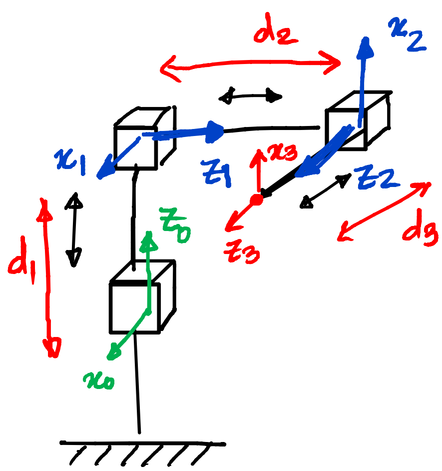
Assignment 3 - Navneet Kaur (18110106)

1. We know that Jacobian is a function of $\vec{q}$. Those configurations which lead to a decrease in the rank of the Jacobian Matrix (J), are known as the singular configurations. In such configurations, certain directions of motions are unattainable.

   Near singularities, there will be no solution for the inverse kinematics problem. The inverse of the Jacobian matrix will not exist, as the determinant of J will be zero.

5. Three-link cartesian manipulator:



	a_i	α_i	d_i	θ_i
q_1	0	$-\pi/2$	d_1	0
q_2	0	$\pi/2$	d_2	$\pi/2$
q_3	0	0	d_3	$-\pi/2$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^3 = A_1 A_2 A_3$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

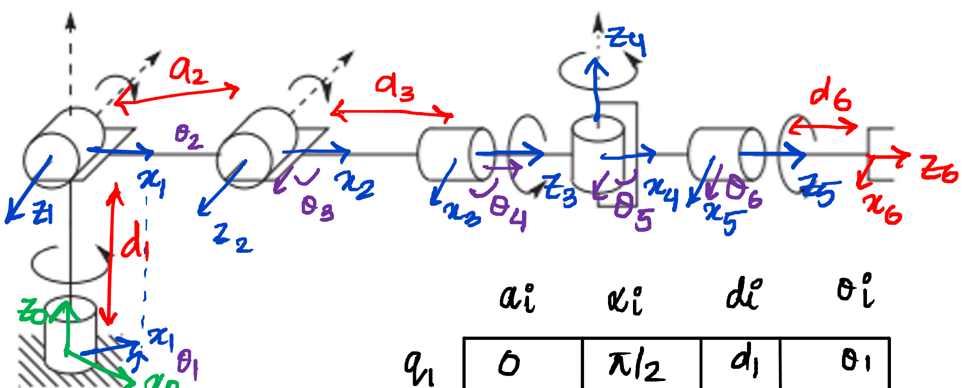
$$= \begin{bmatrix} 0 & 0 & 1 & d_3 \\ -1 & 0 & 0 & d_2 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$r_{11} = 0 \qquad r_{12} = 0 \qquad r_{13} = 1 \qquad d_x = d_3$

$r_{21} = -1 \qquad r_{22} = 0 \qquad r_{23} = 0 \qquad d_y = d_2$

$r_{31} = 0 \qquad r_{32} = -1 \qquad r_{33} = 0 \qquad d_z = d_3$

6. Spherical wrist + three link articulated manipulator



	a_i	α_i	d_i	θ_i
q_1	0	$\pi/2$	d_1	θ_1
q_2	a_2	0	0	θ_2
q_3	a_3	0	0	θ_3
q_4	0	$-\pi/2$	0	θ_4
q_5	0	$\pi/2$	0	θ_5
q_6	0	0	d_6	θ_6

$$A_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^6 = A_1 A_2 A_3 A_4 A_5 A_6 = T_0^3 T_3^6$$

$$T_0^3 = A_1 A_2 A_3$$

$$= \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & a_2c_1c_2 + a_3c_1c_{23} \\ s_1c_{23} & -s_1s_{23} & -c_1 & a_2c_2s_1 + a_3s_1c_{23} \\ s_{23} & c_{23} & 0 & a_2s_2 + a_3s_{23} + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^6 = A_4 A_5 A_6$$

$$= \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^6 = T_0^3 \, T_3^6$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{11} = c_1c_{23}(c_4c_5c_6 - s_4s_6) - c_1s_{23}(s_4c_5c_6 + c_4s_6) + s_1(-s_5s_6)$$

$$= c_1c_4c_5c_6c_{23} - c_1s_4s_6c_{23} - c_1s_4c_5c_6s_{23} - c_1c_4s_6s_{23} - s_1s_5s_6$$

$$= c_1c_5c_6(c_4c_{23} - s_4s_{23}) - c_1s_6(s_4c_{23} + c_4s_{23}) - s_1s_5s_6$$

$$\boxed{r_{11} = c_1c_5c_6c_{234} - c_1s_6s_{234} - s_1s_5s_6}$$

$$r_{12} = c_1c_{23}(-c_4c_5s_6 - s_4c_6) - c_1s_{23}(-s_4c_5s_6 + c_4c_6) + s_1s_5s_6$$

$$= -c_1 c_4 c_5 s_6 c_{23} - c_1 s_4 c_6 c_{23} + c_1 s_4 c_5 s_6 s_{23}$$
$$- c_1 c_4 c_6 s_{23} + s_1 s_5 s_6$$

$$\boxed{r_{12} = -c_1 (c_5 s_6 c_{234} + c_6 s_{234}) + s_1 s_5 s_6}$$

$$r_{13} = c_1 c_{23} (c_4 s_5) - c_1 s_{23} (s_4 s_5) + c_5 s_1$$
$$= c_1 c_4 s_5 c_{23} - c_1 s_4 s_5 s_{23} + s_1 c_5$$

$$\boxed{r_{13} = c_1 s_5 c_{234} + s_1 c_5}$$

$$d_x = c_1 c_{23} (c_4 s_5 d_6) - c_1 s_{23} (s_4 s_5 d_6)$$
$$+ s_1 (c_5 d_6) + (a_2 c_1 c_2 + a_3 c_1 c_{23})$$
$$= c_1 c_4 s_5 d_6 c_{23} - c_1 s_4 s_5 d_6 s_{23} + s_1 c_5 d_6$$
$$+ a_2 c_1 c_2 + a_3 c_1 c_{23}$$

$$\boxed{d_x = a_2 c_1 c_2 + a_3 c_1 c_{23} + d_6 (c_1 s_5 c_{234} + s_1 c_5)}$$

$$r_{21} = s_1 c_{23} (c_4 c_5 c_6 - s_4 s_6)$$
$$- c_1 s_{23} (s_4 c_5 c_6 + c_4 s_6) + c_1 s_5 c_6$$

$$\boxed{r_{21} = s_1 (c_5 c_6 c_{234}) - s_1 s_6 (s_{234}) + c_1 s_5 s_6}$$

$$r_{22} = s_1 c_{23} (- c_4 c_5 s_6 - s_4 c_6)$$
$$- s_1 s_{23} (- s_4 c_5 s_6 + c_4 c_6)$$
$$- c_1 s_5 s_6$$

$$\boxed{r_{22} = -S_1 C_6 S_{234} - S_1 C_5 C_6 C_{234} - C_1 S_5 S_6}$$

$$r_{23} = S_1 C_{23} (C_4 S_5) - S_1 S_{23} (S_4 S_5) - C_1 C_5$$

$$= S_1 C_{23} C_4 S_5 - S_1 S_{23} S_4 S_5 - C_1 C_5$$

$$\boxed{r_{23} = S_1 S_5 C_{234} - C_1 C_5}$$

$$d_y = S_1 C_{23} C_4 S_5 d_6 - S_1 S_{23} S_4 S_5 d_6 - C_1 C_5 d_6 + a_2 C_2 S_1 + a_3 S_1 C_{23}$$

$$= S_1 S_5 C_{234} d_6 - C_1 C_5 d_6 + a_2 C_2 S_1 + a_3 S_1 C_{23}$$

$$\boxed{d_y = a_2 C_2 S_1 + a_3 S_1 C_{23} + d_6 (S_1 S_5 C_{234} - C_1 C_5)}$$

$$r_{31} = S_{23} (C_4 C_5 C_6 - S_4 S_6) + C_{23} (S_4 C_5 C_6 + C_4 S_6)$$

$$= C_4 C_5 C_6 S_{23} - S_4 S_6 S_{23} + S_4 C_5 C_6 C_{23} + C_4 S_6 C_{23}$$

$$\boxed{r_{31} = S_6 C_{234} + C_5 C_6 S_{234}}$$

$$r_{32} = S_{23} (-C_4 C_5 S_6 - S_4 C_6) + C_{23} (-S_4 C_5 S_6 + C_4 C_6)$$

$$= -c_4 c_5 s_6 s_{23} - s_4 c_6 s_{23}$$
$$- s_4 c_5 s_6 c_{23} + c_4 c_6 c_{23}$$

$$\boxed{r_{32} = c_6 c_{234} - c_5 s_6 s_{234}}$$

$$r_{33} = s_{23} c_4 s_5 + c_{23} s_4 s_5 = s_5 s_{234}$$

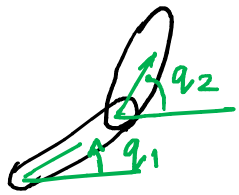
$$\boxed{r_{33} = s_5 s_{234}}$$

$$d_z = s_{23} (c_4 s_5 d_6)$$
$$+ c_{23} (s_4 s_5 d_6)$$
$$+ a_2 s_2 + a_3 s_{23} + d_1$$

$$\boxed{d_z = s_5 d_6 s_{234} + a_2 s_2 + a_3 s_{23} + d_1}$$

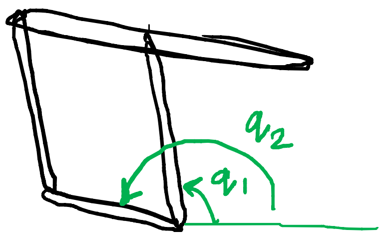
7. 2R Manipulator $\longrightarrow$ Direct drive
$\quad\quad\hookrightarrow$ remotely-driven
$\quad\quad\hookrightarrow$ 5 bar parallelogram

In directly driven, each joint has its motor at its respective origin. Whereas, in remotely driven, both the joints are driven by motors mounted at the base. Thus, the angle moved by the second link will not be affected by the angle swept by the first link.



q_2 will not depend on q_1

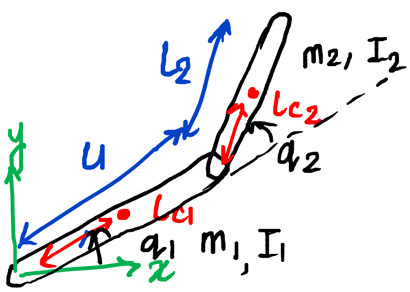
For the 5 bar parallelogram arrangement, the equations of the manipulator are decoupled.



Thus, q_1 and q_2 can be controlled independently in this arrangement as well.

τ_1 will depend only on $d_{11}, \phi_1, \ddot{q}_1$ terms and similarly τ_2 will only depend on $d_{22}, \phi_2, \ddot{q}_2$

8.



l_{c1}, l_{c2} - distance between center of mass of link and previous joint

$\cos q_1 = c_1 \quad \sin q_1 = s_1$

$\cos q_2 = c_2 \quad \sin q_2 = s_2$

$$\vec{v}_{c1} = J_{v_{c1}} \dot{q}$$

$$= \begin{bmatrix} -l_{c1} s_1 & 0 \\ l_{c1} c_1 & 0 \\ 0 & 0 \end{bmatrix} \dot{q}$$

$$\vec{v}_{c2} = J_{v_{c2}} \dot{q}$$

$$= \begin{bmatrix} -l_1 s_1 - l_{c2} s_{12} & -l_{c2} s_{12} \\ l_1 c_1 + l_{c2} c_{12} & l_{c2} c_{12} \\ 0 & 0 \end{bmatrix} \dot{q}$$

For translational KE,

$$TKE = \frac{1}{2} m_1 \vec{v_{c1}}^T \vec{v_{c1}} + \frac{1}{2} m_2 \vec{v_{c2}}^T v_{c2}$$

$$= \frac{1}{2} \dot{q} \left[m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}} \right] \dot{q}$$

$$\omega_1 = \dot{q}_1 \hat{k} \qquad \omega_2 = (\dot{q}_1 + \dot{q}_2) \hat{k}$$

For rotational KE,

$$RKE = \frac{1}{2} \dot{q}^T \left[I_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right] \dot{q}$$

$$D(q) = m_1 J_{V_{C1}}^T J_{V_{C1}} + m_2 J_{V_{C2}}^P J_{V_{C2}}$$

$$+ \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

Thus, we have,

$$\boxed{\begin{aligned} d_{11} &= m_1 l_{c1}^2 + m_2 (l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} c_2) + I_1 + I_2 \\ d_{12} &= d_{21} = m_2 (l_{c2}^2 + l_1 l_{c2} c_2) + I_2 \\ d_{22} &= m_2 l_{c_2}^2 + I_2 \end{aligned}}$$

Christoffel symbols,

$$c_{111} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0$$

$$c_{121} = c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2 l_1 l_{c2} s_2 = h$$

$$c_{221} = \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = m_2 l_1 l_{c2} s_2 = h$$

$$c_{112} = \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2 l_1 l_{c2} s_2 = -h$$

$$c_{122} = c_{212} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0$$

$$c_{222} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0$$

For potential energy,

$$V_1 = m_1 g l_{c1} s_1 \qquad V_2 = m_2 g (l_1 s_1 + l_{c2} s_{12})$$

$$V = V_1 + V_2 = m_1 g l_{c1} s_1 + m_2 g (l_1 s_1 + l_{c2} s_{12})$$

$$\phi_1 = \frac{\partial V}{\partial q_1} = (m_1 l_{c1} + m_2 l_1) g c_1 + m_2 l_{c2} g c_{12}$$

$$\phi_2 = \frac{\partial V}{\partial q_2} = m_2 l_{c2} g c_{12}$$

$$\tau_1 = d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1$$
$$+ c_{221}\dot{q}_2^2 + \phi_1$$

$$\tau_2 = d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + \phi_2$$

10. We have $D(q)$ and $V(q)$

$$K = \frac{1}{2}\sum_{i,j}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

$$\mathcal{L} = K - V = \frac{1}{2}\sum_{ij} d_{ij}(q)\dot{q}_i\dot{q}_j - V(q)$$

Now,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_j d_{kj}(q)\dot{q}_j$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_j d_{kj}(q)\ddot{q}_j + \sum_j \frac{d}{dt} d_{kj}(q)\dot{q}_j$$

$$= \sum_j d_{kj}(q)\ddot{q}_j + \sum_{ij}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{ij}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial V}{\partial q_k}$$

$$\sum_j d_{kj}(q)\ddot{q}_j + \sum_{ij}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_k}\right)\dot{q}_i\dot{q}_j - \frac{\partial V}{\partial q_k} = \tau_k$$

$$\sum_{ij}\left(\frac{\partial d_{kj}}{\partial q_i}\right)\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{ij}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right)\dot{q}_i\dot{q}_j$$

$$\Rightarrow \sum_{ij}\left(\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j$$

$$= \sum_{ij}\underbrace{\frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right)}_{c_{ijk}}\dot{q}_i\dot{q}_j$$

$$\phi_k = \frac{\partial V}{\partial q_k}$$

$$\boxed{\sum_j d_{kj}(q)\ddot{q}_j + \sum_{ij} c_{ijk}(q)\dot{q}_i\dot{q}_j + \phi_k(q) = \tau_k \qquad k = 1, 2, \dots, n}$$