)\big(\cos(x) + \sin(x)\big) = \cos(x) - \sin(x) \quad / -\big(\cos(x) - \sin(x)\big)$$

$$\big(\cos(x) - \sin(x)\big) \cdot \big(\cos(x) + \sin(x)\big) - \big(\cos(x) - \sin(x)\big) = 0$$

$$\big(\cos(x) - \sin(x)\big) \cdot \big(\cos(x) + \sin(x) - 1\big) = 0$$

I. $\cos(x) - \sin(x) = 0 \iff \mathrm{tg}(x) = 1$

$$\left(\sin\left(\frac{\pi}{2} - x\right) = \cos(x) \right)$$

$$x = \frac{\pi}{4} + k \cdot \pi \quad (k \in \mathbb{Z})$$

II. $\cos(x) + \sin(x) - 1 = 0$

$$\cos(x) + \sin(x) = 1 \qquad /(\ )^2$$

$$1 + 2 \cdot \cos(x)\sin(x) = 1$$

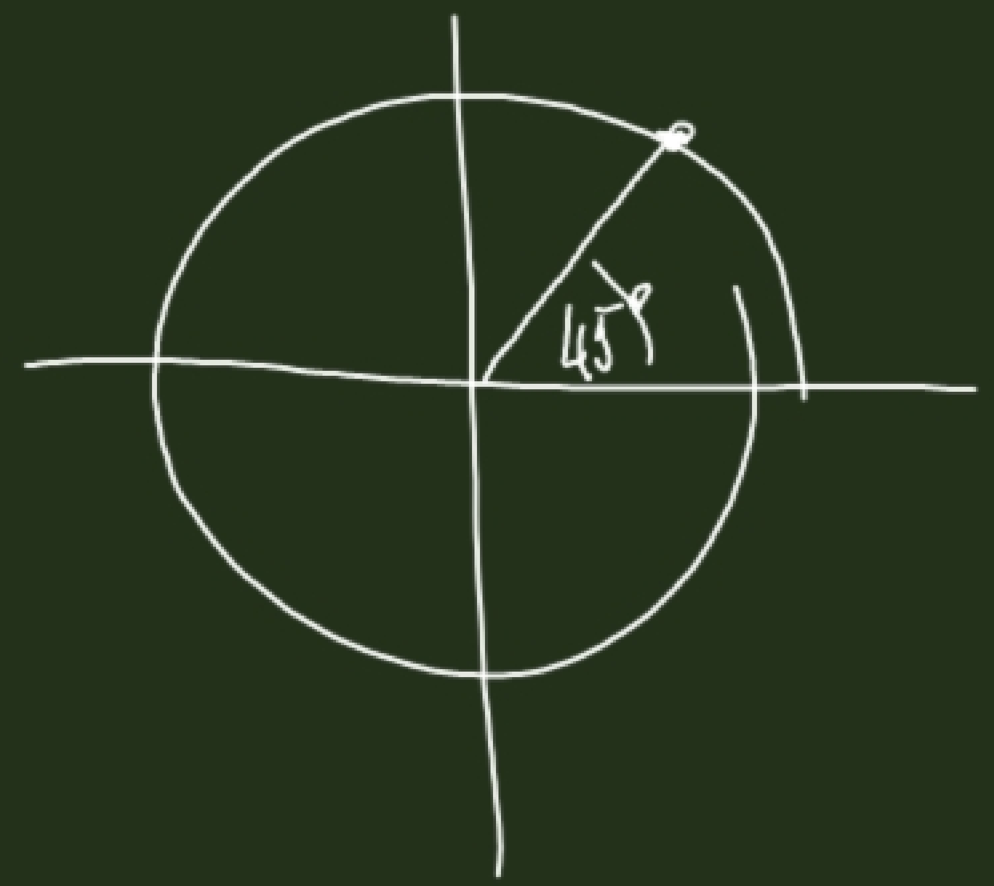
$$\sin(2x) = 0$$

$$2x = k \cdot \pi \quad , \quad x = k \cdot \frac{\pi}{2} \quad (k \in \mathbb{Z}) \quad ?$$

$$\cos(x) + \sin(x) = 1 \qquad / \cdot \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} \cdot \cos(x) + \frac{1}{\sqrt{2}} \cdot \sin(x) = \frac{1}{\sqrt{2}} \qquad \left(\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \right)$$

$$\sin\left(\frac{\pi}{4}\right) \cdot \cos(x) + \cos\left(\frac{\pi}{4}\right) \cdot \cos(x) = \frac{\sqrt{2}}{2}$$



$$\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$x = \ldots\ldots$$

HF

$\underline{7\,(a)}$ $\quad 2\cdot\sin^2(x)-\sin(x)-1>0 \qquad y:=\sin(x),$

$$y\in[-1,1]$$

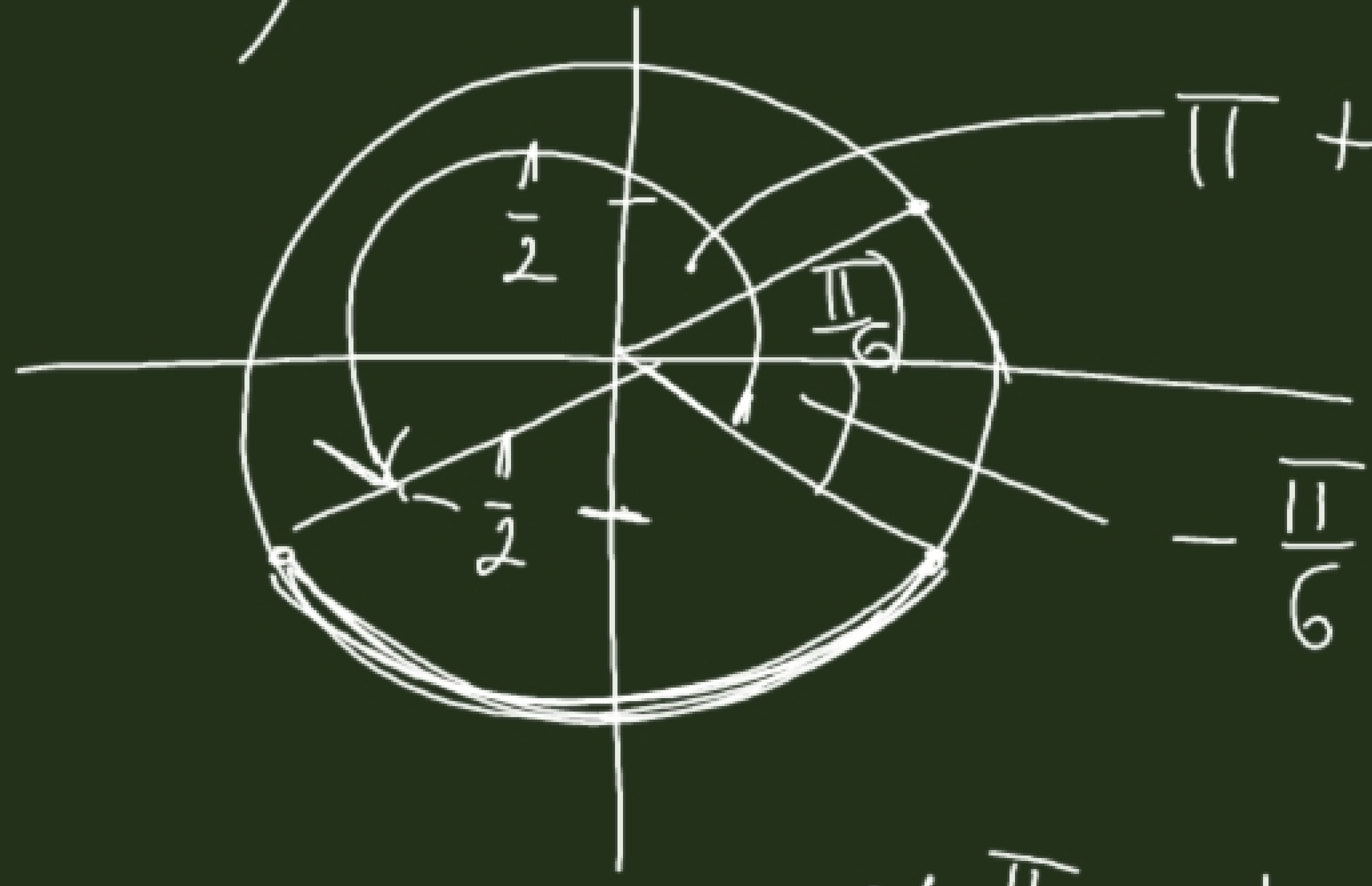
$$2y^2-y-1>0$$

$$y_{1,2}=\frac{1\pm\sqrt{1+8}}{4}=\frac{1\pm3}{4}\begin{cases}1\\ -\frac{1}{2}\end{cases}$$

$$\left(y<-\tfrac{1}{2}\text{ vagy } y>1\right)\cap\left(-1\le y\le 1\right)=$$

$$=\left[-1,-\tfrac{1}{2}\right)$$

$$-1\le\boxed{\sin(x)<-\frac{1}{2}}$$



$$\pi+\frac{\pi}{6}=\frac{7\pi}{6}$$

$$-\frac{\pi}{6}=\frac{11\pi}{6}$$

$$\frac{7\pi}{6}+k\cdot2\pi<x<\frac{11\pi}{6}+k\cdot2\pi$$
$$(k\in\mathbb{Z})$$