1 Curvature Corrections

1.1 Preliminary Setup

Barton et al. give a prescription on how to determine first order corrections - first order in ℓ - on the proportionality between the number of links and the area of the horizon.

First, let's define the horizon $\mathcal{H}$ as

$$\mathcal{H}(x^\mu) = r - 2M = 0, \tag{1}$$

and the spacelike hypersurface Σ as

$$\Sigma(x^\mu) = t^* = 0. \tag{2}$$

The intersection of the two hypersurfaces is $\mathcal{J}$, which represents the horizon at time $t^* = 0$. Evaluating a tensor at $\mathcal{J}$ implies setting $r = 2M$ and $t^* = 0$.

The metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ in EFS coordinates are

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & \frac{2M}{r} & 0 & 0 \\ \frac{2M}{r} & \left(1+\frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{3}$$

and

$$g^{\mu\nu} = \begin{pmatrix} -\left(1+\frac{2M}{r}\right) & \frac{2M}{r} & 0 & 0 \\ \frac{2M}{r} & \left(1-\frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{4}$$

First order curvature corrections must depend on mutually independent invariant scalars of dimension l^{-1}, determined by the geometric setup and evaluated at $\mathcal{J}$. Barton et al identified as all and the only quantities satisfying said conditions: the trace K of the extrinsic curvature of Σ, its component K_m tangential to Σ and orthogonal to $\mathcal{J}$, and the null expansion ϑ of the horizon.

They found the first order correction to be, in 4 dimensions,

$$\Delta a^{(1)} \approx -\ell(0.036\, K + 0.088\, K_m + 0.021\, \theta). \tag{5}$$

1.2 The Vectors

The corrections are derived from the basic dimensionless geometrical objects of our setup. One is obviously the metric. The other two are the vectors n^μ and k^μ, which are the normal vectors to the hypersurfaces involved, i.e. Σ. and $\mathcal{H}$ respectively.

The vector n^μ is a future-pointing, normalised as $n^\mu n_\mu = -1$. Then

$$n_\mu = a\partial_\mu\Sigma = a(1,0,0,0), \tag{6}$$

and

$$n^\mu = g^{\mu\nu}n_\nu = a\left(-1-\frac{2M}{r}, \frac{2M}{r}, 0, 0\right), \tag{7}$$

with $a < 0$ as it is required to be future pointing. Normalisation then clearly imposes $a = -1/\sqrt{(2M/r)+1}$.

The vector k^μ is also future pointing, and normalised such that $[k.n]_{\mathcal{J}} = -1/\sqrt{2}$, where $[...]_{\mathcal{J}}$ means evaluated at $\mathcal{J}$. Then

$$k_\mu = b\,\partial_\mu\mathcal{H} = b(0,1,0,0), \tag{8}$$

and we immediately see that $[k_\mu n^\mu]_{\mathcal{J}} = -1/\sqrt{2}$ sets $b = 1$.

Finally, we define the vector $m^\mu = \sqrt{2}k^\mu - n^\mu$. Evaluated on $\mathcal{J}$, this is tangent to Σ and orthogonal to $\mathcal{J}$.

In table I, the vectors n^μ, k^μ, m^μ and the corresponding covectors are shown in their general form and as evalaluated on $\mathcal{J}$.

NEED TO MAKE TABLE, FOR NOW...

$$\begin{cases} n^\mu = \left(\sqrt{1+\frac{2M}{r}}, -\frac{1}{\sqrt{1+\frac{2M}{r}}}, 0, 0\right) & \mapsto \left(\sqrt{2}, -\frac{-1}{\sqrt{2}}, 0, 0\right) \\ n_\mu = \left(-\frac{1}{\sqrt{1+\frac{2M}{r}}}, 0, 0, 0\right) & \mapsto \left(-\frac{-1}{\sqrt{2}}, 0, 0, 0\right) \end{cases}. \tag{9}$$

$$\begin{cases} k^\mu = (\frac{2M}{r}, 1-\frac{2M}{r}, 0, 0) & \mapsto (1,0,0,0) \\ k_\mu = (0,1,0,0) & \mapsto (0,1,0,0) \end{cases}. \tag{10}$$

$$\begin{cases} m^\mu = \left(\frac{2\sqrt{2}M}{r} - \sqrt{1+\frac{2M}{r}}, \sqrt{2}\left(1-\frac{2M}{r}\right) + \frac{-1}{\sqrt{2}}, 0, 0\right) & \mapsto \left(0, \frac{1}{\sqrt{2}}, 0, 0\right) \\ m_\mu = \left(\frac{1}{\sqrt{1+\frac{2M}{r}}}, \sqrt{2}, 0, 0\right) & \mapsto \left(-\frac{1}{\sqrt{2}}, \sqrt{2}, 0, 0\right) \end{cases}. \tag{11}$$

1.3 Local Geometric Invariants

To evaluate the geometric invariants we need to introduce the "projector" tensors

$$h^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} + n^{\alpha} n_{\beta}, \tag{12}$$

and

$$\sigma^{\alpha}{}_{\beta} = h^{\alpha}{}_{\beta} - m^{\alpha} m_{\beta}. \tag{13}$$

These tensors project vectors defined on Σ and $\mathcal{H}$ on the tangent spaces of said hypersurfaces.

Then, we define the extrinsic curvature as

$$K_{\mu\nu} = (\nabla_{\beta} n_{\alpha}) h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu}. \tag{14}$$

The geometric invariants all evaluated on $\mathcal{J}$, are the trace K of the extrinsic curvature

$$K(\mathcal{J}) = [g^{\mu\nu} K_{\mu\nu}]_{\mathcal{J}}, \tag{15}$$

its component along the $m-$direction

$$K_m(\mathcal{J}) = [K_{\mu\nu} m^{\mu} m^{\nu}]_{\mathcal{J}}, \tag{16}$$

and the null expansion of the horizon

$$\vartheta = (\nabla_{\beta} k_{\alpha}) \sigma^{\alpha\beta}. \tag{17}$$

WRITE DOWN CHRISTOOFEL SYMBOLS OF METRIC (WHICH NO ONE DID BEFORE FOR EFS) AND MATRIX FORM OF INTERMEDIATE STUFF.

In the end, $\vartheta = 0$, as expected as the Schwarzschild black hole is static. The extrinsic curvature is found being

$$K_{\mu\nu} = -\frac{1}{32} \begin{pmatrix} \frac{\sqrt{2}}{M} & \frac{2\sqrt{2}}{M} & 0 & 0 \\ \frac{2\sqrt{2}}{M} & \frac{4\sqrt{2}}{M} & 0 & 0 \\ 0 & 0 & 64M & 0 \\ 0 & 0 & 0 & 64M \sin^2\theta \end{pmatrix}. \tag{18}$$

Then, the trace is

$$K = -\left(1 + \frac{\sqrt{2}}{16}\right) \frac{1}{M} \approx -\frac{1.088}{M}, \tag{19}$$

whereas the component along m^{μ} is

$$K = -\frac{\sqrt{2}}{16} \frac{1}{M} \approx -\frac{0.088}{M}. \tag{20}$$

These result in a correction on the links' proportionality

$$\Delta a^{(1)} \approx 0.0464 \frac{\ell}{M}. \tag{21}$$