1 Sprinkling a Black Hole

Different metric are available to depict a Black Hole. Here we concentrate on Schwarzschild and Eddington-Finkelstein (EF) coordinates. The latter can be divided into two case: original Eddington-Finklestein (EForig), as they were the ones initially proposed by the authors, and "uv" Eddington-Finklestein (EFuv), the more conventionally used ones.

Given a point with Schwarzschild coordinates $x^\mu_S = (t_s, r, \theta, \phi)$, then the ingoing EF coordinates, in the two cases, are defined by a change in the time coordinates, with, for the original,

$$t^* = t_{EForig_{in}} = t_S + 2M \ln \left| \frac{r}{2M} - 1 \right|, \tag{1}$$

and for the tortoise ones:

$$v = t_{EFv} = t_S + r + 2M \ln \left| \frac{r}{2M} - 1 \right|. \tag{2}$$

Given that the in Schwarzschild coordinates the metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{3}$$

where $d\Omega^2 = d\theta^2 + \sin\theta d\phi^2$, in the aforementioned Eddington-Finkelstein coordinates we have

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^{*2} + \frac{4M}{r} dt^* dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2, \tag{4}$$

and

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2. \tag{5}$$

It can be easily shown that these metrics have the same volume element

$$dV = \sqrt{-g_{\mu\nu}}\, dt\, dr\, d\theta\, d\phi = r^2 \sin\theta\, dt\, dr\, d\theta\, d\phi, \tag{6}$$

which is the same as the one of cylindrical coordinates in 3+1 dimensions. This allows for the implementation of the same sprinkling technique previously described.

Finally, it must be noted EForig coordinates have a profound advantage over the other sets of coordinates: a constant-time hypersurface is spacelike. In fact, let us define the constant-time hypersurface Σ as

$$\Sigma(x^\mu) = t^* = const. \tag{7}$$

rsurface is defined by the orientation of the normal vector n^μ, where

$$n_\mu = \partial_\mu \Sigma, \tag{8}$$

and the orientation is given by the scalar product

$$n^\mu n_\mu = g^{\mu\nu} n_\nu n_\mu. \tag{9}$$

More precisely, the hypersurface is spacelike if said product is smaller than 0, it is timelike if larger than 0, it is null if 0.

The covector normal to the constant-time hypersurface is $n_\mu = (1, 0, 0, 0)$. The inverse metric in EForig coordinates is given by

$$g^{\mu\nu} = \begin{pmatrix} -\left(1 + \frac{2M}{r}\right) & \frac{2M}{r} & 0 & 0 \\ \frac{2M}{r} & \left(1 - \frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix}. \tag{10}$$

This results in the product being

$$n^\mu n_\mu = -\left(\frac{2M}{r} + 1\right) < 0 \, \forall r \in \mathbb{R}^+. \tag{11}$$

1.1 Rotating the Angles

Given two events E_1 and E_2, it is always possible to rotate their spatial coordinates to fix their position on the equatorial plane ($\theta = \pi/2$) and having $\pi > \phi_2 > \phi_1 = 0$. This means E_1 will lie on the x-axis, whereas E_2 somewhere on the equatorial plane, with positive y coordinate.

To do so, we require three rotations. In the following, the convention used is to assume them clockwise.

- First, a rotation around the z-axis by an angle $\alpha = -\phi_1$, yielding

$$\begin{aligned} \phi_1 \mapsto \phi_1^{(1)} &= 0 & , \\ \phi_2 \mapsto \phi_2^{(1)} &= \Delta\phi = \phi_2 - \phi_1. & . \end{aligned} \tag{12}$$

- Secondly, a rotation around the y-axis by an angle $\beta = \pi/2 - \theta_1$ to fix E_1 on the x-axis. This rotation reads

$$R_y\left(\frac{\pi}{2} - \theta_1\right) = \begin{pmatrix} \sin\theta_1 & 0 & \cos\theta_1 \\ 0 & 1 & 0 \\ -\cos\theta_1 & 0 & \sin\theta_1 \end{pmatrix}. \tag{13}$$

  Given that the spatial vector of E_2 after the first rotation is

$$v_2^{(1)} = r_2 \begin{pmatrix} \sin\theta_2 \cos\Delta\phi \\ \sin\theta_2 \sin\Delta\phi \\ \cos\theta_2 \end{pmatrix}, \tag{14}$$

  the rotation leads to the new position coordinates

$$v_2^{(2)} = r_2 \begin{pmatrix} \sin\theta_1 \sin\theta_2 \cos\Delta\phi + \cos\theta_1 \cos\theta_2 \\ \sin\theta_2 \sin\phi_2^{(1)} \\ \cos\theta_1 \sin\theta_2 \cos\Delta\phi - \sin\theta_1 \cos\theta_2 \end{pmatrix}. \tag{15}$$

- Finally, a rotation about the x-axis is required to have E_2 lying on the equatorial plane while keeping E_1 on the x-axis. In general,

$$\phi = sign(y) \arccos \frac{x}{\sqrt{x^2 + y^2}}. \tag{16}$$

  As after this rotation $z_2 = 0$, we note that $\sqrt{x_2^2 + y_2^2} = r_2$. As this rotation leaves the x coordinates unchanged, we can already compute $\phi_2^{(3)}$ without requiring to determine the angle of rotation:

$$\phi_2^{(3)} = sign(y_2) \arccos\left(\sin\theta_1 \sin\theta_2 \cos\Delta\phi + \cos\theta_1 \cos\theta_2\right). \tag{17}$$

  Finally, we note that y_2 can always be set positive as, if it were negative, an additional π rotation about the x-axis would invert its sign. Therefore, we conclude we can always set our two points to

$$\begin{aligned} E_1 &= (t_1^*, r_1, \pi/2, 0), \\ E_2 &= (t_2^*, r_2, \pi/2, \arccos\left(\sin\theta_1 \sin\theta_2 \cos\Delta\phi + \cos\theta_1 \cos\theta_2\right)). \end{aligned} \tag{18}$$

1.2 The Null Geodesic Upper Bound, He 2009

The Rideout, He 2009 paper provides an algorithm for computing whether two points in a Black Hole Spacetime are causally connected. Said points can be expressed, in EForig coordinates, as $E_1 = (t_1^*, r_1, \pi/2, 0)$ and $E_2 = (t_2^*, r_2, \pi/2, \varphi_2)$, where the symmetry about $\theta = \pi/2$ (such plane is totally geodesic) has been exploited to de facto eliminate one coordinate and thus simplify the problem. In the following, assume $t_1^* < t_2^*$.

In the paper a line of constant spatial components is taken to be a worldline of an object everywhere in spacetime. This is wrong inside the horizon.Due to the bending of the light-cones occurring inside the horizon, all objects are forced to gradually fall towards the singularity, hence no object can mantain a worldline of constant spatial components. Therefore, in the 2D case ($\varphi_2 = 0$) and in the integral proceeding, an additional bound is required.

1.2.1 The 2D Case

Consider the case $r_1 \geq r_2$, i.e. the first point is further from the singularity than the second. There are three possible cases: both outside the horizon, one in one out, both inside. The paper states that "for both events inside the horizon, the necessary and sufficient condition for their causal relation is still $t_2 \geq t_1 + r_1 - r_2$". This is clearly wrong as it only considers the constant v null line. In fact, in 2D, the spacetime element can be written as

$$ds^2 = -dt^{*2} + dr^2 + \frac{2M}{r}(dt^* + dr)^2, \tag{19}$$

which decomposes into

$$ds^2 = (dt^* + dr)\left[\left(\frac{2M}{r} - 1\right)dt^* + \left(\frac{2M}{r} + 1\right)dr\right]. \tag{20}$$

Setting the spacetime element to 0 gives the null curves, which are the constant v straight-line

$$dt^* + dr = 0 \;\Rightarrow\; t_2^* - t_1^* = r_1 - r_2, \tag{21}$$

and the curved

$$dt^* = \frac{2M + r}{r - 2M}dr \;\Rightarrow\; t_2^* - t_1^* = \left[r + 4M\ln(r - 2M)\right]_{r_1}^{r_2}. \tag{22}$$

Two points both inside the horizon can only be connected iff the second falls inside the light cone of the first, i.e., the second falls within the two null curves previously computed passing through the first:

$$r_1 - r_2 \leq t_2^* - t_1^* \leq r_2 - r_1 + 4M\ln\left(\frac{2M - r_2}{2M - r_1}\right). \tag{23}$$

1.2.2 The 3D and 4D Case

The error in the integral section closely resembles the aforementioned one: ignoring that inside the horizon the light cone bends, such that a curve of constant spatial components is NOT a world line of a particle.

Let's first review the derivation of the required formulas. Fixing $\theta = \pi/2$ leads to the metric, in EFS coordinates:

$$ds^2 = -\left(\frac{2M}{r} - 1\right) dt^{*2} + \frac{4M}{r} dt^* dr + \left(\frac{2M}{r} + 1\right) dr^2 + r^2 d\phi^2, \tag{24}$$

which means any geodesics must satisfy, given an affine parameter λ,

$$\{-1\}_{tl} = -\left(\frac{2M}{r} - 1\right)\left(\frac{dt^*}{d\lambda}\right)^2 + \frac{4M}{r}\frac{dt^*}{d\lambda}\frac{dr}{d\lambda} + \left(\frac{2M}{r} + 1\right)\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2, \tag{25}$$

where the notation $\{x\}_{tl}$ will be used through the derivation to indicate terms that are null for a null geodesic and are x for a timelike geodesic.

Recall from general relativity that, if a metric is independent of the x^α coordinate, then the quantity

$$g_{\alpha\mu}\frac{dx^\mu}{d\lambda} \tag{26}$$

is conserved. As the metric is time and ϕ independent, we can define the conserved energy and angular momentum

$$E^* = g_{0\mu}\frac{dx^\mu}{d\lambda} = \left(\frac{2M}{r} - 1\right)\frac{dt^*}{d\lambda} + \frac{2M}{r}\frac{dr}{d\lambda}, \tag{27}$$

$$L = g_{2\mu}\frac{dx^\mu}{d\lambda} = r^2\frac{d\phi}{d\lambda}. \tag{28}$$

Substituting these expressions in the null geodesic equation, simplifying and rearranging leads to

$$\left(\frac{dr}{d\lambda}\right)^2 = E^{*2} - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} + \left\{\frac{2M}{r} - 1\right\}_{tl}. \tag{29}$$

Then, the full set of equations governing the evolution of the geodesic in terms of the affine parameter is given by combining the conservation relations with this latest result, and reads

$$\frac{dt^*}{d\lambda} = \frac{r}{2M - r}\left(E^* \mp_{dr} \frac{2M}{r}\sqrt{E^{*2} - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} + \left\{\frac{2M}{r} - 1\right\}_{tl}}\right), \tag{30}$$

$$\frac{dr}{d\lambda} = \pm_{dr}\sqrt{E^{*2} - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} + \left\{\frac{2M}{r} - 1\right\}_{tl}}, \tag{31}$$

$$\frac{d\phi}{d\lambda} = \frac{L}{r^2}, \tag{32}$$

where we introduced the notation

$$\pm_{dx} = sign(\Delta x). \tag{33}$$

It is now convenient to get rid of the affine parameter and write all formulas in terms of derivatives over ϕ, by noticing that

$$\frac{dr}{d\phi} = \frac{dr}{d\lambda}\frac{d\lambda}{d\phi} = \frac{r^2}{L}\frac{dr}{d\phi}. \tag{34}$$

Thus, we have

$$\left(\frac{dr}{d\phi}\right)^2 = \eta^2 r^4 - \left(1 - \frac{2M}{r}\right) r^2 + \left\{\left(\frac{2M}{r} - 1\right)\frac{r^4}{L^2}\right\}_{tl}, \tag{35}$$

where we have introduced the ratio

$$\eta = \frac{E^*}{L}, \tag{36}$$

which is here necessarily finite as the $L = 0$ case coincides with the 2D $\varphi_2 = 0$ case. It is computationally advantageous to rewrite this in term of the inverse distance $u = 1/r$ - to reduce the order of the powers involved - such that

$$\frac{du}{d\phi} = \frac{du}{dr}\frac{dr}{d\phi} = -u^2\frac{dr}{d\phi}. \tag{37}$$

The geodesic equation then transforms into

$$\left(\frac{du}{d\phi}\right)^2 = \eta^2 + 2Mu^3 - u^2 + \left\{(2Mu - 1)\frac{1}{L^2}\right\}_{tl}, \tag{38}$$

and, finally, we can write

$$d\phi = \pm_{du}\left(\eta^2 + 2Mu^3 - u^2 + \left\{(2Mu - 1)\,L^{-2}\right\}_{tl}\right)^{-1/2} du. \tag{39}$$

The evolution of time along the geodesic is instead given by

$$\begin{aligned}
\frac{dt^*}{d\phi} &= \frac{dt^*}{d\lambda}\frac{d\lambda}{d\phi} \\
&= \left[\frac{r}{2M - r}\left(E^* - \frac{2M}{r}\frac{dr}{d\lambda}\right)\right]\frac{r^2}{L} \\
&= \frac{r^3}{2M - r}\left[\eta \mp_{dr} \frac{2M}{r}\sqrt{\eta^2 - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2} + \left\{\left(\frac{2M}{r} - 1\right)\frac{1}{L^2}\right\}_{tl}}\,\right] \\
&= \frac{1}{2M/r - 1}\left[\eta r^2 \mp_{dr} 2M\sqrt{\eta^2 r^2 + \left(\frac{2M}{r} - 1\right) + \left\{\left(\frac{2M}{r} - 1\right)\frac{r^2}{L^2}\right\}_{tl}}\,\right],
\end{aligned} \tag{40}$$

which can be written in terms of u as

$$\frac{dt^*}{d\phi} = \frac{1}{u^2(2Mu - 1)}\left[\eta \pm_{du} 2Mu\sqrt{\eta^2 + u^2\,(2Mu - 1) + \left\{(2Mu - 1)\frac{1}{L^2}\right\}_{tl}}\,\right]. \tag{41}$$

Finally, we can obtain dt^* in terms of du:

$$\begin{aligned}\frac{dt^*}{du} &= \frac{dt^*}{d\phi}\frac{d\phi}{du} \\ &= \frac{1}{u^2(2Mu-1)}\left[\frac{\pm_{du}\eta}{\sqrt{\eta^2 + u^2(2Mu-1) + \left\{(2Mu-1)\frac{1}{L^2}\right\}_{tl}}} + 2Mu\right]. \end{aligned} \tag{42}$$

For completeness, the following is in terms of dr

$$\frac{dt^*}{dr} = \frac{1}{(2M/r-1)}\left[\frac{\pm_{dr}\eta r}{\sqrt{\eta^2 r^2 + (2M/r-1) + \left\{(2Mu-1)\frac{r^2}{L^2}\right\}_{tl}}} - \frac{2M}{r}\right]. \tag{43}$$

Then, as we know both the initial and final (u, ϕ), we can use $d\phi(du)$ as given by Eq. 39 to find the required value of η^2 to match the limits. It is important to note that η can be both positive and negative. Then, we can use the above equation with $\pm\eta$ to find the values $t^{*'}_{nc}$ and $t^{*''}_{nc}$ at which the null geodesics reach the spatial coordinates of E_2. Recall that for the null geodesic, the $\{...\}_{tl}$ term vanishes.

When both points are inside the horizon, i.e. $2M > r_1 > r_2$, it is not enough to show a null curve from E_1 reaches the spatial components of E_2 at a time $t^{*'}_{nc} < t^*_2$, as a constant-space curve is not a worldline of a particle inside the horizon. It is necessary to also show that another curve can reach the same spatial coordinates at a time $t^{*''}_{nc} > t^*_2$. The alternative approach, which is to show that there exists a timelike geodesic connecting the two points E_1 and E_2, would require fitting not only η^2 but L^2 as well by considering $dt^*(du)$: this method is computationally more expensive and prone to errors, thus it was excluded.

If both are outside the horizon, $\eta < 0$ is the only way of having the signs agreeing. When both are inside, it is easy to see that the negative and positive values of η respectively give the smaller and larger time t^*_{nc} of intersection with E_2 spatial coordinates.

Two things should now be proved to show this result is consistent with our knowledge: firstly, this equation forces any object inside the horizon to continue its ineluctable fall toward the singularity; secondly, it reproduced our previous results in the 2D limit.

To prove the former, consider a movement opposite to the fall with $r_2 > r_1$, hence $dr > 0$. As $2M > r$ and, obviously, $dt^* > 0$, this would require

$$\frac{+\eta r}{\sqrt{(2M/r-1) + \eta^2 r^2}} > 2M/r. \tag{44}$$

Trivially, this has no solution for negative η, whereas for positive $\eta = |\eta|$, substituting $2M/r = 1 + \delta$, with $\delta > 0$, and rearranging, this yields

$$|\eta| r > |\eta| r(1+\delta)\sqrt{1 + \frac{\delta}{\eta^2 r^2}}, \tag{45}$$

which also, clearly, has no solution.

To evaluate its 2D approximation, consider the$\Delta\phi \to 0$ limit. Given its definition, this leads to $L \to 0$, implying $|\eta| \to \infty$. Then, expanding the dt^*/dr equation for the null geodesic:

$$\begin{aligned}\frac{dt^*}{dr}_{\Delta\phi\to 0} &\approx \frac{r}{2M-r}\left[\pm_{dr}\frac{\eta r}{|\eta r|} - 2M/r\right] \\ &\approx \frac{r}{2M-r}\left[\pm_{dr} sign(\eta) - 2M/r\right]. \end{aligned} \tag{46}$$

The case we are interested in is the one where both lie inside the horizon, and $2M > r_1 > r_2$ hence take $\pm_{dr} \equiv -$. Thus

$$\frac{dt^*}{dr}_{\Delta\phi\to 0} \approx -\frac{r\ sign(\eta) + 2M}{2M-r}. \tag{47}$$

Clearly, the two signs of η reproduce the two geodesics derived for the 2D case, as wanted.

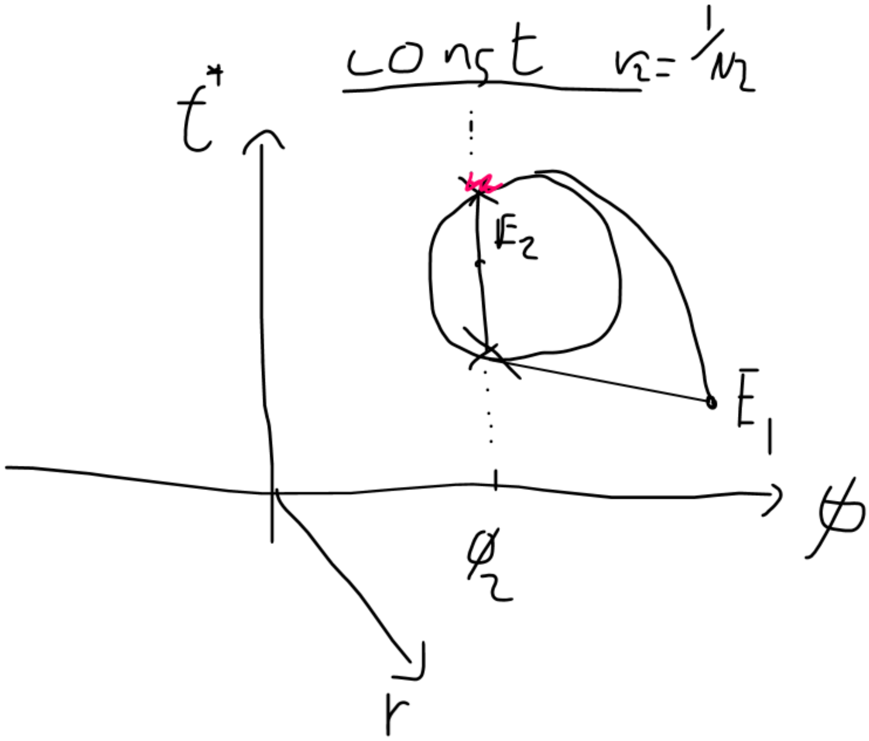

Figure 1: Fig: Sketch of LightCone in Equatorial Plane of 4D Black Hole. The light cone is (poorly) projected onto the $r = r_2$ plane. The idea is that there are an earlier and later null geodesics going from E_1 to γ, which is the curve of constant space components set at those of E_2. If t_2^* is between the intersection of γ with those geodesics, then they are related, otherwise not.