SOLUTIONS

Problem 1

a) Row-echelon

b) Reduced row echelon

c) Neither. The "1" in the (4,1) position violates the rule that there should be only zeros below leading entries. This is a non-zero entry in the first column below the leading entry in Row 1.

d) Row-echelon

e) Row-echelon

f) Reduced Row-echelon

g) Neither. The "1" in the (2,1) position appears in the column below the leading entry in Row 1.

One way to do this:

<u>Problem 2</u>

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

$R_2 \mapsto R_2 - 4R_1$
$R_3 \mapsto R_3 - 6R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix}$$

$R_2 \mapsto \frac{-1}{3} R_2$
$R_3 \mapsto \frac{-1}{5} R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$R_3 \mapsto R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \mapsto R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem3

1) $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{bmatrix}$

First, we convert this to REF:

$R_2 \to R_2 - 3R_1$ $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

This corresponds to the system:

$$x_1 - 2x_2 - x_3 = 3 \quad — ①$$

$$x_3 = -7 \quad — ②$$

Substituting ② in ①:

$$x_1 - 2x_2 + 7 = 3$$

$$\therefore x_1 = -4 + 2x_2$$

Therefore, the system has infinitely

many solutions given by:

$$x_1 = -4 + 2t$$

$$x_2 = t \qquad \text{for any } t \in \mathbb{R}.$$

$$x_3 = -7$$

2) $$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$$

Converting this to REF:

$R_3 \mapsto R_3 + R_1$: $$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix}$$

$R_3 \mapsto R_3 + 4R_2$: $$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We now write the corresponding linear system:

$$x_1 - 7x_2 + 6x_4 = 5 \quad - ①$$

$$x_3 - 2x_4 = -3 \quad - ②$$

② gives: $x_3 = 2x_4 - 3$ (so x_4 is a free variable)

① gives $x_1 = 5 - 7x_2 + 6x_4$

(so x_2 is another free variable)

Thus, the set of solutions is:

$$x_1 = 5 - 7s + 6t$$

$$x_2 = s$$

$$x_3 = 2t - 3$$

$$x_4 = t$$

for any $s, t \in \mathbb{R}$.

Problem 4

The corresponding linear system is obtained by equating the number of atoms of C, H and O on both sides:

C: $\quad x_1 + x_2 = x_4$

O: $\quad x_1 + 2x_2 = x_5.$

H: $\quad 2x_3 = 2x_5 + 4x_4$

This leads to the matrix:

$$\begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -4 & -2 & 0 \end{bmatrix}$$

$R_2 \mapsto R_2 - R_1$ gives:
$R_3 \mapsto R_3/2$

$$\begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 \end{bmatrix},$$

which is in REF.

The associated linear system is:

$$x_1 + x_2 \quad - x_4 \quad = 0 \quad -①$$

$$x_2 \quad + x_4 - x_5 = 0 \quad -②$$

$$x_3 - 2x_4 - x_5 = 0 \quad -③$$

③ gives: $x_3 = 2x_4 + x_5$, so x_4 and x_5 are free variables.

② gives: $x_2 = -x_4 + x_5$

① gives $x_1 = -x_2 + x_4$

$$= -(-x_4 + x_5) + x_4$$

$$= 2x_4 - x_5$$

Let $x_4 = s$, $x_5 = t$.

Therefore
$$x_1 = 2s - t$$
$$x_2 = t - s$$
$$x_3 = 2s + t$$
$$x_4 = s$$
$$x_5 = t.$$
for any $s, t \in \mathbb{R}$.

Since x_1, x_2, x_3, x_4, x_5 represent no. of atoms, the values must be positive integers.

This gives $x_1 = 2s - t > 0$

and $x_2 = t - s > 0$

so we get $s < t < 2s$.

If we let $s = 1$. Then t needs to be between 1 and 2, which is impossible.

So this choice of s does NOT work.

Choose $s = 2$. Then we can take $t = 3$, since it lies between $s(=2)$ and $2s(=4)$.

Thus, one solution is

$$x_1 = 1, \quad x_2 = 1, \quad x_3 = 7, \quad x_4 = 2, \quad x_5 = 3.$$

Note: if s is any integer > 1, we get a valid solution, for example

$s = 3, \; t = 5$ or $s = 3, \; t = 4$.

$x_1 = 1$	$x_1 = 2$
$x_2 = 2$	$x_2 = 1$
$x_3 = 11$	$x_3 = 10$
$x_4 = 3$	$x_4 = 3$
$x_5 = 5$	$x_5 = 4$

Problem 5

We get the linear system:

at nodes

A: $50 + x_2 = 120 + x_1$

B: $150 + x_1 = 80 + x_3$

C: $x_4 = 100 + x_2$

D: $x_3 + 100 = x_4$

leading to the augmented matrix:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 70 \\ 1 & 0 & -1 & 0 & -70 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & -1 & 1 & 100 \end{bmatrix}$$

Convert this into REF:

$R_2 \mapsto R_2 + R_1$:

$R_3 \mapsto R_3 + R_2$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 70 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 100 \\ 0 & 0 & -1 & 1 & 100 \end{bmatrix}$$

$R_4 \mapsto R_4 + R_3$: $\begin{bmatrix} -1 & 1 & 0 & 0 & 70 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

leading to the equivalent system:

$$-x_1 + x_2 = 70 \quad - (1)$$
$$x_2 - x_3 = 0 \quad - (2)$$
$$-x_3 + x_4 = 100 \quad - (3)$$

(3) gives: $x_4 = 100 + x_3$, so x_3 is a free variable.

(2) gives $x_2 = x_3$

(1) gives $x_1 = x_2 - 70$
$= x_3 - 70$.

$\therefore$ Taking $x_3 = t$, we get

$x_1 = t - 70$, $x_2 = t$, $x_3 = t$, $x_4 = t + 100$.

Since $x_i \geq 0$, we can choose t to be any integer ≥ 70. Thus, $x_4 \geq 170$.

Problem 6

$$x_1 + h x_2 = 2$$

$$4x_1 + 8x_2 = k.$$

[sample solution. This is not the only set of correct answers.]

a) Let $h = 2$, $k = 0$.

Then, the equations become:

$$x_1 + 2x_2 = 2 \quad — ①$$

$$4x_1 + 8x_2 = 0.$$

Multiplying ① by 4 we get

$$4x_1 + 8x_2 = 8$$
$$4x_1 + 8x_2 = 0,$$

which clearly has no solution.

b) Let $h = 1$, $k = 12$.

The system becomes

$$x_1 + x_2 = 2$$
$$4x_1 + 8x_2 = 12.$$

Solving, $x_1 = 1, x_2 = 1$ is the only solution.

c) Let $h = 2$, $k = 8$.

The system becomes

$$x_1 + 2x_2 = 2 \quad -①$$
$$4x_1 + 8x_2 = 8 \quad -②$$

② is just $4 \times$ ①.

So we really just have one equation.

$$x_1 + 2x_2 = 2$$

Let $x_2 = t$. Then $x_1 = 2(1-t)$.

Therefore, this system has infinitely

many solutions given by

$$x_1 = 2(1-t)$$
$$x_2 = t$$

for any $t \in \mathbb{R}$.