# Linear Algebra

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# Lecture-1

What is linear algebra? You might have heard things like "calculus and linear algebra are the two fundamental pillars of mathematics"

You have had some exposure to both these, if you have taken math in $11^{th}$ and $12^{th}$.

Linear algebra is the branch of mathematics that deals with things like:

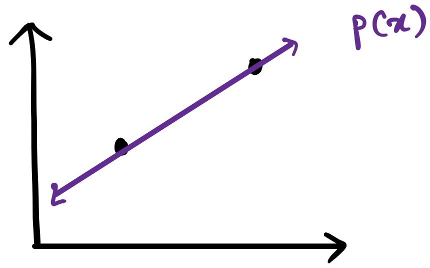
- solving a system of linear equations
- representing data using matrices and 'vectors', and manipulating them towards solving a problem at hand
- creating linear models to work with data - using "linear transformations"

A wide variety of applications come out of concepts in linear algebra ranging from image processing to oil exploration and electrical networks.

There are some interesting theoretical applications too!

- → Polynomial interpolation
- → Fibonacci sequence
- → Powers of matrices

* Polynomial Interpolation:




$n = 2$

Given $n$ distinct values $x_1, x_2, \dots, x_n$ and $n$ values $y_1, y_2, \dots, y_n$, there is a unique polynomial $P$ of degree less than $n$, satisfying

$$P(x_1) = y_1$$
$$P(x_2) = y_2$$
$$\vdots$$
$$P(x_n) = y_n.$$

* Fibonacci numbers:

$F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad \dots$

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 3.$$

What is the $67^{th}$ term in the sequence?

* Approximate solutions to inconsistent linear systems: Method of least squares.

* Powers of matrices: $A$ is a $1000 \times 1000$ matrix. Can we compute $A^{2000}$ efficiently?

Fancier applications:

* Image processing: uses singular value decomposition (SVD)

* Google's search algorithm

# Syllabus for the course

→ Gaussian elimination, echelon forms, system of linear equations, applications. Inverse of a matrix

→ Vector spaces, subspaces, linear independence, basis, dimension, rank & nullity, linear transformations

→ Inner product spaces, Orthogonal vectors and subspaces, projections, least-square approximations, Gram-Schmidt orthonormalization

→ Eigenvalues, eigenvectors, diagonalization, Numerical linear algebra

→ SVD, matrix norms, condition numbers, iterative methods.

Books: 1. Gillbert Strang, "Linear Algebra and its applications".

2. David Lay, Steven Lay, Judi Mcdonald, "Linear Algebra and its applications."

# Linear Systems

At a fundamental level, much of mathematics is about solving equations.

The simplest kind is linear equations.

This is an equation of the form:

coefficients

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

variables

in general we have

eg. 1) $3x + 4y = 7$

$2x - y = 33$

2) $a + b + c = 1$

Here $a_1, a_2, \ldots, a_n, b$ can be real or complex numbers.

A linear system is a collection of one or more such equations.

eg.

$$2x + 3y + 4z = 5$$
$$x - 2y + \frac{3}{8}z = -1$$

is a system of two equations in three variables.

A solution of a linear system is a list which when substituted in the place of variables makes all the equations true.

A linear system can have more than one solution, or no solutions or a unique solution.

$2x + 3y = 1$
$-x - y = 2$

The set of all possible solutions is called a solution set of the linear system

$x + y = 3$

example!

can you write this as a set?

Moreover, two linear systems are said to be equivalent if they have the same solution set.

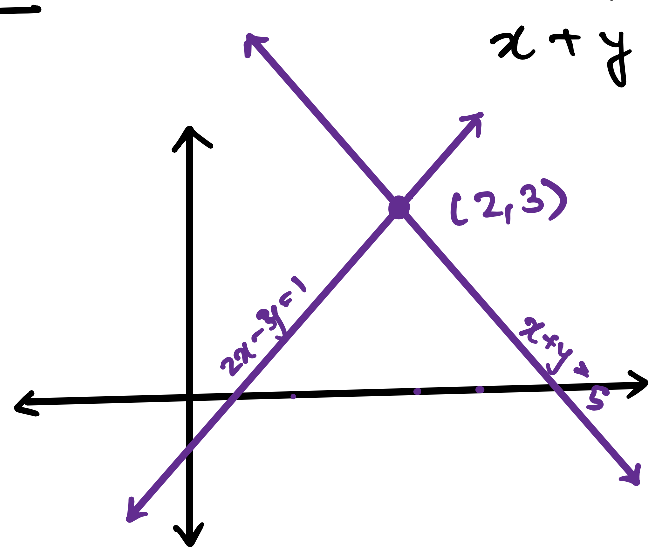
A system of linear equations is said to be <u>consistent</u> if it has either one solution or infinitely many solutions, and <u>inconsistent</u> if it has no solutions.

→ How about $>1$ but $<\infty$ ?

The fact that you cannot have say 3 solutions becomes evident if we think a bit geometrically:

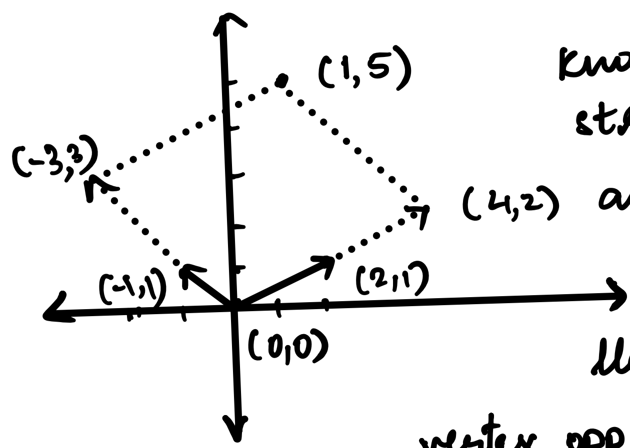
<u>Example</u>: Solve: $2x - y = 1$

$x + y = 5$.



Alternatively, we can rewrite the equns as:

$$x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$$




So we want to know how much to stretch the vectors $\binom{2}{1}$ and $(-1,1)$ so that they complete the parallelogram with the vertex opp. the origin being $(1,5)$.

## Solving a linear system

Which system is easier to solve?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \quad\quad - 5x_3 &= 10 \end{aligned} \quad \text{vs.} \quad \begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned}$$

(I) (II)

Clearly, (II) is easier to solve. Why?