EXERCISES [5] – DERIVATIVES. APPLICATIONS.

5.1 Using the definition, determine the derivatives of the following functions at the given points:

a) $f(x) = \frac{1}{x^3}$ at $x = 1$

b) $f(x) = \sqrt{3x+2}$ at $x = a$

5.2 Find the derivatives of the following functions:

a) $f(x) = x^5 + 4x^3$

b) $f(x) = e^{3x}x^2$

c) $f(x) = \frac{\sin x}{1+\cos x}$

d) $f(x) = \ln(x^2 + x) + xe^{-x} - \frac{1}{x}$

e) $f(x) = \sqrt[3]{x^2 - 1}$

f) $f(x) = \ln(\ln x)$

g) $f(x) = \frac{\sin^2 x}{\sin(x^2)}$

h) $f(x) = \left(\frac{1-\ln x}{x}\right)^2$

i) $f(x) = \frac{x \sin x}{1+x^2}$

5.3 Consider the real function f defined by $f(x) = \begin{cases} \ln(\sqrt{1-x^2}) \ , & -1 < x \leq 0 \\ x + \frac{2}{x}\sin^2(x) \ , & x > 0 \end{cases}$

a) Find the domain of f.

b) Study this function f about the differentiability in its domain.

c) Determine the algebraic expression of f'.

5.4 Consider the real function f defined by $f(x) = \begin{cases} \sqrt{\sin(\pi x) + 1} - 1 \ , & x \leq 0 \\ \frac{e^{\frac{x^2}{3}} - 1}{x} + \frac{kx}{2} \ , & x > 0 \end{cases}$ $(k \in \mathrm{IR})$

a) Find the value of k for which f is a differentiable function at $x = 0$.

b) Write the equation of the tangent line to the graph of f at $x = -1$.

5.5 Let f be differentiable function in IR. Consider another function h defined by:

$$h(x) = f(\sin x + x)$$

a) Is h a differentiable function in IR? Justify your answer.

b) Determine the expression of $h'(x)$.

5.6 Consider $f: \mathrm{IR} \rightarrow \mathrm{IR}^+$ a differentiable function in IR and g another function defined by:

$$g(x) = \ln(f(x^2 + 1))$$

- **a)** Find the derivative of g.
- **b)** Knowing that f' is also a differentiable function, $f(2) = f'(2) = 1$ and $f''(2) = 2$, determine $g''(1)$.

5.7 Let $f: \mathrm{IR}^+ \rightarrow \mathrm{IR}$ be a differentiable function on IR^+. Consider another function g defined by $g(x) = (x-2)e^{f\left(\frac{1}{x}\right)}$.

- **a)** Determine the domain of g and justify the differentiability g in its domain.
- **b)** Determine $g'(x)$.

5.8 Let f and g be continuous functions in $[a,b]$ and differentiable in $]a,b[$. Suppose that $f(a) = f(b)$ and $g(a) = g(b)$. Prove that exists $c \in]a,b[$ such that $f'(c) = g'(c)$.

5.9 Consider f a differentiable function in IR defined by: $f(x) = \begin{cases} 5 - x^2, & x \leq 1 \\ \frac{3}{x} + x, & x > 1 \end{cases}$

Apply the Lagrange's Theorem to the function f on $[0,3]$. Find the value(s) of c mentioned in the theorem.

5.10 Apply the Lagrange's Theorem to the function $f(x) = e^x$ on the interval $[0,x]$ and show that $e^x > x + 1$, for all $x > 0$.

5.11 Consider the real function f with one real variable defined by: $f(x) = \ln x$

- **a)** Show that the Lagrange's Theorem can be applied to the f on any interval $[1,x]$ with $x > 1$.
- **b)** Using the Lagrange's Theorem, prove that $\forall x > 1,\ x - 1 < \ln(x^x) < x^2 - x$.

5.12 Let $g: \mathrm{IR} \rightarrow \mathrm{IR}^+$ be a real function with one real variable, two-times differentiable on IR. Consider the function $f: D \rightarrow \mathrm{IR}$ defined by:

$$f(x) = g(\ln x) + \ln(g(x))$$

- **a)** Find the domain, D, of f and justify its differentiability on D.
- **b)** Knowing that $g(0)$, $g(1)$ and $g(e)$ are local extreme values of g, show that f'' has at least one zero on the interval $[1,e]$.

5.13 Evaluate the following limits using the Cauchy's Rule:

a) $\lim_{x\to 0} \frac{x-tan(x)}{x-\sin(x)}$

b) $\lim_{x\to 0^+} x^{\sin x}$

c) $\lim_{x\to 0^+} (e^x - x)^{\frac{1}{x}}$

5.14 Consider the function defined by: $f(x) = \ln(2x+1)$

a) Find the domain of *f*.

b) Write the Taylor formula of degree 2 of the function *f* centred at *a*=0.

c) Determine an approximate value of $f(0.1)$.

5.15 Using the Mac-Laurin's polynomial of degree 2 of the function $f(x) = e^x$, compute an approximate value for $\sqrt{e}$.

5.16 Consider the function $f(x) = \sin(x)$. Applying Taylor's formula of order 2 to the function *f* at $x = 0$, with Lagrange's error, prove that: $\sin(x) \geq x - \frac{x^3}{6}$, $\forall x \geq 0$.

5.17 Use the first-order Taylor's expansion of the function $f(x) = \cos(x)$ at $x = 0$ with Lagrange's error, to compute $\lim_{x\to 0^+} \frac{\cos(x)-1}{x^2}$.

5.18 Consider the real function f with one real variable defined by: $f(x) = \frac{x}{1+x^2}$

a) Find the domain of the function.

b) Find the points of intersection with the axes.

c) Study the continuity of the function.

d) Study the monotony of the function and the existence of local extreme values.

e) Study the concavity of the function and the existence of inflection points.

f) Study the existence of asymptotes.

g) Sketch the graph of the function.

h) Find the range of the function.

5.19 Answering the same questions of the exercise before, study the following function and sketch its graph:

a) $f(x) = \frac{x-1}{x-2}$

b) $f(x) = (\ln x)^2$

c) $f(x) = \frac{e^{x+1}}{x+3}$

More exercises

5.20 Using the definition, determine the derivatives of the following functions at the given points:

a) $f(x) = 3x - 2$, at $x = 2$

b) $f(x) = \sqrt{x+1}$, at $x = 3$

5.21 Find the derivatives of the following functions:

a) $f(x) = \cos(2x) - 2\sin x$

b) $f(x) = \ln\sqrt{x}$

c) $f(x) = x\ln\left(\frac{1}{x}\right)$

d) $f(x) = \frac{e^{2x}-x^2}{x}$

e) $f(x) = \ln\left(\frac{x-2}{1+3x}\right)$

f) $f(x) = 2e^{x^2-1}$

g) $f(x) = \frac{x}{1+\sqrt{x}}$

5.22 Determine the equation of the tangent line to the curve $y = 3x^2 - 5x + 2$ at $x = 2$.

5.23 Show that the curve $y = 4x^3 + 4x - 2$ does not have a tangent line with a slope equal to 3.

5.24 Determine the points of the curve $y = x^3 - x^2 - x - 1$ that have a horizontal tangent line.

5.25 Prove, using the derivative definition, that:

a) If f is a constant function, that is if $f(x) = c\ \forall x$, then $f'(x) = 0, \forall x$.

b) If f is a function such that $f(x) = mx + b$ then $f'(x) = m, \forall x$.

5.26 Consider the function: $f(x) = \begin{cases} x^2 & , x \leq 0 \\ ax + b & , x > 0 \end{cases}$. Find the values of a and b such that $f'(0)$ exists.

5.27 Consider the function: $f(x) = \begin{cases} \frac{x}{1+e^{\frac{1}{x}}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

a) Determine the domain of f.

b) Study the continuity of the function on the domain.

c) Study the differentiability of f at $x = 0$.

d) Determine the derivative function, $f'(x)$.

5.28 Let f be a real function with one real variable, differentiable on IR. Let g be a function defined by: $g(x) = f(\ln(x^2 + 1))$. Justify that g is also differentiable on IR and determine the expression of $g'(x)$.

5.29 Let $f(x) = x^4$, g a differentiable function on IR and h a function defined by $h(x) = (g \circ f)\big(sen(x)\big)$. Define the derivative function of h.

5.30 Is it possible to apply the Rolle's Theorem to the function $f(x) = \sqrt[3]{(x-1)(x-2)}$ on the interval $[1,2]$? Justify your answer.

5.31 Prove, applying the Rolle's Theorem, That the equation $x^3 - 3x^2 + 3x + b = 0$ ($b \in$ IR) cannot have more than one solution on the interval $[-1,1]$.

5.32 Is it possible to apply the Lagrange's Theorem to the following function at the given intervals? If so, find the value c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

a) $f(x) = \frac{1}{x}$, $[2,3]$

b) $f(x) = \sqrt[3]{x}$, $[-1,1]$

5.33 Evaluate:

a) $\lim_{x \to 0^+} \frac{\ln(\sin(4x))}{\ln(\sin(3x))}$

b) $\lim_{x \to +\infty} x^{\frac{1}{x-1}}$

5.34 For the following functions, write the Taylor formula of degree n, centred at a:

a) $f(x) = \sin^2 x \qquad a = 0, n = 3$

b) $f(x) = \frac{1}{x^2+3}$ $\quad a = 1, n = 2$

c) $f(x) = e^{x^2}$ $\quad a = 0, n = 3$

d) $f(x) = x \sin x$ $\quad a = \pi, n = 2$

5.35 Do the complete study (see exercise 5.18) of the following functions and draw their graphs:

a) $f(x) = x^3 + 8$

b) $f(x) = \frac{1}{x-2}$

c) $f(x) = \sqrt{x-1}$

d) $f(x) = \ln(x^2)$

e) $f(x) = \frac{\ln x}{x}$

Solutions:

5.20 a) $f'(2) = 3$ b) $f'(3) = \frac{1}{4}$

5.21 a) $f'(x) = -2\sin(2x) - 2\cos(x)$ b) $f'(x) = \frac{1}{2x}$ c) $f'(x) = \ln\left(\frac{1}{x}\right) - 1$

d) $f'(x) = \frac{e^{2x}(2x-1)-x^2}{x^2}$ e) $f'(x) = \frac{1}{x-2} - \frac{3}{1+3x}$ f) $f'(x) = 4xe^{x^2-1}$

g) $f'(x) = \frac{2\sqrt{x}+x}{2\sqrt{x}\left(1+\sqrt{x}\right)^2}$

5.22 $y = 7x - 10$

5.23 Show that it is not possible to have a point a such that $f'(a) = 3$.

5.24 5.24$(1, -2)$ and $\left(-\frac{1}{3}, -\frac{22}{27}\right)$

5.26 $a = 0; b = 0$

5.27 a) $Df = \mathrm{IR}$

b) f is continuous on its domain.

c) f is not differentiable at $x = 0$.

d) $f'(x) = \frac{x+e^{\frac{1}{x}}(x+1)}{x\left(1+e^{\frac{1}{x}}\right)^2}$ $\quad \forall x \in \mathrm{IR}\backslash\{0\}$

5.28 g is a differentiable function on its domain (IR) because g is the composition of differentiable functions on their domains (IR): f (differentiable by hypothesis) and $ln(x^2+1)$ that is also differentiable because it is the composition of differentiable functions (ln and polynomial functions).

$g'(x) = \frac{f'(\ln(x^2+1))\, 2x}{x^2+1}$

5.29 $h'(x) = g'\left(\sin^4(x)\right) \times 4\sin^3(x) \times \cos(x)$

5.30 Yes. f verifies all the hypotheses of the Rolle's theorem on the interval $[1,2]$.

5.31 Consider the function $f(x) = x^3 - 3x^2 + 3x + b$ and suppose that f has 2 zeros in $[-1,1]$. Applying the Rolle's theorem, we conclude that f' has another zero besides $x = 1$, which it is not possible. Therefore, f cannot have to zeros in $[-1,1]$ and that means that the given equation cannot have more than one solution in $[-1,1]$.

5.32 a) Yes. $c = \sqrt{6}$. b) No.

5.33 a) 1 b) 1

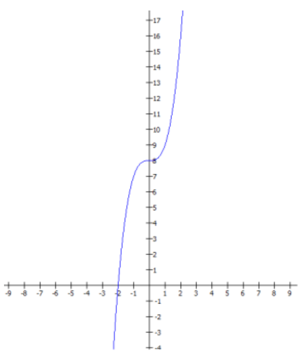
5.34 a) $f(x) = x^2 + R_3(x)$ where $\lim_{x \to 0} \frac{R_3(x)}{|x|^3} = 0$

b) $f(x) = \frac{1}{4} - \frac{1}{8}(x-1) + R_2(x)$ where $\lim_{x \to 1} \frac{R_2(x)}{|x-1|^2} = 0$

c) $f(x) = 1 + x^2 + R_3(x)$ where $\lim_{x \to 0} \frac{R_3(x)}{|x|^3} = 0$

d) $f(x) = -\pi(x-\pi) - (x-\pi)^2 + R_2(x)$ where $\lim_{x \to \pi} \frac{R_2(x)}{|x-\pi|^2} = 0$

5.35 a)



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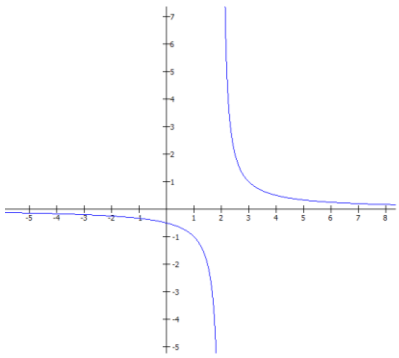

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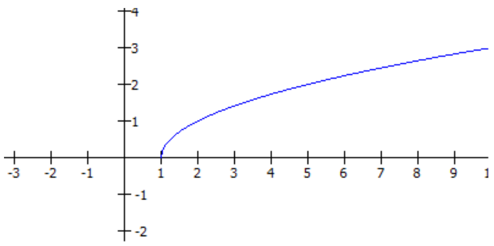

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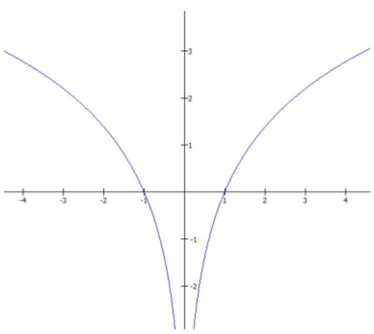

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