Final remarks on invertibility of a square matrix:

<u>The following are equivalent statements:</u>

a) A is an invertible $n \times n$ matrix.

b) $\exists$ an $n \times n$ matrix B such that $AB = BA = I$.

c) A is row equivalent to the $n \times n$ identity matrix

d) $\det A \neq 0$

e) The columns of A form a linearly independent set.

f) The columns of A span $\mathbb{R}^n$.

g) The rank of A is n.

<u>Definition</u> : Given an $n \times n$ matrix A, a number $\lambda \in \mathbb{C}$ such that there is an $n \times 1$ non-zero vector $x \in \mathbb{C}^n$ satisfying

$$(A - \lambda I)x = 0$$

is called an <u>eigenvalue</u> of the matrix A, and the vector x is called its associated <u>eigenvector</u>.

<u>Note</u> : 1) This means that λ is an eigenvalue iff the nullspace of $A - \lambda I$ contains vectors other than the zero vector.

i.e., $\det(A - \lambda I) = 0$.

2) The equation obtained on expanding $\det(A - \lambda I) = 0$ is called the characteristic polynomial of A, and the roots of this polynomial are the eigenvalues.

<u>Summarizing</u>:

a) Compute the determinant of $A - \lambda I$.

b) Find the roots of this polynomial.

c) For each eigenvalue obtained in b), solve $(A - \lambda I)x = 0$. Since the determinant of $A - \lambda I$ is zero, there are solutions other than $x = 0$.

Let us start with a simple example:

1) $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$, so $A - \lambda I = \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix}$

$$\det(A - \lambda I) = (4-\lambda)(-3-\lambda) + 10$$
$$= \lambda^2 - \lambda - 2.$$

$\Rightarrow \lambda_1 = 2$, $\lambda_2 = -1$ are the roots.

- Eigenvector for $\lambda_1 = 2$:

$$(A - \lambda_2 I)\, x = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is an eigenvector.

- Eigenvector for $\lambda_2 = -1$:

$$(A - \lambda_1 I)\, x = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector.

Example 2 : Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Solution Step 1 : Find the eigenvalues of A.

solve : $\det(A - \lambda I) = 0$ to get the

char. poly. as $-\lambda^3 - \lambda^2 + 12\lambda = 0$

$$\Rightarrow -\lambda(\lambda+4)(\lambda-3) = 0$$

roots : $\lambda_1 = 0$, $\lambda_2 = -4$, $\lambda_3 = 3$.

Step 2: Find eigenvectors :

(i) Eigenvector v_1 for $\lambda_1 = 0$:

Solve $(A - \lambda_1 I)v_1 = 0$:

$$\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 0 \quad - (1)$$

$$6x_1 - x_2 \qquad = 0 \quad - (2)$$

$(2) \Rightarrow x_2 = 6x_1$

$\therefore$ (1) becomes $13x_1 + x_3 = 0$

$\Rightarrow x_3 = -13x_1$

$$\therefore \quad v_1 = c_1 \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}.$$

(ii) Eigenvector v_2 for $\lambda_2 = -4$ can be calculated using the same process.

$$v_2 = c_2 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

iii) Eigenvector v_3 for $\lambda_3 = 3$ can be found to be $v_3 = c_3 \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$.

Eigenvalues in special cases:

1. Diagonal matrices: The eigenvalues are the elements on the diagonal.

2. Triangular matrices: same as 1.

Two facts about eigenvalues:

1. Trace of A is the sum of its eigenvalues.
2. Determinant of A is the product of its eigenvalues.

Example 3: Let $A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

Find eigenvalues and eigenvectors of A.

Solution.

Start by computing the characteristic polynomial of A:

$$\begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$= (5-\lambda)(5-\lambda)(2-\lambda) - 4\left(4(2-\lambda)-4\right) + 2\left[8-2(5-\lambda)\right]$$

$$= (5-\lambda)\left[10-7\lambda+\lambda^2-4\right] - 4(4-4\lambda) + 2(-2+2\lambda)$$

$$= (5-\lambda)(\lambda^2-7\lambda+6) + 16(\lambda-1) + 4(\lambda-1)$$

$$= (5-\lambda)(\lambda-1)(\lambda-6) + 16(\lambda-1) + 4(\lambda-1)$$

$$= (\lambda-1)\left((5-\lambda)(\lambda-6)+16+4\right)$$

$$= (\lambda-1)(-\lambda^2+11\lambda-10)$$

$$= -(\lambda-1)(\lambda-1)(\lambda-10).$$

Thus, the eigenvalues are $\lambda = 10, 1, 1$.