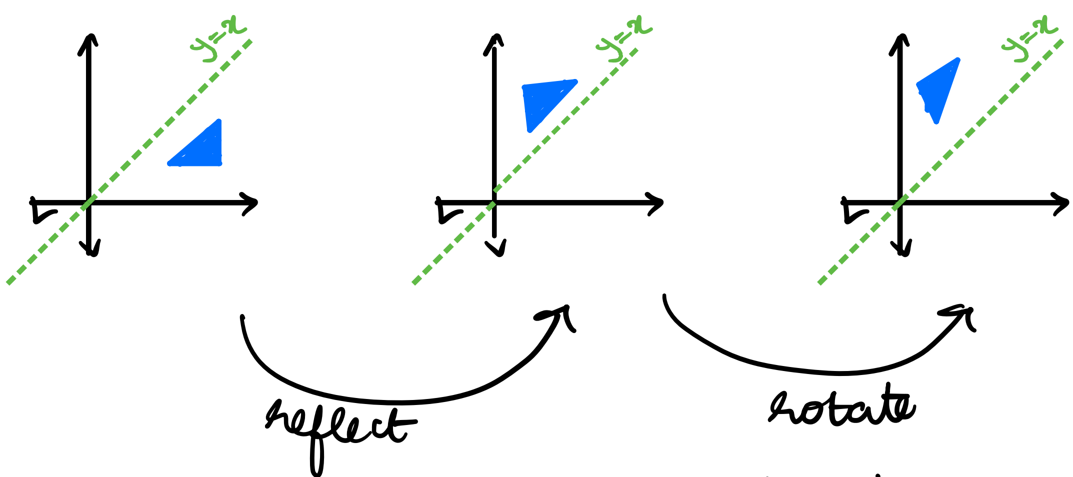
LECTURE-14

Q. What happens when we want to do two (or more) transformations, one after another?

Example: Suppose we want to reflect across the line $y=x$, then rotate by 30°.



Then any point (a,b) in the triangle is changed twice:

First, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$ results in reflection to get $\begin{pmatrix} b \\ a \end{pmatrix}$

Second, $\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} b \\ a \end{pmatrix}$ results in rotation of the reflected point to give $\begin{pmatrix} \frac{\sqrt{3}}{2}b - \frac{a}{2} \\ \frac{b}{2} + \frac{\sqrt{3}a}{2} \end{pmatrix}$.

This can be achieved using one matrix, which is the product of the two matrices corresponding to reflection & rotation, (in that order! Remember, matrix multiplication is NOT commutative in general)

So we have the combined transformation as

$$\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{a}{2} + \frac{\sqrt{3}b}{2} \\ \frac{\sqrt{3}a}{2} + \frac{b}{2} \end{pmatrix}.$$

To repeat, the matrix of the transformation that performs (i) reflection across the line $y = x$

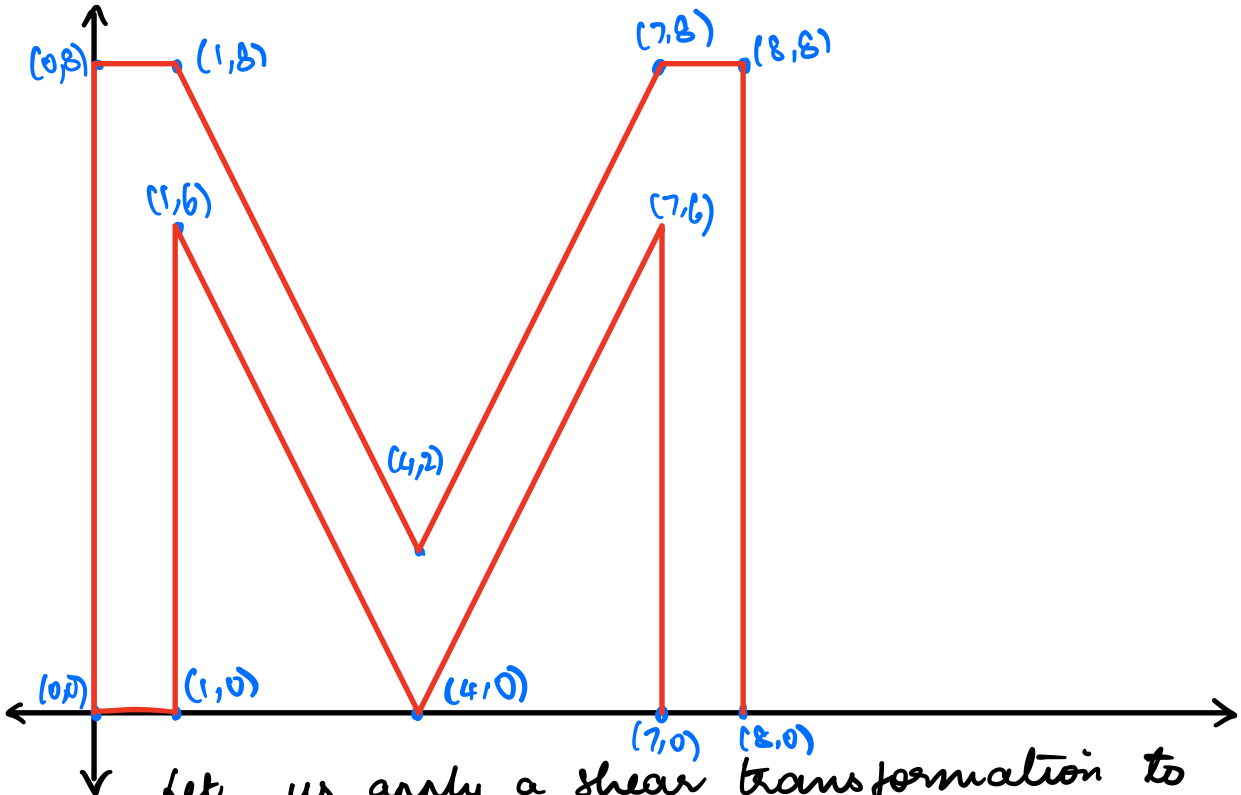
(ii) rotation by 30°

in this order, is:

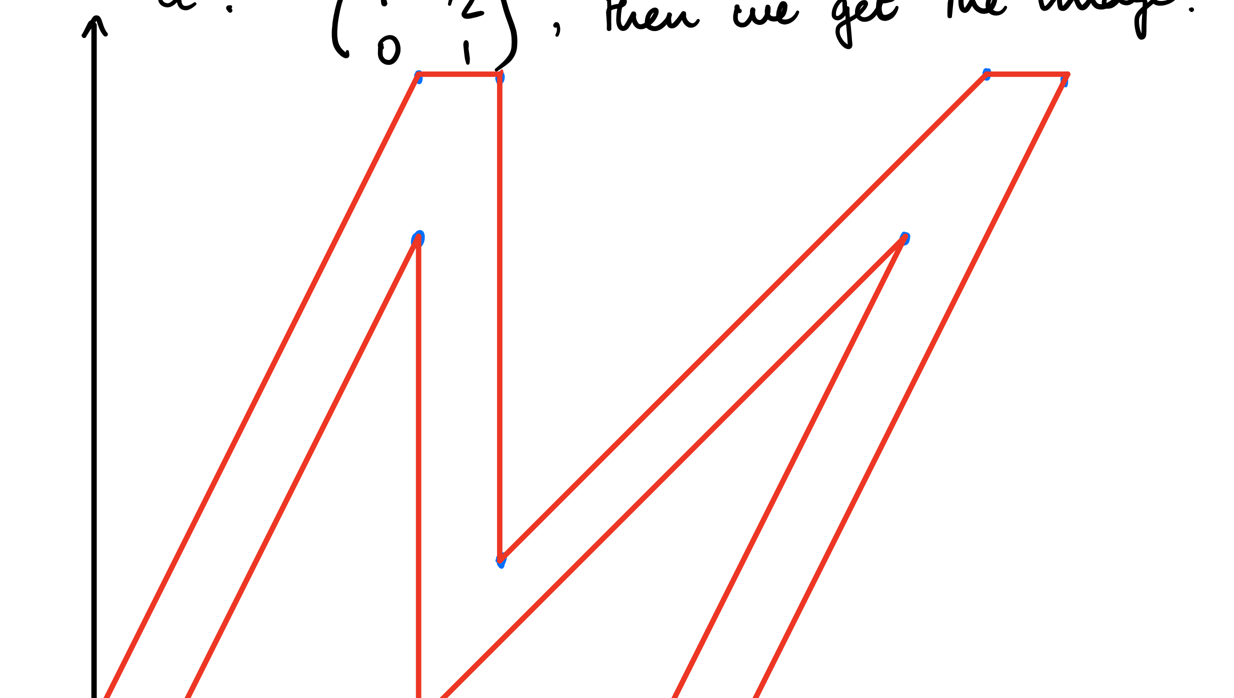
$$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

(note that doing rotation by 30° first, followed by reflection across $y = x$ results in the matrix $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$, which is different from the above matrix!)

Consider the letter M in the diagram:



Let us apply a shear transformation to it: $\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$, then we get the image:



and we obtain the new coordinates by using the shear matrix:

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 4 & 7 & 7 & 8 & 8 & 7 & 4 & 1 & 0 \\ 0 & 0 & 6 & 0 & 6 & 0 & 0 & 8 & 8 & 2 & 8 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 4 & 4 & 10 & 7 & 8 & 12 & 11 & 5 & 5 & 4 \\ 0 & 0 & 6 & 0 & 6 & 0 & 0 & 8 & 8 & 2 & 8 & 8 \end{pmatrix}$$

↳ coordinates of M

Other operations such as rotations, reflections and shearing with a different value of k can be performed. But as we observed earlier, no 2×2 matrix will give us the translation

$$(x, y) \mapsto (x+h_1, y+h_2).$$

To overcome this difficulty, we introduce homogeneous coordinates.

For every point $(x_1, x_2) \in \mathbb{R}^2$, we identify it with the point $(x_1, x_2, 1) \in \mathbb{R}^3$

So in order to translate a point

(x_1, x_2) to $(x_1, x_2) + (h_1, h_2)$

we attempt to find a 3×3 matrix

A^* such that

$$A^* \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 + h_1 \\ x_2 + h_2 \\ 1 \end{pmatrix}.$$

and that

$$A^* = \begin{pmatrix} 1 & 0 & h_1 \\ 0 & 1 & h_2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{works.}$$

<u>Remark</u> Consider the transformation

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

given by a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

i.e., $A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Under homogeneous coordinates,
the image of $(x_1, x_2, 1)$ is $(y_1, y_2, 1)$
and note that

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix}.$$

In other words, we can use homogeneous coordinates to study <u>all</u> the remaining matrix transformations from $\mathbb{R}^2 \to \mathbb{R}^2$.

We just replace the 2×2 matrix A by

$$A^* = \left(\begin{array}{c:c} A & 0 \\ \hdashline 0 & 1 \end{array}\right)$$

So the shear example of the letter M in homogeneous coordinates is as follows:

i) The picture M in matrix form is:

$$\begin{pmatrix} 0 & 1 & 1 & 4 & 7 & 7 & 8 & 8 & 7 & 4 & 1 & 0 \\ 0 & 0 & 6 & 0 & 6 & 0 & 0 & 8 & 8 & 2 & 8 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

extra row → consisting of 1's

ii) The 2×2 matrix $A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$

is replaced by the **3 × 3 matrix**

$$A^* = \left(\begin{array}{cc:c} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right)$$

Note that

$$A^* \begin{pmatrix} 0 & 1 & 4 & 4 & 7 & 7 & 8 & 8 & 7 & 4 & 1 & 0 \\ 0 & 0 & 6 & 0 & 6 & 0 & 0 & 8 & 8 & 2 & 8 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 4 & 7 & 7 & 8 & 8 & 7 & 4 & 1 & 0 \\ 0 & 0 & 6 & 0 & 6 & 0 & 0 & 8 & 8 & 2 & 8 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 3 & 10 & 7 & 8 & 12 & 11 & 5 & 5 & 4 \\ 0 & 0 & 6 & 6 & 0 & 0 & 8 & 8 & 2 & 8 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

↳ (homogeneous) coordinates of M. We get back original coordinates by deleting the row of 1's.

Now let us consider a translation by the vector $(2,3)$.

The matrix under homogeneous coordinates for this translation is given by

$$B^* = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$B^* A^* \begin{pmatrix} 0 & 1 & 1 & 4 & 7 & 7 & 8 & 8 & 7 & 4 & 1 & 0 \\ 0 & 0 & 6 & 0 & 6 & 0 & 0 & 8 & 8 & 2 & 8 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 4 & 10 & 7 & 8 & 12 & 11 & 5 & 5 & 4 \\ 0 & 0 & 6 & 6 & 0 & 0 & 8 & 8 & 2 & 8 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

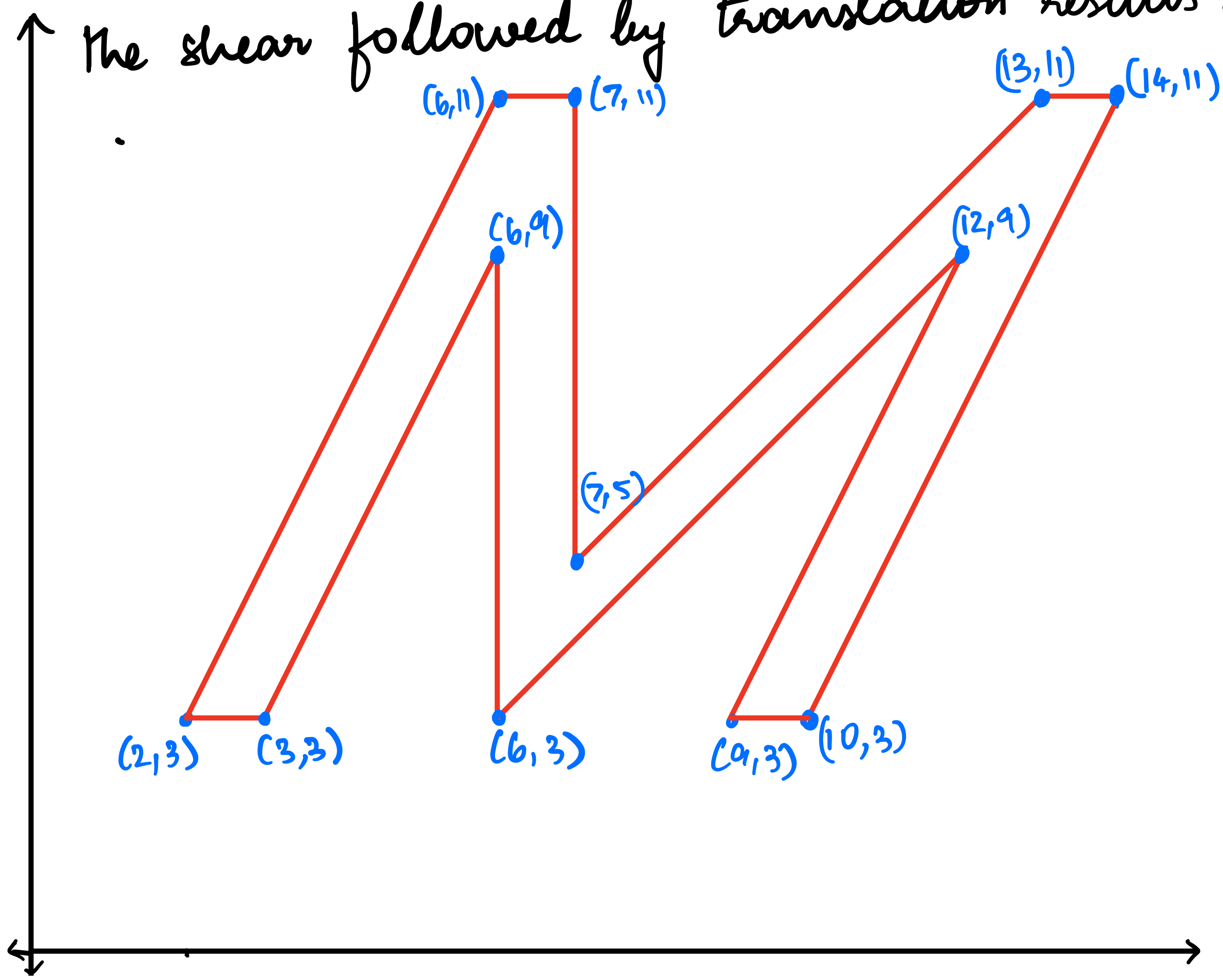
$$= \begin{pmatrix} 2 & 3 & 6 & 12 & 9 & 10 & 14 & 13 & 7 & 7 & 6 \\ 3 & 3 & 9 & 9 & 3 & 3 & 11 & 11 & 5 & 11 & 11 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

giving rise to coordinates in $\mathbb{R}^2$,

displayed as an array

$$\begin{pmatrix} 2 & 3 & 6 & 6 & 12 & 9 & 10 & 14 & 13 & 7 & 7 & 6 \\ 3 & 3 & 9 & 3 & 9 & 3 & 3 & 11 & 11 & 5 & 11 & 11 \end{pmatrix}$$

Hence, the image of the letter M under the shear followed by translation results in .



<u>Example</u>: Under homogeneous coordinates, the transformation representing the foll. sequence of transformations:

a) reflection across the x-axis

b) shear by factor 2 in the direction of the x-axis

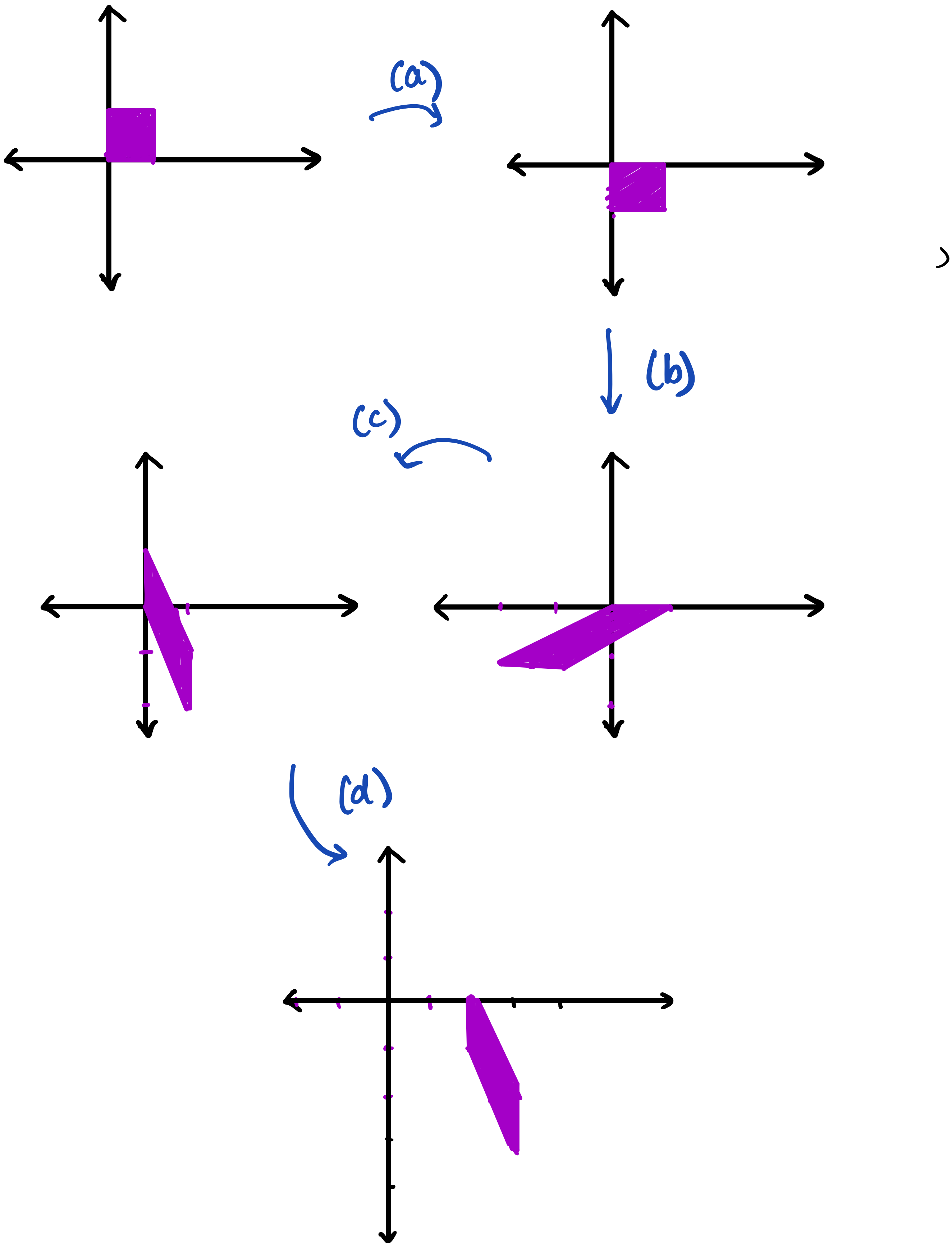
c) anticlockwise rotation by 90°

d) translation by vector $(2,-1)$

has the matrix:

$$\underset{(d)}{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}} \underset{(c)}{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{(b)}{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{(a)}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 0 & 1 & 2 \\ 1 & -2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$


(a)
(b)
(c)
(d)