Ans. $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

8. $Q = (x_1 + x_2 + x_3)^2$

Ans. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

9. $Q = 4x_1x_3 + 2x_2x_3 + x_3^2$

Ans. $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Verify that the following matrices are orthogonal:

10. $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Hint: Show that $AA^T = I$.

11. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12. Prove that the product AB of two symmetric matrices A and B is symmetric if $AB = BA$.

Hint: $(AB)^T = B^T A^T = BA$ since $A = A^T, B = B^T$, so $(AB)^T = BA = AB$ then AB is symmetric. Provided $AB = BA$.

14.7 CANONICAL FORM: or SUM OF THE SQUARES FORM

Of a real quadratic form $Q = X^T AX$ is $Y^T DY$ or

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \qquad (1)$$

which is obtained by an orthogonal transformation $X = PY$. Here P is known as **modal** matrix. D is known as **spectral** matrix. D is a diagonal matrix with the eigen values of A as the diagonal elements.

Let r be the rank of A and n be the number of variables $x_1, x_2, \cdots, x_n$ in the quadratic form. Then:

Index S of a quadratic form is the number of positive square terms in the canonical form.

Signature of a quadratic form is the difference between the number of positive and negative square terms in the canonical form.

Definiteness A real nonsingular quadratic form $Q = X^T AX$ (with $|A| \neq 0$) is said to be

Positive definite: If rank and index are equal i.e., $r = n$, $s = n$ or if all the eigen values of A are positive

Negative definite: If index equals to zero i.e., $r = n, s = 0$ or if all the eigen values of A are negative.

Positive semi-definite: If rank and index are equal but less than n

i.e., $$s = r < n, (|A| = 0)$$

or all eigen values of A are non-negative (≥ 0) and at least one eigen value is zero.

Negative semi definite: If index zero

i.e., $$s = 0, r < n, (|A| = 0)$$

or all eigen values of A are non-positive (≤ 0) and at least one eigen value is zero.

Indefinite: Quadratic form is said to be indefinite in any other case or some eigen values are positive and some eigen values are negative.

Note: If Q is negative definite (semi-definite) then $-Q$ is positive definite (semi-definite).

WORKED OUT EXAMPLES

Example 1: Determine the nature, index and signature of the quadratic form $2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_1x_3 - 4x_2x_3$.

Solution: The real symmetric matrix A associated with the Q.F. is

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$$

Its characteristic equation is

$$\begin{vmatrix} 2-\lambda & 1 & -2 \\ 1 & 2-\lambda & -2 \\ -2 & -2 & 3-\lambda \end{vmatrix}$$

$$= \lambda^3 - 7\lambda^2 + 7\lambda - 1 = 0$$

$$= (\lambda - 1)(\lambda - (3 + \sqrt{8}))(\lambda - (3 - \sqrt{8})) = 0$$

The eigen values are $\lambda = 1, 0.1715, 3.1715$ which are all positive. So Q.F. is positive definite.

Index: 3, Signature : $3 - 0 = 3$.

Example 2: Find the nature, index and signature of Q.F.

$$2x_1x_2 + 2x_1x_3 + 2x_2x_3$$

Solution: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Characteristic equation is

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 3\lambda - 2 = 0 \quad \text{or} \quad (\lambda + 1)^2(\lambda - 2) = 0$$

The eigen values are $2, -1, -1$, some are positive and some are negative. So the Q.F. is indefinite.

Index: 1, Signature: $1 - 2 = -1$.

Example 3: Identify the nature, index and signature of the Q.F.

$$x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$$

Solution: $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = \lambda^2(\lambda - 6) = 0$$

Eigen values are $\lambda = 0, 0, 6$. So Q.F. is positive semi definite.

Index: 3, Signature: 3.

Example 4: Classify the Q.F. and find the index and signature of

$$-3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$$

Solution: $A = \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$

Characteristic equation is

$$\begin{vmatrix} -3-\lambda & -1 & -1 \\ -1 & -3-\lambda & 1 \\ -1 & 1 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 + 9\lambda^2 + 24\lambda + 16 = (\lambda + 1)(\lambda + 4)^2 = 0$$

All the eigen values $-1, -4, -4$, are negative. Q.F. is negative definite.

Index: 0, Signature: $0 - 3 = -3$.

Note:

$$Q = 3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$$

is positive definite.

EXERCISE

Determine the nature, index and signature of th[e] quadratic form:

1. $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_2x_3 - 2x_3x_1 + 2x_1x_2$

Ans. indefinite (eigen value: $1, 1, -2$), index: [illegible] signature: 1

2. $5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x$[illegible]

Ans. positive semi definite (eigen value: 5, 0, [illegible] index: 3, signature: 3

3. $x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_3x_1$

Ans. indefinite (eigen value: $-2, 3, 6$), index: [illegible] signature: 1

4. $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_3x_1 - 2x_1x_2$

Ans. positive definite (eigen value: 2, 3, 6), index: [illegible] signature: 3

5. $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x$[illegible]

Ans. positive semi definite (eigen value: 3, 0, 1[illegible] index: 3, signature: 3

6. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_1x_3$

Ans. positive definite (eigen value: 8, 2, 2), inde[x] 3, signature: 3

7. $-4x_1^2 - 2x_2^2 - 13x_3^2 - 4x_1x_2 - 8x_2x_3 - 4x$[illegible]

Ans. negative definite, index: 0, signature: -3

8. $-3x_1^2 - 3x_2^2 - 7x_3^2 - 6x_1x_2 - 6x_2x_3 - 6x$[illegible]

Ans. negative definite, index: 0, signature -3.

14.8 TRANSFORMATION (REDUCTION) OF QUADRATIC FORM TO CANONICAL FORM

Let Q be the quadratic form given by

$$Q = X^T AX = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \tag{1}$$

The coefficient matrix A is real symmetric therefore has n linearly independent orthonormal set of eigen vectors corresponding n eigen values (which need not necessarily be distinct). Let $\hat{P}$ be normalized modal matrix of A. Then $\hat{P}$ is an orthogonal matrix.

Thus the transformation

$$X = \hat{P}Y \tag{2}$$

is an orthogonal transformation. This transfers the quadratic form Q to canonical form, as follows:

We know that P diagonalizes A. Thus

$$\hat{P}^{-1} A\hat{P} = D$$

$$A = \hat{P}D\hat{P}^{-1} = \hat{P}D\hat{P}^T \tag{3}$$

since $\hat{P}^{-1} = \hat{P}^T$ by virtue of orthogonality

Substituting (3) in (1)

$$Q = X^T AX = X^T \hat{P}D\hat{P}^T X = (X^T \hat{P})(D)(\hat{P}^T X) \tag{4}$$

Pre-multiplying (2) by $\hat{P}^{-1}$, we get

$$\hat{P}^{-1}X = \hat{P}^{-1}\hat{P}Y = Y$$

So

$$Y = \hat{P}^{-1}X = \hat{P}^T X \tag{5}$$

since $\hat{P}^{-1} = \hat{P}^T$

Taking transpose of this equation

$$Y^T = (\hat{P}^T X)^T = X^T \hat{P} \tag{6}$$

Using (5) and (6) in (4), we have

$$Q = X^T AX = Y^T DY$$

$$= [y_1, y_2 \ldots y_n] \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \tag{7}$$

(7) is known as the *canonical form* or "*sum of the squares form*" or "*principal axes form*".

Procedure to Reduce Quadratic Form to Canonical Form

1. Identify the real symmetric matrix associated with the quadratic form Q.
2. Determine the eigen values of A.
3. The required canonical form is given by (7)
4. Form the modal matrix containing the n eigen vectors of A as n column vectors. Normalize. Then

$$X = \hat{P}Y$$

is the required orthogonal transformation which reduces Q.F. to C.F.

WORKED OUT EXAMPLES

Example 1: Find the orthogonal transformation which transforms the quadratic form

$$x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$$

to canonical form (or "sum of squares form" or "principal axes form). Determine the index, signature and nature of the quadratic form.

Solution: Let $X = [x_1 x_2 x_3]^T$, $Y = [y_1 y_2 y_3]^T$. Let P be the non-singular orthogonal matrix, containing the (three) eigen vectors of the coefficient matrix A of the given quadratic form. Then $X = \hat{P}Y$ is the required non-singular linear transformation that transforms (reduces) the given quadratics form to canonical form. Here $\hat{P}$ is the normalized modal matrix P. The coefficient matrix A of the given quadratic form is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I|$$

$$= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} \begin{matrix} = \lambda^3 - 7\lambda^2 + 14\lambda - 8 \\ = (\lambda-1)(\lambda-2)(\lambda-4) = 0 \end{matrix}$$

So there are three distinct real eigen values $\lambda = 1, 2, 4$ of A.

For $\lambda = 1$,

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{matrix} \sim \begin{matrix} 2x_2 = x_3 \\ x_2 = 2x_3 \end{matrix}$$

$\therefore \quad x_2 = x_3 = 0, x_1 = \text{arbitrary},$

The eigen vector X_1 associated with $\lambda = 1$ is

$$X_1 = [1 \quad 0 \quad 0]^T$$

For $\lambda = 2$,

$$\begin{matrix} -x_1 + 0 + 0 = 0 \\ x_2 - x_3 = 0 \\ -x_2 + x_3 = 0 \end{matrix} \quad \therefore \quad \begin{matrix} x_1 = 0 \\ x_2 = x_3 \end{matrix}$$

$$X_2 = [0 \quad 1 \quad 1]^T$$

For $\lambda = 3$,

$$\begin{matrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{matrix} \sim \begin{matrix} x_1 = 0 \\ x_2 = -x_3 \end{matrix}$$

$$X_3 = [0 \quad 1 \quad -1]^T$$

Thus the modal matrix P is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The norm of the eigen vector X_1 is

$$\|X_1\| = \sqrt{1^2 + 0 + 0} = 1,$$

$$\|X_2\| = \sqrt{0 + 1^2 + 1^2} = \sqrt{2},$$

$$\|X_3\| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Then the normalized modal matrix $\hat{P}$ is

$$\hat{P} = \begin{bmatrix} \frac{1}{1} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

To find inverse of P:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \begin{matrix} R_{32(-1)} \\ R_{3(-\frac{1}{2})} \\ R_{23(-1)} \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

Thus

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and the normalized P^{-1} is

$$\hat{P}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Diagonalization:

$$\hat{P}^{-1} A \hat{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\times \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Then

$$\hat{P}^{-1} A \hat{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D = \text{diagonal matrix}$$

with the eigen values of A as the diagonal elements.

Transformation (reduction) to canonical form:

Quadratic form (Q.F.)

$$Q = x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$$

$$= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X^T A X$$

Put $X = \hat{P}Y$ and $X^T = (\hat{P}Y)^T = Y^T \hat{P}^T$.
So $Q = X^T A X = Y^T \hat{P}^T A \hat{P} Y = Y^T (\hat{P}^T A \hat{P}) Y$

But we know that $\hat{P}$ is an orthogonal matrix because

$$\hat{P}\hat{P}^T = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus

$$\hat{P}^T = \hat{P}^{-1}$$

So $\quad$ $$\text{Q.F.} = X^T A X = Y^T(\hat{P}^{-1} A \hat{P})Y$$

But through diagonalization

$$\hat{P}^{-1} A \hat{P} = D$$

Therefore

$$Q = X^T A X = Y^T D Y$$

$$= [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [y_1 \quad 2 \cdot y_2 \quad 4y_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 2y_2^2 + 4y_3^2$$

This is the required canonical form (or sum of squares form).

Orthogonal transformation:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \hat{P}Y = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

so $x_1 = y_1$; $x_2 = \frac{1}{\sqrt{2}}(y_2 + y_3)$, $x_3 = \frac{1}{\sqrt{2}}(y_2 - y_3)$ is the orthogonal transformation which reduces the Q.F. to canonical form.

Index is 3 for the Q.F. since the number of positive terms in the canonical form is 3. i.e., $S = 3$, rank r is 3. The number of variables is $n = 3$.

Signature of Q.F. is $2s - r = 6 - 3 = 3$ (difference between number of positive and negative terms in C.F.).

The given Q.F. is *positive definite* because $r = 3 = n$ and $s = 3 = n$.

Example 2: **By Lagrange's reduction transform the quadratic form $X^T AX$ to sum of the squares form for**

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}$$

Solution:

$$\text{Q.F.} = X^T AX = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{Q.F.} = [x_1+2x_2+4x_3 \quad 2x_1+6x_2-2x_3 \quad 4x_1-2x_2+18x_3] \times$$

$$\times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 + 6x_2^2 + 18x_3^2 + 4x_1x_2 + 8x_1x_3 - 4x_2x_3$$

$$= [x_1^2 + 4x_1(x_2 + 2x_3)] + 6x_2^2 + 18x_3^2 - 4x_2x_3$$

$$= [x_1^2 + 4x_1(x_2 + 2x_3) + 2^2(x_2 + 2x_3)^2]$$

$$-2^2(x_2 + 2x_3)^2 + 6x_2^2 + 18x_3^2 - 4x_2x_3$$

$$= [x_1 + 2(x_2 + 2x_3)]^2 + 2x_2^2 + 2x_3^2 - 20x_2x_3$$

$$= [x_1 + 2(x_2 + 2x_3)]^2 + 2[x_2^2 - 10x_2x_3] + 2x_3^2$$

$$= [x_1 + 2(x_2 + 2x_3)]^2$$

$$+2[x_2^2 - 10x_2x_3 + 5^2x_3^2] - 2 \cdot 5^2x_3^2 + 2x_3^2$$

$$= [x_1 + 2(x_2 + 2x_3)]^2 + 2[x_2 - 5x_3]^2 - 48x_3^2$$

$$\text{Q.F.} = y_1^2 + 2y_2^2 - 48y_3^2$$

where

$$y_1 = x_1 + 2(x_2 + 2x_3),$$

$$y_2 = x_2 - 5x_3,$$

$$y_3 = x_3.$$

Index: $S = 2$, $(n = 3, r = 3)$,
Signature: $2s - r = 2 \cdot 2 - 3 = 1$ (or $2 - 1 = 1$).

EXERCISE

Transform (reduce) the quadratic form to canonical form (or "sum of squares form" or "principal axes form") by orthogonal transformation.

State matrix for transformation (i.e., modal matrix).

1. $17x_1^2 - 30x_1x_2 + 17x_2^2$

Ans. $A = \begin{bmatrix} 17 & -15 \\ -15 & +17 \end{bmatrix}$, $\lambda = 2, 32$,

$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}$, C.F.: $2y_1^2 + 32y_2^2$

2. $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_2x_3 + 2x_1x_3 - 2x_1x_2$

Ans. $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}, \lambda = 2, 3, 6,$

$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$, C.F.: $2y_1^2 + 3y_2^2 + 6y_3^2$

3. $5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 6x_1x_2 + 14x_1x_3$

Ans. $A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}, \lambda = 5, \frac{121}{3}, 0,$

$P = \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{16}{11} \\ 0 & 1 & \frac{1}{11} \\ 0 & 0 & 1 \end{bmatrix}$, C.F.: $5y_1^2 + \frac{121}{3} y_2^2$

4. $2(x_1x_2 + x_2x_3 + x_3x_1)$; nature of Q.F.

Ans. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & +1 \\ 1 & +1 & 0 \end{bmatrix}, \lambda = 2, -1, -1,$

$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix},$

C.F.: $2y_1^2 - y_2^2 - y_3^2$

Nature: Indefinite

5. $2(x_1^2 + x_1x_2 + x_2^2)$

Ans. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \lambda = 1, 3,$

$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, C.F.: $y_1^2 + 3y_2^2$

6. $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$, find index

Ans. $A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}, \lambda = 1, -1, -1,$

$P = \begin{bmatrix} a & -3b & \frac{11c}{17} \\ 0 & b & \frac{2b}{17} \\ 0 & 0 & c \end{bmatrix}$

where $a = 1/\sqrt{2}, b = 1/\sqrt{17}, c = \sqrt{(17/81)}$,

C.F.: $y_1^2 - y_2^2 - y_3^2$, Index = 1

7. $3x_1^2 - 2x_2^2 - x_3^2 - 4x_1x_2 + 12x_2x_3 + 8x_1x_3$

Ans. $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix}, \lambda = 3, 6, -9,$

$P = \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$, C.F.: $3y_1^2 + 6y_2^2 - 9$

8. $8x_1^2 + 7x_2^2 + 3x_3^2 + 12x_1x_2 + 4x_1x_3 - 8x$ find the rank, index, signature and nature.

Ans. $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}, \lambda = 3, 0, 15,$

$P = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix}$, C.F.: $3y_1^2 + 15y$

rank of Q.F.: 2, index: 2, signature 2, positive definite.

Lagrange's reduction

9. $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

Ans. $(x_1 - 2x_2 + 4x_3)^2 - 2(x_2 - 4x_3)^2 + 9x_3^2$

10. $2x_1^2 + 5x_2^2 + 19x_3^2 - 24x_4^2 + 8x_1x_2 + 12x_1x_3 + 8x_1x_4 + 18x_2x_3 - 8x_2x_4 - 16x_3x_4$

Ans. $2(x_1 + 2x_2 + 3x_3 + 2x_4)^2 - 3(x_2 + x_3 + 4x_4)^2 + 4(x_3 - 2x_4)^2$

11. $2x_1^2 + 7x_2^2 + 5x_3^2 - 8x_1x_2 - 10x_2x_3 + 4x_1x_3$

Ans. $2(x_1 - 2x_2 - x_3)^2 - (x_2 + x_3)^2 + 4x_3^2$

12. Coefficient matrix $A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -1 & 4 & 6 & 4 \\ 0 & 6 & 11 & 8 \\ 2 & 4 & 8 & 8 \end{bmatrix}$

Hint: QF: $x_1^2 + 4x_2^2 + 11x_3^2 + 8x_4^2 - 2x_1x_2 + 4x_1x_4 + 12x_2x_3 + 8x_2x_4 + 16x_3x_4$

Ans. $(x_1 - x_2 + 2x_3)^2 + 3(x_2 + 2x_3 + 2x_4)^2 - (x_3 + 4x_4)^2 + 8x_4^2$.