Imperial College
London

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The Entropy of a Schwarzschild black hole in Causal Set Theory

MSci Project Viva - Theoretical Physics Group

Author: Vid Homšak
Partner: Stefano Veroni
Supervisor: Prof Fay Dowker

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Overview

Credits to my partner Stefano Veroni, my ride-or-die.

Presentation Outline:

- Causal Set Theory
- Simulating causal sets in Schwarzschild spacetime
- Black Hole Entropy $\rightarrow$ Entropy in Causal Sets
- Results and conclusions

Discrete theory of Quantum Gravity

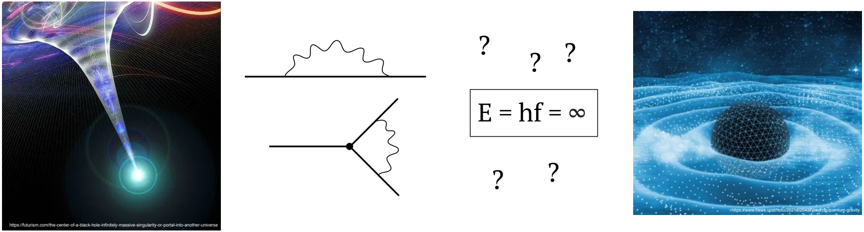


Figure 1: The infinities otherwise dealt with by applying the renormalisation resurface in naive attempts to quantise gravity.

Causal Set Approach to Quantum Gravity

- Hawking-King-McCarthy-Malament (HKMM) theorem:

  GEOMETRY = ORDER + NUMBER!

- Spacetime - macroscopic approximation to an underlying causal discrete structure.
- Causal set:
  discrete, partially ordered set of events C with order relation $\prec$.

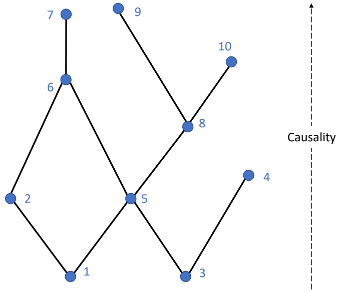
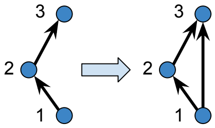


Figure 2: Causal Set

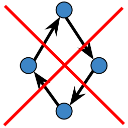
Causal Set Properties

A causal set (causet) satisfies ($\forall\, x, y, z \in C$):

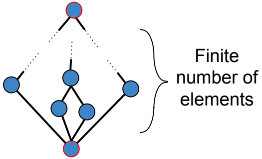
- *Transitivity:* $x \prec y \wedge y \prec z \Rightarrow x \prec z$.
- *Acyclicity (irreflexivity):* $x \not\prec x$
- *Local finiteness:* $|\{z|\ x \prec z \prec y\}|$ is finite.



Transitivity



Acyclicity



Local finiteness

How to Simulate Causal Sets?

Poisson Sprinkling!

1. Pick the sprinkling density $\rho = \ell^{-4}$.
2. Distribute $N = \text{Poiss}(\rho V)$ points uniformly such that the probability of having n points in a volume V is

$$\text{Poiss}(n; V) = \frac{(\rho V)^n}{n!} e^{-\rho V}.$$

   Note: $dV(x) \propto \sqrt{-g_{\mu\nu}(x)}$.
3. Connect the elements according to the causal structure of the spacetime region.

Simulation Framework

- Schwarzschild spacetime - **computationally expensive**!
  $\Rightarrow$ No extensive studies of Schwarzschild causets until now
- **created** highly-parallelised simulation framework in `C++` running on Imperial's HPC
  $\Rightarrow$ x15000 speed up vs. single-core Python
- Need high density - sprinkle into a 4D cylinder
  $\Rightarrow$ Causal set cardinality $\sim 750k \rightarrow$ 1TB of RAM :)

GitHub repository: `vidh2000/MSciSchwarzschildCausets`

Causality in Schwarzschild spacetime

- Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt_S^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- Using well-behaved Eddington-Finkelstein (EF-original) coordinates:

$$\left(t^* = t_S + 2M \ln\left|\frac{r}{2M} - 1\right|, r, \theta, \phi\right)$$

yields a metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^{*2} + \frac{4M}{r} dt^* dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2$$

Determining the causality between two events in EF-original coordinates

Imagine two events E_1 and E_2 in 4D EF-original coordinates.

1. Rotate coord. system s.t. $\theta_1 = \theta_2 = \pi/2 \to$ reduces to 3D.
2. Null geodesics then satisfy, (for affine parameter λ)

$$\left(\frac{2M}{r} - 1\right)\left(\frac{dt^*}{d\lambda}\right)^2 + \frac{4M}{r}\frac{dt^*}{d\lambda}\frac{dr}{d\lambda} + \left(\frac{2M}{r} + 1\right)\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

Time and ϕ independent metric $\Rightarrow$ conserved "energy" and "angular momentum"

$$E^* = \left(\frac{2M}{r} - 1\right)\frac{dt^*}{d\lambda} + \frac{2M}{r}\frac{dr}{d\lambda}, \qquad L = r^2\frac{d\phi}{d\lambda}.$$

Determining the causality between two events in EF-original coordinates

3. Inserting these into the geodesic equation and rearranging yields (for $u = 1/r$)

$$\frac{d\phi}{du} = \pm_{du} \left(\eta^2 + 2Mu^3 - u^2\right)^{-1/2}$$

$$\frac{dt^*}{du} = \frac{1}{u^2(2Mu-1)} \left[\frac{\pm_{du}\eta}{\sqrt{\eta^2 + u^2(2Mu-1)}} + 2Mu\right]$$

where $\eta = E^*/L$ and $\pm_{du} = \mathrm{sign}(\Delta u)$

4. Using boundary conditions (ϕ, u) and $d\phi/du \to$ fit for η^2
5. Using $dt^*/du \to$ find t^*_+, t^*_- when null geodesic reaches E_2

Conditions for determining the causality

Hence $E_1 \prec E_2$ iff:

- $t_2^* \in [t_-^*, t_+^*]$
- pair satisfies statements that allow determining if $E_1 \prec E_2$

Can solve analytically in (1+1)D, yielding lower bound

$$t_2^* - t_1^* = r_1 - r_2$$

and upper bound

$$t_2^* - t_1^* = [r + 4M \ln(r - 2M)]_{r_1}^{r_2}$$

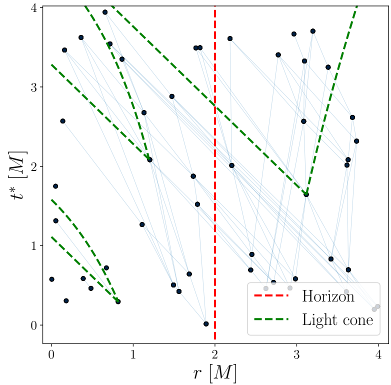


Figure 3: Connections form only within the light cones, (1+1)D.

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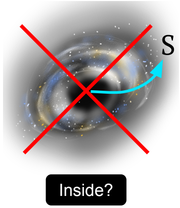
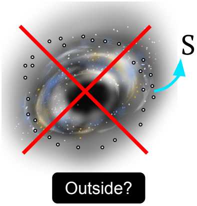
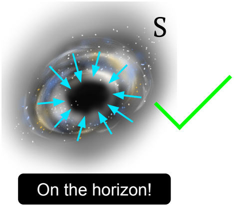
The Mystery of the Black Hole Entropy

Black hole thermodynamics:
$\Rightarrow$ Hawking and Bekenstein in 1970s - black holes radiate!

"If your theory is found to be against the second law of thermodynamics, I can give you no hope; there is nothing for it but to collapse in deepest humiliation." Arthur Eddington, 1915.







Black Hole Entropy in Causal Set Theory

- The entropy of a black hole with event horizon area A

$$S_{BH} = \frac{1}{4}\frac{A}{\ell_p^2}, \qquad \ell_p = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35}\ \mathrm{m}.$$

- Entropy $\Leftrightarrow$ microstates
- Horizon $\equiv$ empty spacetime $\Rightarrow$ spacetime molecules?
- Assume N_{mol} independent molecules, then Gibbs entropy is

$$S = -N_{mol}\sum p_n \ln(p_n),$$

where p_n is probability for molecule to be in n^{th} state.

Horizon Molecules

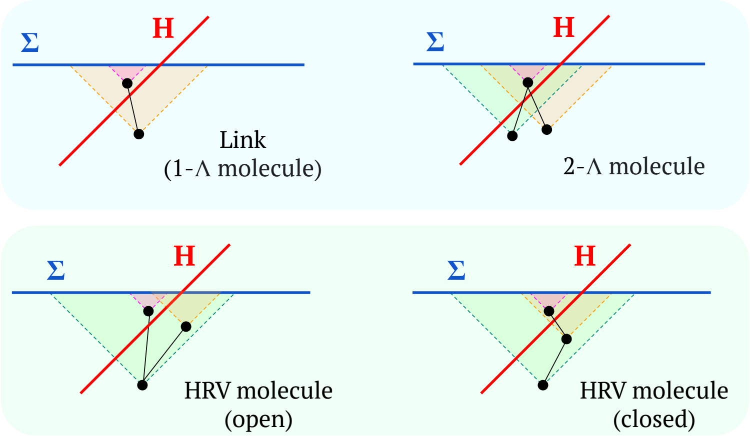


Figure 4: (Up) n-Λ molecules. (Down) HRV molecules.

Molecule counting: Overview

- Horizon $\mathcal{H} : r = r_S$ and a space-like hypersurface $\Sigma : t^* = const. \Leftarrow$ defines the time of "measuring entropy"
- The probability a region ΔR of space with volume V is occupied by n events

$$P(n\ in\ \Delta R) = \frac{(\rho V)^n}{n!} e^{-\rho V}$$

- Locally flat $\Rightarrow$ assume Rindler horizon for analytic calculations

$$\mathcal{H} : t = x, \quad \Sigma : t = constant$$

Molecule counting: HRV molecules

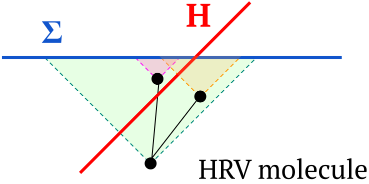
- Integrating over infinitesimal volume elements yields

$$\begin{aligned}\langle N_{HRV}\rangle &= \rho^3 \int_{I^-(\mathcal{J})} dV_p\ V_{in}^{+}(p)\ V_{out}^{-}(p)\ e^{-\rho V^{+}(p)} \\ &= \frac{3\sqrt{3}}{70}\sqrt{\rho}A\end{aligned}$$

- Expressing in terms of Planck length

$$\langle N_{HRV}\rangle \approx \frac{0.07423}{\ell^2}\frac{A}{l_p^2},$$

which is our own analytic result.



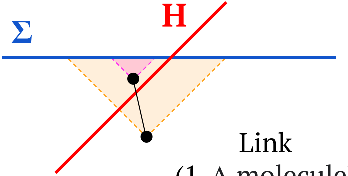
Molecule counting: Links and n-Λ molecules

- Barton et. al gives the scaling of links as

$$\langle N_{link} \rangle = \frac{\sqrt{3}}{10 l^2} \frac{A}{l_p^2}$$

- Counting n-Λ gives that

$$N_{link} = \sum_{n=1}^{\infty} n N_{n\text{-}\Lambda}$$



Link
(1-Λ molecule)

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Results: n-Λ counting

- Entropy scales as

$$S_\Lambda = 0.0929 \pm 0.0006 \frac{A}{\ell^2},$$

where ℓ is discreteness scale.

- Confirms analytical results for the scaling of single-link molecules
- Analytical calculations for curvature corrections yield

$$\Delta S_\Lambda = 0.173 \pm 0.001 \frac{\sqrt{A}}{\ell}.$$

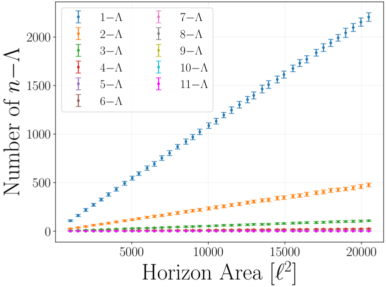

Figure 5: Number of n-Λ molecules scales perfectly with horizon area.

When discreteness scale becomes physical

- Setting $S_\Lambda = S_{BH}$, noting that $S_{BH} = \frac{1}{4}\frac{A}{\ell_p^2}$ yields

$$\ell = (0.610 \pm 0.002)\ell_p$$

- The implied minimum "step" in time and space
  $\Rightarrow \Delta x_{min} \sim 10^{-34}$m
  $\Rightarrow \Delta t_{min} \sim 10^{-44}$s

Entropy really lives on the Horizon?

- Elements in molecules have $t_{min} < -3l \approx 10^{-44}$s
  $\Rightarrow$ entropy defined at a certain point in time.
- Elements in molecules straddle the horizon, $\Delta r_{max} < 2l \approx 10^{-34} m$
  $\Rightarrow$ black hole entropy lives **on the horizon**.

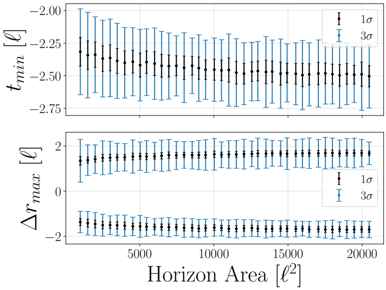

Figure 6: Horizon molecules are very localised in space and time.

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Results: n-Λ distribution

- $n - \Lambda$ molecules distributed as a falling exponential

$$p_n \propto e^{-bn}, \quad b = 1.52 \pm 0.01$$

- Each link adds an integral
- Assuming $p_n \propto e^{-E_n/k_B T}$ we obtain

$$E_n \approx \frac{3n}{2} k_B T,$$

$\sim$ energy of n-atomic molecules in 3D space.

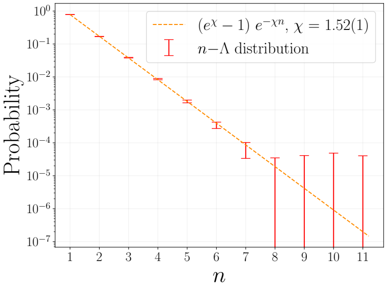

Figure 7: Probability of n-Λ versus n. The distribution of molecules is well described by a falling exponential.

Results: HRV molecules

- Analytically derived number scaling agrees with simulation,

$$N_{HRV} = 0.0742\frac{A}{\ell^2}.$$

- Majority HRV molecules are the *open* type

$$p_{open} = 0.972 \pm 0.008, \qquad p_{closed} = 0.0276 \pm 0.0002$$

- Entropy obtained via HRV counting scales as

$$S_{HRV} = 0.0098 \pm 0.0003\frac{A}{\ell^2},$$

yielding a discreteness scale $\ell = 0.198 \pm 0.003\ell_p$.

Conclusion

- Created **our own** extremely efficient CST simulation framework - also allows one to study causets in Schwarzschild
- For the **first time** showed that entropy of Schwarzschild black hole in CST
  - scales proportional to area
  - lives on the horizon
- Analytically derived
  - expected number of HRV molecules as a function of area
  - curvature corrections for link-counting in Schwarzschild
- Possible extensions:
  - discretise more complex black holes
  - investigate other entropy models and use the framework to study Ricci scalar, dimension estimators...

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Literature

1. Hawking SW, King AR, McCarthy PJ. 1976. J. Math. Phys. 17:174-181.
2. Malament DB. 1977. J. Math. Phys. 18:1399-1404.
3. Bombelli L, et al. 1987. Phys. Rev. Lett. 59:521-524.
4. Bekenstein JD. 1972. Lett. Nuovo Cimento 4:737–740.
5. Hawking SW. 1975. Commun.Math. Phys. 43:199–220.
6. Dou D, Sorkin RD. 2003. Found.Phys. 33:279-296.
7. He S, Rideout D. 2009. Class.Quant.Grav. 26:125015.
8. Barton C, et al. 2019. Phys. Rev. D 100:126008.
9. Benincasa DMT. 2013. PhD Thesis. Imperial College London.
10. Sorkin RD, Yazdi YK. 2018. Class. Quantum Grav. 35:074004.