Generating Networks to Target Assortativity via Archimedean Copula Graphons

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Motivation I

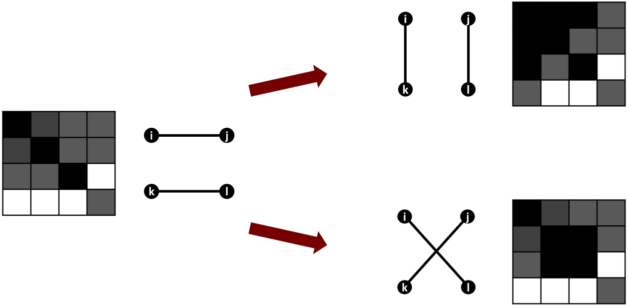


Figure 1: Example of double edge swap rewiring for a network

The aim of the paper is to address the question:

Can we generate a random graph to a target level of assortativity, without having to rewire?

Paper link: `https://arxiv.org/abs/2503.03061`

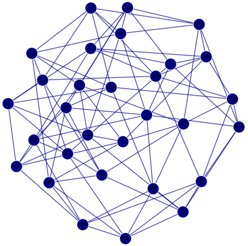
Simple graphs

A simple graph $G = (V, E)$, also referred to as $G(n, m)$:

- $|V| = n$ is the size (number of nodes)
- $|E| = m$ is the number of edges.

  Given $A = \{A_{ij}\}$ the adjacency matrix,

- A is symmetric $A_{ii} = 0$
- $A_{ij} = 0$ or $1, i \neq j$, no multiedges.



Homomorphism densities

The homomorphism density between simple graphs [2], for $V(F) < V(G)$:

$$t\left(F,G\right) := \frac{\hom(F,G)}{n^m} \tag{1}$$

as well as the isomorphisms, t_{inj} or $|F|$, follows:

$$t_{\mathrm{inj}}\left(F,G\right) := \frac{\mathrm{inj}(F,G)}{(n)_m} \tag{2}$$

where $(n)_m$ is the falling factorial.

Assortativity Coefficient I

- The (degree) assortativity coefficient is a graph parameter r_S, it is the Pearson correlation coefficient applied to the pairs of excess degrees on each edge of a network [3, 4, 1, 5]:

$$r = \frac{|P_2|(|P_{3/2}| + C - |P_{2/1}|)}{3|S_3| - |P_2|(|P_{2/1}| - 1)} \tag{3}$$

where $C = \frac{3|C_3|}{|P_2|}$ is the clustering coefficient and $|P_{r/s}| := |P_r|/|P_s|$ and

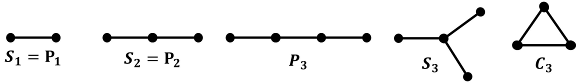


Figure 2: Star and path networks in the combinatorial expression of Newman's assortativity coefficient

Graphons I

- A graphon $W \in \mathcal{W}$ is an integrable function $W : [0,1]^2 \to [0,1]$, $u_i, u_j \in [0,1]$

$$A_{ij}|\mathbf{u} \sim \mathit{Bernoulli}\left(W(u_i, u_j)\right) \tag{4}$$

- The homomorphism density of F_G for a given graphon is expressed as:

$$t(F, W) = \int \prod_{i \in V(F)} dx_i \prod_{ij \in E(F)} W(x_i, x_j) \tag{5}$$

Note that $t_{\mathrm{inj}}(F, W) = t(F, W)$ since all maps of i occur with probability 1.

Graphons II

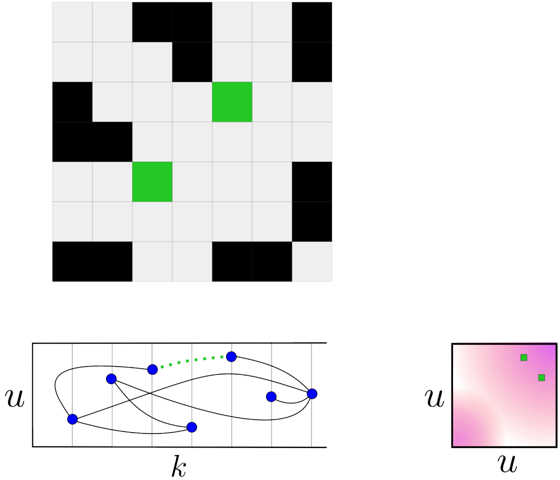


Figure 3: Illustrative example of a graphon, source: https://arxiv.org/abs/1512.03099, CC BY 3.0

Archimedean copulas

General form of an Archimedean copula:

$$C\left(u_1, u_2\right) = \varphi_\theta^{[-1]}\left(\varphi_\theta(u_1) + \varphi_\theta(u_2)\right) \qquad \text{for } u_i \in [0, 1]$$

where φ_θ and $\varphi_\theta^{[-1]}$ are defined are generator and the (pseudo) generator inverse.

Copulas II

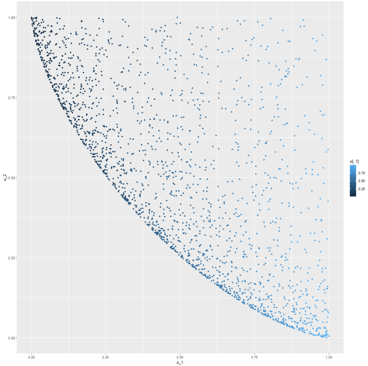


Figure 4: Clayton copula $\theta = -0.7$

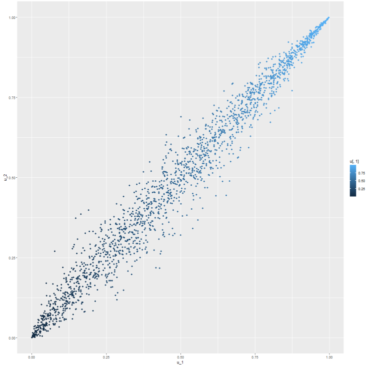


Figure 5: Gumbel copula $\theta = 10$

Copula Graphons I

- Let G be a simple graph with graphon W with assumed copula structure C. The adjacency matrix A of G is as follows:

$$A_{ij}|\mathbf{u} \sim Bernoulli\left(C(u_i, u_j)\right) \tag{6}$$

  where u_i, u_j are uniform i.i.d random variables.

- We can express several graph statistics of the copula graphon in terms of the generator function.

Theorem (Edge Density of the Copula Graphon)

Under the copula graphon framework for dense graphs, the edge density, $|P_1| = t(P_1, W)$ *is:*

$$\frac{1}{2} - \int_1^{\varphi_\theta^{-1}(1)} \int_1^{\varphi_\theta^{-1}(1)} \varphi_\theta(x)\, \varphi_\theta'(x+y)\, \varphi_\theta'(y)\, dx\, dy. \qquad (7)$$

Assortativity Theorems I

By using (5), we can relate the degree-degree assortativity with the graphon function.

Theorem (Degree Assortativity Coefficient with Graphon Homomorphism Densities)

Let $W \in \mathcal{W}$ be a graphon on n nodes. The degree-degree assortativity coefficient characterised by homomorphism counts is:

$$r_W = \frac{(n-3)t(P_3, W) + \frac{3nt(C_3,W)}{(n-1)(n-2)} - \frac{(n-2)t(P_2,W)^2}{t(P_1,W)}}{(n-3)3t(S_3, W) + t(P_2, W) - \frac{(n-2)t(P_2,W)^2}{t(P_1,W)}}. \tag{8}$$

Algorithm 1: Generative Pseudo Algorithm for a copula graphon or copula density graphon to desired r

Input: Function $F^\star$ which is bivariate Archimedean copula function C or bivariate Archimedean copula density function c; parameter θ; n number of nodes

Init: Generate an empty adjacency matrix A with n rows and n columns

for each $i = 1$ to j **do**

Generate A_{ij} which satisfies the following:

$$\mathbb{P}(A_{ij} = 1 | u_1, u_2) = F^\star(u_i, u_j) \tag{9}$$

Replace lower triangular matrix with transpose of upper triangular matrix.

Set diagonal= 0 **end for** set matrix $A = (A_{ij})$

Return: Simple graph W_G generated from A.

Synthetic Example: Zero assortativity networks

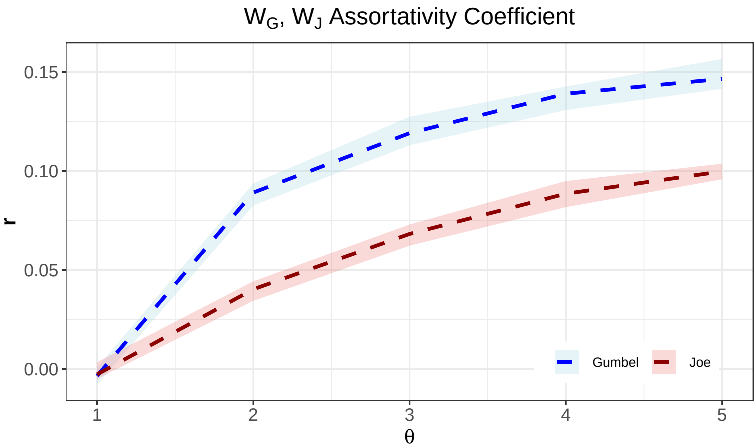


Figure 6: Assortativity coefficient range by W_G, W_J for $\theta \in [1, 5]$ for neutral networks. Blue, red line is the average assortativity coefficient of W_G, W_J respectively, over 10 repetitions, the minimum and maximum within the range is the shaded region

Synthetic Example: Assortative networks

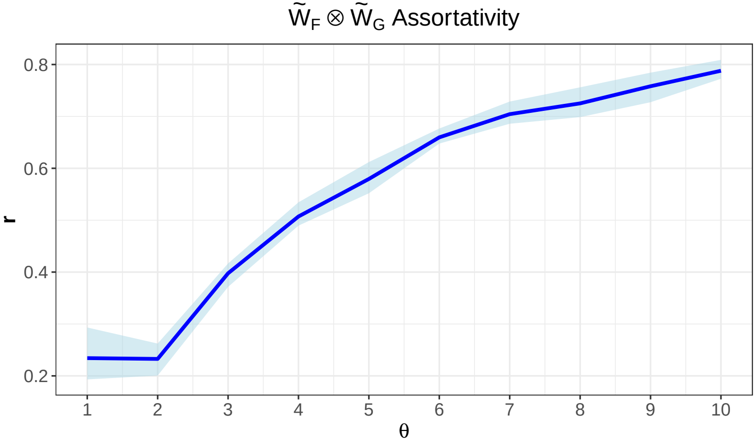


Figure 7: Assortativity coefficient range through $\tilde{W}_J \otimes \tilde{W}_G$ for $\theta_J = 2$, $\theta_G \in [1, 10]$. Blue line is the average assortativity coefficient over 10 repetitions, the minimum and maximum within the range is the shaded region

Synthetic Example: Disassortative networks

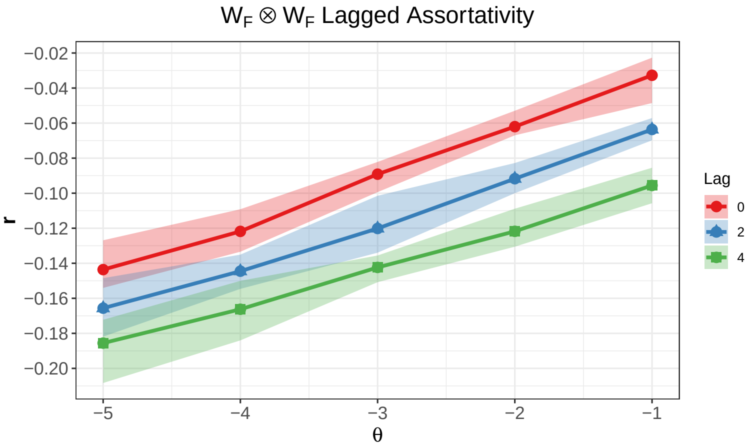


Figure 8: Assortativity coefficient range through $\tilde{W}_F \otimes \tilde{W}_F$ for $\theta_F \in [1, 5]$ at different lags. Assortativity coefficient is averaged over 10 repetitions. Line is the average assortativity coefficient over 10 repetitions, the minimum and maximum within the range is the shaded region

Conclusion and discussion

- We have developed a method using copulas and graphons to generate a network to meet a target level of assortativity without using rewiring.
- We conducted extensive experiments to explore the possible assortativity different within the parameter space of selected:
  - copula graphons,
  - copula density graphons and their
  - copula tensor products.

Thank you