Solution 1 Since, columns of rotational matrices comprises direction cosines of one axis of one refrecens system with respect to other refrence system. We can write

$$i_0^2 + j_0^2 + k_0^2 = 1$$

and $$i_1^2 + j_1^2 + k_1^2 = 1$$

Let's denote a general rotational matrix (R_0^1), who relates b/w coordinate system $O_0x_0y_0z_0$ and $O_1x_1y_1z_1$.

$$R_0^1 = \begin{bmatrix} i_0 \cdot i_1 & j_0 \cdot i_1 & k_0 \cdot i_1 \\ i_0 \cdot j_1 & j_0 \cdot j_1 & k_0 \cdot j_1 \\ \underbrace{i_0 \cdot k_1}_{c_1} & \underbrace{j_0 \cdot k_1}_{c_2} & \underbrace{k_0 \cdot k_1}_{c_3} \end{bmatrix}$$

c_1, c_2 and c_3 are column vectors of R_0^1.

Now let's perform

$$c_1^T \cdot c_2 = \begin{bmatrix} i_0 i_1 & i_0 j_1 & i_0 k_1 \end{bmatrix} \begin{bmatrix} j_0 i_1 \\ j_0 j_1 \\ j_0 k_1 \end{bmatrix} = i_0 \cdot j_0 (1) = 0$$

$$c_2^T \cdot c_3 = \begin{bmatrix} j_0 i_1 & j_0 j_1 & j_0 k_1 \end{bmatrix} \begin{bmatrix} k_0 i_1 \\ k_0 j_1 \\ k_0 k_1 \end{bmatrix} = j_0 \cdot k_0 (1) = 0$$

similarly

$$c_3^T \cdot c_1 = \begin{bmatrix} k_0 i_1 & k_0 j_1 & k_0 k_1 \end{bmatrix} \begin{bmatrix} i_0 i_1 \\ i_0 j_1 \\ i_0 k_1 \end{bmatrix} = k_0 \cdot i_0 (1) = 0$$

$\Rightarrow$ column vectors of rotational matrix R_0^1 are orthogonal to each other.

Solution 2 Since, we know that rotation matrices are orthogonal. So, we can use properties of orthogonalities.

- By defination of orthogonal matrices

$$\left(R_0^1\right)^T = \left(R_0^1\right)^{-1}$$

On multiplying R_0^1 on both sides

$$R_0^1 \left(R_0^1\right)^T = R_0^1 \left(R_0^1\right)^{-1}$$

$$\Rightarrow R_0^1 \left(R_0^1\right)^T = I$$

On taking determinant on both sides

$$\det\left(R_0^1 (R_0^1)^T\right) = \det(I)$$
$$= 1$$

$$\Rightarrow \left[\det(R_0^1)\right]^2 = 1 \qquad \left(\because \det(R_0^1) = \det({R_0^1}^T\right)$$

$$\Rightarrow \boxed{\det(R_0^1) = \pm 1}$$

<u>Soution 5</u> we have to prove that $R S(a) R^T = S(Ra)$

- On observing carefully one can interprete both LHS & RHS as some operator, which can be operated over some vector for linear transformation of vector.
- Let's assume some arbitrary vector b such that $b \in \mathbb{R}^3$.

Now, let's apply operator of LHS on vector b.

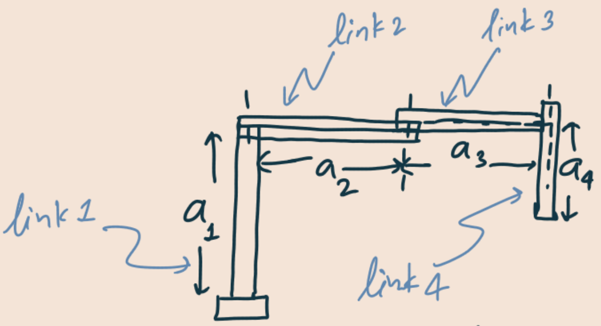
$$\begin{aligned} LHS(b) &= R S(a) R^T \, (b) \\ &= R S(a) R^T b \\ &= R\left(a \times R^T b\right) && \left(\because S(a)p = a \times p\right) \\ &= R(a) \times R(R^T b) && \left(\because R(p \times q) = R(p) \times R(q)\right) \\ &= R(a) \times (R R^T b) && \left(\because R R^T = I \text{ since } R \text{ is orthogonal matrix}\right) \\ &= R(a) \times b \\ &= S(R(a)) b \\ &= S(R(a)) \, (b) \\ &= RHS(b) \end{aligned}$$

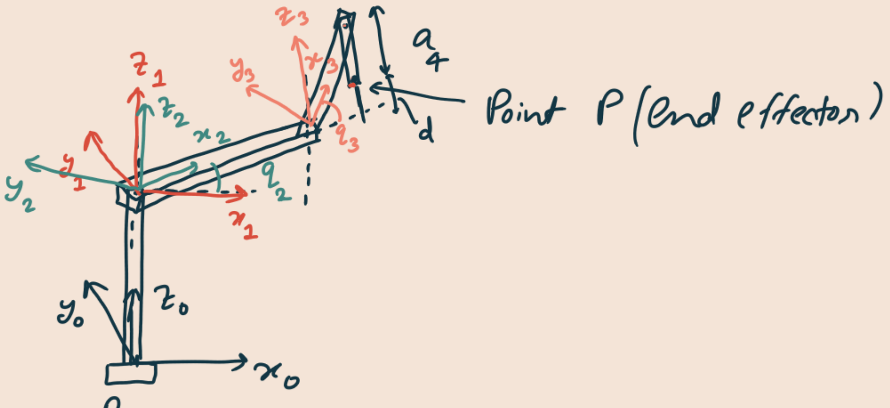
Hence proof.

$$\Rightarrow \boxed{R S(a) R^T = S(R(a))}$$

<u>Solution 6</u> Homogenous transformation for RRP SCARA configuration:

Schematics:





- a_1, a_2, a_3, a_4 are link lengths.

Base

$$\begin{bmatrix} P_0 \\ 1 \end{bmatrix} = H_0^1 H_1^2 H_2^3 \begin{bmatrix} P_3 \\ 1 \end{bmatrix} \quad \text{——— (1)}$$

$$R_0^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; d_0^1 = \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} \Rightarrow H_0^1 = \begin{bmatrix} R_0^1 & d_0^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1^2 = \begin{bmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; d_1^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow H_1^2 = \begin{bmatrix} cq_2 & -Sq_2 & 0 & 0 \\ Sq_2 & cq_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2^3 = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; d_2^3 = \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} \Rightarrow H_2^3 = \begin{bmatrix} cq_3 & -Sq_3 & 0 & a_2 \\ Sq_3 & cq_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} a_3 \\ 0 \\ -a_4 - d \end{bmatrix}$$

from equation (1)

$$\begin{bmatrix} P_0 \\ 1 \end{bmatrix} = H_0^1 H_1^2 H_2^3 \begin{bmatrix} P_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_2 & -Sq_2 & 0 & 0 \\ Sq_2 & cq_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_3 & -Sq_3 & 0 & a_2 \\ Sq_3 & cq_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ 0 \\ -a_4 - d \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} cq_2 & -Sq_2 & 0 & 0 \\ Sq_2 & cq_2 & 0 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_3 & -Sq_3 & 0 & a_2 \\ Sq_3 & cq_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ 0 \\ -a_4 - d \\ 1 \end{bmatrix}$$

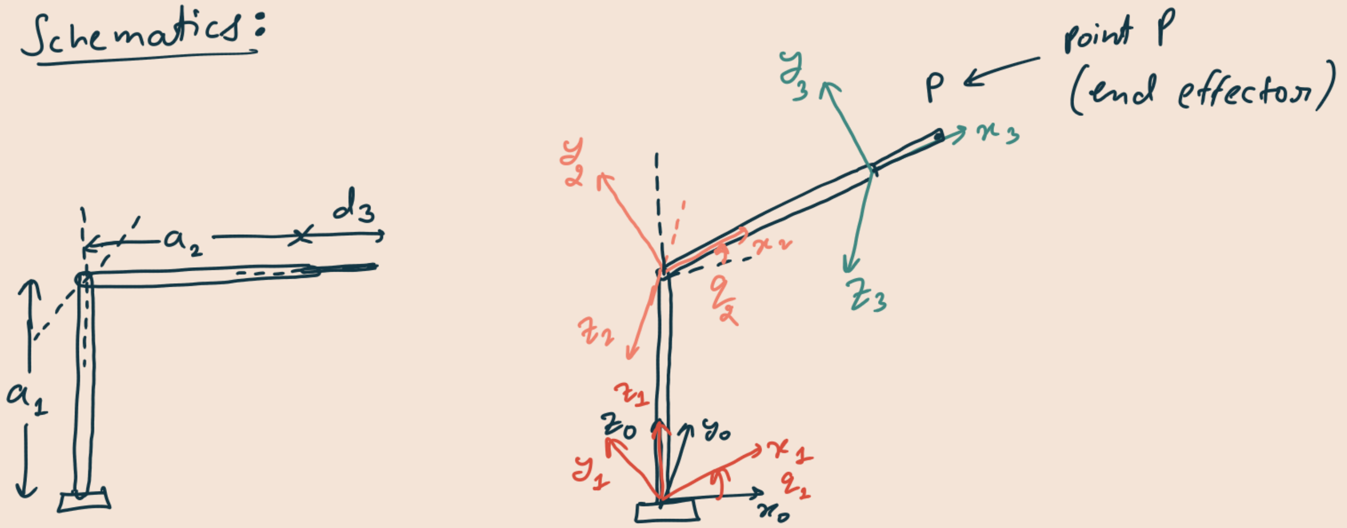
$$= \begin{bmatrix} cq_2 & -Sq_2 & 0 & 0 \\ Sq_2 & cq_2 & 0 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_3 cq_3 + a_2 \\ a_3 Sq_3 \\ -a_4 - d \\ 1 \end{bmatrix} = \begin{bmatrix} (a_2 + a_3 cq_3) cq_2 - a_3 Sq_2 Sq_3 \\ (a_2 + a_3 cq_3) Sq_2 + a_3 Sq_3 cq_2 \\ -a_4 - d + a_1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{P_0 = \begin{bmatrix} (a_2 + a_3 cq_3) cq_2 - a_3 Sq_2 Sq_3 \\ (a_2 + a_3 cq_3) Sq_2 + a_3 Sq_3 cq_2 \\ -a_4 - d + a_1 \end{bmatrix}}$$

Solution 7 Python code is attached with this PDF file.

Solution 8 Homogenous transformation for Stanford type RRP configuration

Schematics:



- a_1, a_2 are link lengths, d_3 is the extension at joint 3.

$$\begin{bmatrix} P_0 \\ 1 \end{bmatrix} = H_0^1 H_1^2 H_2^3 \begin{bmatrix} P_3 \\ 1 \end{bmatrix}$$

Now, $R_0^1 = \begin{bmatrix} cq_1 & -sq_1 & 0 \\ sq_1 & cq_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $d_0^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow H_0^1 = \begin{bmatrix} R_0^1 & d_0^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} cq_1 & -sq_1 & 0 & 0 \\ sq_1 & cq_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$R_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\pi_2 & -s\pi_2 \\ 0 & s\pi_2 & c\pi_2 \end{bmatrix} \begin{bmatrix} cq_2 & -sq_2 & 0 \\ sq_2 & cq_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} cq_2 & -sq_2 & 0 \\ sq_2 & cq_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} cq_2 & -sq_2 & 0 \\ 0 & 0 & -1 \\ sq_2 & cq_2 & 0 \end{bmatrix}$$

$$d_1^2 = \begin{bmatrix} 0 \\ 0 \\ a_1 \end{bmatrix} \Rightarrow H_1^2 = \begin{bmatrix} R_1^2 & d_1^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} cq_2 & -sq_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ sq_2 & cq_2 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2^3 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $d_2^3 = \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} \Rightarrow H_2^3 = \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

and $P_3 = \begin{bmatrix} d_3 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} P_0 \\ 1 \end{bmatrix} = H_0^1 H_1^2 H_2^3 P_3$$

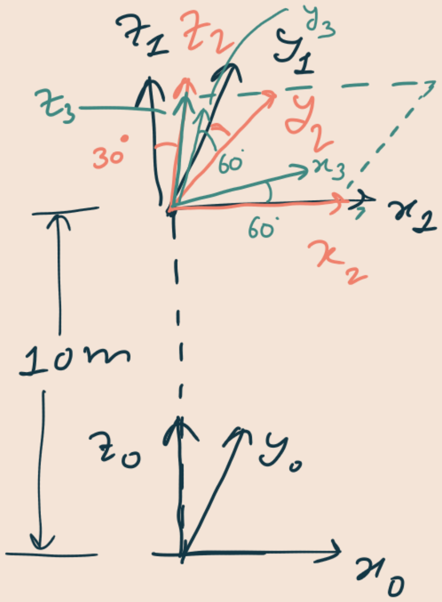
$$\Rightarrow \begin{bmatrix} P_0 \\ 1 \end{bmatrix} = \begin{bmatrix} cq_1 & -sq_1 & 0 & 0 \\ sq_1 & cq_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cq_2 & -sq_2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ sq_2 & cq_2 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} cq_1 cq_2 & -cq_1 sq_2 & sq_1 & 0 \\ sq_1 cq_2 & -sq_1 sq_2 & -cq_1 & 0 \\ sq_2 & cq_2 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_3 + a_2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (d_3+a_2)\, cq_1 cq_2 \\ (d_3+a_2)\, sq_1 cq_2 \\ (d_3+a_2)\, sq_2 + a_1 \\ 1 \end{bmatrix} \Rightarrow \boxed{P_0 = \begin{bmatrix} (d_3+a_2)\, cq_1 cq_2 \\ (d_3+a_2)\, sq_1 cq_2 \\ (d_3+a_2)\, sq_2 + a_1 \end{bmatrix}}$$

Python code for above formulation is attached with this PDF file

Solution (9)



- In order to find out the position vector of obstacle with respect to base frame we apply consecutive homogenous transformation in the logical sequence.

$$H_0^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_{\pi/6} & -S_{\pi/6} & 0 \\ 0 & S_{\pi/6} & C_{\pi/6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{\pi/3} & -S_{\pi/3} & 0 & 0 \\ S_{\pi/3} & C_{\pi/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} & 0 \\ \sqrt{3}/4 & \frac{1}{4} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

We know that

$$\begin{bmatrix} P_0 \\ 1 \end{bmatrix} = H_0^1 H_1^2 \begin{bmatrix} P_2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} P_0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} & 0 \\ \sqrt{3}/4 & \frac{1}{4} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} P_0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} & 0 \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{2} & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3/2 \\ \frac{3\sqrt{3}}{2} + 10 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{P_0 = \begin{bmatrix} 0 \\ -3/2 \\ 10 + \frac{3\sqrt{3}}{2} \end{bmatrix}}$$

Solution 10   Types of gearbox used in robotics application

- Spur gears
- Planetary gears
- Bevel gears
- Worm Gears
- Ball screw
- Idler gears
- Compound spur gears

- Spur gear ⇒ This type of gear can be used when we parallel driven and driving shafts. Very high gear ratio can not be acheive in this type of gear relative to other gear boxes, due to its simplest construction.

- Planetary gear ⇒ This gear is also known as "Epicyclic" gear. It consists three main parts: Ring gear, planet gears and central gear (sun gear). Any of this gear can be made stationary and different configuration can be acheived. Best part about this type of gear is that a very high gear ratio can be acheived. Also it takes very small space to accomodate.

- Bevel gears ⇒ This type of gearboxes are capable of transfering the torque / rotations in the shafts perpendicular to each other. This type of gearbox should be very strong due to its construction geometry. Again, very high gear ration can't be obtaine in this type of gearbox.

- Warm Gear ⇒ This type of gear are non-back drivable. They can only transfer torque/rotation in one direction, reverse is not possible due to its construction geometry. Although a high gear ratio can be acheived.

- **Ball Screw ⇒** This type of gears are lead screw with small bearing balls in the nuts. It serves the purpose of converting rotational motion into linear translation motion. These type of gears are highly efficient and have long lifespan.

- **Idler gears ⇒** This type of gears are generally used to reverse the direction of rotation only. It does not changes the gear ratio or in other words gear ratio is one.

- **Compound spur gear ⇒** This can be thought as extended version of simple spur gears. It's only used to increase the gear ratio. For doing so special arrangements of tooths is to be done.

Solution (11) Jacobian for RRP SCARA configuration

- We can use the results obtained in the problem (6) to quickly proceed ahead.
- Variables used for subsequent developement are same as of problem (6)
- Coordinates of end effectors (P_0) with respect to base frame is as follows:

$$P_0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (a_2 + a_3 c_{q_3}) c_{q_2} - a_3 s_{q_2} s_{q_3} \\ (a_2 + a_3 c_{q_3}) s_{q_2} + a_3 s_{q_3} c_{q_2} \\ -a_4 - d + a_1 \end{bmatrix}$$

- Note that we had q_2, q_3 and d as joint variables.
- We know that,

$$\dot{x} = J_v \dot{q}$$

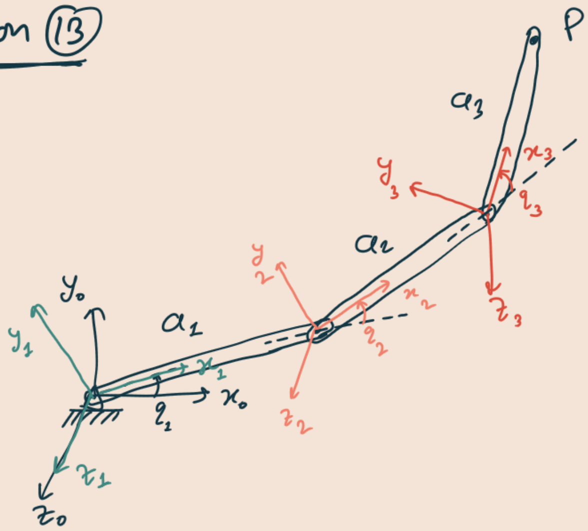
where, $\dot{x} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$ and $\dot{q} = \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \dot{d} \end{bmatrix}$

$$J_v = \begin{bmatrix} \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} & \frac{\partial x}{\partial d} \\ \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} & \frac{\partial y}{\partial d} \\ \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} & \frac{\partial z}{\partial d} \end{bmatrix} = \begin{bmatrix} -s_{q_2}(a_2 + a_3 c_{q_3}) - a_3 c_{q_2} s_{q_3} & (a_2 - a_3 s_{q_3}) c_{q_2} - a_3 s_{q_2} c_{q_3} & 0 \\ c_{q_2}(a_2 + a_3 c_{q_3}) - a_3 s_{q_2} s_{q_3} & (a_2 - a_3 s_{q_3}) s_{q_2} + a_3 c_{q_2} c_{q_3} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\boxed{J_v = \begin{bmatrix} -s_{q_2}(a_2+a_3c_{q_3}) - a_3 c_{q_2} s_{q_3} & (a_2 - a_3 s_{q_3}) c_{q_2} - a_3 s_{q_2} c_{q_3} & 0 \\ c_{q_2}(a_2+a_3c_{q_3}) - a_3 s_{q_2} s_{q_3} & (a_2 - a_3 s_{q_3}) s_{q_2} + a_3 c_{q_2} c_{q_3} & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$

Solution (12) Pythone code for obtaining the Jacobian of problem (10) is attached with this file.

Solution (13)



$$R_0^1 = \begin{bmatrix} c_{q_1} & -s_{q_1} & 0 \\ s_{q_1} & c_{q_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad d_0^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad H_0^1 = \begin{bmatrix} R_0^1 & d_0^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{q_1} & -s_{q_1} & 0 & 0 \\ s_{q_1} & c_{q_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1^2 = \begin{bmatrix} c_{q_2} & -s_{q_2} & 0 \\ s_{q_2} & c_{q_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad d_1^2 = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}, \quad H_1^2 = \begin{bmatrix} R_1^2 & d_1^2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{q_2} & -s_{q_2} & 0 & a_1 \\ s_{q_2} & c_{q_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2^3 = \begin{bmatrix} c_{q_3} & -s_{q_3} & 0 \\ s_{q_3} & c_{q_3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad d_2^3 = \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix}, \quad H_2^3 = \begin{bmatrix} R_2^3 & d_2^3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{q_3} & -s_{q_3} & 0 & a_2 \\ s_{q_3} & c_{q_3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} a_3 \\ 0 \\ 0 \end{bmatrix}$$

- P_0 is position of end effector with respect to base frame $O x_0 y_0 z_0$.

$$\Rightarrow \begin{bmatrix} P_0 \\ 1 \end{bmatrix} = H_0^1 H_1^2 H_2^3 \begin{bmatrix} P_3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{q_1} & -S_{q_1} & 0 & 0 \\ S_{q_1} & c_{q_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{q_2} & -S_{q_2} & 0 & a_1 \\ S_{q_2} & c_{q_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{q_3} & -S_{q_3} & 0 & a_2 \\ S_{q_3} & c_{q_3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{q_1+q_2} & -S_{q_1+q_2} & 0 & a_1 c_{q_1} \\ S_{q_1+q_2} & c_{q_1+q_2} & 0 & a_1 S_{q_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_3 c_{q_3} + a_2 \\ a_3 S_{q_3} \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (a_2 + a_3 c_{q_3}) c_{q_1+q_2} - a_3 S_{q_1+q_2} S_{q_3} + a_1 c_{q_1} \\ (a_2 + a_3 c_{q_3}) S_{q_1+q_2} + a_3 S_{q_3} c_{q_1+q_2} + a_1 S_{q_1} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_2 c_{q_1} + a_2 c_{q_1+q_2} + a_3 c_{q_1+q_2+q_3} \\ a_1 S_{q_1} + a_2 S_{q_1+q_2} + a_3 S_{q_1+q_2+q_3} \\ 0 \end{bmatrix}$$

Now we know that

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = J_v \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \quad \text{where,} \quad J_v = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{bmatrix}$$

$$\boxed{J_v = \begin{bmatrix} -a_1 S_{q_1} - a_2 S_{q_1+q_2} - a_3 S_{q_1+q_2+q_3} & -a_2 S_{q_1+q_2} - a_3 S_{q_1+q_2+q_3} & -a_3 S_{q_1+q_2+q_3} \\ a_1 c_{q_1} + a_2 c_{q_1+q_2} + a_3 c_{q_1+q_2+q_3} & a_2 c_{q_1+q_2} + a_3 c_{q_1+q_2+q_3} & a_3 c_{q_2+q_2+q_3} \\ 0 & 0 & 0 \end{bmatrix}}$$

since, it is simple case we can write orientation of end effector (ϕ) as follows

$$\phi_z = q_1 + q_2 + q_3 \quad , \quad \psi_x = 0 \quad , \quad \theta_y = 0$$

$\Rightarrow$ We know $\omega = J_\omega \dot{q}$

where, $\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$, $\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$, $J_\omega = \begin{bmatrix} \frac{\partial \psi_x}{\partial q_1} & \frac{\partial \psi_x}{\partial q_2} & \frac{\partial \psi_x}{\partial q_3} \\ \frac{\partial \theta_y}{\partial q_1} & \frac{\partial \theta_y}{\partial q_2} & \frac{\partial \theta_y}{\partial q_3} \\ \frac{\partial \phi_z}{\partial q_1} & \frac{\partial \phi_z}{\partial q_2} & \frac{\partial \phi_z}{\partial q_3} \end{bmatrix}$

$$\boxed{J_\omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}$$

Solution (14) Python code for problem (13) is attached with this PDF file.