Mathematics for Computer Engineers (MCE)

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Contents

Chapter 1

Numerical Linear Algebra

1.1 Foundations

Numerical Linear Algebra (NLA) is the study of numerical methods to find approximate solutions of real-time problems to save time.

1.1.1 Elementary Row Operations

The elementary row operations that can be performed on a matrix are as follows:

- **Scaling**: A row R_i can be scaled by a scalar λ as $R_i \longrightarrow \lambda R_i$. This scales the determinant of the matrix by λ.
- **Vector addition**: A row vector R_j scaled by a scalar λ can be added to a row vector R_i as $R_i \longrightarrow R_i + \lambda R_j$. This has no effect on the determinant of the matrix.
- **Swap**: Two rows R_i and R_j can be swapped as $R_i \longleftrightarrow R_j$. This negates the determinant of the matrix.

Row Echelon Form

Transforming a matrix to a upper triangular matrix by elementary row transformations produces what is called the "row echelon form" of the matrix.

For example,

$$\begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix} \sim \begin{bmatrix} \mathbf{a'_{11}} & \cdots & a'_{1i} & \cdots & a'_{1n} \\ 0 & \cdots & \mathbf{a'_{2i}} & \cdots & a'_{2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \mathbf{a'_{nn}} \end{bmatrix}$$

Pivot elements The first non-zero elements of every row of a matrix in row echelon form. In the above example, $a'_{11}, a'_{2i}, \dots, a'_{nn}$ form the pivot elements.

1.1.2 Matrix Augmentation

An *augmented matrix* $A|B$ is the matrix obtained when a matrix B is appended at the end of matrix A.

For example, consider $A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{F})$[1] and $B := \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{F})$

The augmented matrix $A|B$ is given by,

$$A|B = \left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & b_{11} & b_{12} \\ a_{21} & a_{22} & a_{23} & b_{21} & b_{22} \end{array}\right] \in \mathcal{M}_{2\times 5}(\mathbb{F})$$

1.1.3 Eigenvalues and Eigenvectors

Eigenvectors

A matrix performs a linear transformation when it is multiplied with a vector. The vectors which are only scaled but not rotated when multiplied by a given matrix are known as the *eigenvectors* of that matrix.

Eigenvalues

The factor by which a given eigenvector is scaled on multiplication with its matrix is known as the *eigenvalue* of the given matrix for that particular eigenvector.

$$A\underset{\uparrow\,\text{Eigenvector}}{\mathbf{x}} = \underset{\uparrow\,\text{Eigenvalue}}{\lambda}\mathbf{x} \tag{1.1}$$

1.1.4 Scalar Product

If vectors $\mathbf{u} := \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}^\top$ and $\mathbf{v} := \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^\top \in \mathbb{R}^n$, then their scalar product (or dot product) is defined as,

$$\mathbf{u} \cdot \mathbf{v} := u_1v_1 + u_2v_2 + \cdots + u_nv_n \tag{1.2}$$

Norm

The norm of a vector $\mathbf{v} := \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^\top \in \mathbb{R}^n$ is defined as:

$$\|\mathbf{v}\| := \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{1.3}$$

1.2 Linear System of Equations

A *system of equations* is said to be *linear* if each equation is a *linear combination* of all the the unknowns.

An $n \times n$ system is as follows:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n} & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n} & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn} & = & b_n \end{array}$$

[1] $\mathbb{F}$ represents the scalar field, which may be $\mathbb{R}$ or $\mathbb{C}$.

The above system can be represented as $AX = B$, where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad A \in \mathcal{M}_{n\times n}(\mathbb{F})$$

$$\text{and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \qquad B \in \mathbb{F}^n$$

1.3 Direct Methods

Direct methods are the methods which produce exact results. These methods are typically more expensive computationally.

1.3.1 Gauss Elimination

In Gaussian elimination method, we perform elementary row operations on the augmented matrix $A|B$ to convert it into an *upper triangular matrix.*

$$\therefore A|B = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{nn} & b'_n \end{array}\right]$$

Upon back-substitution, we get

$$a'_{nn}x_n = b'_n \implies \boxed{x_n = \frac{b'_n}{a'_{nn}}}$$

$$a'_{(n-1)(n-1)}x_{n-1} + a'_{(n-1)n}x_n = b'_{n-1} \text{ and so on.}$$

The time complexity of Gauss elimination is $\mathcal{O}\big(2n^3/3\big)$ for a system of order n.

Q.1. Solve using Gauss elimination:

$$\begin{aligned} x + 2y + z &= 3 \\ 2x + 3y + 3z &= 10 \\ 3x - y + 2z &= 13 \end{aligned}$$

We have the system of equations,

$$\begin{aligned} 1x + 2y + 1z &= 3 \\ 2x + 3y + 3z &= 10 \\ 3x - 1y + 2z &= 13 \end{aligned}$$

The above system can be written as,

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix}$$

By Gauss elimination method,

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right] &\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right] & \begin{pmatrix} R_2 \longrightarrow R_2 - 2R_1 \\ R_3 \longrightarrow R_3 - 3R_1 \end{pmatrix} \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] & \left(R_3 \longrightarrow R_3 - 7R_1\right) \end{aligned}$$

Upon back-substitution, we get

$$\begin{aligned} -8z = -24 &\implies \boxed{z = 3} \\ -y + z = 4 &\implies \boxed{y = -1} \\ x + 2y + z = 3 &\implies \boxed{x = 2} \end{aligned}$$

Therefore, $\boxed{(x, y, z) = (2, -1, 3)}$. **(Ans.)**

1.3.2 Gauss-Jordan Elimination

In Gauss-Jordan elimination, we perform elementary row operations on the augmented matrix $A|B$ to convert it into a *diagonal matrix.*

$$\begin{aligned} \therefore A|B &= \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \\ &\sim \left[\begin{array}{cccc|c} a'_{11} & 0 & \cdots & 0 & b'_1 \\ 0 & a'_{22} & \cdots & 0 & b'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{nn} & b'_n \end{array}\right] \end{aligned}$$

Upon back-substitution, we get

$$
\begin{aligned}
a'_{nn}x_n = b'_n &\implies \boxed{x_n = \frac{b'_n}{a'_{nn}}} \\
a'_{(n-1)(n-1)}x_{n-1} = b'_{n-1} &\implies \boxed{x_{n-1} = \frac{b'_{n-1}}{a'_{(n-1)(n-1)}}} \\
&\vdots \\
a'_{22}x_2 = b'_2 &\implies \boxed{x_2 = \frac{b'_2}{a'_{22}}} \\
a'_{11}x_1 = b'_1 &\implies \boxed{x_1 = \frac{b'_1}{a'_{11}}}
\end{aligned}
$$

The time complexity of Gauss-Jordan elimination is $\mathcal{O}\big(2n^3/3\big)$ for a system of order n.

Q.2. Solve using Gauss-Jordan elimination:

$$
\begin{aligned}
x + 2y + z &= 3 \\
2x + 3y + 3z &= 10 \\
3x - y + 2z &= 13
\end{aligned}
$$

We have the system of equations,

$$
\begin{aligned}
1x + 2y + 1z &= 3 \\
2x + 3y + 3z &= 10 \\
3x - 1y + 2z &= 13
\end{aligned}
$$

The above system can be written as,

$$
\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix}
\begin{bmatrix} x \\ y \\ z \end{bmatrix}
=
\begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix}
$$

By Gauss-Jordan elmination,

$$
\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]
&\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right]
&& \begin{pmatrix} R_2 \longrightarrow R_2 - 2R_1 \\ R_3 \longrightarrow R_3 - 3R_1 \end{pmatrix} \\
&\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right]
&& \big(R_3 \longrightarrow R_3 - 7R_1\big) \\
&\sim \left[\begin{array}{ccc|c} 8 & 16 & 0 & 0 \\ 0 & -8 & 0 & 8 \\ 0 & 0 & -8 & -24 \end{array}\right]
&& \begin{pmatrix} R_1 \longrightarrow 8R_1 + R_3 \\ R_2 \longrightarrow 8R + 2R_3 \end{pmatrix} \\
&\sim \left[\begin{array}{ccc|c} 8 & 0 & 0 & 16 \\ 0 & -8 & 0 & 8 \\ 0 & 0 & -8 & -24 \end{array}\right]
&& \big(R_1 \longrightarrow R_1 + 2R_2\big)
\end{aligned}
$$

Upon back-substitution, we get

$$\begin{aligned} -8z = -24 &\implies \boxed{z = 3} \\ -8y = 8 &\implies \boxed{y = -1} \\ 8x = 16 &\implies \boxed{x = 2} \end{aligned}$$

Therefore, $\boxed{(x, y, z) = (2, -1, 3)}$. **(Ans.)**

1.3.3 LU Decomposition

We have the linear system of equations $AX = B$. We can decompose the coefficient matrix A as $A = LU$, where L is a lower triangular matrix while U is an upper triangular matrix.

The time complexity of LU decomposition is $\mathcal{O}(n^3/3 + n^2)$ for a system of order n, which is slightly better than Gauss elimination.

$A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix.

Therefore,

$$\begin{aligned} AX = B &\implies (LU)X = B \\ &\implies LUX = B \\ &\therefore LY = B, \text{ where } Y \coloneqq UX \end{aligned}$$

$$\implies L \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \text{ and } A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

We can define the matrices L and U slightly differently depending on the leading diagonal, which corresponds to two methods.

Crout's Method

$$A = LU, \text{ where } L \coloneqq \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \text{ and } U \coloneqq \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Doolittle's Method

$$A = LU, \text{ where } L \coloneqq \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \text{ and } U \coloneqq \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Q.3. Solve using LU decomposition:

$$
\begin{aligned}
2x + 3y + z &= -1 \\
5x + y + z &= 9 \\
3x + 2y + 4z &= 11
\end{aligned}
$$

We have the system of equations,

$$
\begin{aligned}
2x + 3y + 1z &= -1 \\
5x + 1y + 1z &= 9 \\
3x + 2y + 4z &= 11
\end{aligned}
$$

The above system can be written as,

$$
\begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}, \text{ or } AX = B.
$$

Using Crout's method,

Let $A := LU$, where $L := \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and $U := \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$.

By LU decomposition,

$$
\begin{aligned}
A &= LU \\
\therefore \begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} &= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \\
\therefore \begin{bmatrix} 2 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} &= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}
\end{aligned}
$$

Upon comparing like terms and evaluating, $L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & -13/2 & 0 \\ 3 & -5/2 & 40/13 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 3/13 \\ 0 & 0 & 1 \end{bmatrix}$.

Now,

$$
\begin{aligned}
AX &= B \\
\implies (LU)X = B &\implies L(UX) = B \\
\therefore LY &= B \qquad \left(\text{where } Y = UX := \begin{bmatrix} a & b & c \end{bmatrix}^\top\right)
\end{aligned}
$$

$$
\therefore \begin{bmatrix} 2 & 0 & 0 \\ 5 & -13/2 & 0 \\ 3 & -5/2 & 40/13 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}
$$

Upon back-substitution, we get

$$\begin{aligned} 2a = -1 &\implies a = -0.5 \\ 5a - (^{13}/_{2})b = 9 &\implies b = -^{23}/_{13} \\ 3a - (^{5}/_{2})b + (^{40}/_{3})c = 11 &\implies c = ^{21}/_{8} \end{aligned}$$

$$\implies Y = \begin{bmatrix} -\frac{1}{2} & -\frac{23}{13} & \frac{21}{8} \end{bmatrix}^{\top} = UX$$

$$\therefore \begin{bmatrix} 1 & ^{3}/_{2} & ^{1}/_{2} \\ 0 & 1 & ^{3}/_{13} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -^{1}/_{2} \\ -^{23}/_{13} \\ ^{21}/_{8} \end{bmatrix}$$

Upon back-substitution, we get

$$\begin{aligned} z = ^{21}/_{8} &\implies \boxed{z = ^{21}/_{8}} \\ y + (^{3}/_{13})z = -^{23}/_{13} &\implies \boxed{y = -^{19}/_{8}} \\ x + (^{3}/_{2})y + (^{1}/_{2})z = -^{1}/_{2} &\implies \boxed{x = ^{7}/_{4}} \end{aligned}$$

Therefore, $\boxed{(x, y, z) = \left(\frac{7}{4}, -\frac{19}{8}, \frac{21}{8}\right)}$. **(Ans.)**

Q.4. Solve using Crout's method as well as Doolittle's method:

$$\begin{aligned} 3x + 5y + 2z &= 8 \\ 8y + 2z &= -7 \\ 6x + 2y + 8z &= 26 \end{aligned}$$

We have the system of equations,

$$\begin{aligned} 3x + 5y + 2z &= 8 \\ 0x + 8y + 2z &= -7 \\ 6x + 2y + 8z &= 26 \end{aligned}$$

The above system can be written as,

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}, \text{ or } AX = B.$$

Using Crout's method,

Let $A = LU$, where $L := \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and $U := \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$.

By LU decomposition,

$$
\begin{aligned}
A &= LU \\
\therefore \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} &= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \\
\therefore \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} &= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}
\end{aligned}
$$

Upon comparing like terms and evaluating, $L = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 6 & -8 & 6 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 5/3 & 2/3 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$.

Now,

$$
\begin{aligned}
AX &= B \\
\implies (LU)X &= B \implies L(UX) = B \\
\therefore LY &= B \qquad \left(\text{where } Y = UX \coloneqq \begin{bmatrix} a & b & c \end{bmatrix}^\top\right) \\
\therefore \begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 6 & -8 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}
\end{aligned}
$$

Upon back-substitution, we get

$$
\begin{aligned}
3a = 8 &\implies a = 8/3 \\
8b = -7 &\implies b = -7/8 \\
6a - 8b + 6c = 26 &\implies 1/2
\end{aligned}
$$

$$
\implies Y = \begin{bmatrix} \frac{8}{3} & -\frac{7}{8} & \frac{1}{2} \end{bmatrix}^\top = UX
$$

$$
\therefore \begin{bmatrix} 1 & 5/3 & 2/3 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8/3 \\ -7/8 \\ 1/2 \end{bmatrix}
$$

Upon back-substitution, we get

$$
\begin{aligned}
z = 1/2 &\implies \boxed{z = 1/2} \\
y + (1/4)z = -7/8 &\implies \boxed{y = -1} \\
x + (5/3)y + (2/3)z = 8/3 &\implies \boxed{x = 4}
\end{aligned}
$$

Using Doolittle's method,

Let $A = LU$, where $L := \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U := \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$.

By LU decomposition,

$$\begin{aligned} A &= LU \\ \therefore \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \\ \therefore \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} &= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} \end{aligned}$$

Upon comparing like terms and evaluating, $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}$.

Now,

$$\begin{aligned} AX &= B \\ \implies (LU)X = B &\implies L(UX) = B \\ \therefore LY &= B \qquad \left(\text{where } Y = UX := \begin{bmatrix} a & b & c \end{bmatrix}^\top\right) \\ \therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix} \end{aligned}$$

Upon back-substitution, we get

$$\begin{aligned} a &= 8 \\ b &= -7 \\ 2a - b + c = 26 &\implies c = 3 \end{aligned}$$

$$\implies Y = \begin{bmatrix} 8 & -7 & 3 \end{bmatrix}^\top = UX$$

$$\therefore \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}$$

Upon back-substitution, we get

$$
\begin{aligned}
6z = 3 &\implies \boxed{z = 1/2} \\
8y + 2z = -7 &\implies \boxed{y = -1} \\
3x + 5y + 2z = 8 &\implies \boxed{x = 4}
\end{aligned}
$$

Therefore, $\boxed{(x, y, z) = \left(4, -1, \frac{1}{2}\right)}$. **(Ans.)**

Cholesky's Method

If A is a symmetric and positive definite[2] matrix, then $A = LL^\top$, where $L := \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$.

1.4 Indirect Methods

As seen above, the direct methods are computationally very expensive, so for most practical applications we prefer indirect methods to approximate the solution within a reasonable tolerance.

1.4.1 Gauss-Jacobi Method

Consider the linear system of equations:

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n} & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n} & = & b_2 \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn} & = & b_n
\end{array}
$$

If the absolute values of pivot (diagonal) coefficients are greater than the absolute value of the sum of other coefficients in the same row, then we can apply the Gauss-Jacobi algorithm to approximate the solution.

$$
\begin{array}{ccccccccc}
|a_{11}| & > & |a_{12}| & + & |a_{13}| & + & \dots & + & |a_{1n}| \\
|a_{22}| & > & |a_{21}| & + & |a_{23}| & + & \dots & + & |a_{2n}| \\
\vdots & & \vdots & & \vdots & & \ddots & & \vdots \\
|a_{nn}| & > & |a_{n1}| & + & |a_{n2}| & + & \dots & + & \left|a_{n(n-1)}\right|
\end{array}
$$

We shall start with an initial guess for the solution $\begin{bmatrix} x_1^{(0)} & x_2^{(0)} & \cdots & x_n^{(0)} \end{bmatrix}^\top$ and follow the

[2] $\forall \mathbf{x} \in \mathbb{R}^n \; \mathbf{x}^\top A \mathbf{x} > 0$. In other words, all eigenvalues are real and positive.

recurrence relation:

$$
\begin{aligned}
x_1^{(k+1)} &:= \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)}\right)\\
x_2^{(k+1)} &:= \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}\right)\\
\vdots \quad & \quad \vdots\\
x_n^{(k+1)} &:= \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \cdots - a_{n(n-1)}x_{n-1}^{(k)}\right)
\end{aligned}
$$

1.4.2 Gauss-Seidel Method

The Gauss-Seidel algorithm is an optimization to the Gauss-Jacobi algorithm, where the variables are updated with updated values of the previous computed variables within the same iteration.

$$
\begin{aligned}
x_1^{(k+1)} &:= \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)}\right)\\
x_2^{(k+1)} &:= \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}\right)\\
\vdots \quad & \quad \vdots\\
x_n^{(k+1)} &:= \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \cdots - a_{n(n-1)}x_{n-1}^{(k+1)}\right)
\end{aligned}
$$

Q.5. Solve using Gauss-Jacobi and Gauss-Seidel methods:

$$
\begin{aligned}
10x - 5y - 2z &= 3\\
4x - 10y + 3z &= -3\\
x + 6y + 10z &= -3
\end{aligned}
$$

We can observe that:

$$
\begin{aligned}
|10| &> |-5| + |-2|\\
|-10| &> |4| + |3|\\
|10| &> |1| + |6|
\end{aligned}
$$

Thus, the solution to the given system can be approximated using Gauss-Jacobi method.

The recurrence relation would be of the form:

$$
\begin{aligned}
x^{(k+1)} &= \frac{1}{10}\left(3 + 5y^{(k)} + 2z^{(k)}\right)\\
y^{(k+1)} &= \frac{1}{10}\left(3 + 4x^{(k)} + 3z^{(k)}\right)\\
z^{(k+1)} &= -\frac{1}{10}\left(3 + x^{(k)} + 6y^{(k)}\right)
\end{aligned}
$$

Let the initial approximation be $x^{(0)} := 0, y^{(0)} := 0, z^{(0)} := 0$.

Using Gauss-Jacobi method,

Iteration 1

$$\begin{aligned} x^{(1)} &= \frac{1}{10}\left(3 + 5y^{(0)} + 2z^{(0)}\right) = 0.3 \\ y^{(1)} &= \frac{1}{10}\left(3 + 4x^{(0)} + 3z^{(0)}\right) = 0.3 \\ z^{(1)} &= -\frac{1}{10}\left(3 + x^{(0)} + 6y^{(0)}\right) = -0.3 \end{aligned}$$

Iteration 2

$$\begin{aligned} x^{(2)} &= \frac{1}{10}\left(3 + 5y^{(1)} + 2z^{(1)}\right) = 0.39 \\ y^{(2)} &= \frac{1}{10}\left(3 + 4x^{(1)} + 3z^{(1)}\right) = 0.33 \\ z^{(2)} &= -\frac{1}{10}\left(3 + x^{(1)} + 6y^{(1)}\right) = -0.51 \end{aligned}$$

Iteration 3

$$\begin{aligned} x^{(3)} &= \frac{1}{10}\left(3 + 5y^{(2)} + 2z^{(2)}\right) = 0.363 \\ y^{(3)} &= \frac{1}{10}\left(3 + 4x^{(2)} + 3z^{(2)}\right) = 0.303 \\ z^{(3)} &= -\frac{1}{10}\left(3 + x^{(2)} + 6y^{(2)}\right) = -0.537 \end{aligned}$$

Iteration 4

$$\begin{aligned} x^{(4)} &= \frac{1}{10}\left(3 + 5y^{(3)} + 2z^{(3)}\right) = 0.3441 \\ y^{(4)} &= \frac{1}{10}\left(3 + 4x^{(3)} + 3z^{(3)}\right) = 0.2841 \\ z^{(4)} &= -\frac{1}{10}\left(3 + x^{(3)} + 6y^{(3)}\right) = -0.5181 \end{aligned}$$

Iteration 5

$$\begin{aligned} x^{(5)} &= \frac{1}{10}\left(3 + 5y^{(4)} + 2z^{(4)}\right) = 0.33840 \\ y^{(5)} &= \frac{1}{10}\left(3 + 4x^{(4)} + 3z^{(4)}\right) = 0.28221 \\ z^{(5)} &= -\frac{1}{10}\left(3 + x^{(4)} + 6y^{(4)}\right) = -0.50487 \end{aligned}$$

Iteration 6

$$\begin{aligned} x^{(6)} &= \frac{1}{10}\left(3 + 5y^{(5)} + 2z^{(5)}\right) \approx 0.3401 \\ y^{(6)} &= \frac{1}{10}\left(3 + 4x^{(5)} + 3z^{(5)}\right) \approx 0.2839 \\ z^{(6)} &= -\frac{1}{10}\left(3 + x^{(5)} + 6y^{(5)}\right) \approx -0.5031 \end{aligned}$$

Therefore, our solution is $\boxed{\begin{bmatrix} x & y & z \end{bmatrix}^\top \approx \begin{bmatrix} 0.3401 & 0.2839 & -0.5031 \end{bmatrix}^\top}$

by Gauss-Jacobi in 6 iterations. **(Ans.)**

Using Gauss-Seidel method,

Iteration 1

$$
\begin{aligned}
x^{(1)} &= \frac{1}{10}\left(3 + 5y^{(0)} + 2z^{(0)}\right) = 0.3 \\
y^{(1)} &= \frac{1}{10}\left(3 + 4x^{(1)} + 3z^{(0)}\right) = 0.42 \\
z^{(1)} &= -\frac{1}{10}\left(3 + x^{(1)} + 6y^{(1)}\right) = -0.582
\end{aligned}
$$

Iteration 2

$$
\begin{aligned}
x^{(2)} &= \frac{1}{10}\left(3 + 5y^{(1)} + 2z^{(1)}\right) = 0.3936 \\
y^{(2)} &= \frac{1}{10}\left(3 + 4x^{(2)} + 3z^{(1)}\right) = 0.28284 \\
z^{(2)} &= -\frac{1}{10}\left(3 + x^{(2)} + 6y^{(2)}\right) = -0.509064
\end{aligned}
$$

Iteration 3

$$
\begin{aligned}
x^{(3)} &= \frac{1}{10}\left(3 + 5y^{(2)} + 2z^{(2)}\right) = 0.3396072 \\
y^{(3)} &= \frac{1}{10}\left(3 + 4x^{(3)} + 3z^{(2)}\right) = 0.28312368 \\
z^{(3)} &= -\frac{1}{10}\left(3 + x^{(3)} + 6y^{(3)}\right) = -0.503834928
\end{aligned}
$$

Iteration 4

$$
\begin{aligned}
x^{(4)} &= \frac{1}{10}\left(3 + 5y^{(3)} + 2z^{(3)}\right) \approx 0.3408 \\
y^{(4)} &= \frac{1}{10}\left(3 + 4x^{(4)} + 3z^{(3)}\right) \approx 0.2852 \\
z^{(4)} &= -\frac{1}{10}\left(3 + x^{(4)} + 6y^{(4)}\right) \approx -0.5052
\end{aligned}
$$

Therefore, our solution is $\boxed{\begin{bmatrix} x & y & z \end{bmatrix}^\top \approx \begin{bmatrix} 0.3408 & 0.2852 & -0.5052 \end{bmatrix}^\top}$
by Gauss-Seidel in 4 iterations. **(Ans.)**

1.4.3 Power method to find Eigenvalues

Dominant eigenvalue

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n eigenvalues of a square matrix of order n.

If $|\lambda_1| > |\lambda_i| \, \forall i \in [\![2, n]\!]$ then λ_1 is called the **dominant eigenvalue** and its corresponding eigenvector is called the **dominant eigenvector**.

Power method

Let A be a square matrix of order n. Start with an initial guess: say $\mathbf{x}_0$ is the first approximation of the dominant eigenvector.

Let $\mathbf{x}_0 \coloneqq \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^\top$. We'll follow the recurrence relation:

$$\mathbf{x}_{k+1} \coloneqq A\mathbf{x}_k \xrightarrow{\textit{Scale}} \frac{1}{m^{(k+1)}} \mathbf{x}^{(k+1)}, \tag{1.4}$$

where $m^{(k+1)}$ is the magnitude of the highest component in the vector $\mathbf{x}^{(k+1)}$. We scale the vector each iteration so that the values do not explode and precision is maintained while using floating point arithmetic.

Theorem 1. *If $\mathbf{x}$ is an eigenvector of a matrix A, then its corresponding eigenvalue is given by*

$$\lambda = \frac{A\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{A\mathbf{x} \cdot \mathbf{x}}{\|\mathbf{x}\|^2},$$

Q.6. Find the approximated dominant eigenvalue and eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}.$$

Consider the initial guess $\mathbf{x}_0 \coloneqq \begin{bmatrix} 1.00 & 1.00 & 1.00 \end{bmatrix}^\top$.

For the sake of our sanity, we will stay precise upto 2 decimal places.

$$\therefore \mathbf{x}_1 \coloneqq A\mathbf{x}_0 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 3.00 \\ 1.00 \\ 5.00 \end{bmatrix} \xrightarrow{\textit{Scale}} \begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix}$$

$$\therefore \mathbf{x}_2 \coloneqq A\mathbf{x}_1 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 1.00 \\ 1.00 \\ 2.20 \end{bmatrix} \xrightarrow{\textit{Scale}} \begin{bmatrix} 0.45 \\ 0.45 \\ 1.00 \end{bmatrix}$$

$$\therefore \mathbf{x}_3 \coloneqq A\mathbf{x}_2 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.45 \\ 0.45 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 1.35 \\ 1.55 \\ 2.80 \end{bmatrix} \xrightarrow{\textit{Scale}} \boxed{\underbrace{\begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}}_{\text{Dominant eigenvector}}}$$

(Ans.)

From theorem 1, the dominant eigenvalue λ is given by

$$
\begin{aligned}
\lambda &= \frac{A\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} \\
&= \frac{\begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix} \cdot \begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}}{\begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix} \cdot \begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}} = \frac{\begin{bmatrix} 1.50 \\ 1.50 \\ 3.00 \end{bmatrix} \cdot \begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}}{\begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix} \cdot \begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}} \\
&= \frac{(1.50)(0.50) + (1.50)(0.50) + (3.00)(1.00)}{(0.50)^2 + (0.50)^2 + (1.00)^2} \\
&= \frac{0.75 + 0.75 + 3.00}{0.25 + 0.25 + 1.00} \\
&= \frac{4.50}{1.50} \implies \boxed{\lambda \approx 3.00}
\end{aligned}
$$

(Ans.)

Chapter 2

Quadratic Forms

2.1 Foundations

2.1.1 Rank of a Matrix

The rank of a matrix is the number of non-zero rows it has when it is in Row-Echelon form.

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \cdots & 0 \\ 0 & \alpha_{22} & \alpha_{23} & \alpha_{24} & \cdots & 0 \\ 0 & 0 & 0 & \alpha_{34} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_{(n-1)(n)} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n}$$

The above matrix has $n-1$ non-zero rows in it's reduced row echelon form and hence has rank $n-1$.

2.1.2 Column Space

The column space of a matrix A, is just the subspace spanned by it's columns. It is also known as the Image or the Range of the matrix A.

$$\text{Col Space}(A) = \text{Im}(A) = \{A \cdot \mathbf{x} \,|\, \mathbf{x} \in V\}$$

where A is a transformation from V to W.

Why is the Image of A a subspace?

Consider a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Since the vector $A \cdot \mathbf{x}$ is given as $\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$, which is essentially just a *linear combination* of the columns of A. The set of all elements $A \cdot \mathbf{x}$ is just a subspace in W.

2.1.3 Null Space

The null space of a matrix A, is the set of vectors which are mapped to $\mathbf{0_W}$ when transformed by A. It is also known as the Kernel of the matrix A.

$$\text{Null Space}(A) = \text{rank}(A) = \{\mathbf{x} \in V \mid A \cdot \mathbf{x} = \mathbf{0_W}\}$$

where A is a transformation from V to W.

2.1.4 Similarity and Diagonalisability

Similarity

A matrix A is said to be *similar* to another matrix B, if there exists and invertible matrix P such that:

$$B = P^{-1}AP$$

The similarity relation is an equivalence relation.

Diagonalisability

A matrix A is said to be *diagonalisable* iff it is similar to a diagonal matrix D.

$$\underset{\text{Diagonal Matrix}}{D} = \underset{\text{Similar}}{P^{-1}}AP$$

The most important theorem about diagonalisability is as follows:
Theorem 2. *An $n \times n$ matrix A is diagonalisable if and only if there is an invertible matrix P given by*

$$P = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$$

where the X_k are eigenvectors of A.

We accept this theorem without a proof.

2.2 Quadratic Forms

The expression of the form $Q(\mathbf{x}) = k_{11}x_1^2 + k_{12}x_1x_2 + k_{13}x_1x_3 + \cdots + k_{22}x_2^2 + \cdots + k_{nn}x_n^2$, where the degree of each term is 2, is called a *quadratic form*.

Every Quadratic Form (QF) can be represented in the following way:

$$Q : \mathbb{R}^n \to \mathbb{R}$$

$$Q(\mathbf{x}) = \mathbf{x}^\top \underset{\text{Associated Matrix}}{A}\, \mathbf{x}$$

2.2.1 Associated matrices for common vector spaces

The vector space $\mathbb{R}^2$

For the vector space $\mathbb{R}^2$, with vectors of the form $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$, consider the associated matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} Q(\mathbf{x}) \end{bmatrix} &= \mathbf{x}^\top A\mathbf{x} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} \\ \therefore \begin{bmatrix} k_{11}x^2 + k_{12}x_1x_2 + k_{22}x^2 \end{bmatrix} &= \begin{bmatrix} a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2 \end{bmatrix} \end{aligned}$$

Comparing similar terms, we get $a_{11} = k_{11}$, $a_{12} + a_{21} = k_{12}$ and $a_{22} = k_{22}$. We prefer symmetric matrices as they have a few special properties which will help us. Hence, we can say that the associated matrix A is

$$A = \begin{bmatrix} k_{11} & {}^{k_{12}}/_{2} \\ {}^{k_{12}}/_{2} & k_{22} \end{bmatrix}$$

The vector space $\mathbb{R}^3$

Following a similar process for $\mathbb{R}^2$, the associated matrix for a QF in $\mathbb{R}^3$ is

$$A = \begin{bmatrix} k_{11} & {}^{k_{12}}/_{2} & {}^{k_{13}}/_{2} \\ {}^{k_{12}}/_{2} & k_{22} & {}^{k_{23}}/_{2} \\ {}^{k_{13}}/_{2} & {}^{k_{23}}/_{2} & k_{33} \end{bmatrix}$$

2.2.2 Canonical () Form

The canonical form of a QF is a corresponding QF where only the square terms exist. e.g. $f(x, y) = ax^2 + by^2$

To convert a given QF into it's canonical form, we convert the associated matrix into a similar diagonal matrix. Since we have taken A(the associated matrix) to be a symmetric matrix, it is always diagonalisable by the Spectral Theorem.

It is important to note that for a given QF, there exists a unique canonical form of the form $\lambda_1x_1^2 + \lambda_2x_2^2 + \cdots + \lambda_nx_n^2 = \sum_{i=1}^{n} \lambda_ix_i^2$ where λ_i are the eigenvalues of the matrix A.

2.2.3 Definiteness of Eigenvalues

We define the nature of a quadratic form $\mathbf{x}^\top A\mathbf{x}$ as follows:

- Positive definite: $\mathbf{x}^\top A\mathbf{x} > 0\,; \forall\mathbf{x} \neq 0$
- Positive semi-definite: $\mathbf{x}^\top A\mathbf{x} \geq 0\,; \forall\mathbf{x} \neq 0$
- Negative definite: $\mathbf{x}^\top A\mathbf{x} < 0\,; \forall\mathbf{x} \neq 0$
- Negative semi-definite: $\mathbf{x}^\top A\mathbf{x} \leq 0\,; \forall\mathbf{x} \neq 0$

- Indefinite: $\mathbf{x}^\top A\mathbf{x} > 0 \wedge \mathbf{x}^\top A\mathbf{x} < 0$

Alternatively, we can determine the nature from the eigenvalues of the associated matrix:

- Positive definite: $\forall\lambda.\lambda > 0$
- Positive semi-definite: $\forall\lambda.\lambda \geq 0$
- Negative definite: $\forall\lambda.\lambda < 0$
- Negative semi-definite: $\forall\lambda.\lambda \leq 0$
- Indefinite: $\lambda > 0 \wedge \lambda < 0$

Sylvester's Criterion

Sylvester's Criterion provides a straightforward method to determine whether a given real, symmetric matrix A is positive definite, negative definite, or indefinite. Instead of computing eigenvalues, which can be computationally expensive, this criterion relies on **Leading Principal Minors**, making it a more efficient alternative.

Leading Principal Minors. A *Leading Principal Minor* (LPM) of a matrix is the determinant of its top-left $k \times k$ submatrix. For an $n \times n$ symmetric matrix A, the k-th LPM is defined as:

$$\Delta_k = \det(A_k), \quad k = 1, 2, \ldots, n,$$

where A_k is the submatrix formed by taking the first k rows and columns of A.

For example, for a 3×3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

the LPMs are:

$$\Delta_1 = a_{11}, \quad \Delta_2 = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \Delta_3 = \det(A).$$

Why Use Sylvester's Criterion? While eigenvalues can also determine definiteness, computing them is much more challenging:

- Finding eigenvalues involves solving the characteristic equation $\det(A - \lambda I) = 0$, which is a polynomial of degree n.
- Polynomials of degree greater than 4 do not have general solutions (as per Abel-Ruffini theorem). Thus, eigenvalues can only be explicitly computed for matrices up to 4×4.
- In contrast, Sylvester's Criterion only requires computing a series of determinants, which is computationally inexpensive for most practical applications.

The Criterion. Sylvester's Criterion states:

- A is **positive definite** if all the LPMs are positive:

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \ldots, \quad \Delta_n > 0.$$

- A is **negative definite** if the signs of the LPMs alternate, starting with $-$:

$$\Delta_1 < 0, \quad \Delta_2 > 0, \quad \Delta_3 < 0, \quad \ldots.$$

- A is **indefinite** if the sequence of LPMs does not meet either of the above criteria.

2.3 Principal Axes Theorem

If A is an $n \times n$ symmetric matrix associated with the quadratic form $\mathbf{x}^\top A\mathbf{x}$ and if Q is an orthonormal matrix such that $Q^{-1}AQ = Q^\top AQ = D$, then the change of variable $\mathbf{x} = Q\mathbf{y}$ transforms the quadratic form $\mathbf{x}^\top A\mathbf{x}$ into the quadratic form $\mathbf{y}^\top D\mathbf{y}$.

$$\mathbf{x}^\top A\mathbf{x} \mapsto \mathbf{y}^\top D\mathbf{y} = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$$

Proof.

$$\begin{aligned} \mathbf{x}^\top A\mathbf{x} &= (Q\mathbf{y})^\top A(Q\mathbf{y}) \\ &= (\mathbf{y}^\top Q^\top)A(Q\mathbf{y}) \\ &= \mathbf{y}(Q^\top AQ)\mathbf{y} \\ \therefore \mathbf{x}^\top A\mathbf{x} &= \mathbf{y}^\top D\mathbf{y} \end{aligned}$$

□

Note:

Rank:- Number of non-zero eigenvalues/Rank of associated matrix.
Index:- Number of positive eigenvalues
Signature:- Difference between the number of positive and number of negative eigenvalues.

Q.1. Determine the nature, rank, index, signature of the quadratic form: $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_2x_3 - 2x_3x_1 + 2x_1x_2$. Also transform it into it's canonical form.

The associated matrix of the above quadratic form is $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Finding the eigenvalues of A using the characteristic equation.

$$\begin{aligned} &\det(A - \lambda I) = 0 \\ &\therefore \det\left(\begin{bmatrix} 1-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{bmatrix}\right) = 0 \\ &\therefore \lambda^3 - 6\lambda^2 + 8\lambda + 2 = 0 \\ &\therefore \lambda \in \{-0.214\ldots, 3.675\ldots, 2.539\ldots\} \end{aligned}$$

The above QF is $\boxed{\text{indefinite}}$, has $\boxed{\text{rank} = 3}$, has $\boxed{\text{index} = 2}$ and $\boxed{\text{signature} = 1}$. **(Ans.)**

Converting the above QF to canonical form, we get $\mathbf{x}^\top A\mathbf{x} \mapsto \mathbf{y}^\top D\mathbf{y}$, where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

Thus, the canonical form is $\boxed{(-0.214\ldots)y_1^2 + (3.675\ldots)y_2^2 + (2.539\ldots)y_3^2}$. **(Ans.)**

2.4 Cholesky's Factorization

In section 1.3.3, we defined a numerical method to find the exact solution of a system of linear equations by decomposing a symmetric, positive definite matrix into $LL^\top$ where L is a lower triangular matrix.

Q.2. Decompose the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}$ using Cholesky factorization.

Finding the eigenvalues of A using the characteristic equation.

$$\begin{aligned}
&\det(A - \lambda I) = 0 \\
&\therefore \det\left(\begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 8-\lambda & 22 \\ 3 & 22 & 82-\lambda \end{bmatrix}\right) = 0 \\
&\therefore \lambda^3 - 91\lambda^2 + 249\lambda - 36 = 0 \\
&\therefore \lambda \in \{88.1808\ldots, 2.665\ldots, 0.153\ldots\}
\end{aligned}$$

Since, all the eigenvalues are positive in nature, A is a positive definite matrix. A is also symmetric in nature. A can be decomposed to $LL^\top$.

Consider, $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$.

$$\begin{aligned}
\because LL^\top &= A \\
\therefore A &= LL^\top \\
\therefore \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} &= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} \\
\therefore \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} &= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}
\end{aligned}$$

Comparing similar terms, we find the values:

$$\begin{aligned}
l_{11} &= \sqrt{1} = 1 \\
l_{21} &= {}^{2}\!/_{1} = 2 \\
l_{31} &= {}^{3}\!/_{1} = 3 \\
l_{22} &= \sqrt{8 - 2^2} = \sqrt{4} = 2 \\
l_{32} &= {}^{(22-6)}\!/_{2} = {}^{16}\!/_{2} = 8 \\
l_{33} &= \sqrt{82 - 3^2 - 8^2} = \sqrt{9} = 3
\end{aligned}$$

(We consider only the positive roots.)

Thus, $A = LL^\top$, $\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix}}$. **(Ans.)**

Q.3. Transform $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$ into it's canonical form and find the change of variable.

The associated matrix of the above quadratic form is $A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$.

Finding the eigenvalues of A using the characteristic equation.

$$\begin{aligned}
&\det(A - \lambda I) = 0 \\
&\therefore \det\left(\begin{bmatrix} 1-\lambda & 6 & -2 \\ 6 & 1-\lambda & -4 \\ -2 & -4 & -3-\lambda \end{bmatrix}\right) = 0 \\
&\therefore \lambda^3 - 63\lambda - 162 = 0 \\
&\therefore \lambda \in \{-6, -3, 9\}
\end{aligned}$$

Finding the eigenvectors of A:

For $\lambda = -6$:

$$(A + 6I) \cdot \mathbf{X_1} = \mathbf{0}_{\mathbb{R}^3}$$

$$\therefore \begin{bmatrix} 8 & 6 & -2 \\ 6 & 7 & -4 \\ -2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From Cramer's rule, $\mathbf{X_1} = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}^\top$

For $\lambda = -3$:

$$(A + 3I) \cdot \mathbf{X_2} = \mathbf{0}_{\mathbb{R}^3}$$

$$\therefore \begin{bmatrix} 5 & 6 & -2 \\ 6 & 4 & -4 \\ -2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From Cramer's rule, $\mathbf{X_2} = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^\top$

For $\lambda = 9$:

$$(A - 9I) \cdot \mathbf{X_3} = \mathbf{0}_{\mathbb{R}^3}$$

$$\therefore \begin{bmatrix} -7 & 6 & -2 \\ 6 & -8 & -4 \\ -2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From Cramer's rule, $\mathbf{X_3} = \begin{bmatrix} -2 & -2 & 1 \end{bmatrix}^\top$

We know that for diagonalisability, there must exist an invertible matrix P(here Q), such that $P^{-1}AP$. (From theorem 2)

If P is orthogonal then, $P^{-1} = P^\top$. Hence, the orthogonal matrix $Q = \begin{bmatrix} \mathbf{X_1} & \mathbf{X_2} & \mathbf{X_3} \end{bmatrix}$ is the required matrix. If we normalise the vectors, then we get the eigenvalues as the coefficients of the canonical QF.

The canonical form of the above equation is $\boxed{-6y_1^2 - 3y_2^2 + 9y_3^2}$. **(Ans.)**

The change of variable is $\boxed{\mathbf{x} = \begin{bmatrix} -1/3 & 2/3 & -2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} \mathbf{y}}$. **(Ans.)**

2.5 Constraint Optimisation

If we have been given a quadratic form $\mathbf{x}^\top A\mathbf{x}$ with the constraint, $\|\mathbf{x}\| = 1$, then:

- The largest (maximum) value of $Q(\mathbf{x})$ is found to be $\lambda_{\max}$ (the maximum eigenvalue) for the corresponding eigenvector scaled so that it's norm is 1.
- The smallest (minimum) value of $Q(\mathbf{x})$ is found to be $\lambda_{\min}$ (the minimum eigenvalue) for the corresponding eigenvector scaled so that it's norm is 1.

Q.4. Find the minimum and maximum value of $Q(\mathbf{x}) = x_1^2 + 4x_1x_2 - 2x_2^2$ with the constraint $\mathbf{x}^\top\mathbf{x} = 1$.

The associated matrix of the above quadratic form is $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

Finding the eigenvalues of A using the characteristic equation.

$$\det(A - \lambda I) = 0$$

$$\therefore \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}\right) = 0$$

$$\therefore \lambda^2 + \lambda - 6 = 0$$

$$\therefore \lambda \in \{-3, 2\}$$

Finding the eigenvectors of A:

For $\lambda = -3$:

$$(A + 3I)\mathbf{X_1} = \mathbf{0}_{\mathbb{R}^2}$$
$$\therefore \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From Cramer's Rule: $\mathbf{X_1} = \begin{bmatrix} 1 & -2 \end{bmatrix}^\top$. Normalising the vector: $\mathbf{X_1} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}^\top$.

For $\lambda = 2$:

$$(A - 2I)\mathbf{X_2} = \mathbf{0}_{\mathbb{R}^2}$$
$$\therefore \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From Cramer's Rule: $\mathbf{X_2} = \begin{bmatrix} 2 & 1 \end{bmatrix}^\top$. Normalising the vector: $\mathbf{X_2} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}^\top$.

Thus, from the theory of constraint optimisation, we know that the maximum value of $Q(\mathbf{x})$ is 2 for $\mathbf{x} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ and the minimum value of $Q(\mathbf{x})$ is -3 for $\mathbf{x} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$. **(Ans.)**

Chapter 3

Inner Product Spaces

3.1 Foundations

Throughout this chapter we will go through the already-known concepts regarding vectors in 3-dimensional space and generalise them for any valid vector space.

3.1.1 Vectors

A *vector* is the most important mathematical structure for Computer Scientists and Engineers. The notion of a vector in the mathematical sense is very different from what you would expect vectors to be in the traditional sense of the word (magnitude and direction). In mathematics, a vector is simply a structure which usually contains more than one element (usually numbers) in a specific predefined format. This format may be a list, grid or anything else.

3.1.2 Vector Spaces

A *vector space* is a set of elements (usually called *vectors*) defined over a field $\mathbb{F}$ and having the operations of addition(+) and scalar multiplication($\cdot$) (from the field)[1]

Vector spaces follow a few basic properties which have been discussed in short here.

Consider a generic vector space V and the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and the scalars k, l, m from the field $\mathbb{F}$.

Closure Properties

1. Vector Addition is closed in V.
   $\mathbf{u} + \mathbf{v} \in V$
2. Scalar multiplication of vectors is closed in V.
   $k \cdot \mathbf{u} \in V$

Addition Properties

3. Vector addition is commutative. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
4. Vector addition is associated. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{u}) + \mathbf{w}$

[1] The addition and scalar multiplication operations are purely notational. They may be defined in any specific way that we would like.

5. There exists a special vector $\mathbf{0_V} \in V$ for all $\mathbf{u} \in V$ such that: $\mathbf{u} + \mathbf{0_V} = \mathbf{u}$
(Existence of additive identity).
6. For all $\mathbf{u} \in V$, there exist $-\mathbf{u} \in V$ such that: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0_V}$
(Existence of additive inverse).

Multiplication Properties

7. Compatibility with field multiplication. $k \cdot (l \cdot \mathbf{u}) = (kl) \cdot \mathbf{u}$
8. Distribution of scalar multiplication over vector addition. $k \cdot (\mathbf{u} + \mathbf{v}) = k \cdot \mathbf{u} + k \cdot \mathbf{v}$
9. Distribution of scalar multiplication over field addition. $(k + l) \cdot \mathbf{u} = k \cdot \mathbf{u} + l \cdot \mathbf{u}$
10. There exists a special scalar $1_V \in \mathbb{F}$ for all $\mathbf{u} \in V$ such that: $1_V \cdot \mathbf{u} = \mathbf{u}$
(Existence of multiplicative identity).

3.1.3 The vector space $\mathbb{R}^n$

The vector space defined over the set of Real Numbers ($\mathbb{R}$) with the vectors of the form $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^\top$, is commonly known as the n-dimensional *Euclidean Space* when equipped with the standard dot product. This is one of the most familiar and widely studied vector spaces in mathematics.

3.1.4 Length of a vector

The length of a vector for 3-dimensional Euclidean Space over a real field is known defined as the distance of the point to which the vector points from the origin of the vector space and is equal to $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$ for a vector $\vec{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

3.2 Inner Product

The inner product is the generalization of the dot product defined for *n-dimensional Euclidean Space* for any generic vector space.

Formally, the inner product is a function defined for a vector space over a field $\mathbb{F}$ that takes in an ordered pair of vectors, $(\mathbf{u}, \mathbf{v})$ and assigns to them, a number $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{F}$:

$$f : V \times V \to \mathbb{F}$$

$$f(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ is just a notational placeholder for the inner product.

3.2.1 Rules for Inner Products

The following rules must be followed by a binary function for it to be considered a valid inner product.

1. **Linearity**: $\langle \lambda \cdot \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \lambda \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.
2. **Conjugate Symmetry**: $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.
3. **Non-negativity**: $\forall u \in V \ \langle \mathbf{u}, \mathbf{u} \rangle \geq 0_{\mathbb{F}} \quad \wedge \quad \langle \mathbf{u}, \mathbf{u} \rangle = 0_{\mathbb{F}} \iff \mathbf{u} = \mathbf{0_V}$.

3.2.2 Inner Product Space

A vector space which defines a valid inner product is known as an Inner Product Space.

3.3 A few common Inner Product Spaces

Let us look at a few inner product spaces and their corresponding inner products.

3.3.1 Inner Product for the vector space $\mathbb{C}^n$

Consider the vector space where the vectors are defined as a list of n complex numbers as follows: $\mathbf{z} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix}^\top$ and $\mathbf{w} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}^\top$ where $\forall z_i, w_i \in \mathbb{C}$.
The inner product is defined as follows:

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1\overline{w_1} + z_2\overline{w_2} + z_3\overline{w_3} + \cdots + z_n\overline{w_n}$$

We can easily verify that this function satisfies the above mentioned rules, hence it is a valid inner product.

The standard dot product for n-dimensional Euclidean space over a real field is just a special case of this exact inner product. (Remember that for real numbers $\overline{r} = r$)

3.3.2 Inner Product for the vector space of continuous real-valued functions

The set of continuous real-valued functions constitute an inner product space with the standard inner product as follows:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} f(x) \cdot g(x) \cdot \mathrm{d}x$$

If the function f and/or g are not defined for the domain $[-1, 1]$, then we just take the integration for the largest possible common domain of definition.

Let us verify the validity of this function as the inner product:

1. Linearity:

$$\begin{aligned}
\langle \lambda\mathbf{f} + \mathbf{g}, \mathbf{h} \rangle &= \int_{-1}^{1} (\lambda f(x) + g(x)) \cdot h(x) \cdot \mathrm{d}x \\
&= \int_{-1}^{1} \lambda f(x) \cdot h(x) + g(x) \cdot h(x) \cdot \mathrm{d}x \\
&= \int_{-1}^{1} \lambda f(x) \cdot h(x)\,\mathrm{d}x + \int_{-1}^{1} g(x) \cdot h(x) \cdot \mathrm{d}x \\
&= \lambda \int_{-1}^{1} f(x) \cdot h(x)\,\mathrm{d}x + \int_{-1}^{1} g(x) \cdot h(x) \cdot \mathrm{d}x \\
\therefore \langle \lambda\mathbf{f} + \mathbf{g}, \mathbf{h} \rangle &= \lambda \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle
\end{aligned}$$

2. Conjugate Symmetry:

$$
\begin{aligned}
\langle \mathbf{f}, \mathbf{g} \rangle &= \int_{-1}^{1} f(x) \cdot g(x) \cdot \mathrm{d}x \\
&= \int_{-1}^{1} g(x) \cdot f(x) \cdot \mathrm{d}x \\
&(\because \text{ Integration of real-valued functions is real valued.}) \\
&= \overline{\int_{-1}^{1} g(x) \cdot f(x) \cdot \mathrm{d}x} \\
\therefore \langle \mathbf{f}, \mathbf{g} \rangle &= \overline{\langle \mathbf{g}, \mathbf{f} \rangle}
\end{aligned}
$$

3. Non-negativity:

$$
\begin{aligned}
\langle \mathbf{f}, \mathbf{f} \rangle &= \int_{-1}^{1} f(x) \cdot f(x) \cdot \mathrm{d}x \\
&= \int_{-1}^{1} (f(x))^2 \cdot \mathrm{d}x \\
&(\because \text{Area under the curve of a positive function is always positive} \\
&\quad \text{and is zero only when the curve is 0.}) \\
\therefore \langle \mathbf{f}, \mathbf{f} \rangle &\geq 0 \wedge \langle \mathbf{f}, \mathbf{f} \rangle = 0 \iff \mathbf{f}(x) = 0
\end{aligned}
$$

Since, all three conditions are satisfied, the above defined function is a valid inner product on the inner product space of real-valued continuous functions defined in the domain $[-1, 1]$.

Q.1. Suppose that for a vector space over the set of continuous real-valued functions defined for the domains $[0, \infty)$, $[0, 1]$ has an inner product candidate as follows:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{0}^{1} |f(x) \cdot g(x)| \cdot \mathrm{d}x$$

Is it an inner product space?

The given function is an inner product if it satisfies the three properties as defined before.

1. Linearity:

$$
\begin{aligned}
\langle \lambda\mathbf{f} + \mathbf{g}, \mathbf{h} \rangle &= \int_{0}^{1} |(\lambda f(x) + g(x)) \cdot h(x)| \cdot \mathrm{d}x \\
&= \int_{0}^{1} |\lambda(f(x) \cdot h(x)) + (g(x) \cdot h(x))| \cdot \mathrm{d}x \\
\lambda \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle &= \lambda \int_{0}^{1} |f(x) \cdot h(x)| \cdot \mathrm{d}x + \int_{0}^{1} |g(x) \cdot h(x)| \cdot \mathrm{d}x \\
&= \int_{0}^{1} |\lambda f(x) \cdot h(x)| + |g(x) \cdot h(x)| \cdot \mathrm{d}x
\end{aligned}
$$

We know that $\exists x, y \in \mathbb{R}; |x + y| \neq |x| + |y|$.

$\therefore \langle \lambda\mathbf{f} + \mathbf{g}, \mathbf{h} \rangle$ is not always equal to $\lambda \langle \mathbf{f}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbf{h} \rangle$.

Thus, the given function is not an inner product. **(Ans.)**

Note:

Whenever the problem statement does not specify a specific inner product, we make use of the standard inner products defined in these sections.

Q.2. Find inner product of x and x^2 on $[-1, 1]$

Let $f(x) = x$ and $g(x) = x^2$

$$\begin{aligned}
\langle \mathbf{f}, \mathbf{g} \rangle &= \int_{-1}^{1} f(x) \cdot g(x) \cdot \mathrm{d}x \\
&= \int_{-1}^{1} x \cdot x^2 \cdot \mathrm{d}x \\
&= \int_{-1}^{1} x^3 \cdot \mathrm{d}x \\
&= \left[x^4/4\right]_{-1}^{1} \\
&= 1/4 - 1/4 = \boxed{0}
\end{aligned}$$

Thus the inner product of x and x^2 is $\boxed{\langle \mathbf{x}, \mathbf{x^2} \rangle = 0}$. **(Ans.)**

3.3.3 Inner Product for the vector space of $\mathcal{M}_{m\times n}(\mathbb{C})$

The set of $m \times n$ matrices over a complex field constitute an inner product space with the standard inner product as follows:

$$\langle \mathbf{A}, \mathbf{B} \rangle = \operatorname{tr}(BA^*)$$

where A^* is the transposed conjugate of the matrix A.

Let us verify the validity of this function as the inner product:

1. Linearity:

$$\begin{aligned}
\langle \lambda\mathbf{A} + \mathbf{B}, \mathbf{C} \rangle &= \operatorname{tr}(C \cdot (\lambda A + B)^*) \\
&= \operatorname{tr}(C(\lambda A^* + B^*)) = \operatorname{tr}(C \cdot \lambda A^* + CB^*) \\
&= \operatorname{tr}(\lambda CA^*) + \lambda(CB^*) = \lambda \operatorname{tr}(CA^*) + \operatorname{tr}(CB^*) \\
\therefore \langle \lambda\mathbf{A} + \mathbf{B}, \mathbf{C} \rangle &= \lambda \langle \mathbf{A}, \mathbf{C} \rangle + \langle \mathbf{B}, \mathbf{C} \rangle
\end{aligned}$$

2. Conjugate Symmetry:

$$\begin{aligned}
\langle \mathbf{A}, \mathbf{B} \rangle &= \operatorname{tr}(BA^*) \\
&= \overline{\left(\overline{\operatorname{tr}(BA^*)}\right)} \\
&= \overline{(\operatorname{tr}((BA^*)^*))} \\
&= \overline{(\operatorname{tr}((A^*)^*(B)^*))} \\
&= \overline{(\operatorname{tr}(AB^*))} \\
\langle \mathbf{A}, \mathbf{B} \rangle &= \overline{\langle \mathbf{B}, \mathbf{A} \rangle}
\end{aligned}$$

Conjugate of conjugate is the number itself

3. Non-negativity:

$$
\begin{aligned}
\langle \mathbf{A}, \mathbf{A} \rangle &= \operatorname{tr}(AA^*) \\
&= \operatorname{tr}\left(\begin{bmatrix} \sum_{i=1}^{n} |a_{1i}|^2 & \cdots & \cdots & \cdot \\ \vdots & \sum_{i=1}^{n} |a_{2i}|^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \cdot & \cdots & \cdots & \sum_{i=1}^{n} |a_{mi}|^2 \end{bmatrix}_{m \times n} \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^2 \\
\therefore \langle \mathbf{A}, \mathbf{A} \rangle \geq 0 \wedge \langle \mathbf{A}, \mathbf{A} \rangle = 0 &\iff \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m \times n}
\end{aligned}
$$

Sum of squares of elem

Since, all three conditions are satisfied, the above defined function is a valid inner product on the inner product space of complex-elemented matrices of order $m \times n$.

3.3.4 Inner Product for the vector space of $\mathcal{P}^n[X]$

The set of complex-valued polynomials of degree less than or equal to n constitute an inner product space. Assuming the vectors to be of the form:

$$
\mathbf{p}(\mathbf{x}) = p_0 + p_1 x + p_2 x^2 + \cdots p_n x^n = \begin{bmatrix} 1 & x & x^2 & \cdots & x^n \end{bmatrix} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}
$$

The standard inner product is defined as follows:

$$
\langle \mathbf{p}, \mathbf{q} \rangle = p_0 q_0 + p_1 q_1 + p_2 q_2 + \cdots + p_n q_n = \sum_{i=0}^{n} p_i q_i
$$

Note:

Since real-valued polynomials are continuous real-valued functions as well, we can use the standard inner product for the vector space of continuous real valued functions as well.

Since we can represent the polynomials as column matrices, the vector space is just converted to an $(n+1)$-dimensional Euclidean Space over a complex field with basis as $\{1, x, x^2, \cdots, x^n\}$ instead of $\{x_1, x_2, x_3, \cdots, x_{n+1}\}$. The function $\langle \mathbf{p}, \mathbf{q} \rangle$ also has the same output as the inner product for that vector space.

We know that the inner product for n-dimensional Euclidean Space is valid and hence, the inner product candidate for the vector space of polynomials of degree less than or equal to n is also valid.

3.4 Norm (magnitude)

The *norm* of a vector is the generalisation of the concept of *length* of a vector. We define the norm $\|\mathbf{u}\|$ of a vector $\mathbf{u}$ as follows:

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

Note that this reduces to the well-known identity $|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$ for Euclidean Space.

Q.3. Find the norm of $f(x) = x$ on $[-1, 1]$ for the vector space:

(i) Real-valued continuous functions

(ii) $\mathcal{P}^2[x]$

(i) Real-valued continuous functions

$$\begin{aligned}
\|\mathbf{f}\| &= \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle} \\
&= \sqrt{\int_{-1}^{1} f(x) \cdot f(x) \cdot \mathrm{d}x} = \sqrt{\int_{-1}^{1} x \cdot x \cdot \mathrm{d}x} \\
&= \sqrt{\int_{-1}^{1} x^2 \, \mathrm{d}x} \\
&= \sqrt{[x^3/3]_{-1}^{1}} = \sqrt{1/3 - (-1)/3} \\
&= \boxed{\sqrt{2/3}}
\end{aligned}$$

(Ans.)

(ii) $\mathcal{P}^2[x]$

$$\begin{aligned}
\|\mathbf{f}\| &= \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle} \\
&= \sqrt{p_0 \cdot p_0 + p_1 \cdot p_1 + p_2 \cdot p_2} = \sqrt{0.0 + 1.1 + 0.0} \\
&= \sqrt{1} = \boxed{1}
\end{aligned}$$

(Ans.)

Q.4. If $\mathbf{u} = \left(\frac{-3}{5}, \frac{4}{5} \right)$, $\|\mathbf{u}\| = ?$

For vector space $\mathbb{R}^2$, inner product is defined as $\sqrt{u_1 \cdot v_1 + u_2 \cdot v_2}$ for vectors $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. So,

$$\begin{aligned}
\|\mathbf{u}\| &= \sqrt{u_1^2 + u_2^2} \\
&= \sqrt{\left(\frac{-3}{5} \right)^2 + \left(\frac{4}{5} \right)^2} \\
&= \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{\cancel{25}}{\cancel{25}}} = \sqrt{1} \\
\|\mathbf{u}\| &= \boxed{1}
\end{aligned}$$

(Ans.)

3.4.1 Unit Vector

Let V be an inner product space. A vector $\mathbf{u} \in V$ is said to be a unit vector iff it's norm is 1.

$$\forall \mathbf{u} \in V/\{\mathbf{0_V}\}; \frac{\mathbf{u}}{\|\mathbf{u}\|} \text{ is a unit vector.}$$

Proof. Consider the vector $\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$. It is a unit vector iff $\|\mathbf{v}\| = 1$.

$$\begin{aligned}
\|\mathbf{v}\| &= \left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| \\
&= \left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\rangle \\
&= \frac{1}{\|\mathbf{u}\|} \cdot \frac{1}{\|\mathbf{u}\|} \cdot \langle \mathbf{u}, \mathbf{u} \rangle \\
&= \frac{1}{\|\mathbf{u}\|^2} \cdot \left(\sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} \right)^2 \\
&= \frac{1}{\|\mathbf{u}\|^2} \cdot \|\mathbf{u}\|^2 \\
\therefore \|\mathbf{v}\| &= 1
\end{aligned}$$

□

e.g. $(1, 2, 3)^\top \in \mathbb{R}^n$

$$\text{Unit}((1, 2, 3)^\top) = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)^\top$$

3.5 Orthogonality

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be orthogonal iff $\langle \mathbf{u}, \mathbf{v} \rangle = 0_{\mathbb{F}}$.

The concept of orthogonality denotes "perpendicularity" of some sort.

3.5.1 Orthogonal Set

A set of vectors is said to be an *Orthogonal Set* if all the distinct pairs of vectors in that set are orthogonal to each other.

3.5.2 Orthogonal Basis

An orthogonal set with the minimum number of linearly independent elements required to span the entire space is known as an *Orthogonal Basis*.

3.5.3 Orthonormal Basis

An orthogonal basis where all the elements are unit vectors is known as an *Orthonormal Basis*.

3.5.4 Orthogonal Complement

The *Orthogonal Complement* of a subspace W of a vector space V is another subspace containing elements of V that are orthogonal to all the elements of W simultaneously. It is referred to as the **perp** or **perpendicular complement** informally and is denoted by $W^{\perp}$.

Q.5. Is the set $S = \{(2,1,-1),(0,1,1),(1,-1,1)\}$ an orthogonal basis? If yes, convert it to an orthonormal basis.

Consider the vectors $\mathbf{v_1} = (2,1,-1)$, $\mathbf{v_2} = (0,1,1)$, and $\mathbf{v_3} = (1,-1,1)$.

$$\begin{aligned}
\langle \mathbf{v_1}, \mathbf{v_2} \rangle &= 2\cdot 0 + 1\cdot 1 + (-1)\cdot 1 = 0 + 1 - 1 = 0\\
\langle \mathbf{v_2}, \mathbf{v_3} \rangle &= 0\cdot 1 + 1\cdot(-1) + 1\cdot 1 = 0 - 1 + 1 = 0\\
\langle \mathbf{v_3}, \mathbf{v_1} \rangle &= 1\cdot 2 + (-1)\cdot 1 + 1\cdot(-1) = 2 - 1 - 1 = 0
\end{aligned}$$

$\boxed{S \text{ is an orthogonal set}}$. Consider S' to be the orthonormal version of S. **(Ans.)**

$$\begin{aligned}
\|\mathbf{v_1}\| &= \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{4+1+1} = \sqrt{6}\\
\|\mathbf{v_2}\| &= \sqrt{0^2 + 1^2 + 1^2} = \sqrt{0+1+1} = \sqrt{2}\\
\|\mathbf{v_3}\| &= \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{1+1+1} = \sqrt{3}\\
\therefore S' &= \left\{ \frac{1}{\sqrt{6}}(2,1,-1), \frac{1}{\sqrt{2}}(0,1,1), \frac{1}{\sqrt{3}}(1,-1,1) \right\}\\
\therefore S' &= \boxed{\left\{ \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \right\}}
\end{aligned}$$

(Ans.)

Q.6. Is $S_1 = \{\sin(nx) \mid n = 1,2,3,\dots \text{ on } [0,\pi]\}$ an orthogonal set? Is it an orthonormal set?

Consider any two positive integers m and n which are not equal to each other. Calculating the inner product for $f(x) = \sin mx$ and $g(x) = \sin nx$.

$$\begin{aligned}
\langle \mathbf{f}, \mathbf{g} \rangle &= \int_0^{\pi} f(x)\cdot g(x)\cdot \mathrm{d}x = \int_0^{\pi} \sin(mx)\cdot\sin(nx)\cdot\mathrm{d}x\\
&= \int_0^{\pi} \frac{-1}{2}(\cos((m+n)x) - \cos((m-n)x))\cdot \mathrm{d}x\\
&= -0.5\left(\left[\frac{\sin((m+n)x)}{(m+n)} - \frac{\sin((m-n)x)}{(m-n)}\right]_0^{\pi}\right)\\
&= -0.5\left(\frac{\sin((m+n)\pi) - \sin(0)}{(m+n)} - \frac{\sin((m-n)\pi) - \sin(0)}{(m-n)}\right)\\
&= -0.5((0-0)/(m+n) - (0-0)/(m-n))\\
\therefore \langle \mathbf{f}, \mathbf{g} \rangle &= \boxed{0}
\end{aligned}$$

Since, the inner product of any two non-equal vectors is 0, $\boxed{\text{the set } S_1 \text{ is an orthogonal set}}$. **(Ans.)**

Consider the norm of a random vector $f(x) = \sin(mx)$.

$$\begin{aligned}
\|\mathbf{f}\| &= \int_0^{\pi} f(x) \cdot g(x) \cdot \mathrm{d}x \\
&= \int_0^{\pi} \frac{-1}{2}(\cos((m+m)x) - \cos((m-m)x)) \cdot \mathrm{d}x \\
&= -0.5\bigg(\int_0^{\pi} \cos(2mx) - 1 \, \mathrm{d}x\bigg) \\
&= 0.5\bigg(\int_0^{\pi} 1 - \cos(2mx) \, \mathrm{d}x\bigg) \\
&= 0.5\left(x - \frac{\sin(2mx)}{2m}\right)_0^{\pi} \\
&= 0.5\left([\pi - 0] - \frac{\sin(2m\pi) - \sin(0)}{2m}\right) = 0.5\left(\pi - \frac{0-0}{2m}\right) \\
&= 0.5\pi
\end{aligned}$$

Since, the norm of every element in the S_1 is 0.5π, the set $\boxed{S_1 \text{ is not an orthonormal set}}$. **(Ans.)**

Q.7. Find two vectors with norm 1 which are orthogonal to $(3, -4)$ in $\mathbb{R}^2$.

Consider an orthogonal vector to $\vec{a} = (3, -4)$ in $\mathbb{R}^2$ to be of the form $\vec{b} = (x, y)$. Now since, we know that the two vectors are orthogonal,

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= 0 \\
\therefore 3x - 4y &= 0 \\
\therefore y &= \frac{3}{4}x
\end{aligned}$$

The above equation provides a line in $\mathbb{R}^2$ but we only require unit vectors so:

$$\begin{aligned}
\left\|\vec{b}\right\| &= 1 \\
\therefore \sqrt{x^2 + y^2} &= 1 \\
\therefore \sqrt{x^2 + \left(\frac{3}{4}x\right)^2} &= 1 \\
\therefore \sqrt{\frac{25}{16}x^2} &= 1 \\
\therefore \frac{25}{16}x^2 &= 1^2 \\
\therefore x^2 &= \frac{16}{25} \\
\therefore x &= \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}
\end{aligned}$$

From the above equation $y = \frac{3}{4}x$, $y = \pm\frac{3}{5}$. Thus the two unit vectors orthogonal to $(3, -4)$

in $\mathbb{R}^2$ are $\boxed{\left(\frac{4}{5}, \frac{3}{5}\right) \text{ and } \left(-\frac{4}{5}, -\frac{3}{5}\right)}$. **(Ans.)**

3.6 Orthogonal Projections

The orthogonal projection of a vector **u** onto **v** is defined as follows:

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \cdot \mathbf{v}$$

To understand this formula, let us take a look at 2-Dimensional Euclidean Space in Cartesian (Rectangular) Coordinates. In the above diagram, we can see that the vector $\vec{u}$ casts a "shadow"

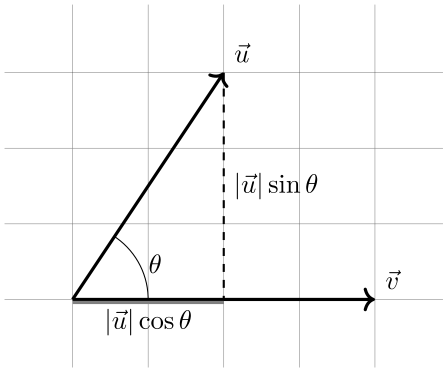


Figure 3.1: Orthogonal Projection of $\vec{u}$ onto $\vec{v}$.

onto the vector **v** of length $|\vec{u}| \cos\theta$, i.e. the component of $\vec{u}$ in the direction of $\vec{v}$ has the length $|\vec{u}| \cos\theta$ where θ is the angle between the two vectors. This length is just called the scalar projection of $\vec{u}$ onto $\vec{v}$.

If we need the vector in this direction, we already have $\vec{v}$ but it is not of the required length, so we convert it into a unit vector by dividing it by it's length and then multiply the length of the "shadow" with this new unit vector to get our required vector.

The vector (say $\vec{w}$) perpendicular to $\vec{v}$ and going from the tip of the projection of $\vec{u}$ onto $\vec{v}$ to the tip of $\vec{u}$ is the other vector which forms up $\vec{u}$. By the triangle law of addition of vectors, we can say that:

$$\vec{w} + \text{proj}_{\vec{v}}\vec{u} = \vec{u}$$
$$\therefore \vec{w} = \vec{u} - \text{proj}_{\vec{v}}\vec{u}$$

Thus, the orthogonal projection of **u** onto **v** is just the component of the vector **u** in the direction of **v** i.e. $\boxed{\text{proj}_{\mathbf{v}}(\mathbf{u})}$ and the component of **u** orthogonal to **v** is $\boxed{\mathbf{u} - \text{proj}_{\mathbf{v}}(\mathbf{u})}$

Q.8. Find $\text{proj}_{\vec{v}}\vec{u}$ where $\vec{u} = (1, 0, 5)$ and $\vec{v} = (3, 1, -7)$:

$\|\vec{v}\| = \sqrt{1^2 + 0^2 + 5^2} = \sqrt{26}$

$\vec{u} \cdot \vec{v} = 3 \cdot 1 + 1 \cdot 0 + 5 \cdot (-7) = 3 - 35 = -32$

$\text{proj}_{\vec{v}}\vec{u} = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \cdot \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\sqrt{26}^2} \cdot \vec{v} = -\frac{32}{26} \cdot (1, 0, 5) = \boxed{\left(-\frac{16}{13}, 0, -\frac{80}{13}\right)}$ **(Ans.)**

Q.9. Find the orthogonal complement $W^\perp$ of $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| 2x - y = 0;\ x, y \in \mathbb{R} \right\}$ and give it's basis.

Consider the elements of $W^\perp$ to be of the form $\begin{bmatrix} u \\ v \end{bmatrix}$. Since the elements of $W^\perp$ are orthogonal to W, we can say that:

$$\begin{aligned}
&\begin{bmatrix} u & v \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \\
&\therefore ux + vy = 0 \\
&\therefore ux + v(2x) = 0 \\
&\therefore v = -0.5u
\end{aligned}$$

The elements of the orthogonal complement $W^\perp$ are of the form $\begin{bmatrix} u \\ v \end{bmatrix}$ where $v = -0.5u$, i.e. $u \cdot \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$.

Thus the orthogonal complement of W is $\boxed{W^\perp = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \middle| 2v + u = 0;\ u, v \in \mathbb{R} \right\}}$ with the basis $\boxed{\left\{ \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} \right\}}$. **(Ans.)**

3.6.1 Orthogonal Projection on a Subspace

Given a subspace W with an orthogonal basis $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3} \cdots, \mathbf{w_n}\}$, the orthogonal projection of $\mathbf{u}$ onto W is:

$$\text{proj}_W \mathbf{u} = \text{proj}_{\mathbf{w_1}}(\mathbf{u}) + \text{proj}_{\mathbf{w_2}}(\mathbf{u}) + \text{proj}_{\mathbf{w_3}}(\mathbf{u}) + \cdots + \text{proj}_{\mathbf{w_n}}(\mathbf{u}) = \sum_{i=1}^{n} \text{proj}_{\mathbf{w_i}}(\mathbf{u})$$

.

We can think of orthogonal projection onto a subspace as casting a "shadow" onto the subspace when an overhead light source is present.

Q.10. Let W be the plane in $\mathbb{R}^3$ with equation $x - y + 2z = 0$. Find the orthogonal projection of $\begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^\top$ onto W.

By trial and error, we know that $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top$ lies in W. Another vector in W perpendicular to this vector forms a basis of W.

Let the other vector be of the form $\begin{bmatrix} x & y & z \end{bmatrix}^\top$ but by the equation of the plane, we know that $x = y - 2z$. The other vector becomes $\begin{bmatrix} y - 2z & y & z \end{bmatrix}^\top$.

Taking the dot product,

$$\begin{bmatrix} y-2z & y & z \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\therefore (y-2z)(1) + (y)(1) + (z)(0) = 0$$

$$\therefore y - 2z + y = 0$$

$$\therefore y = z$$

$$\therefore \text{ The other vector becomes } \begin{bmatrix} -y & y & y \end{bmatrix}^\top = y \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^\top$$

We know that one of the orthogonal basis of W is $\left\{ \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top, \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \right\}^\top$.

Finding the projection of $\begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^\top$ on $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top$

$$\begin{aligned} \mathrm{proj}_{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top} \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^\top &= \frac{3(1) - 1(1) + 2(0)}{\sqrt{1^2 + 1^2 + 0^2}^2} \cdot \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top \\ &= \frac{2}{2} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top \\ &= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top \end{aligned}$$

Finding the projection of $\begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^\top$ on $\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^\top$

$$\begin{aligned} \mathrm{proj}_{\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^\top} \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^\top &= \frac{3(-1) - 1(1) + 2(1)}{\sqrt{(-1)^2 + 1^2 + 1^2}^2} \cdot \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^\top \\ &= -\frac{2}{3} \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^\top \\ &= \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \end{bmatrix}^\top \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{proj}_W \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^\top &= \mathrm{proj}_{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top} \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^\top + \mathrm{proj}_{\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^\top} \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^\top \\ &= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top + \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \end{bmatrix}^\top \\ &= \begin{bmatrix} \frac{5}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}^\top \end{aligned}$$

Thus the orthogonal projection of $\begin{bmatrix} 3 & -1 & 2 \end{bmatrix} \top$ on the plane $x - y + 2z = 0$ is $\boxed{\begin{bmatrix} \frac{5}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}^\top}$.

(Ans.)

3.7 Graham-Schmidt Orthogonalisation Process

The Graham-Schmidt orthogonalisation process is used to convert a set of linearly independent vectors to an orthogonal set of vectors.

Let $\{x_1, x_2, x_3, \ldots, x_k\}$ be the basis of a subspace W of k dimensions. Then we consider an orthogonal basis as the following set $\{v_1, v_2, v_3, \ldots, v_k\}$ where:

$$
\begin{aligned}
v_1 &= x_1 \\
v_2 &= x_2 - \text{proj}_{v_1} x_2 \\
v_3 &= x_3 - \text{proj}_{v_1} x_3 - \text{proj}_{v_2} x_3 \\
&\vdots \\
v_k &= \underbrace{x_k - \left(\text{proj}_{v_1} x_k + \text{proj}_{v_2} x_k + \text{proj}_{v_3} x_k + \cdots + \text{proj}_{v_{k-1}} x_k\right)}_{\text{The vector component of } x_k \text{ perpendicular to the subspace formed by first } k-1 \text{ vectors}}
\end{aligned}
$$

Note:

Even though the bold notation or arrow notation has not been used, the terms x_i and v_i still represent vectors over here.

Q.11. Construct Orthonormal basis for the subspace $W = \text{span}\left\{\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\2\\1\\2\end{bmatrix}\right\}$

Using the Graham-Schmidt Orthogonalisation Process:

$$\mathbf{v_1} = \mathbf{x_1} = \begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}^\top$$

Finding the value of $\mathbf{v_2}$,

$\langle \mathbf{x_2}, \mathbf{v_1} \rangle = 2(1) + 1(-1) + 0(-1) + 1(1) = 2 - 1 + 0 + 1 = 2$

$\|\mathbf{v_1}\| = \sqrt{1^2 + (-1)^2 + (-1)^2 + 1^2} = \sqrt{1+1+1+1} = 2$

$$
\begin{aligned}
\mathbf{v_2} &= \mathbf{x_2} - \text{proj}_{\mathbf{v_1}}(\mathbf{x_2}) \\
&= \mathbf{x_2} - \frac{\langle \mathbf{x_2}, \mathbf{v_1} \rangle}{\|\mathbf{v_1}\|^2} \cdot \mathbf{v_1} \\
&= \begin{bmatrix}2 & 1 & 0 & 1\end{bmatrix}^\top - \frac{2}{2^2}\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}^\top \\
&= \begin{bmatrix}2 & 1 & 0 & 1\end{bmatrix}^\top - \frac{1}{2}\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}^\top \\
\therefore \mathbf{v_2} &= \begin{bmatrix}1.5 & 1.5 & 0.5 & 0.5\end{bmatrix}^\top
\end{aligned}
$$

Finding the value of$\mathbf{v_3}$,

$\langle \mathbf{x_3}, \mathbf{v_1} \rangle = 2(1) + 2(-1) + 1(-1) + 2(1) = 2 - 2 - 1 + 2 = 1$

$\langle \mathbf{x_3}, \mathbf{v_2} \rangle = 2(1.5) + 2(1.5) + 1(0.5) + 2(0.5) = 3 + 3 + 0.5 + 1 = 7.5$

$\|\mathbf{v_2}\| = \sqrt{1.5^2 + 1.5^2 + 0.5^2 + 0.5^2} = \sqrt{2.25 + 2.25 + 0.25 + 0.25} = \sqrt{5}$

$$\begin{aligned}
\mathbf{v_3} &= \mathbf{x_3} - \text{proj}_{\mathbf{v_1}}(\mathbf{x_3}) - \text{proj}_{\mathbf{v_2}}(\mathbf{x_3}) \\
&= \mathbf{x_3} - \frac{\langle \mathbf{x_3}, \mathbf{v_1} \rangle}{\|\mathbf{v_1}\|^2} \cdot \mathbf{v_1} - \frac{\langle \mathbf{x_3}, \mathbf{v_2} \rangle}{\|\mathbf{v_2}\|^2} \cdot \mathbf{v_2} \\
&= \begin{bmatrix} 2 & 2 & 1 & 2 \end{bmatrix}^\top - \frac{1}{2^2}\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}^\top - \frac{7.5}{\sqrt{5}^2}\begin{bmatrix} 1.5 & 1.5 & 0.5 & 0.5 \end{bmatrix}^\top \\
&= \begin{bmatrix} 2 & 2 & 1 & 2 \end{bmatrix}^\top - \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}^\top - \frac{3}{2}\begin{bmatrix} 1.5 & 1.5 & 0.5 & 0.5 \end{bmatrix}^\top \\
&= \begin{bmatrix} 2 & 2 & 1 & 2 \end{bmatrix}^\top - \begin{bmatrix} 0.25 & -0.25 & -0.25 & 0.25 \end{bmatrix}^\top - \begin{bmatrix} 2.25 & 2.25 & 0.75 & 0.75 \end{bmatrix}^\top \\
\therefore \mathbf{v_3} &= \begin{bmatrix} -0.5 & 0 & 0.5 & 1 \end{bmatrix}^\top
\end{aligned}$$

$\|\mathbf{v_3}\| = \sqrt{(-0.5)^2 + 0^2 + 0.5^2 + 1^2} = \sqrt{0.25 + 0 + 0.25 + 1} = \sqrt{1.5}$

Thus, we have found an orthogonal set of vectors.

To convert it to orthonormal set, $\mathbf{w_i} = {}^{\mathbf{v_i}}/_{\|\mathbf{v_i}\|}$

$\mathbf{w_1} = {}^{\mathbf{v_1}}/_{\|\mathbf{v_1}\|} = {}^{[1 \quad -1 \quad -1 \quad 1]^\top}/_{2} = \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}^\top$

$\mathbf{w_2} = {}^{\mathbf{v_2}}/_{\|\mathbf{v_2}\|} = {}^{[1.5 \quad 1.5 \quad 0.5 \quad 0.5]^\top}/_{\sqrt{5}} = \begin{bmatrix} {}^3/_{2\sqrt{5}} & {}^3/_{2\sqrt{5}} & {}^1/_{2\sqrt{5}} & {}^1/_{2\sqrt{5}} \end{bmatrix}^\top$

$\mathbf{w_3} = {}^{\mathbf{v_3}}/_{\|\mathbf{v_3}\|} = {}^{[-0.5 \quad 0 \quad 0.5 \quad 1]^\top}/_{\sqrt{1.5}} = \begin{bmatrix} -{}^1/_{\sqrt{6}} & 0 & {}^1/_{\sqrt{6}} & {}^2/_{\sqrt{6}} \end{bmatrix}^\top$

3.7.1 QR-Factorisation

We use QR-Factorisation to find the linear dependence of a set of vectors. For QR-Factorisation, we decompose the matrix A into $Q \cdot R$ where:

- Q is an orthogonal $Q^\top = Q^{-1}$ or unitary $Q^* = Q^{-1}$.
- R is an Upper Triangular Square Matrix.

Q can be easily found out by applying Graham-Schmidt Process to the columns of A and R is found out as follows.

$$\begin{aligned}
\because A &= Q \cdot R \\
\therefore Q^* \cdot A &= Q^* \cdot Q \cdot R \\
\therefore R &= Q^* \cdot A
\end{aligned}$$

Premultiplying by Q^*

Q.12. Find a $QR-$Factorisation of $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

For the given matrix A, the matrix Q can easily be found out by performing Graham-Schmidt Orthogonalisation on the columns of A.

$$Q = \begin{bmatrix} 1 & 1.5 & -0.5 \\ -1 & 1.5 & 0 \\ -1 & 0.5 & 0.5 \\ 1 & 0.5 & 1 \end{bmatrix}$$

(Reason: Ye to kahi dekhe dekhe hue hai)

$$
\begin{aligned}
R &= Q^* \cdot A \\
&= \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1.5 & 1.5 & 0.5 & 0.5 \\ -0.5 & 0 & 0.5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1+1+1+1 & 2-1+0+1 & 2-2-1+2 \\ 1.5-1.5-0.5-0.5 & 3+1.5+0+0.5 & 3+3+0.5+1 \\ -0.5-0-0.5+1 & -1+0+0+1 & -1+0+0.5+2 \end{bmatrix} \\
\therefore R &= \begin{bmatrix} 4 & 2 & 1 \\ 0 & 5 & 7.5 \\ 0 & 0 & 1.5 \end{bmatrix}
\end{aligned}
$$

3.7.2 Best Approximation of a Vector

Let $\mathbf{v}$ be a vector and V be the vector space in which $\mathbf{v}$ resides. Let W be a subspace of V. The best approximation of $\mathbf{v}$ onto W is defined as the vector in W whose distance from $\mathbf{v}$ is the least from any other vector in W.

$$|\mathbf{v} - \text{Best Approximation}| \leq |\mathbf{v} - \mathbf{w}|$$

$$\forall \mathbf{w} \in W$$

It has been found that the best approximation of a vector in a subspace is the projection of that vector onto that subspace itself. i.e. Best Projection $= \text{proj}_W(\mathbf{v})$.

Q.13. Find the best approximation of $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ in the plane $W = \text{span}\left\{ \vec{w_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{w_2} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \right\}$

The vectors $\vec{w_1}$ and $\vec{w_2}$ are not orthogonal in nature because $\vec{w_1} \cdot \vec{w_2} = 5 \cdot 1 + 2 \cdot 2 + (-1) \cdot (-1) = 10$.

Applying the Graham-Schmidt Orthogonalisation Process,

Finding the value of $\vec{v_1}$,

$$\vec{v_1} = \vec{w_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Finding the value of $\vec{v_2}$,

$$\vec{v_2} = \vec{w_2} - \frac{\vec{w_2} \cdot \vec{w_1}}{|w_1|^2} \cdot \vec{v_1} = \vec{w_2} - \frac{\vec{w_2} \cdot \vec{v_1}}{|v_1|^2} \cdot \vec{v_1} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} - \frac{10}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 10/3 \\ -4/3 \\ 2/3 \end{bmatrix}$$

Now to find the best approximation, we find the orthogonal projection of $\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ on the plane formed by $\vec{v_1}$ and $\vec{v_2}$.

$$\text{proj}_{\vec{v_1}}\vec{u} = \frac{3(1)+2(2)+5(-1)}{6}\begin{bmatrix}1\\2\\-1\end{bmatrix} = \begin{bmatrix}1/3\\2/3\\-1/3\end{bmatrix}$$

$$\text{proj}_{\vec{v_2}}\vec{u} = \frac{3(10/3)+2(-4/3)+5(2/3)}{40/3}\begin{bmatrix}10/3\\-4/3\\2/3\end{bmatrix} = \begin{bmatrix}8/3\\-16/15\\8/15\end{bmatrix}$$

$$\text{proj}_W\vec{u} = \text{proj}_{\vec{v_1}}\vec{u} + \text{proj}_{\vec{v_2}}\vec{u} = \begin{bmatrix}1/3\\2/3\\-1/3\end{bmatrix} + \begin{bmatrix}8/3\\-16/15\\8/15\end{bmatrix} = \begin{bmatrix}9/3\\-6/15\\3/15\end{bmatrix} = \begin{bmatrix}3\\-0.4\\0.2\end{bmatrix}$$

Thus, the projection of $\begin{bmatrix}3\\2\\5\end{bmatrix}$ on the subspace W is $\boxed{\begin{bmatrix}3\\-0.4\\0.2\end{bmatrix}}$. **(Ans.)**

3.7.3 Least Square Method

In many cases, there is a direct linear relation between a measured input value and the measured output value of a system. While devising such a linear relation, we must know the *slope* and *y-intercept* of the line we want to graph. We can never find the exact linear relation as there may be many errors while measuring or due to the inherent variability of data, what we CAN do however, is get as close as possible to the line. This is where the *Least Square Method* comes in.

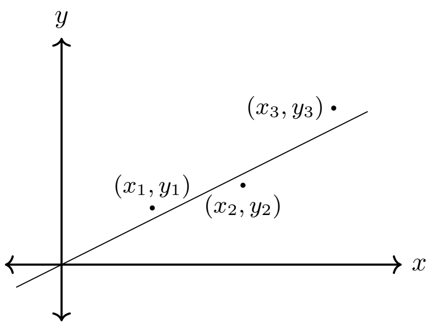


Figure 3.2: Least Square Method: $y = a + bx$

The least square method basically seeks to minimise the value of the sum of squares of the absolute errors i.e. $\boxed{\text{minimise } \sum_{i=1}^{k}(y_i - (a+bx_i))^2}$ where k is the number of samples taken.

$$
\begin{aligned}
\sum_{i=1}^{k}(y_i-(a+bx_i))^2 &= (y_1-(a+bx_1))^2+(y_2-(a+bx_2))^2+\cdots+(y_k-(a+bx_k))^2 \\
&= \left\| \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} - \begin{bmatrix} a+bx_1 \\ a+bx_2 \\ \vdots \\ a+bx_k \end{bmatrix} \right\|^2 \\
&= \left\| \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}}_{\mathbf{y}} - \underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_k \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\mathbf{u}} \right\|^2 \\
&= \left\|\mathbf{y}-A\mathbf{u}\right\|^2
\end{aligned}
$$

So we must minimise the value of $\left\|A\mathbf{u}-\mathbf{y}\right\|^2$ instead. We can consider that $A\mathbf{u}$ is just the vector subspace formed by the transforming matrix A for all possible values of $\mathbf{u}$ and for a particular value, $A\mathbf{u}$ is the best approximation of $\mathbf{y}$.

$$
\begin{aligned}
&\because A\mathbf{u} = \mathrm{proj}_{\mathrm{Im}(A)}\mathbf{y} \\
&\therefore \mathbf{y}-A\mathbf{u} \perp \mathbf{a_j} && \forall \mathbf{a_j} \in \mathrm{Im}(A) \\
&\therefore \langle \mathbf{y}-A\mathbf{u}, \mathbf{a_j}\rangle = 0_{\mathbb{F}} \\
&\therefore \langle \mathbf{a_j}, \mathbf{y}-A\mathbf{u}\rangle = 0_{\mathbb{F}} && \text{In this case, } \mathbb{F}=\mathbb{C} \\
&\therefore \mathbf{a_j}^* \left(\mathbf{y}-A\mathbf{u}\right) = 0_{\mathbb{C}} = 0 && \text{Since } \langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^\top \cdot \mathbf{v};\ \forall \mathbf{u},\mathbf{v} \in \mathbb{C}^k \\
&\therefore A^* \left(\mathbf{y}-A\mathbf{u}\right) = \mathbf{0}_{\mathbb{C}^\mathbf{k}} \\
&\therefore A^*\mathbf{y}-A^*A\mathbf{u} = \mathbf{0}_{\mathbb{C}^\mathbf{k}} \\
&\therefore A^*\mathbf{y} = A^*A\mathbf{u} && \text{Pre-multiplying by } \left(A^*A\right)^{-1} \\
&\therefore \mathbf{u} = \left(A^*A\right)^{-1}A^*\mathbf{y}
\end{aligned}
$$

Thus, the value of $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ is given by $\left(A^*A\right)^{-1}A^*\mathbf{y}$

Q.14. Find the Best Fitted Line using Least Square Method from the set S of samples given as follows: $S = \{(-2,4), (-1,2), (0,1), (2,1), (3,1)\}$

Applying the Least Square Method to find the best fitted line, the line has the equation $y = a + bx$, where a and b are given as follows:

$$
\begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{u} = (A^*A)^{-1}A^*\mathbf{y}
$$

where,

- $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_5 \end{bmatrix}^\top = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 & 3 \end{bmatrix}^\top$
- $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_5 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 & 1 & 1 \end{bmatrix}^\top$

Therefore, $A^* = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_5 \end{bmatrix}$.

Therefore, $A^*A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 18 \end{bmatrix}$.

$|A^*A| = 5 \times 18 - 2 \times 2 = 90 - 4 = 86$

Therefore, $(A^*A)^{-1} = \dfrac{1}{|A^*A|} \cdot \operatorname{adj}(A^*A) = \dfrac{1}{86} \begin{bmatrix} 18 & -2 \\ -2 & 5 \end{bmatrix}$

Therefore, $A^*\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$

Thus, $\mathbf{u} = (A^*A)^{-1}A^*\mathbf{y} = \dfrac{1}{86} \begin{bmatrix} 18 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 9 \\ -5 \end{bmatrix} = \dfrac{1}{86} \begin{bmatrix} 172 \\ -43 \end{bmatrix} = \begin{bmatrix} 2 \\ -0.5 \end{bmatrix}$

Hence, the $\boxed{\text{required line is } y = 2 - 0.5x}$. **(Ans.)**

Chapter 4

Complex Analysis

4.1 Foundations

4.1.1 Imaginary Unit

The imaginary unit is denoted by i. It is defined as the number with the property that $\boxed{i^2 := -1}$.

4.1.2 Complex Numbers

The set of complex numbers $\mathbb{C}$ is defined as follows,

$$\mathbb{C} := \{z \,|\, z := x + iy : x, y \in \mathbb{R}\}$$

$z = x + iy$ is called the *Cartesian* (or *rectangular*) form of the complex number z. x is called the *real part* of z (denoted as $\Re(z)$ or $\mathrm{Re}(z)$), while y is known as its *imaginary part* (denoted as $\Im(z)$ or $\mathrm{Im}(z)$).

If $\Im(z) = 0$, then $z = x + 0i = x \implies z \in \mathbb{R}$. Therefore, $\mathbb{R} \subset \mathbb{C}$.

4.1.3 Euler's Formula

Euler's formula is given by,

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{4.1}$$

Therefore, sometimes $e^{i\theta}$ or $\cos\theta + i\sin\theta$ may be denoted as $\operatorname{cis}\theta$.

Consider a complex number $z := x + iy$.

$$\begin{aligned} z &= x + iy \\ &= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right) \end{aligned}$$

Multiply and divide by $\sqrt{x^2+y^2}$

Let $r := \sqrt{x^2+y^2}$ and θ be the angle such that $\cos\theta := x/r$ and $\sin\theta := y/r$.

$$\begin{aligned} \therefore z &= r(\cos\theta + i\sin\theta) \\ z &= re^{i\theta} \end{aligned}$$

From eq. (4.1)

$z = re^{i\theta}$ is called the *Euler* (or *polar*) form of the complex number z. r is called the *modulus* of z (denoted as $|z|$), while θ is known as its *argument* (denoted as $\arg(z)$).

Euler's Identity

Euler's identity is a famous identity that brings the five most important mathematical constants together in one equation.

$$e^{i\pi} + 1 = 0$$

Coordinate Conversion

$$x + iy = \text{hypot}(x, y)e^{i \cdot \text{atan2}(y,x)} \tag{4.2}$$

$$re^{i\theta} = r\cos\theta + i \cdot r\sin\theta \tag{4.3}$$

where $\text{hypot}(x, y) := \sqrt{x^2 + y^2}$ and $\text{atan2}(y, x)$ is a special piece-wise function that returns the principal argument $\text{Arg}(x + iy)$.

4.1.4 ArGand Plane

While real numbers require only a number line, complex numbers shall be represented using a complex plane (also called the Argand plane), since there are two degrees of freedom.

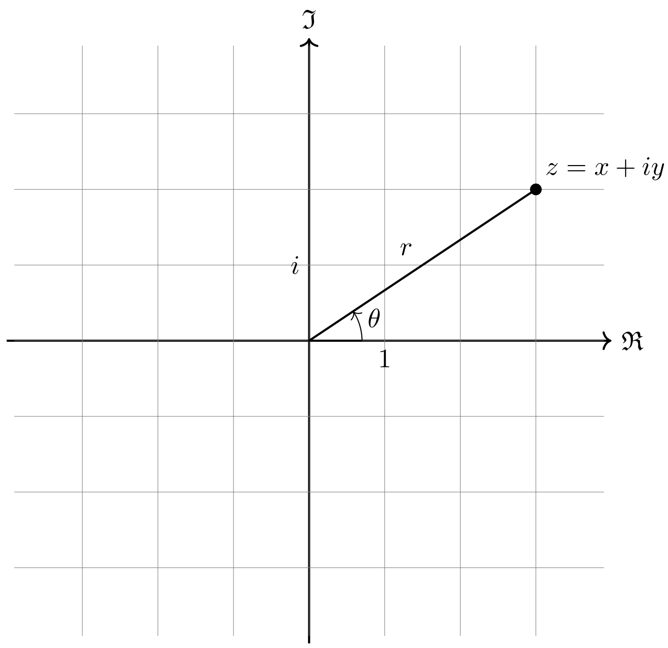


Figure 4.1: The Argand plane

The modulus of z, $r := |z|$, describes the distance of z from 0. The argument of z, $\theta := \arg(z)$, describes the angle made by z with the positive real axis.

Since coterminal angles can imply multiple values for $\arg(z)$, we confine the range of the principal argument of z in the interval $(-\pi, \pi]$. The principal argument of a complex number z is denoted as $\text{Arg}(z)$.

When $\arg(z) = n\pi, n \in \mathbb{Z}$, $\sin\theta = 0 \implies \Im(z) = 0 \implies z \in \mathbb{R}$.

4.1.5 Complex Conjugate

The complex conjugate $\overline{z}$ of a complex number z is defined as the reflection of z about the real axis.

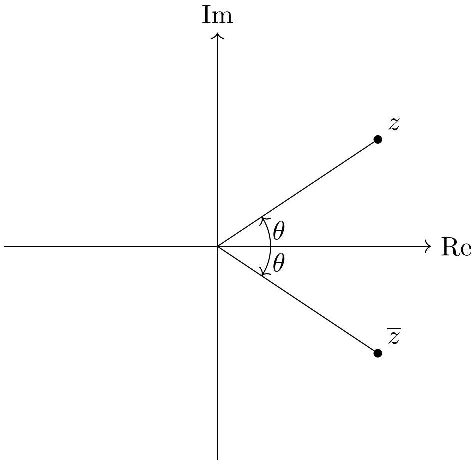


Figure 4.2: Complex conjugate

$$z = x + iy \iff \overline{z} = x - iy \tag{4.4}$$

$$z = re^{i\theta} \iff \overline{z} = re^{-i\theta} \tag{4.5}$$

Properties of the conjugate

1. $\overline{\overline{z}} = z$
2. $z = \overline{z} \iff \Im(z) = 0 \iff z \in \mathbb{R}$
3. $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$
4. $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$
5. $z\overline{z} = |z|^2$
6. $\Re(z) = \dfrac{z + \overline{z}}{2}$
7. $\Im(z) = \dfrac{z - \overline{z}}{2i}$

4.1.6 Equality of Complex Numbers

Rectangular Coordinates

Two complex numbers $z_1 \coloneqq x_1 + iy_1$ and $z_2 \coloneqq x_2 + iy_2$ are said to be equal iff $x_1 = x_2$ and $y_1 = y_2$.

Polar Coordinates

Two complex numbers $z_1 \coloneqq r_1 e^{i\theta_1}$ and $z_2 \coloneqq r_2 e^{i\theta_2}$ are said to be equal iff $r_1 = r_2$ and $\exists n \in \mathbb{Z} : \theta_1 = \theta_2 + 2n\pi$.

4.1.7 Powers of Complex Numbers

Consider $z := re^{i\theta}$. Exponentiating z to an index n can be interpreted as rotating z about 0 by the angle 'θ' $(n-1)$ times and scaling this rotated vector to a length of r^n.

Q.1 If $z = \sqrt{3} + i$, find z^8.

By eq. (4.2),

$$\begin{aligned} |z| &:= \operatorname{hypot}(\sqrt{3}, 1) = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2 \\ \operatorname{Arg}(z) &:= \operatorname{atan2}(1, \sqrt{3}) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \end{aligned}$$

$$\begin{aligned} \implies z &= 2e^{i\pi/6} \quad \textit{Exponentiate to } 8^{\text{th}} \textit{ power} \\ \therefore z^8 &= \left(2e^{i\pi/6}\right)^8 \\ &= 2^8 \operatorname{cis}\left(i\frac{\pi}{6} \cdot 8\right) \\ &= 256e^{i4\pi/3} \quad \textit{Apply Euler's formula} \\ &= 256\left(-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) \\ \implies z^8 &= \boxed{-128 - 128\sqrt{3}i} \end{aligned}$$

(Ans.)

4.1.8 Roots of Complex Numbers

Consider $z := re^{i\theta}$ to be an n^{th} root of the complex number $w := Re^{i\phi}$.

$$\begin{aligned} \implies z^n &:= Re^{i\phi} \\ \therefore \left(re^{i\theta}\right)^n &= Re^{i\phi} \\ \therefore r^n e^{in\theta} &= Re^{i\phi} \end{aligned}$$

From §4.1.6, we can deduce that:

$$r^n = R \implies \boxed{r = R^{1/n}} \tag{4.6}$$

$$\exists m \in \mathbb{Z} : n\theta = \phi + 2m\pi \implies \boxed{\theta = \frac{\phi}{n} + \frac{2m\pi}{n},\ m \in [\![0, n-1]\!]} \tag{4.7}$$

By the fundamental theorem of algebra, the polynomial equation $z^n = w$ should have n principal solutions. Therefore, we constrain m to the interval $\{0, 1, \ldots, n-1\} := [\![0, n-1]\!]$.

Q.2. Find all the fifth roots of unity.

Consider $z := re^{i\theta} \in \mathbb{C}$ s.t. $z^5 := 1$.

We can express 1 in polar form as $e^{2i\pi}$. From eq. (4.6),

$$r^5 = 1 \implies \boxed{r = 1}$$

From eq. (4.7),

$$\theta = \frac{2\pi}{5} + \frac{2m\pi}{5} \implies \boxed{\theta = \frac{2(m+1)\pi}{5},\ m \in [\![0,4]\!]}$$

Therefore,

$$z \in \boxed{\left\{e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5}, 1\right\}}$$

(Ans.)

Q.3. Find all values of $(-8i)^{1/3}$.

$$\begin{aligned} -8i &= 0 - 8i \\ &= 8(0 - i) \\ -8i &= \boxed{8e^{-i\pi/2}} \end{aligned}$$

Consider $z \coloneqq re^{i\theta} \in \mathbb{C}$ s.t. $z^3 \coloneqq -8i$.

From eq. (4.6),

$$r^3 = 8 \implies \boxed{r = 2}$$

From eq. (4.7),

$$\theta = -\frac{\pi/2}{3} + \frac{2m\pi}{3} \implies \boxed{\theta = \frac{(4m-1)\pi}{6},\ m \in [\![0,2]\!]}$$

Therefore,

$$\begin{aligned} z &\in \left\{2e^{-i\pi/6}, 2e^{i3\pi/6}, 2e^{i7\pi/6}\right\} \\ &\in \boxed{\left\{\sqrt{3} - i, 2i, -\sqrt{3} - i\right\}} \end{aligned}$$

(Ans.)

Q.4. Evaluate $\left(-8 - 8\sqrt{3}i\right)^{1/4}$.

Consider $z \coloneqq re^{i\theta} \in \mathbb{C}$ s.t. $z^4 = -8 - 8\sqrt{3}i$

$$\begin{aligned} \because z^4 &= -8 - 8\sqrt{3}i \\ \therefore r^4 e^{i(4\theta)} &= 8(-1 - \sqrt{3}i) \\ &= 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ \therefore r^4 e^{i(4\theta)} &= 16e^{i(-2\pi/3)} \end{aligned}$$

From eq. (4.6),

$$r^4 = 16 \implies \boxed{r = 2}$$

From eq. (4.7),

$$\theta = \frac{-{}^{2\pi}/_{3}}{4} + \frac{2m\pi}{4} \implies \boxed{\theta = \frac{(3m-1)\pi}{6}, m \in [\![0,3]\!]}$$

Therefore,

$$\begin{aligned} z &\in \left\{2e^{-i\pi/6}, 2e^{i\pi/3}, 2e^{i5\pi/6}, 2e^{i4\pi/3}\right\} \\ &\in \boxed{\left\{\sqrt{3}-i, 1+\sqrt{3}i, -\sqrt{3}+i, -1-\sqrt{3}i\right\}} \end{aligned}$$

(Ans.)

Q.5. Find the cube roots of $3+4i$.

Consider $z := re^{i\theta}$ s.t. $z^3 = 3+4i$

We can represent the complex number $(3+4i)$ as $\left(5e^{i\arctan({}^{4}/_{3})}\right)$.

From eq. (4.6),

$$r^3 = 5 \implies \boxed{r = \sqrt[3]{5}}$$

From eq. (4.7),

$$\boxed{\theta = \frac{\arctan({}^{4}/_{3}) + 2m\pi}{3}, m \in [\![0,2]\!]}$$

Therefore,

$$z \in \boxed{\left\{\sqrt[3]{5}e^{\frac{i}{3}\arctan\left(\frac{4}{3}\right)}, \sqrt[3]{5}e^{\frac{i}{3}\left(\arctan\left(\frac{4}{3}\right)+2\pi\right)}, \sqrt[3]{5}e^{\frac{i}{3}\left(\arctan\left(\frac{4}{3}\right)+4\pi\right)}\right\}}$$

(Ans.)

4.2 Complex Functions

Let set $S \subseteq \mathbb{C}$.

A function f defined on set S is a rule that assigns each element of $z \in S$ to a complex number $w \in S$.

$$\underset{\text{Dependent variable}\uparrow}{w} := f(\underset{\uparrow\text{Independent variable}}{z})$$

By convention, we denote $z := x+iy$, whereas we denote $w := u+iv$. Therefore, complex functions can be visualized as mappings from an xy space to a uv space.

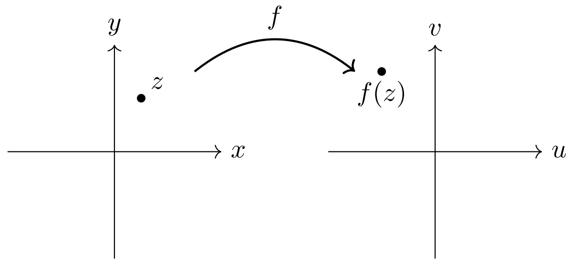


Figure 4.3: A complex function f mapping xy space to uv space

For example, consider the function $f(z) \coloneqq z^2 + iz$. For $z \coloneqq x + iy$,

$$\begin{aligned} w &\coloneqq f(z) \\ &= z^2 + iz \\ &= (x+iy)^2 + i(x+iy) \\ &= x^2 - y^2 + 2xyi + ix - y \\ \implies u + iv &= (x^2 - y^2 - y) + i(2xy + x) \end{aligned}$$

Therefore, $u = x^2 - y^2 - y$ and $v = 2xy + x$, which means u and v are themselves also functions of x and y.

Q.6. If $f(z) \coloneqq z^2 + 3z$, then compute $f(1+3i)$.

$$\begin{aligned} f(1+3i) &\coloneqq (1+3i)^2 + 3(1+3i) \\ &= 1^2 - 3^2 + (2 \cdot 1 \cdot 3)i + 3 + 9i \\ \therefore f(1+3i) &= \boxed{-5 + 15i} \end{aligned}$$

(Ans.)

4.2.1 Limits

The following is the definition of a limit for real-valued functions:

$$\lim_{x \to x_0} f(x) = L \text{ } means \ \forall \varepsilon > 0 \ \exists \delta > 0 : |f(x) - L| < \varepsilon \impliedby |x - x_0| < \delta.$$

The limit is said to exist and equal to L when $\lim\limits_{x \to {x_0}^+} f(x) = \lim\limits_{x \to {x_0}^-} f(x) = L$ as shown in fig. 4.4.

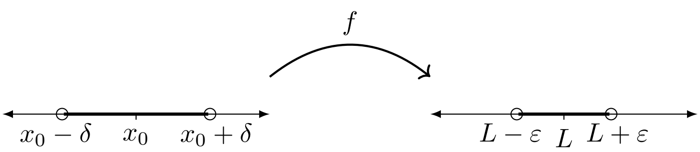


Figure 4.4: Definition of a limit

Complex Definition

$$\lim_{z \to z_0} f(z) = l \text{ } means \ \forall \varepsilon > 0 \ \exists \delta > 0 : |f(z) - l| < \varepsilon \impliedby |z - z_0| < \delta^1. \tag{4.8}$$

[1] $|z - z_0| < \delta$ represents a disc (excluding circumference) in the complex plane centered at z_0 with radius δ.

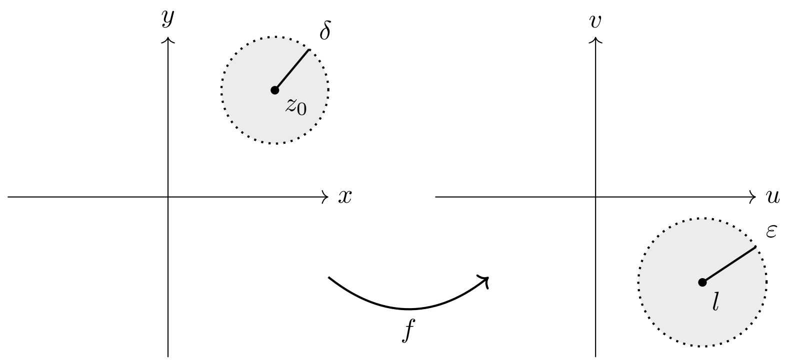


Figure 4.5: Complex definition of a limit

Unlike real numbers, a complex number can be approached from any direction, as seen in fig. 4.5. Therefore, the complex limit exists and is equal to l only when the limit from every direction is equal and equal to l.

Q.7. Let $f(z) \coloneqq \frac{z}{\overline{z}}$. Find $\lim\limits_{z \to 0} f(z)$.

Let $z \coloneqq x + iy \implies \overline{z} \coloneqq x - iy$.

$$\therefore f(z) = \frac{x + iy}{x - iy}$$
$$\implies \lim_{z \to 0} \frac{z}{\overline{z}} = \lim_{(x,y) \to (0,0)} \frac{x + iy}{x - iy}$$

Let us assume a line of approach $y \coloneqq mx$ to the point $(0,\ 0)$ where m is the slope of the line ($\because y \to 0 \implies mx \to 0 \implies x \to 0$).

$$\begin{aligned}\therefore \lim_{z \to 0} \frac{z}{\overline{z}} &= \lim_{x \to 0} \frac{x + imx}{x - imx}\\ &= \lim_{x \to 0} \frac{\cancel{x}(1 + im)}{\cancel{x}(1 - im)}\\ &= \lim_{x \to 0} \frac{1 + im}{1 - im}\\ &= \frac{1 + im}{1 - im}\end{aligned}$$

For different directions of approach, the value of m changes so the value of limit changes as well. The limit $\boxed{\text{does not exist}}$. **(Ans.)**

Q.8. $f(z) \coloneqq |z|^2$. Find $\lim\limits_{z \to 0} f(z)$.

Let $z \coloneqq x + iy \implies |z| = \sqrt{x^2 + y^2}$

$$\therefore \lim_{z \to 0} |z|^2 = \lim_{(x,y) \to (0,0)} (x^2 + y^2)$$

Let us assume a line of approach $y \coloneqq mx$ to the point $(0,\ 0)$ where m is the slope of the line ($\because y \to 0 \implies mx \to 0 \implies x \to 0$).

$$\begin{aligned}
\lim_{z\to 0}|z|^2 &= \lim_{x\to 0}(x^2+m^2x^2)\\
&= \lim_{x\to 0} x^2(1+m^2)\\
&= 0\cdot(1+m^2) \qquad \textit{Applying the limit}\\
&\implies \boxed{\lim_{z\to 0}|z|^2 = 0}
\end{aligned}$$

(Ans.)

Q.9. Suppose $f(z) \coloneqq \dfrac{i\cdot\overline{z}}{2}$. Show that $\lim\limits_{z\to 1} f(z) = \dfrac{i}{2}$.

Proof. Let us assume that the limit of the given function as $z \to 1$ is equal to $\frac{i}{2}$. According to eq. (4.8):

$$\forall\varepsilon>0\ \exists\delta>0 : \left|\frac{i\cdot\overline{z}}{2}-\frac{i}{2}\right| < \varepsilon \impliedby |z-1|<\delta$$

Let $z \coloneqq x+iy$

$\overline{z} = x-iy$

$$\begin{aligned}
\left|\frac{i}{2}(\overline{z}-1)\right| &= \left|i\left(\frac{x-1}{2}-i\frac{y}{2}\right)\right|\\
&= \left|\frac{y}{2}+i\frac{x-1}{2}\right|\\
&= \frac{1}{2}\cdot\sqrt{y^2+(x-1)^2}\\
\therefore \varepsilon &> \frac{1}{2}\cdot\sqrt{y^2+(x-1)^2}\\
\therefore 2\varepsilon &= \sqrt{y^2+(x-1)^2}+c_1 \qquad (c_1>0)
\end{aligned}$$

$$\begin{aligned}
|z-1| &= |(x-1)+iy|\\
&= \sqrt{(x-1)^2+y^2}\\
\therefore \delta &> \sqrt{(x-1)^2+y^2}\\
\therefore \delta &= \sqrt{(x-1)^2+y^2}+c_2 \qquad (c_2>0)\\
\therefore \delta &= 2\varepsilon - c_1 + c_2
\end{aligned}$$

Since there exists a valid relation between ε and δ,

$\boxed{\text{The limit of the given function as } z \text{ approaches 1 is } i/2.}$ ∎

4.2.2 Continuity

A complex-valued function f is said to be *continuous* at z_0 iff the limit of the function at z_0 exists and equals the value of the function at z_0. This can be defined as follows:

$$\lim_{z\to z_0} f(z) = f(z_0)\ \textit{means}\ \forall\varepsilon>0\ \exists\delta>0 : |f(z)-f(z_0)| < \varepsilon \impliedby |z-z_0|<\delta.$$

Q.10. Consider a function f as follows,

$$f(z) \coloneqq \begin{cases} \dfrac{\Re(z^2)}{|z|^2}, & z \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Is f continuous?

Consider $z \coloneqq x + iy$.

When we expand the function for $z \neq 0$, we get $f(z) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Let us assume a line of approach from a random direction specified by the equation $y = mx$. Thus the function f is

$$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{x^2 - m^2 x^2}{x^2 + m^2 x^2} = \lim_{x \to 0} \frac{\cancel{x^2}(1 - m^2)}{\cancel{x^2}(1 + m^2)} = \frac{1 - m^2}{1 + m^2}$$

After applying the limit, the final limit depends on the direction of approach (m). Hence the limit does not exist, and the function is $\boxed{\text{not continuous}}$ at $z = 0$. **(Ans.)**

4.2.3 Differentiability

The ratio of the change in the value of the function $f(z)$ to the change in the value of the complex variable z is known as its *derivative*. It is defined as follows,

$$f'(z) \coloneqq \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{4.9}$$

Iff this limit exists, f is differentiable

An alternative definition is given by,

$$f'(z) \coloneqq \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{4.10}$$

Q.11. Is $f(z) \coloneqq |z|^2$ differentiable?

By eq. (4.9),

$$\begin{aligned} f'(z) &\coloneqq \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &\coloneqq \lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)\overline{(z + \Delta z)} - z\overline{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\overline{z} + \Delta\overline{z}) - z\overline{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\cancel{z\overline{z}} + z\Delta\overline{z} + \overline{z}\Delta z + \Delta z\Delta\overline{z} - \cancel{z\overline{z}}}{\Delta z} \\ &= \lim_{\Delta z \to 0} z\frac{\Delta\overline{z}}{\Delta z} + \overline{z} + \Delta\overline{z} \qquad \Delta z \to 0 \iff \Delta\overline{z} \to 0 \\ &= \overline{z} + z \lim_{\Delta z \to 0} \frac{\Delta\overline{z}}{\Delta z} \end{aligned}$$

Assuming that Δz approaches 0 from the x-axis, $\Delta\overline{z} = \Delta z$, the given limit equals $(\overline{z} + z)$.

Assuming that Δz approaches 0 from the y-axis, $\Delta\overline{z} = -\Delta z$, the given limit becomes equals $(\overline{z} - z)$.

Since, the value of the derivative is different for different directions of approach, the function is not differentiable at every point. **(Ans.)**

Necessary condition for differentiability

Assume $z \coloneqq x + iy$ is a complex number and f is a complex-valued function s.t. $f(x, y) \coloneqq u(x, y) + i \cdot v(x, y)$. If f is differentiable at $z = z_0$ then it means that the below limit exists.

$$f'(z) \coloneqq \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{\Delta x + i\,\Delta y}$$

Assuming that $\Delta z \to 0$ from the x-axis *i.e.* $\Delta y = 0$, the above limit becomes:

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \\ &= \frac{\partial}{\partial x} f(x_0, y_0) = \frac{\partial}{\partial x} u(x_0, y_0) + i \frac{\partial}{\partial x} v(x_0, y_0) \\ &= u_x + i v_x \end{aligned}$$

Assuming that $\Delta z \to 0$ from the y-axis *i.e.* $\Delta x = 0$, the above limit becomes:

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{i\,\Delta y} \\ &= \frac{i}{i^2} \cdot \frac{\partial}{\partial y} f(x_0, y_0) = -i \left(\frac{\partial}{\partial y} u(x_0, y_0) + i \frac{\partial}{\partial y} v(x_0, y_0) \right) \\ &= v_y - i u_y \end{aligned}$$

We know that for a complex limit to exist, it must be equal for all directions of approach, so from the above two equations we can say,

$$u_x = v_y \text{ and } v_x = -u_y \tag{4.11}$$

The above equations are known as the **Cauchy-Riemann (CR) equations** and they hold true for every complex differentiable function.

Sufficient conditions for differentiability

For a complex function f to be complex differentiable at $z = z_0$, it should follow the following two conditions:

i) The function satisfies the CR equations.

ii) The partial derivatives u_x, v_x, u_y and v_y are continuous in the neighborhood of z_0.

4.2.4 Holomorphic Functions

A complex function is said to be *holomorphic* in an open subset U of $\mathbb{C}$ if it is complex differentiable for every point in U.

4.2.5 Analytic Functions

A complex function is said to be *analytic* at a given point z_0 of its domain if it can be locally represented as a convergent power series around z_0. That is $\exists R > 0$ such that $f(z)$ can be written as $\sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad \forall z|z - z_0| < R$

If a complex function is holomorphic in a given region of its domain then it is *analytic* for every point in its domain.

Q.12. Check the analyticity of the function $f(z) := |z|^2$.

Given that $f(z) := |z|^2$.

Therefore, $f(z) = f(x, y) = x^2 + y^2 \implies u(x, y) = x^2 + y^2 \wedge v(x, y) = 0$.
Since $u_x = 2x$ and $v_y = 0$, CR equations are not satisfied. Therefore, the function is not holomorphic and therefore $\boxed{\text{not analytic}}$. **(Ans.)**

Q.13. Show that if $f(z)$ and $\overline{f(z)}$ are both analytic then f is a constant function.

Proof. Let $f(z) := u + iv \iff \overline{f(z)} = u - iv$. If $f(z)$ is analytic then it means that the CR equations are satisfied: $u_x = v_y$ and $v_x = -u_y$.

Similarly if $\overline{f(z)}$ is analytic then we get $u_x = (-v)_y \implies u_x = -v_y$ and $(-v)_x = -u_y \implies v_x = u_y$.

From the above two results we know that $u_x = -u_x$ and $v_x = -v_x$. This can occur iff $u_x = 0 \wedge v_x = 0$.

$$\therefore f'(z) := u_x + iv_x = 0 \implies f(z) = c, \text{ where } c \text{ is a constant.}$$

Hence, if a complex function and its conjugate are both analytic then we can say that the function is constant. □

4.2.6 Entire Functions

A complex function is said to be *entire* if it is holomorphic for the entire complex plane $\mathbb{C}$.

Q.14. Check whether $f(z) := (z^2 - 2)e^{-x}e^{-iy}$ is an entire function or not.

$$\begin{aligned}
f(z) &:= (z^2 - 2)e^{-x}e^{-iy} \\
&= e^{-x}(x^2 - y^2 - 2 + 2ixy)(\cos y - i \ \sin y) \\
&= e^{-x}((x^2 - y^2 - 2)\cos y + 2xy \sin y) + i\, e^{-x}(2xy \cos y - (x^2 - y^2 - 2)\sin y)
\end{aligned}$$

Now comparing with the CR equations we can say that:

$$\begin{aligned}
u(x, y) &= e^{-x}((x^2 - y^2 - 2)\cos y + 2xy \sin y) \\
\therefore u_x &= (2x \cos y + 2y \sin y)e^{-x} - ((x^2 - y^2 - 2)\cos y + 2xy \sin y)e^{-x} \\
&= ((y^2 + 2x - x^2 + 2)\cos y + (2y - 2xy)\sin y)e^{-x} \\
\therefore u_y &= (-2y \cos y - (x^2 - y^2 - 2)\sin y + 2x \sin y + 2xy \cos y)e^{-x} \\
&= ((y^2 + 2x - x^2 + 2)\sin y - (2y - 2xy)\cos y)e^{-x}
\end{aligned}$$

and

$$
\begin{aligned}
v(x,y) &= e^{-x}((2xy)\cos y - (x^2-y^2-2)\sin y) \\
\therefore v_x &= (2y\cos y - 2x\sin y)e^{-x} - (2xy\cos y - (x^2-y^2-2)\sin y)e^{-x} \\
&= ((2y-2xy)\cos y + (x^2-2x-y^2-2)\sin y)e^{-x} \\
\therefore v_y &= (2x\cos y - 2xy\sin y - \ -2y\sin y - (x^2-y^2-2)\cos y)e^{-x} \\
&= ((y^2+2x-x^2+2)\cos y + (2y-2xy)\sin y)e^{-x}
\end{aligned}
$$

As we can clearly see, $u_x = v_y$, $v_x = -u_y$ and the partial derivatives u_x, v_x, u_y and v_y are continuous over the entire complex plane[2]. Therefore, the function is holomorphic over the entire complex plane and $\boxed{\text{is entire}}$. **(Ans.)**

4.2.7 Harmonic Functions

A real valued function in two variables say x and y is said to be *harmonic* if it's first and second order derivatives exist, these derivatives are continuous and they satisfy the Laplace equation:

$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{4.12}
$$

Harmonic Conjugates

If two real valued harmonic functions u and v exist such that the function $u+iv$ is analytic in nature, then v is known as the harmonic conjugate of u.

Q.15. Find the analytic function $f(z) := u + iv$ whose real part is defined as follows: $u = \sin x \cdot \cosh y$.

Since f is analytic, we can say that it satisfies the CR equations, *i.e.* $u_x = v_y = \cos x \cdot \cosh y$ and $u_y = -v_x = \sin x \cdot \sinh y$.

$$
\begin{aligned}
\frac{\partial v}{\partial y} &= \cos x \cdot \cosh y \\
\therefore \int \mathrm{d}v &= \int \cos x \cdot \cosh y \,\mathrm{d}y \\
\therefore v &= \cos x \int \cosh y \,\mathrm{d}y \\
\therefore v &= \cos x \cdot \sinh y + c_1(x)
\end{aligned}
\qquad
\begin{aligned}
\frac{\partial v}{\partial x} &= -\sin x \cdot \sinh y \\
\therefore \int \mathrm{d}v &= \int -\sin x \cdot \sinh y \,\mathrm{d}x \\
\therefore v &= \sinh y \int -\sin x \,\mathrm{d}x \\
\therefore v &= \sinh y \cdot \cos x + c_2(y)
\end{aligned}
$$

Notice that the constants of integration are actually functions of x and y respectively. This is because while taking a partial derivative, the other variables are considered as constants and consequently the partial derivatives of their functions evaluate to zero.

Now on comparing the two values of v which we have evaluated, we get the equation $c_1(x) = c_2(y)$. This can only happen iff $c_1(x) = c_2(y) = k$ where $k \in \mathbb{C}$.

Therefore, the entire function f is

$$
\boxed{f(x,y) := \sin x \cdot \cosh y + i(\cos x \cdot \sinh y + k) \text{ where } k \in \mathbb{C}}
$$

[2]because they are compositions of polynomial, exponential and trigonometric sine and cosine functions.

(Ans.)

Alternatively,
$u_x = \cos x \cdot \cosh y$ and $u_y = \sin x \cdot \sinh y$.

From CR equations, we can deduce that $v_x = -u_y = -\sin x \cdot \sinh y$ and $v_y = u_x = \cos x \cdot \cosh y$.

Now,

$$
\begin{aligned}
v_y &\coloneqq \frac{\partial v}{\partial y} \\
\implies \frac{\partial v}{\partial y} &= \cos x \cdot \cosh y \\
\therefore \int \mathrm{d}v &= \int \cos x \cdot \cosh y \, \mathrm{d}y \\
\therefore v &= \cos x \cdot \sinh y + \phi(x), \text{ where } \phi(x) \text{ is a function independent of } y.
\end{aligned}
$$

Differentiating v with respect to x,

$$
\begin{aligned}
v_x &= \frac{\partial}{\partial x}(\cos x \cdot \sinh y + \phi(x)) \\
\therefore v_x &= -\sin x \cdot \sinh y + \phi'(x)
\end{aligned}
$$

However, we know from CR equations that $v_x = -\sin x \cdot \sinh y \implies \phi'(x) = 0 \implies$ $\boxed{\phi(x) = k}$, where $k \in \mathbb{C}$ is a constant[3].

Therefore, the entire function f is

$$\boxed{f(x, y) \coloneqq \sin x \cdot \cosh y + i(\cos x \cdot \sinh y + k) \text{ where } k \in \mathbb{C}}$$

(Ans.)

The above procedure can be summarised with the following algorithm:

Algorithm 1: Obtaining analytic function when real part is known

Data: Real part u is given.
1. Find u_x and u_y.
2. Using CR equations, obtain v_x and v_y.
3. Integrate v_y or v_x with respect to y or x respectively.
4. Differentiate v obtained in step 3 with respect to x or y depending on the variable taken in step 3.
5. Compare v_x or v_y depending on the variable chosen in step 3.

Result: $f(z) \coloneqq u + iv$.

Q.16. If $v \coloneqq 4x^3y - 4xy^3$, then find the analytic function $f(z) \coloneqq u + iv$.

Given $v \coloneqq 4x^3y - 4xy^3$, therefore $v_x = 12x^2y - 4y^3$ and $v_y = 4x^3 - 12xy^2$.

[3] $\phi(x)$ was independent of y, therefore $\phi'(x) = 0$ iff it is a constant.

Algorithm 2: Obtaining analytic function when imaginary part is known

Data: Imaginary part v is given.
1 Find v_x and v_y.
2 Using CR equations, obtain u_x and u_y.
3 Integrate u_y or u_x with respect to y or x respectively.
4 Differentiate u obtained in step 3 with respect to x or y depending on the variable taken in step 3.
5 Compare u_x or u_y depending on the variable chosen in step 3.
Result: $f(z) \coloneqq u + iv$.

From the CR equations, we can say that $u_x = v_y$ and $v_x = -u_y$.

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= 4x^3 - 12xy^2 \\
\therefore \mathrm{d}u &= (4x^3 - 12xy^2)\,\mathrm{d}x \\
\therefore \int \mathrm{d}u &= \int 4x^3 - 12xy^2\,\mathrm{d}x \qquad \textit{Integrating both sides} \\
\therefore u &= x^4 - \frac{12}{2}x^2y^2 + \phi(y) = x^4 - 6x^2y^2 + \phi(y)
\end{aligned}
$$

Now consider the partial derivative of u with respect to y.

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(x^4 - 6x^2y^2 + \phi(y)\right) \\
u_y &= -12x^2y + \phi'(y) \\
u_y &= -(12x^2y^2 - \phi'(y))
\end{aligned}
$$

We know from the CR equations that $v_x = -u_y$. Substituting the values of v_x and u_y we find that $\phi'(y) = 4y^3$. Therefore $\phi(y) = \int 4y^3\,\mathrm{d}y = y^4 + c$ where $c \in \mathbb{C}$.

Therefore the function f is equal to $\boxed{(x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3) + c}$ or

$$\boxed{f(z) = z^4 + c \text{ where } c \in \mathbb{C}}$$

(Ans.)

4.2.8 Milne-Thomson's Method

It is another method provided by the scientist *Milne-Thomson* to directly solve for an analytic function $f(z)$ when only its real (or imaginary) part is known.

Algorithm 3: Finding analytic $f(z)$ when $u(x, y)$ or $v(x, y)$ is given

Data: Real (u) or imaginary (v) part is given.
1 $f'(z) \coloneqq u_x - iu_y$ or $f'(z) \coloneqq v_y + iv_x$ from CR.
2 In the partial derivatives, substitute $x = z$ and $y = 0$[4].
3 Integrate $f'(z)$ now obtained with respect to z to get $f(z)$.
Result: $f(z) \coloneqq u + iv$

[4]The reason being that $x = \dfrac{z + \overline{z}}{2}$ and $y = \dfrac{z - \overline{z}}{2}$, which is true even when x and $y \notin \mathbb{R}$ meaning that z and $\overline{z}$ are independent and can be equated to each other.

Q.17. If $v := 4x^3y - 4xy^3$, then find the analytic function $f(z) := u + iv$ using Milne-Thompson's method.

Given $v := 4x^3y - 4xy^3$, therefore $v_x = 12x^2y - 4y^3$ and $v_y = 4x^3 - 12xy^2$.

We know from the CR equations that $u_x = v_y$ and $v_x = -u_y$. Thus $f'(x) = v_y + iv_x$.

Substituting $x = z$ and $y = 0$ in v_x and v_y, $f'(z) = (4z^3 - 0) + i(0 - 0) = 4z^3$.

Upon integration w.r.t. z, we get $f(z) = \int 4z^3 \,\mathrm{d}z = \boxed{z^4 + c \text{ where } c \in \mathbb{C}}$. **(Ans.)**

Finding the entire function when sum or difference of the real and imaginary parts is given

Consider a complex number $w := u + iv$ which is the result of a complex analytic function f. i.e. $f(z) = u + iv$.

Now consider the function $\Phi(z) := (1 + i)f(z)$. This function is also complex differentiable for all points in a given domain of f meaning it is holomorphic for the same region as f and hence is analytic as well. Now $\Phi(z) := (1 + i)f(z) = (u - v) + i(u + v)$.

Since, we know the sum(or difference) of f, it means we know the real(or imaginary) part of a complex analytic function and hence we can apply Milne-Thomson's method to find Φ. Then we can just divide by $(1 + i)$ to get the original function f.

Q.18. Find the analytic function $f(z) = u + iv$ when $u + v = \dfrac{2\sin 2x}{e^y + e^{-y} - 2\cos 2x}$.

We know that,

$$\begin{aligned}(1 + i)f(z) &= (u - v) + i(u + v) \\ &:= U + iV\end{aligned}$$

Thus, $V := u + v = \dfrac{2\sin 2x}{e^y + e^{-y} - 2\cos 2x}$.

Differentiating,

$$\begin{aligned}\therefore V_x &= \frac{\left(e^{2y} + e^{-2y} - 2\cos 2x\right)4\cos 2x - 2\sin 2x(4\sin 2x)}{\left(e^{2y} + e^{-}2y - 2\cos 2x\right)^2} \\ &= \frac{\left(e^{2y} + e^{-2y}\right)4\cos 2x - 8\cos^2 2x - 8\sin^2 2x}{\left(e^{2y} + e^{-}2y - 2\cos 2x\right)^2} \\ \therefore V_x &= \frac{\left(e^{2y} + e^{-2y}\right)4\cos 2x - 8}{\left(e^{2y} + e^{-}2y - 2\cos 2x\right)^2}\end{aligned}$$

$$\therefore V_y = -\frac{2\sin 2x(2e^{2y} - 2e^{-2y})}{\left(e^{2y} + e^{-}2y - 2\cos 2x\right)^2}$$

By Milne-Thompson's method,

$$
\begin{aligned}
(1+i)f'(z) &= V_y(z,0) + iV_x(z,0) \\
&= -\frac{2\sin 2z(2-2)}{(1+1-2\cos 2z)^2} + i\left(\frac{(1+1)4cos2z - 8}{(1+1-2\cos 2z)^2}\right) \\
&= i\left(\frac{8(\cos 2z - 1)}{(2-2\cos 2z)^2}\right) \\
&= -8i\left(\frac{\cancel{1-\cos 2z}}{4(1-\cos 2z)^{\cancel{2}}}\right) \\
&= -\frac{2i}{1-\cos 2z} \\
&= -\frac{i}{\sin^2 z} \\
\therefore (1+i)f'(z) &= -i \cdot \csc^2 z \\
\implies (1+i)f(z) &= -\int i \cdot \csc^2 z \, \mathrm{d}z
\end{aligned}
$$

$\therefore (1+i)f(z) = i\cot z + c$, where c is the arbitrary constant of integration.

$$\therefore f(z) = \frac{i\cot z}{1+i} + \frac{c}{1+i}$$

Let $k := \dfrac{c}{1+i}$ be another constant.

$$\therefore \boxed{f(z) = \left(\frac{1+i}{2}\right)\cot z + k}$$

(Ans.)

4.2.9 Bijectivity of Complex Functions

Consider a complex function $f : A \to B$. The function f is said to be *bijective* if it follows the two following conditions:

- **Injectivity** (one-one): $\forall z_1, z_2 \in A \ f(z_1) = f(z_2) \implies z_1 = z_2$ *i.e.* each image has *one* and *only one* pre-image.
- **Surjectivity** (onto): $\forall w \in B \ \exists z \in A : w = f(z)$ *i.e.* Range of $f = B$.

Q.19. Restrict the domain and range such that the function $\exp(z),\ z \in \mathbb{C}$ is a bijective.

Assume that the function $f : \mathbb{C} \to \mathbb{C}$ is defined as: $f(z) := \exp(z)$.

$$
\begin{aligned}
z := x + iy \implies f(z) &= \exp(x+iy) \\
&= \exp(x) \cdot \exp(iy)
\end{aligned}
$$

For $f(z) = e^z$ to be an onto function:
Let the codomain of f be $\mathbb{C} \setminus \{0\}$ because e^z can never be zero.

For $f(z) = e^z$ to be a one-one function, $e^{z_1} = e^{z_2} \implies z_1 = z_2$.

Consider $z_1 \coloneqq x_1 + iy_1$ and $z_2 \coloneqq x_2 + iy_2$.

$$\begin{aligned}\therefore e^{z_1} = e^{z_2} &\implies e^{x_1+iy_1} = e^{x_2+iy_2} \\ &\implies e^{x_1}e^{iy_1} = e^{x_2}e^{iy_2}\end{aligned}$$

This is true iff $x_1 = x_2$ and $y_1 = y_2 + 2n\pi,\ n \in \mathbb{Z}$. The function is clearly not injective.

Let domain of f be $\{x + iy \mid y \in [0,\ 2\pi)\,,\ x \in \mathbb{R}\}$.

Domain of f:	$\{x + iy \mid y \in [0,\ 2\pi)\,,\ x \in \mathbb{R}\}$
Codomain of f:	$\mathbb{C} \setminus \{0\}$

(Ans.)

Q.20. Restrict the domain and range such that the function $z^2,\ z \in \mathbb{C}$ is a bijective.

Assume that the function $f : \mathbb{C} \to \mathbb{C}$ is defined as: $f(z) \coloneqq z^2$.

$$z \coloneqq x + iy \implies f(z) = z^2 = (x^2 - y^2) + i(2xy)$$

$f(z) = z^2$ is already an an onto function because if z is defined in the polar form $(re^{i\theta})$ then $f(z)$ becomes $r^2e^{i(2\theta)}$. We can choose any arbitrarily small or large r and θ to obtain every complex number in $\mathbb{C}$.

For $f(z) \coloneqq z^2$ to be a one-one function: $(z_1)^2 = (z_2)^2 \implies z_1 = z_2$ must be true. But:

$$\begin{aligned}(z_1)^2 = (z_2)^2 &\implies (z_1)^2 - (z_2)^2 = 0 \\ &\therefore (z_1 - z_2)(z_1 + z_2) = 0 \\ &\therefore z_1 = z_2 \text{ or } z_1 = -z_2\end{aligned}$$

To make f one to one, we must select two quadrants such that z and $-z$ do not appear in the domain together but also such that most of the complex plane is covered by the range. Selecting the set $\{x + iy \mid x > 0\} \cup \{x + iy \mid x = 0,\ y \geq 0\}$ as the domain, we get the entire complex plane $\mathbb{C}$ as the range.

Domain of f:	$\{x + iy \mid x > 0\} \cup \{x + iy \mid x = 0,\ y \geq 0\}$
Codomain of f:	$\mathbb{C}$

(Ans.)

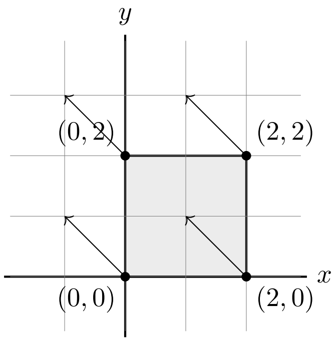
4.2.10 Some Special Mappings

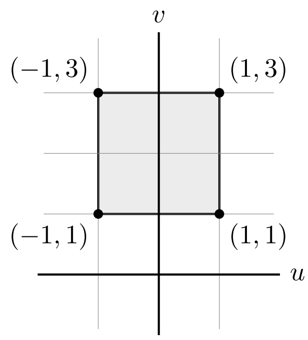
A few of the characteristic complex functions or *mappings* are widely used and hence, have been given special names. They are as follows:

- Translation Map: $f(z) \coloneqq z + c$ where $c \in \mathbb{C}$, *translates* all the points in the complex plane by the vector $\begin{bmatrix}\mathrm{Re}(x) & \mathrm{Im}(x)\end{bmatrix}^\top$.
- Rotation and Scaling Map: $f(z) \coloneqq cz$ where $c \in \mathbb{C}$, applies a linear transformation characterised by $\begin{bmatrix}|c|\cos(\arg(c)) & -|c|\sin(\arg(c)) \\ |c|\sin(\arg(c)) & |c|\cos(\arg(c))\end{bmatrix}$ to all the points in the complex plane.

- Inverse Map: $f(z) \coloneqq {}^1\!/\!_z$ is an important function for many reasons, but it's most apparent useful property is that it is the inverse of itself.

Q.21. Find the image of the rectangle (in the complex plane) bound by the lines $x = 0$, $y = 0$, $x = 2$, $y = 2$ when the function f which is defined as $f(z) \coloneqq z - (1 - i)$ is applied to every point in the complex plane.





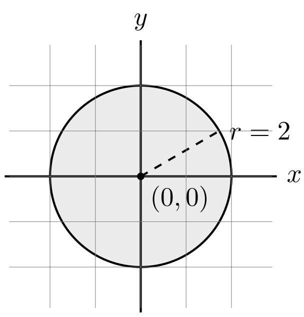
As we can see, the points of the rectangle are now bound by the lines $x = -1$, $x = 1$, $y = 1$ and $y = 3$. **(Ans.)**

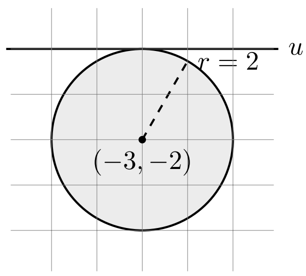
Q.22. Find the image of the circle (in the complex plane) which follows the equation $|z| = 2$ when the function f which is defined as $f(z) \coloneqq z - (3 + 2i)$ is applied to every point in the complex plane.

The output of the function f will be another complex number (say w).

$$\because w = f(z) \coloneqq z - (3 + 2i) \implies z = w + (3 + 2i)$$
$$\therefore |z| = 2 \implies |w + (3 + 2i)| = 2 \iff |w - (-3 - 2i)| = 2$$

The output of the function$f$ when applied to the points inside a circle is another circle of the same radius but transferred to the point $(-3, -2)$ in the complex plane. **(Ans.)**



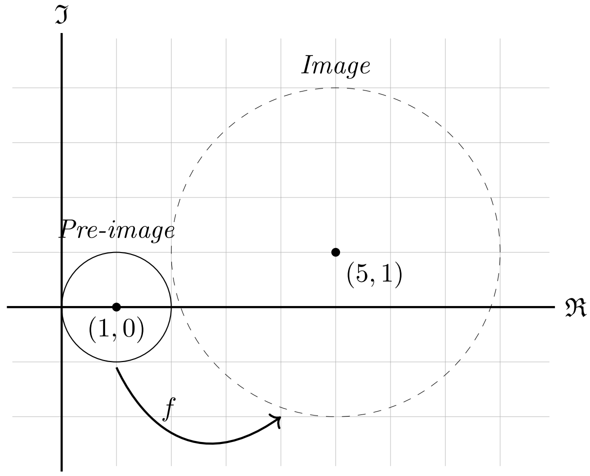


Q.23. Consider the function $f : \mathbb{C} \to \mathbb{C}$ where $f(z) \coloneqq 3z + (2 + i)$ and its effect on the region in the complex plane which follows the following relation. $|z - 1| = 1$.

$$f(z) \coloneqq 3z + (2+i)$$
$$\therefore f(x+iy) \coloneqq (3x+2) + i(3y+1)$$
$$\therefore (\text{Let } f(x+iy) = (u+iv))$$
$$\therefore (u+iv) \coloneqq (3x+2) + i(3y+1)$$
$$\therefore x = \frac{u-2}{3} \text{ and } y = \frac{v-1}{3}$$

$$|z-1| = 1 \implies \sqrt{(x-1)^2+y^2} = 1$$
$$\implies (x-1)^2 + y^2 = 1^2$$
$$\implies \left(\frac{u-5}{3}\right)^2 + \left(\frac{v-1}{3}\right)^2 = 1$$
$$\implies (u-5)^2 + (v-1)^2 = 3^2$$

which is another circle centered at $(5,1)$ and with radius 3. **(Ans.)**



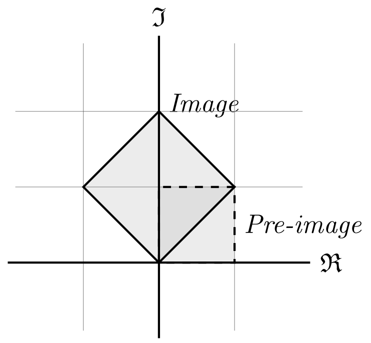
Q.24. Consider the function $f : \mathbb{C} \to \mathbb{C}$ where $f(z) \coloneqq (1+i) \cdot z$ and its effect on the region in the complex plane bound by $x = 0$, $y = 0$, $x = 1$ and $y = 1$.

Let $z \coloneqq x + iy$ and $f \coloneqq u + iv$. Therefore,

$$f(z) \coloneqq (1+i) \cdot (x+iy) \implies (u+iv) = (x-y) + i(x+y) \implies u = x - y \text{ and } v = x + y$$

On solving this linear system of equations we get $x = (u+v)/2$ and $y = (v-u)/2$ which we substitute in the different equations governing the shape of the (square) region.

1. $x = 0 \implies (u+v)/2 = 0 \implies u + v = 0 \implies u = -v$
2. $y = 0 \implies (v-u)/2 = 0 \implies v - u = 0 \implies u = v$
3. $x = 1 \implies (u+v)/2 = 1 \implies u + v = 2 \implies v = -u + 2$
4. $y = 1 \implies (v-u)/2 = 1 \implies v - u = 2 \implies v = u + 2$



Another square which is rotated by 45^o about the origin (in the counterclockwise direction) and has $\sqrt{2}$ times larger sides is the final image. **(Ans.)**

Q.25. Find out what happens to the circle $|z| = 1$ in the complex plane when we apply the transformation $f \coloneqq 1/z$ to the entire plane.

For $w = f(z) := \dfrac{1}{z} = \dfrac{\overline{z}}{|z|}$; $u + iv := \dfrac{x}{|z|} + i\dfrac{-y}{|z|} \implies x = |z| \cdot u$ and $y = -|z| \cdot v$.

For the points (z_0) on the circle where $|z| = 1$:
$|z| = 1 \implies \sqrt{x^2 + y^2} = 1 \implies x^2 + y^2 = 1 \implies |z_0|(u^2 + v^2) = 1 \implies u^2 + v^2 = 1.$

If we have a closer look at the equation $\boxed{w = \dfrac{1}{|z|}(x - iy)}$ we can see that every point in the complex plane is flipped along the real-axis and is scaled by the multiplicative inverse of its distance from origin.

In simpler words, all points have been reflected along the real axis and points outside the circle $|z| = 1$ are mapped to locations inside the circle and vice-versa. **(Ans.)**

Q.26. Find the image of the circle in the complex plane described by the equation $|z - 3| = 1$ when the inversion map $1/z$ is applied to the entire plane.

Let the set $K := \{k : |k - 3| = 1,\ k \in \mathbb{C}\}$ represent the set of values for which we are finding the image of the function and w be the corresponding image of a given element in K. $w \in \operatorname{im}(K) \iff f^{-1}(w) \in K \iff 1/w \in K$.

$$
\begin{aligned}
\because 1/w \in K \implies & |{}^1\!/w - 3| = 1 \\
\therefore & |{}^{(1-3w)}\!/w| = 1 && \text{(Since } |{}^a\!/b| = {}^{|a|}\!/{|b|},\ \forall a, b \in \mathbb{C}) \\
\therefore & |1 - 3w| = |w| \implies |1 - 3w|^2 = |w|^2 \\
\therefore & (1 - 3w)\overline{(1 - 3w)} = w \cdot \overline{w} \\
\therefore & (1 - 3w)(1 - 3\overline{w}) = w \cdot \overline{w} \\
\therefore & 1 - 3(w + \overline{w}) + 9w \cdot \overline{w} = w \cdot \overline{w} \\
\therefore & 1 - 3 \cdot 2\operatorname{Re}(w) + 8w \cdot \overline{w} = 0 \\
\therefore & 1 - 6u + 8(u^2 + v^2) = 0 && \text{(Let } w := u + iv) \\
\therefore & 8u^2 - 6u + {}^9\!/8 + 8v^2 + 1 = {}^9\!/8 && \text{(Completing the square)} \\
\therefore & (2\sqrt{2}u - \tfrac{3}{2\sqrt{2}})^2 + 8v^2 = 1/8 \\
\therefore & (u - {}^3\!/8)^2 + v^2 = ({}^1\!/8)^2 && \text{(Dividing by 8 on both sides)}
\end{aligned}
$$

The circle present at $(3, 0)$ with radius 1 gets mapped to another circle at $({}^3\!/8, 0)$ with a radius of ${}^1\!/8$. i.e.

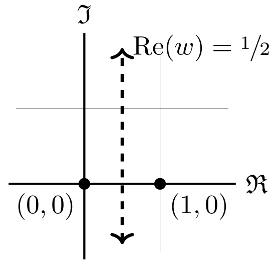
$\boxed{K := \{k : |k - 3| = 1,\ k \in \mathbb{C}\} \text{ becomes } W := \{w : |w - {}^3\!/8| = {}^1\!/8,\ w \in \mathbb{C}\}}$ **(Ans.)**

Q.27. Find the image of the circle $|z - 1| = 1$ under $w = 1/z$.

Let the set $K := \{k : |k - 1| = 1,\ k \in \mathbb{C}\}$ represent the set of values for which we are finding the image of the function and w be the corresponding image of a given element in K. $w \in \operatorname{im}(K) \iff f^{-1}(w) \in K \iff 1/w \in K$.

$$
\begin{aligned}
1/w \in K \implies & |{}^1\!/w - 1| = 1 \\
& |1 - w| = |w| && \left(\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right) \\
& |w - 1| = |w|
\end{aligned}
$$

w is the set of all points whose distance from the point $(1, 0)$ and $(0, 0)$ is the same. This describes a line which passes through the midpoint of these two points and whose direction is perpendicular to that of the line joining these two points.



$\boxed{w \in \{z \in \mathbb{C} : \mathrm{Re}(z) = {}^1\!/_2\}}$ **(Ans.)**

Q.28. Find the image of the line $\{x + iy \in \mathbb{C} : x = y\}$ under $w = 1/z$.

Let the set K represent the set of values for which we are finding the image of the function and w be the corresponding image of a given element in K. $w \in \mathrm{im}(K) \iff f^{-1}(w) \in K \iff 1/w \in K$.

$$\begin{aligned}
\because 1/w \in K &\implies \mathrm{Re}({}^1\!/_w) = \mathrm{Im}({}^1\!/_w) \\
&\therefore \frac{u}{u^2 + v^2} = \frac{-v}{u^2 + v^2} \qquad \left(\because \frac{1}{z} = \frac{\overline{z}}{|z|^2}\right) \\
&\therefore u = -v
\end{aligned}$$

The line $x = y$ gets mapped onto $x = -y$ under inversion mapping.

$\boxed{w \in \{u + iv \in \mathbb{C} : u = -v\}}$ **(Ans.)**

Q.29. Prove that the inversion mapping $1/z$ always maps a straight line or a circle onto another straight line or a circle.

Proof. Let $Q(x, y) = A(x^2 + y^2) + Bx + Cy + D$ be a polynomial in x and y. If the two variables are the real and imaginary parts of a complex number respectively then:

$Q(x, y) = 0$ represents $\begin{cases} \text{a circle} & \forall A \neq 0 \\ \text{a straight line} & A = 0 \end{cases}$ on the complex plane. $(B, C, D \in \mathbb{R})$

$$\begin{aligned}
&Q(x, y) = 0 \\
&\therefore A(x^2 + y^2) + Bx + Cy + D = 0 \\
&\therefore A(z \cdot \overline{z}) + B\left(\frac{z + \overline{z}}{2}\right) + C\left(\frac{z - \overline{z}}{2}\right) + D = 0 \qquad \textit{Inversion} \\
&\therefore \frac{A}{w \cdot \overline{w}} + \frac{B}{2}\left(\frac{1}{w} + \frac{1}{\overline{w}}\right) + \frac{C}{2}\left(\frac{1}{w} - \frac{1}{\overline{w}}\right) + D = 0 \\
&\therefore \frac{A}{w \cdot \overline{w}} + \frac{B}{2}\left(\frac{w + \overline{w}}{w \cdot \overline{w}}\right) + \frac{C}{2}\left(\frac{\overline{w} - w}{w \cdot \overline{w}}\right) + D = 0 \\
&\therefore D(w \cdot \overline{w}) - C\left(\frac{w - \overline{w}}{2}\right) + B\left(\frac{w + \overline{w}}{2}\right) + A = 0 \qquad w := u + iv \\
&\therefore D(u^2 + v^2) - Cv + Bu + A = 0
\end{aligned}$$

This curve is of the same form as $Q(x, y) = 0$ and represents a circle or a straight line depending on the value of D. □

4.2.11 Orthogonal Trajectories

Consider an analytic function $f(z) \coloneqq u(x,y) + iv(x,y)$, then the equations $u(x,y) = c_1$ and $v(x,y) = c_2$ represent the level curves in the complex plane where u and v take a specific value.

Let us find the slopes of the tangent to these curves:

$$\begin{aligned} & u(x,y) = c_1 && v(x,y) = c_2 \\ \therefore & \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y = 0 && \therefore \frac{\partial v}{\partial x}\,\mathrm{d}x + \frac{\partial v}{\partial y}\,\mathrm{d}y = 0 \\ \therefore & \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{u=c_1} = -\frac{u_x}{u_y} && \therefore \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{v=c_1} = -\frac{v_x}{v_y} \\ \therefore & m_u = -\frac{u_x}{u_y} && \therefore m_v = -\frac{v_x}{v_y} \end{aligned}$$

For an analytic function, the CR equations are followed which means that the product $m_u \cdot m_v$ becomes -1. Which means that they are perpendicular(*orthogonal*) to each other.

4.3 Conformal Mapping

Linear Fractional Transformation (LFT), Möbius transformation.

> *"Every möbius tranformation is conformal but not every conformal mapping is-*
> SHUT UP!!! Biden Blast

$$\text{Möbius transformations} \subset \text{Conformal maps}$$

Q.30. Find the möbius transformation such that the points $-1, 0, 1$ are mapped onto the points $-i, 1, i$ respectively.

Let $w \coloneqq \dfrac{az+b}{cz+d}$ be the required transformation for some $a, b, c, d \in \mathbb{C}$.

$$-1 \mapsto -i \implies -i = \frac{a(-1)+b}{c(-1)+d}$$
$$\therefore -i = \frac{-a+b}{-c+d} \implies \boxed{-a+b-ci+di=0} \longrightarrow (1)$$

$$0 \mapsto 1 \implies 1 = \frac{a(0)+b}{c(0)+d}$$
$$\therefore 1 = \frac{b}{d} \implies \boxed{b-d=0} \longrightarrow (2)$$

$$1 \mapsto i \implies i = \frac{a(1)+b}{c(1)+d}$$
$$\therefore i = \frac{a+b}{c+d} \implies \boxed{a+b-ci-di=0} \longrightarrow (3)$$

From (1), (2) and (3),

$$\begin{bmatrix} -1 & 1 & -i & i \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -i & -i \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By Gauss elimination,

$$\therefore \begin{bmatrix} -1 & 1 & -i & i \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -i & -i \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & -i & i \\ 0 & 1 & 0 & -1 \\ 0 & 2 & -2i & 0 \end{bmatrix} \qquad \Big(R_3 \longrightarrow R_3 + R_1\Big)$$

$$\sim \begin{bmatrix} -1 & 1 & -i & i \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2i & 2 \end{bmatrix} \qquad \Big(R_3 \longrightarrow R_3 - 2R_2\Big)$$

Upon back-substitution, we get

$$\begin{aligned} -2ci + 2d &\implies \boxed{d = ci} \\ b - d = 0 &\implies \boxed{b = d = ci} \\ -a + b - ci + di &\implies \boxed{a = \cancel{ci} - \cancel{ci} + ci^2 = -c} \end{aligned}$$

$$\therefore w \coloneqq \frac{az + b}{cz + d} = \frac{-\cancel{c}z + (-\cancel{c}i)}{\cancel{c}z + (-\cancel{c}i)} = \boxed{\frac{-z - i}{z - i}}$$

(Ans.)

4.3.1 Cross Ratio

A transformation which maps the points z_1, z_2, z_3 to w_1, w_2, w_3 respectively is

$$\frac{(z_1 - z_2)(z_3 - z)}{(z_2 - z_3)(z - z_1)} = \frac{(w_1 - w_2)(w_3 - w)}{(w_2 - w_3)(w - w_1)} \tag{4.13}$$

Theorem 3. *A möbius transformation is uniquely determined by the assignment of three points z_1, z_2 and z_3 and their corresponding distinct images w_1, w_2 and w_3.*

Q.31. Determine the LFT that maps the points $2, i, -2$ onto the points $1, i, -1$ respectively using cross ratio.

Given $(z_1 = 2 \mapsto w_1 = 1)$, $(z_2 = i \mapsto w_2 = i)$ and $(z_3 = -2 \mapsto w_1 = -1)$.

$$\text{Computing } \frac{z_1 - z_2}{z_2 - z_3} = \frac{2 - i}{i + 2} = \frac{(2 - i)^2}{2^2 + 1^2} = \frac{3 - 2i}{5}$$

$$\text{Computing } \frac{w_1 - w_2}{w_2 - w_3} = \frac{1 - i}{i + 1} = \frac{(1 - i)^2}{1^2 + 1^2} = \frac{-2i}{2} = -i$$

From eq. (4.13),

$$\frac{(3-2i)(-2-z)}{5(z-2)} = \frac{-i(-1-w)}{w-1}$$

$$\therefore \frac{(3-2i)(z+2)}{5i(z-2)} = \frac{-(w+1)}{w-1}$$

$$\therefore \frac{(\cancel{-}3i\cancel{-}2)(z+2)}{5(z-2)} = \frac{\cancel{-}(w+1)}{w-1}$$

$$\therefore \frac{(3i+2)(z+2)}{5(z-2)} = \frac{w+1}{w-1} = \frac{\cancel{w-1}}{\cancel{w-1}} + \frac{2}{w-1}$$

$$\therefore \frac{(3i+2)(z+2)}{5(z-2)} = 1 + \frac{2}{w-1}$$

(Go to next column)

$$\frac{(3i+2)(z+2)}{5(z-2)} - 1 = \frac{2}{w-1}$$

$$\therefore \frac{(3i+2-5)z+2(3i+2+5)}{5(z-2)} = \frac{2}{w-1}$$

$$\therefore \frac{w-1}{2} = \frac{5z-10}{(3i-3)z+(6i+14)}$$

$$\therefore w = \frac{10z-20}{(3i-3)z+(6i+14)} + 1$$

$$\therefore \boxed{w = \frac{(3i+7)z+(6i-6)}{(3i-3)z+(6i+14)}} \textbf{(Ans.)}$$

4.3.2 Inverse Möbius Transformation

Consider an LFT $w = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$.

The inverse transformation is given by,

$$z = \frac{-dw+b}{cw-a},\ (-d)(-a)-bc \neq 0 \implies ad-bc \neq 0 \tag{4.14}$$

Therefore, if w is an LFT, then its inverse is also an LFT. Such a transformation is said to be *bilinear*.

Q.32. Can we define a bilinear transformation for the transformation $\dfrac{4z+2}{2z+1}$?

A bilinear transform is defined for any LFT if and only if $ad-bc \neq 0$, for the given function $ad-bc = 4\cdot 1 - 2\cdot 2 = 0$. Hence, $\boxed{\text{we cannot define a bilinear transformation}}$ for the given LFT. **(Ans.)**

Linear fractional transformations as matrices

Consider representing the transformation $w = \dfrac{az+b}{cz+d}$ by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Therefore, the transformation is an LFT iff $ad-bc \neq 0$ *i.e.* $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \neq 0$.

The inverse transformation can be represented by the *negative adjugate* of the above matrix, *i.e.* $-\operatorname{adj}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$.

4.3.3 Fixed Points

Fixed point of a transformation $f(z) = \dfrac{az+b}{cz+d}$ can be obtained by putting $f(z) = z$.

Q.33. Find the fixed point of the transformation $w = \dfrac{3z-4}{z+3}$.

To find the fixed point of the given transformation, put $f(z) = z$.

$$
\begin{aligned}
z &= \frac{3z-4}{z+3} \\
\therefore z(z+3) &= 3z-4 \implies z^2+3z = 3z-4 \\
\therefore z^2 + \cancel{3z} &= \cancel{3z} - 4 \implies z^2 = -4 \\
\therefore z &= 2i \lor z = -2i
\end{aligned}
$$

Thus, $\boxed{z \in \{2i, -2i\}}$. **(Ans.)**

4.4 Complex Integration

4.4.1 Line Integral

A line integral is an integral that is evaluated on a function along a curve. The line integral of the function $f(z)$ along the curve C is denoted by $\oint_C f(z)\,\mathrm{d}z$.

Q.34. Evaluate $\oint_0^{2+i} (\overline{z})^2\,\mathrm{d}z$

(i) along the line $x = 2y$.

$$
\begin{aligned}
I &= \oint_0^{2+i} (\overline{z})^2\,\mathrm{d}z = \oint_{(x,y)=(0,0)}^{(x,y)=(2,1)} (x^2-y^2-2ixy)(\mathrm{d}x + i\,\mathrm{d}y) \\
&\text{(Since, we travel along the given line: } x = 2y \implies \mathrm{d}x = \mathrm{d}y) \\
&= \oint_0^1 (4y^2 - y^2 - 4iy^2)(\mathrm{d}y)(2+i) = (3-4i)(2+i)\oint_0^1 y^2\,\mathrm{d}y \\
&= (10-5i)\left[\frac{y^3}{3}\right]_0^1 \\
I &= \frac{10-5i}{3}
\end{aligned}
$$

Thus, the solution of the given line integral is $\boxed{\dfrac{10-5i}{3}}$. **(Ans.)**

(ii) along the line $y = 0$ followed by the line $x = 2$.

$$
\begin{aligned}
I &= \oint_{0}^{2+i} (\overline{z})^2 \,\mathrm{d}z = \oint_{(x,y)=(0,0)}^{(x,y)=(2,1)} (x^2 - y^2 - 2ixy)(\mathrm{d}x + i\,\mathrm{d}y) \\
&= \oint_{(x,y)=(0,0)}^{(x,y)=(2,0)} (x^2 - y^2 - 2ixy)(\mathrm{d}x + i\,\mathrm{d}y) + \oint_{(x,y)=(2,0)}^{(x,y)=(2,1)} (x^2 - y^2 - 2ixy)(\mathrm{d}x + i\,\mathrm{d}y)
\end{aligned}
$$

(For the path $y = 0$, $\mathrm{d}y = 0$ and the path $x = 2$, $\mathrm{d}x = 0$)

$$
\begin{aligned}
&= \oint_{0}^{2} x^2\,\mathrm{d}x + i \oint_{0}^{1} (4 - y^2 - 4iy)\,\mathrm{d}y \\
&= \left[\frac{x^3}{3}\right]_0^2 + i\left[4y - \frac{y^3}{3} - 2iy^2\right]_0^1 = \frac{8}{3} + 2 + i\left(4 - \frac{1}{3}\right) \\
I &= \frac{14}{3} + \frac{11i}{3}
\end{aligned}
$$

Thus, the solution of the given line integral is $\boxed{14/3 + 11/3i}$. **(Ans.)**

Q.35. Evaluate $\oint_{1-i}^{2+i} 2x + iy + 1\,\mathrm{d}z$ along the straight line from the point $1 - i$ to the point $2 + i$.

The given straight line passes through the points $(1, -1)$ and $(2, 1)$. The slope of this line is $(1 - (-1))/(2 - 1) = 2/1 = 2$. The intercept of this line can be found out by substituting either of the two given points in the formula $y = 2x + c \implies c = y - 2x$. If we substitute $(2, 1)$ in the given formula, then we get $c = -3$.

We also require the relation between the rates of changes of the variables x and y. So upon differentiating the given equation of the line w.r.t. x and separating the variables, we get $\mathrm{d}y = 2\,\mathrm{d}x$.

$$
\begin{aligned}
I &= \oint_{1-i}^{2+i} 2x + iy + 1\,\mathrm{d}z = \oint_{(x,y)=(1,-1)}^{(x,y)=(2,1)} (2x + 1 + iy)(\mathrm{d}x + i\,\mathrm{d}y) \\
&= \oint_{x=1}^{x=2} (2x + 1 + i(2x - 3))(1 + 2i)\,\mathrm{d}x \\
&= (1 + 2i) \oint_{1}^{2} (2x + 1 + i(2x - 3))\,\mathrm{d}x \\
&= (1 + 2i)\left[(x^2 + x) + i(x^2 - 3x)\right]_1^2 \\
I &= (1 + 2i)\left((6 - 2i) - (2 - 2i)\right) = (1 + 2i)(4) = 4 + 8i
\end{aligned}
$$

Thus, the solution of the given line integral is $\boxed{4 + 8i}$. **(Ans.)**

Q.36. Evaluate $\oint_0^{1+i} z^2\,\mathrm{d}z$ along the parabola $x = y^2$.

$$I = \oint_0^{1+i} z^2\,\mathrm{d}z = \oint_{(x,y)=(0,0)}^{(x,y)=(1,1)} (x^2 - y^2 + 2ixy)(\mathrm{d}x + i\,\mathrm{d}y)$$

(Since we travel along the given parabola: $x = y^2 \implies \mathrm{d}x = 2y\,\mathrm{d}y$)

$$= \oint_{y=0}^{y=1} (y^4 - y^2 + 2iy^3)(2y + i)\,\mathrm{d}y = -\oint_0^1 (2y^5 - 2y^3 - 2y^3 + i(y^4 - y^2 + 4y^4))\,\mathrm{d}y$$

$$= \oint_0^1 (2y^5 - 4y^3 + i(5y^4 - y^2))\,\mathrm{d}y = \left[\frac{y^6}{3} - y^4 + i\left(y^5 - \frac{y^3}{3}\right)\right]_0^1 = \left[\frac{1}{3} - 1 + i\left(1 - \frac{1}{3}\right)\right]$$

$$I = -\frac{2}{3} + i\frac{2}{3}$$

Thus, the solution of the given line integral is $\boxed{-\frac{2}{3} + i\frac{2}{3}}$. **(Ans.)**

Q.37. Evaluate the integral of the function $f(z) = x^2 + ixy$ from $A(1,1)$ to $B(2,4)$ along the curve $x = t$ and $y = t^2$.

For the given curve, ($x = t$ and $y = t^2$) $\implies$ ($\mathrm{d}x = \mathrm{d}t$ and $\mathrm{d}y = 2t\,\mathrm{d}t$. Thus, $\mathrm{d}z = (1 + 2it)\,\mathrm{d}t$. The function $f(z)$ becomes $f(t) = t^2 + it^3$

For the points A and B, $t = 1$ and $t = 2$ respectively. Thus,

$$I = \oint_{AB} f(z)\,\mathrm{d}z = \oint_1^2 f(t)(1 + 2it)\,\mathrm{d}t$$

$$= \oint_1^2 (t^2 + it^3)(1 + 2it)\,\mathrm{d}t = \oint_1^2 (t^2 - 2t^4 + i(3t^3))\,\mathrm{d}t = \left[\frac{t^3}{3} - \frac{2}{5}t^5 + i\frac{3}{4}t^4\right]_1^2$$

$$= \left[\left(\frac{8}{3} - \frac{64}{5} + 12i\right) - \left(\frac{1}{3} - \frac{2}{5} + \frac{3}{4}\right)\right] = -\frac{151}{15} + i\frac{45}{4}$$

Thus, the solution of the given line integral is $\boxed{-\frac{151}{15} + i\frac{45}{4}}$. **(Ans.)**

When the curve is a circle

Whenever we encounter a closed curve (e.g. circle), we can usually parameterize them in some way such that the integral goes from a closed line integral in the Cartesian coordinates to an open one in the parameterized coordinates.

For a circle, this parameter is θ, the angle which is made with the direction parallel to the positive x-axis. Thus, for a circle with radius r, centered at (c_0, c_1), the value of x and y is $c_0 + r\cos\theta$ and $c_1 + r\sin\theta$ and the value of $\boxed{\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y = r(-\sin\theta + i\cos\theta)\,\mathrm{d}\theta = rie^{i\theta}\,\mathrm{d}\theta}$.

Q.38. Evaluate $\oint_C \overline{z} + 2z \,\mathrm{d}z$ where C is the unit circle.

$$I = \oint_C \overline{z} + 2z \,\mathrm{d}z = \oint_C 3x + iy \,\mathrm{d}z$$

(For the unit circle, centered at origin, $(c_0, c_1) = (0,0)$ and $r = 1$)

$$= \oint_{-\pi}^{\pi} (3\cos\theta + i\sin\theta)(-\sin\theta + i\cos\theta)\,\mathrm{d}\theta$$

$$= \oint_{-\pi}^{\pi} (-4\cos\theta\sin\theta + i(3\cos^2\theta - \sin^2\theta))\,\mathrm{d}\theta$$

$$= \oint_{-\pi}^{\pi} -2\sin(2\theta) + i(4\cos^2\theta - 1)\,\mathrm{d}\theta$$

($\because \sin(2\theta)$ is an odd function and $\oint \cos^2\theta = \frac{\theta}{2} + \frac{\cos(2\theta)}{4} + c$)

$$= 0 + [i2\theta + i\cos(2\theta) - i\theta]_{-\pi}^{\pi} = i(\pi + 1) - i(-\pi + 1) = i2\pi$$

Thus, the value of the given line integral is $\boxed{2\pi i}$. **(Ans.)**

Q.39. Evaluate $\oint_C \frac{2z+5}{z}\,\mathrm{d}z$ where C is the lower half of the circle $|z| = 2$.

$$I = \oint_C \frac{2z+5}{z}\,\mathrm{d}z = \oint_C 2 + \frac{5}{z}\,\mathrm{d}z = \oint_C 2 + \frac{5\overline{z}}{|z|^2}\,\mathrm{d}z$$

(Multiplying and dividing by $\overline{z}$)

$$= \oint_C 2 + \frac{5}{2^2}\overline{z}\,\mathrm{d}z = \oint_C 2 + 1.25\overline{z}\,\mathrm{d}z$$

$\because$ (For the given curve $|z| = 2$)

(Assuming we travel in the counterclockwise direction, $\theta \in [-\pi, 0]$)

$$= \oint_{-\pi}^{0} (2 + (1.25)(2\cos\theta - 2i\sin\theta))\, 2(-\sin\theta + i\cos\theta)\,\mathrm{d}\theta$$

$$= \oint_{-\pi}^{0} (4 + 5\cos\theta - 5i\sin\theta)(-\sin\theta + i\cos\theta)\,\mathrm{d}\theta$$

$$= \oint_{-\pi}^{0} (-4\sin\theta \cancel{-5\cos\theta\sin\theta} + \cancel{5\cos\theta\sin\theta} + 4i\cos\theta + 5i\cos^2\theta + 5i\sin^2\theta)\,\mathrm{d}\theta$$

$$= \oint_{-\pi}^{0} 4(-\sin\theta + i\cos\theta) + 5i\,\mathrm{d}\theta$$

$$= 4\left[\cos\theta + i\sin\theta\right]_{-\pi}^{0} + 5i\left[\theta\right]_{-\pi}^{0} = 4(2) + 5i\pi$$

Thus, the solution to the given line integral is $\boxed{8 + 5\pi i}$. **(Ans.)**

Q.40. Evaluate $\oint_C \frac{\mathrm{d}z}{(z-3)^4}$ where $C : |z - 3| = 4$.

Let $w := r_w e^{i\phi} \in \mathbb{C}$ such that $z - 3 = w$. Then the associated circle $|z - 3| = 4$ becomes

$|w| = 4$ and the infinitesimal $\mathrm{d}z$ becomes $\mathrm{d}w$. $(\because w = z - 3 \implies \mathrm{d}w = \mathrm{d}z)$

$$
\begin{aligned}
I &= \oint_C \frac{\mathrm{d}z}{(z-3)^4} \implies \oint_{C'} \frac{\mathrm{d}w}{w^4} \\
&= \oint_{\phi=-\pi}^{\phi=\pi} (r_w \cdot e^{i\phi})^{-4}\,\mathrm{d}\phi = 4^{-4} \oint_{-\pi}^{\pi} (e^{-4i\phi})\,\mathrm{d}\phi \\
&= \frac{1}{256}\left[\frac{e^{-4i\phi}}{-4i}\right]_{-\pi}^{\pi} = \frac{i}{1024}(e^{-i4\pi} - e^{i4\pi}) = \frac{i}{1024}(1-1) \\
I &= 0
\end{aligned}
$$

Thus, the solution to the given line integral is $\boxed{0}$. **(Ans.)**

Q.41. Evaluate $\oint_C \overline{z}\,\mathrm{d}z$ where C is the left half of the unit circle from $z = -i$ to $z = i$.

$$
\begin{aligned}
I &= \oint_C \overline{z}\,\mathrm{d}z = \oint_{\theta=3\pi/2}^{\theta=\pi/2} (e^{-i\theta})\,\mathrm{d}(e^{i\theta}) \\
&= \oint_{3\pi/2}^{\pi/2} ie^{-i\theta}e^{i\theta}\,\mathrm{d}\theta = i\oint_{3\pi/2}^{\pi/2} 1\,\mathrm{d}\theta \\
&= i[\theta]_{3\pi/2}^{\pi/2} = -i\pi
\end{aligned}
$$

Thus, the solution to the given line integral is $\boxed{-i\pi}$. **(Ans.)**

4.4.2 Cauchy's Theorem

Types of Curves

To study Cauchy's theorem, we must know about the types of curves we can encounter in the complex plane:

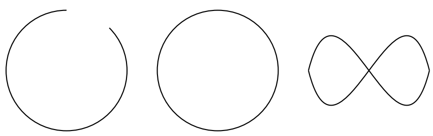


Figure 4.6: (From left to right) Simple open, Simple closed, not simple but closed

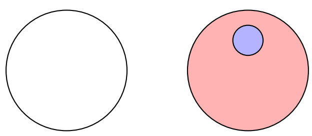


Figure 4.7: (From left to right) Simple connected, multiple connected

Theorem 4. *If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at each point within and on a simple closed curve C then the integral of $f(z)$ along the closed curve C is zero.*

$$\oint_C f(z)\,\mathrm{d}z = 0 \tag{4.15}$$

4.4.3 Cauchy's Integral Formula

If $f(z)$ is analytic within and on a simple closed curve C and z_0 is any point within C then

$$\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z = 2\pi i \cdot f(z_0) \tag{4.16}$$

Q.42. Evaluate the integral of $^1/_z$ along the curve $|z-2| = 2$.

The given function $f(z) = 1/z$ is not defined for $z = 0$ which is in the given region $|z-2| = 2$. Since a circle is a simple closed curve, we can apply Cauchy's integral formula for $f(z) = \frac{g(z)}{z-z_0} = \frac{1}{w} \implies g(z) = 1$. From eq. (4.16), $\oint_C f(z)\,\mathrm{d}z = 2\pi i \cdot g(z_0) = 2\pi i \cdot 1 = 2\pi i$.

Thus, the solution to the given line integral is $\boxed{2\pi i}$. **(Ans.)**

Q.43. Evaluate $\oint_C \frac{z}{z^2-3z+2}\,\mathrm{d}z$ where $C : |z-1| = {}^1/_2$.

The given function $f(z) = \frac{z}{z^2-3z+2}$ is not defined for $z = 1$ and $z = 2$ out of which, $z = 1$ does exist in the given region $|z-1| = {}^1/_2$. Since a circle is a simple closed curve, we can apply Cauchy's integral formula for $f(z) = {}^{g(z)}/_{z\,-\,z_0} = {}^z/_{(z\,-\,1)(z\,-\,2)} \implies g(z) = {}^z/_{z\,-\,2}$. From eq. (4.16), $\oint_C f(z)\,\mathrm{d}z = 2\pi i \cdot g(z_0) = 2\pi i \cdot g(1) = 2\pi i \cdot {}^1/_{-\,1} = -2\pi i$.

Thus, the solution to the given line integral is $\boxed{-2\pi i}$. **(Ans.)**

Q.44. If $f(z) = \frac{z}{z^2-5z+6}$, evaluate $\oint_C f(z)\,\mathrm{d}z$ where $C : |z-2| = 2$.

The given function $f(z) = \frac{z}{z^2-5z+6}$ is not defined for $z = 2$ and $z = 3$ both of which exist in the given region $|z-2| = 2$. Since a circle is a simple closed curve, we can apply Cauchy's integral formula:

$$\begin{aligned}
I &= \oint \frac{z}{z^2-5z+6}\,\mathrm{d}z = \oint \frac{z}{(z-2)(z-3)}\,\mathrm{d}z \\
&\text{(Decompose it into partial fractions)} \\
&= \oint_C -\frac{2}{z-2} + \frac{3}{z-3}\,\mathrm{d}z \\
&= 2\pi i \cdot g(z_0) + 2\pi i \cdot h(z_0) \\
&= -4\pi i + 6\pi i \\
I &= 2\pi i
\end{aligned}$$

Thus, the solution to the given line integral is $\boxed{2\pi i}$. **(Ans.)**

4.4.4 Generalized Cauchy's Integral Formula

If $f(z)$ is analytic within and on a simple closed curve C and z_0 is any point in the region then

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = \frac{2\pi i}{n!} \cdot \frac{\mathrm{d}^n f}{\mathrm{d}z^n}(z_0) \tag{4.17}$$

Q.45. Evaluate $\oint_C \frac{z}{z^2-4z+4}\,\mathrm{d}z$ where $C : |z-2| = 3$.

From eq. (4.17),

$$I = \oint_C \frac{z}{z^2 - 4z + 4}\,\mathrm{d}z = \oint_C \frac{z}{(z-2)^2}\,\mathrm{d}z \qquad \left(\text{Which is of the form: } \frac{f(z)}{(z-z_0)^{n+1}}\right)$$

$$= \frac{2\pi i}{1!} \cdot \frac{\mathrm{d}f}{\mathrm{d}z} = 2\pi i \cdot (1) = 2\pi i$$

Thus, the solution to the given line integral is $\boxed{2\pi i}$. **(Ans.)**

Chapter 5

Elementary Number Theory

5.1 Foundations

5.1.1 Integers

The set of integers consists of the numbers zero (0), counting numbers $(1, 2, 3, \dots)$ and their additive inverses $(-1, -2, -3, \dots)$. It is denoted as $\mathbf{Z}$ or $\mathbb{Z}$.

$$\mathbb{Z} := \{\dots, -3 - 2 - 1, 0, 1, 2, 3, \dots\}$$

- The set of integers under the addition $(+)$ operation is *closed*, *associative*, has an *identity* element (0), has *inverses* and *commutative*.
- The set of integers under the multiplication $(\times)$ operation is *closed*, *associative*, has an *identity* element (1) and *commutative*.

Moreover, multiplication *distributes* over addition in $\mathbb{Z}$.

The set of positive integers is denoted by $\mathbb{Z}^+$ or $\mathbb{N}$ (natural numbers) and negative integers by $\mathbb{Z}^-$. A discrete range $\{a, a+1, a+2, \dots, b-2, b-1, b\}$ is denoted by $[\![a, b]\!]$.

In this chapter, every variable is assumed to be an integer unless stated otherwise.

5.1.2 Divisibility

For two integers $a \neq 0$ and b, if $b = aq$ for some integer q, then a is said to be a *factor* or *divisor* of b, and b is said to be *divisible* by a or a *multiple* of a. This is denoted by $a|b$, which is read as "a divides b".

$$a|b \iff \exists q \in \mathbb{Z} : b = aq \tag{5.1}$$

If $b = aq + r$ for some integers q and $r \in [\![1, |a| - 1]\!]$, then b is said to be *not divisible* by a and is denoted by $a \nmid b$.

$$a \nmid b \iff \exists q \in \mathbb{Z}\ \exists r \in [\![1, |a| - 1]\!] : b = aq + r \tag{5.2}$$

In the expression $b = aq + r$, b is called the *dividend*, a is called the *divisor*, q is called the *quotient* and r is called the *remainder*.

Note:

The set $[a, b] \wedge \mathbb{Z}$ is referred to as $[\![a, b]\!]$ commonly throughout this book.

Q.1. Prove that the square of an even number is even and the square of an odd number is odd.

Proof. We will prove the statement by cases.

Case 1 Consider an even number $a \coloneqq 2q$. Squaring both sides, we obtain $a^2 = 4q^2 = 2(2q^2)$, which is even.

$\therefore$ The square of an even number is even. $\longrightarrow (i)$

Case 2 Consider an odd number $a \coloneqq 2q + 1$. Squaring both sides, we obtain $a^2 = 4q^2 + 4q + 1 = 2(2q^2 + 2q) + 1$, which is odd.

$\therefore$ The square of an odd number is odd. $\longrightarrow (ii)$

From (i) and (ii), we can deduce that the square of an even number is even and the square of an odd number is odd. $\square$

Q.2. Prove that the square of an odd integer can be written in the form $8k + 1$.

Proof. Consider an odd integer $a \coloneqq 2q+1$. Squaring both sides, we obtain $a^2 = 4q^2+4q+1 = 4q(q+1)$. Since q and $q+1$ are consecutive integers, $q(q+1)$ is even, i.e. $q(q+1) \coloneqq 2k$ for some integer k.

Therefore, $a^2 = 4 \cdot 2k + 1 = 8k + 1 \implies$ the square of any odd integer can be written in the form $8k + 1$. $\square$

Q.3. Prove that the cube of any integer has one of the forms: $9k, 9k + 1$ or $9k + 8$.

Proof. We will prove the statement by cases.

Case 1 Consider any integer a of the form $a \coloneqq 3q$. Cubing both sides, we obtain $a^3 = 27q^3 = 9(3q^3) \coloneqq 9k$ for some integer $k \longrightarrow (i)$.

Case 2 Consider any integer a of the form $a \coloneqq 3q + 1$. Cubing both sides, we obtain $a^3 = 27q^3 + 27q^2 + 9q + 1 = 9(3q^3 + 3q^2 + q) + 1 \coloneqq 9k + 1$ for some integer $k \longrightarrow (ii)$.

Case 3 Consider any integer a of the form $a \coloneqq 3q + 2$. Cubing both sides, we obtain $a^3 = 27q^3 + 54q^2 + 36q + 1 = 9(3q^3 + 6q^2 + 4q) + 8 \coloneqq 9k + 8$ for some integer $k \longrightarrow (iii)$.

From (i), (ii) and (iii), we can deduce that the cube of any integer has one of the forms: $9k, 9k + 1$ or $9k + 8$. $\square$

5.1.3 Prime Numbers

A *prime number* is a positive integer that has exactly two distinct positive factors — 1 and itself. The first few primes are 2, 3, 5, 7, 11, ...

A number that has more than two distinct positive divisors is said to be composite. The first few composites are 4, 6, 8, 9, 10, ...

1 is said to be neither prime nor composite, as it has exactly one positive divisor.

$$p \text{ is prime } \iff p > 1 \land (\forall d \in \mathbb{N}\ d|p \implies d = 1 \lor d = p)$$

Cardinality of Primes

There are infinitely many prime numbers. It can be proven using Euclid's proof.

Proof. Suppose for contradiction, there are only r prime numbers: $p_1, p_2, \ldots, p_r$.

Construct the integer $n := \prod_{i=1}^{r} p_i + 1 = p_1 p_2 \cdots p_r + 1$. Since by assumption there are only finitely many primes, n must be composite, *i.e.* at least one prime p_i must divide n.

Comparing with $b = aq+r$, n is the dividend, each p_i can be considered a divisor, each $\left(\frac{\prod_{j=1}^{r} p_j}{p_i}\right)$ can be considered a quotient and 1 is the remainder.

However, from eq. (5.2) we can see that $\forall p_i\ p_i \nmid n$. Therefore, there are no factors of n other than 1 and itself $\implies n$ is prime.

Now if n is added to the set of primes $\{p_1, p_2, \ldots, p_r, n\}$, we can iterate this process and always produce more prime numbers. Therefore, there are *infinitely many primes.* $\square$

5.2 Modular Arithmetic

5.2.1 Modulus Operator

The *modulus* operator is used to find the remainder when an integer a is divided by another integer b. It is denoted as $a \,\%\, b$ or $a \bmod b$, and is read as "a modulo b".

Modulo m, integers can be imagined as a clock with m numbers labeled from 0 to $m-1$. This is because the remainders cycle every m numbers.

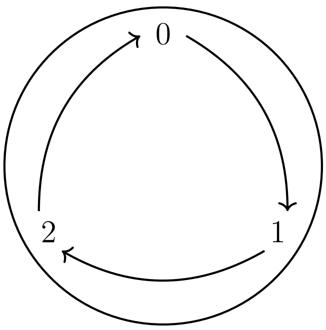


Figure 5.1: Modulo 3

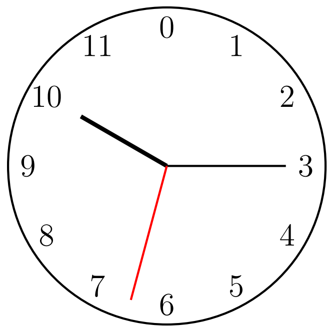


Figure 5.2: Modulo 12

Q.4. Show that $3a^2 - 1$ is not a perfect square.

Proof. Observe that $(3a^2 - 1) \bmod 3 = (3(a^2 - 1) + 2) \bmod 3 = 2$.

Now, consider squaring any integer modulo 3:

$(3q)^2 \bmod 3 = 9q^2 \bmod 3 = 0 \longrightarrow (i)$.

$(3q+1)^2 \bmod 3 = (9q^2 + 6q + 1) \bmod 3 = (3(3q^2 + 2q) + 1) \bmod 3 = 1 \longrightarrow (ii)$.

$(3q+2)^2 \bmod 3 = (9q^2 + 12q + 4) \bmod 3 = (3(3q^2 + 4q + 1) + 1) \bmod 3 = 1 \longrightarrow (iii)$.

From (i), (ii) and (iii), we can deduce that the square of any integer has a remainder of either 0 or 1 modulo 3, i.e. $3a^2 - 1$ *cannot be the square of any integer.* $\square$

5.2.2 Properties of Divisibility

For all integers a, b, c and d the following properties hold:

- $a|0, \; 1|a, \; a|a$
- $a|1 \iff a = \pm 1$
- $(a|b \wedge c|d) \implies ac|bd$
- $(a|b \wedge b|c) \implies a|c$
- $(a|b \wedge b|a) \implies |a| = |b|$
- $(a|b \wedge b \neq 0) \implies |a| \leq |b|$

  Proof.

$$\begin{aligned} a|b &\implies b := aq \; q \neq 0 \\ &\therefore |b| = |aq|, \; |q| \geq 1 \\ &\therefore |b| = |a||q|, \; |q| \geq 1 \implies \boxed{|b| \geq |a|} \end{aligned}$$

$\square$

- $(a|b \wedge c|d) \implies \exists x \; \exists y : a|bx + cy$

  Proof. $a|b \iff b := aq_1$, $a|c \iff c := aq_2$

  Now, $bx + cy = aq_1x + aq_2y = a(q_1x + q_2y) \implies \boxed{a|bx + cy}$. $\square$

  Corollary 1. *If* $a|b_k$, *where* $k = 0, 1, 2, \ldots, n$ *then* $a|(b_0x_0 + b_1x_1 + \cdots + b_nx_n)$.

5.2.3 Division Algorithm

Given integers a and b (not both zero). Then there exists a unique ordered pair of integers q and r such that:

$$a = bq + r, 0 \leq r < b$$

Proof. Consider a set $S := \{a - bx \,:\, a - bx \geq 0 \wedge x \in \mathbb{Z}\}$.

S is a non-empty set of non-negative integers.

Therefore, by the well-ordering principle, there exists a least element $r \in S$. So, $r = a - bq \geq 0$.

We need to show that $r < b$. For contradiction, assume $r \geq b$.

Now,

$$\begin{aligned} & r \geq b \\ \implies & r - b \geq 0 \\ \implies & a - bq - b \geq 0 \\ \implies & a - (q+1)b \geq 0 \implies a - (q+1)b \in S \end{aligned}$$

Or, $r - b \in S$.

$\because b > 0,\ r - b < r$, which implies that $r - b$ is an element lesser than r, which contradicts our claim that r is the least element.

Therefore, by contradiction $r < b$.

Now to prove uniqueness:
Let us consider that for the integers a and b, there exist two pairs of ordered pairs (q_1, r_1) and (q_2, r_2) such that $a = bq_1 + r_1$ and $a = bq_2 + r_2$ where $r_1, r_2 \in [0, b)$.

Consider the integer $|r_1 - r_2|$,

$$\begin{aligned}
|r_1 - r_2| &= |bq_1 - bq_2| \\
&= |b||q_1 - q_2| \\
&= b|q_1 - q_2|
\end{aligned}$$

But we know that $r_1, r_2 \in [\![0, b-1]\!]$. Therefore, $|r_1 - r_2| < b$.

$$\begin{aligned}
\because |r_1 - r_2| &< b \\
\therefore b|q_1 - q_2| &< b \\
\therefore |q_1 - q_2| &< 1
\end{aligned}$$

But since, $q_1, q_2 \in \mathbb{Z}$. Therefore the only possibility is $|q_1 - q_2| = 0 \implies (q_1 = q_2) \wedge (r_1 = r_2)$.

Hence, we have proved the uniqueness as well. $\square$

Theorem 5. *Given two non-zero integers a and b, there exist integers x and y such that $ax+by = \gcd(a, b)$.*

Proof. Consider a set $S \coloneqq \{ax + by : ax + by > 0 \wedge x, y \in \mathbb{Z}\}$.

Therefore, by the well-ordering principle, there exists a least element $d \in S$. So $d = au + bv > 0$. Now we must prove that $d = \gcd(a, b)$.

By the division algorithm,

$$\begin{aligned}
a &= dq + r && , 0 \leq r < d \\
r &= a - dq \\
&= a - (au + bv)q \\
&= a(1 - uq) + b(vq) \\
\therefore r &\in S && \because r \geq 0
\end{aligned}$$

But since $r < d$, it means that r is the minimum element of S, which contradicts our original assumption. It means that the only possible value of r is 0 i.e. $r \notin S$.

This means that $a = dq$, i.e. $d|a$. Similarly, we can prove that $d|b$.

Any other common divisor of a and b of the form $ax + by$ which is greater than d does not exist. We can easily prove this by contradiction.

Thus, we have proved that the $\gcd(a, b)$ is of the form $ax + by$ where x and y are integers, $\square$

Corollary 2. *If the greatest common divisors of a and b is equal to 1, then there exist integers*

x and y such that $ax + by = 1$.

5.3 Greatest Common Divisor

If a and b are arbitrary integers then a positive integer d is said to be the Greatest Common Divisor (GCD) (or Highest Common Factor (HCF)) of a and b if d satisfies the following conditions:

(i) $d|a \wedge d|b$

(ii) $c|a \wedge c|b \implies c \leq d$

We denote this as $\gcd(a, b) = d$.

Q.5. Evaluate $\gcd(-12, 30)$. Prime factorizing,

$$\begin{aligned} |-12| &= 12 = 2 \times \boxed{2 \times 3} \\ |30| &= 30 = \boxed{2 \times 3} \times 5 \end{aligned}$$

Therefore, the GCD of -12 and 30 is $\boxed{2 \times 3 = 6}$. **(Ans.)**

If $\gcd(a, b) = 1$, then a and b are said to be *relatively prime* or *coprime*.
Theorem 6. *If $a|c$ and $b|c$ with $\gcd(a, b) = 1$, then $ab|c$.*

Proof. Since $a|c$, we can say that $c = ak_1$ and since $b|c$, we can say that $c = bk_2$.

Let $a = p_{11}p_{12}p_{13}\cdots p_{1n}$ and $b = p_{21}p_{22}p_{23}\cdots p_{2m}$. Where p_{ij}; $i \in \{1, 2\}$; $j \in [\![1, \max(m, n)]\!]$ are the prime factors of a and b.

Now we can say that

$$\begin{aligned} c &= p_{11}^{k_{11}} p_{12}^{k_{12}} p_{13}^{k_{13}} \cdots p_{1n}^{k_{1n}} \times k_1 = k_1 \times \prod_{j=1}^{n} p_{1j}^{k_{ij}} \\ c &= p_{21}^{k_{21}} p_{22}^{k_{22}} p_{23}^{k_{23}} \cdots p_{2m}^{k_{2m}} \times k_2 = k_2 \times \prod_{j=1}^{m} p_{2j}^{k_{2j}} \end{aligned}$$

Since, $\gcd(a, b) = 1$, it means that none of the prime factors of a are equal to prime factors of b. It means that:

$$\begin{aligned} c &= \left(\prod_{j=1}^{n} p_{1j}^{k_{1j}}\right) \times \left(\prod_{j=1}^{m} p_{2j}^{k_{2j}}\right) \times k_3 \\ \therefore c &= abk_3 \\ \therefore ab &\mid c \end{aligned}$$

Hence, we have proved that $ab|c$. □

5.3.1 Properties of the GCD

Least Common Multiple (LCM)

$$\operatorname{lcm}(a, b) = \frac{a \times b}{\gcd(a, b)} \tag{5.3}$$

Euclid's Lemma

If $a|bc$ with $\gcd(a,b) = 1$ then $a|c$.

Proof.

$$\because a|bc$$
$$\therefore bc = ak_1$$

But, for any $k_2 \in \mathbb{N}$.

$$\because \gcd(a,b) = 1$$
$$\therefore a \neq bk_2 \land b \neq ak_2$$

Therefore, we can say $c = ak_3$ i.e. $a|c$. $\square$

5.3.2 Euclidean Algorithm

Let a and b be two integers whose greatest common divisor d can be obtained by applying the division algorithm repeatedly applying the division algorithm to a and b.

$$\gcd(a,b) = \gcd(b, a \bmod b) \tag{5.4}$$

$$\begin{aligned}
a &= bq + r && ,0 \le r < b \\
\therefore b &= rq_1 + r_1 && ,0 \le r_1 < r \\
\therefore r &= r_1 q_2 + r_2 && ,0 \le r_2 < r_1 \\
\therefore r_1 &= r_2 q_3 + r_3 && ,0 \le r_3 < r_2 \\
&\vdots \\
\therefore r_{n-1} &= r_n q_{n+1} + r_{n+1} && ,0 \le r_{n+1} < r_n \\
\therefore r_n &= r_{n+1} q_{n+2} && ,\text{ for some } n+2,\ r_{n+2} = 0 \\
\therefore \gcd(a,b) &= r_{n+1}
\end{aligned}$$

5.4 Linear Diophantine Equations

The equation

$$ax + by = c,\ x, y \in \mathbb{Z}^+ \cup \{0\}$$

where a, b and c are integers is called a *linear Diophantine equation.*

Theorem 7. *The linear Diophantine equation $ax + by = c$ has a solution iff $d|c$ where $d := \gcd(a,b)$.*

If (x_0, y_0) is any particular solution of $ax + by = c$ then all other solutions are given by:

$$x := x_0 + \left(\frac{b}{d}\right)t,\ y := y_0 - \left(\frac{a}{d}\right)t$$

where t is an arbitrary integer.

Q.6. A theater charges \$1.80 for adult admission and \$0.75 for children admission for a particular evening; the total receipts were \$90. Assuming more adults than children were present, how many were there in the theater?

Suppose there were x adults and y children. According to the given conditons,

$$1.80x + 0.75y = 90, \ x > y \geq 0$$

or, $180x + 75y = 9000$
or, $12x + 5y = 600 \longrightarrow (i)$.

By Euclidean algorithm,

$$\begin{aligned} 12 &= 5 \times 2 + 2 \\ 5 &= 2 \times 2 + \boxed{1} \\ 2 &= \boxed{1} \times 2 \end{aligned}$$

Therefore, $\gcd(12, 5) = 1$.

Expressing the GCD as a linear combination of a and b,

$$\begin{aligned} 1 &= 5 - 2 \times 2 \\ &= 5 - (12 - 5 \times 2) \times 2 \\ &= 5 - 12 \times 2 + 5 \times 4 \\ \therefore 1 &= 5(5) + 12(-2) \end{aligned}$$

Therefore, $12(-2) + 5(5) = 1 \implies 12(-1200) + 5(3000) = 600 \longrightarrow (ii)$.

From (i) and (ii), $x_0 = -1200$, $y_0 = 3000$

The solution to the Diophantine equation is given by,

$$x = x_0 + \left(\frac{5}{1}\right)t \qquad\qquad y = y_0 - \left(\frac{12}{1}\right)t$$

$$\therefore x = -1200 + 5t \qquad\qquad \therefore y = 3000 - 12t$$

Given $x > y \geq 0$

$$\begin{aligned} x &> y \\ \therefore -1200 + 5t &> 3000 - 12t \\ \therefore 17t &> 4200 \\ \therefore t &> \frac{4200}{17} \implies t \geq 248 \end{aligned} \qquad\qquad \begin{aligned} y &\geq 0 \\ \therefore 3000 - 12t &\geq 0 \\ \therefore 12t &\leq 3000 \\ \therefore t &\leq \frac{3000}{12} \implies t \leq 250 \end{aligned}$$

Therefore, $248 \le t \le 250 \implies t \in \{248, 249, 250\}$.

$$\begin{aligned} t = 248 \implies x &= -1200 + 5 \times 248 = \boxed{40} \\ y &= 3000 - 12 \times 248 = \boxed{24} \end{aligned}$$

$$\begin{aligned} t = 249 \implies x &= -1200 + 5 \times 249 = \boxed{45} \\ y &= 3000 - 12 \times 249 = \boxed{12} \end{aligned}$$

$$\begin{aligned} t = 250 \implies x &= -1200 + 5 \times 250 = \boxed{50} \\ y &= 3000 - 12 \times 250 = \boxed{0} \end{aligned}$$

There were $\boxed{\text{40 adults and 24 children}}$, or $\boxed{\text{45 adults and 12 children}}$, or $\boxed{\text{50 adults and no children}}$ in the theater. **(Ans.)**

5.5 Fundamental Theorem of Arithmetic

Theorem 8. *Every positive integer greater than 1 is either prime or a product of primes, whose representation is unique upto the order of the factors in the product.*

$$n := p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_r^{k_r}, \text{ where } \{k_i\}_{i=1}^r \text{ are positive integers and } \{p_i\}_{i=1}^r \text{ are primes.}$$

Q.7. Find the GCD of $3^3 \cdot 5^2 \cdot 7$ and $2^3 \cdot 3^2 \cdot 5 \cdot 7^2$.

Since, we already have the numbers present in the form of prime factors. We can make use of these to find the GCD of these two numbers.

We just take the intersection of the prime factors of the two numbers.

$$\begin{aligned} \gcd(3^3 \cdot 5^2 \cdot 7, 2^3 \cdot 3^2 \cdot 5 \cdot 7^2) &= 2^{\min(3,0)} \cdot 3^{\min(3,2)} \cdot 5^{\min(2,1)} \cdot 7^{\min(1,2)} \\ &= 2^0 \cdot 3^2 \cdot 5 \cdot 7 \\ &= 315 \end{aligned}$$

The GCD of the two numbers is $\boxed{315}$. **(Ans.)**

5.6 Theory of Congruences

Let n be a fixed positive integer. Two integers a and b are said to be *congruent modulo* n iff $n|(a-b)$ and is written as $a \equiv b \pmod n$.

5.6.1 Properties of Congruence

- $a \equiv a \pmod n$ (Reflexitvity)
- $a \equiv b \pmod n \implies b \equiv a \pmod n$ (Symmetricity)
- $a \equiv b \pmod n \land b \equiv c \pmod n \implies a \equiv c \pmod n$ (Transitivity)

- $a \equiv b \pmod{n} \wedge c \equiv d \pmod{n} \implies a + c \equiv b + d \pmod{n}$
  Corollary 3. $a \equiv b \pmod{n} \implies a + c \equiv b + c \pmod{n}$
- $a \equiv b \pmod{n} \wedge c \equiv d \pmod{n} \implies ac \equiv bd \pmod{n}$
  Corollary 4. $a \equiv b \pmod{n} \implies ac \equiv bc \pmod{n}$
- $a \equiv b \pmod{n} \implies \forall k \in \mathbb{N} : a^k \equiv b^k \pmod{n}$

  Proof.

$$\begin{aligned} a \equiv b (\operatorname{mod} n) &\implies n | a - b \\ &\implies n | \, (a-b) \times \underbrace{\left(a^{k-1} + a^{k-2}b + \cdots + ab^{k-2} + b^{k-1}\right)}_{\in \mathbb{Z}} \\ &\implies n | \left(a^k - b^k\right) \\ \therefore a \equiv b (\operatorname{mod} n) &\implies a^k \equiv b^k (\operatorname{mod} n) \end{aligned}$$

□

Q.8. Show that $41 | 2^{40} - 1$.

To show that $41 | 2^{40} - 1$, we must show that $2^{40} \equiv 1 \pmod{41}$.

$$\begin{aligned} \text{Consider, } 2^{10} &\equiv 1024 \pmod{41} \\ \therefore 2^{10} &\equiv 40 \pmod{41} \\ \therefore 2^{10} &\equiv -1 \pmod{41} \\ \therefore 2^{20} &\equiv (-1)^2 \pmod{41} \\ \therefore 2^{20} &\equiv 1 \pmod{41} \\ \therefore 2^{40} &\equiv 1^2 \pmod{41} \\ \therefore 2^{40} &\equiv 1 \pmod{41} \end{aligned}$$

Taking the remainder of 1024/41

Thus, from the definition of the modular congruency, it follows that $\boxed{41 | 2^{40} - 1}$ **(Ans.)**

Q.9. Show that $97 | 2^{48} - 1$.

To show that $97 | 2^{48} - 1$, we must show that $2^{48} \equiv 1 \pmod{97}$.

$$\begin{aligned} \text{Consider, } 2^{12} &\equiv 4096 \pmod{97} \\ \therefore 2^{12} &\equiv 22 \pmod{97} \\ \therefore 2^{24} &\equiv 22^2 \pmod{97} \\ \therefore 2^{24} &\equiv 484 \pmod{97} \\ \therefore 2^{24} &\equiv 96 \pmod{97} \\ \therefore 2^{24} &\equiv -1 \pmod{97} \\ \therefore 2^{48} &\equiv (-1)^2 \pmod{97} \\ \therefore 2^{48} &\equiv 1 \pmod{97} \end{aligned}$$

Taking the remainder of 4096/97

Taking the remainder of 484/97

Thus, from the definition of the modular congruency, it follows that $\boxed{97 | 2^{48} - 1}$ **(Ans.)**

Q.10. Show that $89 | 2^{44} - 1$.

To show that $89|2^{44}-1$, we must show that $2^{44} \equiv 1 \pmod{89}$.

$$\begin{aligned} \text{Consider, } 2^{11} &\equiv 2048 \pmod{89} \\ \therefore 2^{11} &\equiv 1 \pmod{89} \quad \textit{Taking the remainder of 2048/89} \\ \therefore 2^{44} &\equiv 1^4 \pmod{89} \\ \therefore 2^{44} &\equiv 1 \pmod{89} \end{aligned}$$

Thus, from the definition of the modular congruency, it follows that $\boxed{89|2^{44}-1}$ **(Ans.)**

5.6.2 Residue Classes

The *residue class* or *congruence class* of n is denoted as $[n]$ and is congruent to the set $\{0, 1, 2, \ldots, |n|-1\}$ modulo n.

5.6.3 Solution of Linear Congruences

$$ax \equiv b \pmod{n}$$

$$\begin{aligned} ax \equiv b \pmod{n} &\implies n|ax-b \\ &\implies ax-b = ny \qquad \text{For some } y \in \mathbb{N} \\ &\implies ax-ny = b \end{aligned}$$

Now, we must just find a solution to the above linear diophantine equation and the value of x is our required answer.

Q.11. Solve:

(i) $25x \equiv 15 \pmod{29}$

We need to find the solution of the linear Diophantine Equation $25x - 29y = 15$.

By Euclidean Algorithm,

$$\begin{aligned} 29 &= 1 \times 25 + 4 \\ 25 &= 6 \times 4 + \boxed{1} \\ 4 &= 4 \times \boxed{1} \end{aligned}$$

Therefore, $\gcd(25, 29) = 1$.

Expressing the GCD as a linear combination of a and b.

$$\begin{aligned} 1 &= 25 - 4(6) \\ &= 25 - (29-25)(6) \\ \therefore 1 &= 25(7) - 29(6) \quad \textit{Multiply by 15} \\ \therefore 15 &= 25(105) - 29(90) \end{aligned}$$

$x_0 = 105$ is a solution of the above equation.

To find a solution in the complete residue class of 29. Consider,

$$\begin{aligned}
x &= x_0 + \frac{b}{d} \cdot t & x &\in [\![0, n-1]\!] \wedge t \in \mathbb{Z} \\
&= 105 + \frac{-29}{1} \cdot t & x &\in [\![0, 28]\!] \\
&= 105 - 29t & x &\in [\![0, 28]\!]
\end{aligned}$$

Therefore we say,

$$\begin{aligned}
&0 \le 105 - 29t \le 28 \\
&\therefore -105 \le -29t \le -77 \\
&\therefore \frac{105}{29} \ge t \ge \frac{77}{29} \\
&\therefore 2.655 \le t \le 3.620 \\
&\therefore t = 3
\end{aligned}$$

We now substitute, $t = 3$ in our above equation for x.

$$\begin{aligned}
x &= 105 - 29(3) \\
&= 105 - 87 \\
\therefore x &= 18
\end{aligned}$$

We can easily verify that the 25×18 is congruent to $15 \pmod{29}$ and that the answer is $\boxed{x = 18}$. **(Ans.)**

(ii) $6x \equiv 15 \pmod{21}$

We need to find the solution of the linear Diophantine Equation $6x - 21y = 15$.

By Euclidean Algorithm,

$$\begin{aligned}
21 &= 3 \times 6 + \boxed{3} \\
6 &= 2 \times \boxed{3}
\end{aligned}$$

Therefore, $\gcd(6, 21) = 3$.

Expressing the GCD as a linear combination of a and b.

$$\begin{aligned}
3 &= 21 - 6(3) \\
15 &= 21(5) - 6(15) \\
15 &= 6(-15) - 21(-5)
\end{aligned}$$

Multiply by 15

$x_0 = -15$ is a solution of the above equation.

To find a solution in the complete residue class of 21. Consider,

$$
\begin{aligned}
x &= x_0 + \frac{b}{d} \cdot t && x \in [\![0, n-1]\!] \land t \in \mathbb{Z} \\
&= -15 + \frac{-21}{3} \cdot t && x \in [\![0, 20]\!] \\
&= -15 - 7t && x \in [\![0, 20]\!]
\end{aligned}
$$

Therefore we say,

$$
\begin{aligned}
&0 \leq -15 - 7t \leq 20 \\
&\therefore 15 \leq -7t \leq 35 \\
&\therefore \frac{15}{-7} \geq t \geq \frac{35}{-7} \\
&\therefore -5 \leq t \leq -2.142 \\
&\therefore t \in \{-5, -4, -3\}
\end{aligned}
$$

We now substitute, $t = -5, -4$ and -3 in our above equation for x.

$$
\begin{aligned}
x &= -15 - 7(-5) \\
&= -15 + 35 \\
\therefore x &= 20
\end{aligned}
$$

$$
\begin{aligned}
x &= -15 - 7(-4) \\
&= -15 + 28 \\
\therefore x &= 13
\end{aligned}
$$

$$
\begin{aligned}
x &= -15 - 7(-3) \\
&= -15 + 21 \\
\therefore x &= 6
\end{aligned}
$$

We can easily verify that 6×6, 6×13 and 6×20 are congruent to $15 \pmod{21}$. Thus, the answer is $\boxed{x \in \{6, 13, 20\}}$. **(Ans.)**

5.6.4 Chinese Remainder Theorem (CRT)

The Chinese Remainder Theorem (CRT) helps us to solve systems of linear equations of the form

$$
\begin{aligned}
&x \equiv b_1 \pmod{n_1} \\
&x \equiv b_2 \pmod{n_2} \\
&\ \vdots \\
&x \equiv b_r \pmod{n_r} \\
&\text{where } \forall i, j;\ i \neq j;\ \gcd(n_i, n_j) = 1
\end{aligned}
$$

There exists a solution of the form $\boxed{\overline{x} = b_1 N_1 x_1 + b_2 N_2 x_2 + \cdots + b_r N_r x_r}$ where

$$N_i = \frac{N}{n_i} \qquad\qquad N = n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i$$

and x_i is the inverse of $N_i \pmod{n_i}$ i.e., $N_i x_i \equiv 1 \pmod{n_i}$

which is unique modulo the integer $n_1 n_2 n_3 \cdots n_r$

Q.12. Solve the system: $x \equiv 2 \pmod 3$, $x \equiv 4 \pmod 5$, $x \equiv 2 \pmod 7$.

The pairwise GCDs of $3, 5$ and 7 are all 1. Thus we can apply the CRT.

There exists a solution of the form $\overline{x} = b_1 N_1 x_1 + b_2 N_2 x_2 + \cdots + b_r N_r x_r$. The value of N is $3 \times 5 \times 7 = 105$. Thus, the value of N_i are $N_1 = {}^{105}/_{3} = 35$, $N_2 = {}^{105}/_{5} = 21$ and $N_3 = {}^{105}/_{7} = 15$.

Now let us find the values of x_i,

$$\begin{aligned} N_1 x_1 &\equiv 1 \pmod{n_1} & N_2 x_2 &\equiv 1 \pmod{n_2} & N_3 x_3 &\equiv 1 \pmod{n_3} \\ \therefore 35 x_1 &\equiv 1 \pmod 3 & \therefore 21 x_2 &\equiv 1 \pmod 5 & \therefore 15 x_3 &\equiv 1 \pmod 7 \\ \therefore x_1 &\equiv 2 \pmod 3 & \therefore x_2 &\equiv 1 \pmod 5 & \therefore x_3 &\equiv 1 (\bmod 7) \end{aligned}$$

Now the solution

$$\begin{aligned} \overline{x} &= b_1 N_1 x_1 + b_2 N_2 x_2 + b_3 N_3 x_3 \\ &= (2 \cdot 35 \cdot 2) + (4 \cdot 21 \cdot 1) + (2 \cdot 15 \cdot 1) \\ &= 140 + 84 + 30 \\ \overline{x} &= 254 \end{aligned}$$

Thus, the solution of the above system of linear congruences is $\overline{x} = 254 \pmod{105}$ or $\boxed{\overline{x} \equiv 44 \pmod{105}}$. **(Ans.)**

5.6.5 System of Linear Congruences

Consider a set of linear equations,

$$\begin{aligned} a_1 x &\equiv b_1 \pmod{n_1} \\ a_2 x &\equiv b_2 \pmod{n_2} \\ &\vdots \\ a_r x &\equiv b_r \pmod{n_r} \end{aligned}$$

We can solve this system of linear congruency as follows,

1. First solve the independent linear congruency and find their solutions congruent modulo their respective n_i.

2. We will receive the system of linear congruency as follows:

$$\begin{aligned} x &\equiv B_1 \pmod{n_1} \\ x &\equiv B_2 \pmod{n_2} \\ &\vdots \\ x &\equiv B_r \pmod{n_r} \end{aligned}$$

3. Now we can make use of the CRT to evaluate this system of linear congruency.

5.6.6 Linear Congruences in Two Variables

Consider the system of linear congruency as follows:

$$\begin{aligned} ax + by &\equiv r \pmod{n} \\ cx + dy &\equiv s \pmod{n} \end{aligned}$$

It has a unique solution if $\gcd(ad - bc, n) = 1$.

Q.13. Find the solution of the system of linear congruences,

$$2x + 3y \equiv 1 \pmod{7} \qquad (1)$$
$$5x + 9y \equiv 3 \pmod{7} \qquad (2)$$

Since, the GCD of $ad - bc = 2 \times 9 - 3 \times 5 = 18 - 15 = 3$ and $n = 7$ is 1. There exists a unique solution to this system of linear equations.

Multiplying (1) by 3: $6x + 9y \equiv 3 \pmod{7}$ (3)

Subtracting (2) from (3):

$$\begin{aligned} (6x + 9y) - (5x + 9y) &\equiv (3 - 3) \pmod{7} \\ \therefore x &\equiv 0 \pmod{7} \end{aligned}$$

Now substituting this value in (1). Thus,

$$\begin{aligned} 7(0) + 3y &\equiv 1 \pmod{7} \\ \therefore 3y &\equiv 1 \pmod{7} \\ \therefore y &\equiv 5 \pmod{7} \end{aligned}$$

Q.14. Prove that the numbers of the sequence $1, 11, 111, 1111, \ldots$ are not perfect squaares.

Each integer of the sequence is of the form $n \equiv 3 \pmod{4}$ i.e. $n = 4k + 3$ where $k \in \mathbb{Z}$.

The relation congruent modulo 4 partitions the set of integers into the 4 equivalence classes $\{n : n = 4k,\ k \in \mathbb{Z}\}$, $\{n : n = 4k + 1,\ k \in \mathbb{Z}\}$, $\{n : n = 4k + 2,\ k \in \mathbb{Z}\}$ and $\{n : n = 4k + 3,\ k \in \mathbb{Z}\}$.

Consider the squares of all these forms,

$$\begin{aligned}
n^2 &= (4k)^2 = 16k^2 = 4(4k^2) = 4c_0 && c_0 \in \mathbb{Z} \\
n^2 &= (4k+1)^2 = 16k^2 + 8k + 1 = 4(4k^2 + 2k) + 1 = 4c_1 + 1 && c_1 \in \mathbb{Z} \\
n^2 &= (4k+2)^2 = 16k^2 + 16k + 4 = 4(4k^2 + 4k + 1) = 4c_2 && c_2 \in \mathbb{Z} \\
n^2 &= (4k+3)^2 = 16k^2 + 24k + 9 = 4(4k^2 + 4k + 2) + 1 = 4c_3 + 1 && c_3 \in \mathbb{Z}
\end{aligned}$$

None of the squares of integers are congruent to 3 modulo 4. Hence, we can say that no elements of the sequence are perfect squares.

Acronyms

CR Cauchy-Riemann 59–64, 71

CRT Chinese Remainder Theorem 93–95

GCD Greatest Common Divisor 86, 88, 89, 91, 92, 94, 95

HCF Highest Common Factor 86

Jojo Aaditya Joil 1

LCM Least Common Multiple 86

LFT Linear Fractional Transformation 71–73

MCE Mathematics for Computer Engineers 1

NLA Numerical Linear Algebra 5

QF Quadratic Form 22, 23, 25, 26

RRG Rupak R. Gupta 1

$$\boxed{\begin{bmatrix} \mathbb{R} & \Re \\ \mathscr{g} & \infty \end{bmatrix} \wedge \begin{pmatrix} \mathscr{j} & \mathscr{j} \end{pmatrix} \cdot \begin{pmatrix} o \\ o \end{pmatrix}}$$

The End.