<u>Recall</u> : Two linear systems are said to be equivalent if they have the same solution set.

Let us see some simple examples.

(A)	(B)	(C)
(i) $2x + 3y = 5$	(i) $2x + 3y = 5$	(i) $3x + 2y = 7$
(ii) $x - y = 2$	(ii) $-x + y = -2$	(ii) $x - y = 2$

→ First, direct computation shows that the three systems have the same solution.

Q. Is there a way we could have seen this coming *without* solving each one of them?

A. Yes! Observe that (A) and (B) must have the same solution because eqn (ii) of (A) and eqn (ii) of (B) are just multiples of each other, and multiplying an equation on both sides by a constant (in this case -1) does not change the solution set. Meanwhile, eqns (i) of (A) and (i) of (B) are identical.

What about (A) and (C)?
A little thinking will reveal that
eqn (i) of (C) = eqn (i) of (A) + eqn (ii) of (A).
Meanwhile eqn (ii) in both systems match.

Try to see for yourself that this process also does not change the solution set. Think of it like this: Adding the two eqns and replacing eqn (i) by this sum does not introduce any new condition on x and y.

So, taking (i) $2x+3y=5$

(ii) $x-y=2$

(iii) $3x+2y=7 \rightarrow$ (i) + (ii),

solving ANY two out of the (iii) equations gives us the same solution : $x=\frac{11}{5}$, $y=\frac{1}{5}$.

Why bother with equivalent linear systems?

Idea: Given a linear system, if we know of an equivalent linear system that is easier to solve, that would be an advantage! Recall the example from last time:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \quad - 5x_3 &= 10 \end{aligned} \qquad \text{vs.} \qquad \begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned}$$

(I) (II)

These two are equivalent (check by solving!), and (II) is easier to solve. What makes it easier? The "triangular" form of the coefficients. In general, we want to bring our matrices (arising from linear systems) into a form that makes it easy for say, a computer to be told how to solve. This brings us to...

<u>Defn</u>. A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows containing all zeros:

2. Each leading entry of a row (i.e., first non-zero entry starting from the left) is in a column to the right of the leading entry of the row above it.

3. All entries in a column below a leading entry are zeros.

Exercise: Which of the following matrices are in REF? (Row Echelon Form)

(1) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ ✗

(2) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$ ✗

(3) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

(4) $\begin{bmatrix} 3 & 4 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ✗

(5) $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 4 & 1 \\ 1 & 0 & 0 & 4 \end{bmatrix}$ ✗

(6) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ ✗

(7) $\begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$ ✗

(8) $\begin{bmatrix} 4 & 5 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$ ✓

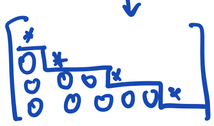
(9) $\begin{bmatrix} 1 & -1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ ✓

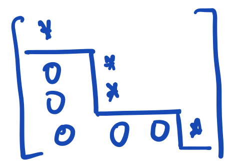
(10) $\begin{bmatrix} 2 & 2 & 1 & 2 & 1 \\ 0 & 0 & 3 & 3 & 1 \\ 0 & 2 & 0 & 0 & 1 \end{bmatrix}$ ✗

(11) $\begin{bmatrix} 0 & 11 & 3 & 1 & 1 & 2 & 4 & 6 \\ 0 & 0 & 7 & 2 & 6 & 3 & 5 & 6 \\ 0 & 0 & 0 & 3 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 6 \end{bmatrix}$

In each case that is NOT in REF, can you identify which rule is violated?

Tip: To see if a matrix is in REF, we look for a "step" pattern—where steps can be "wide", but cannot be "tall".





If a matrix in echelon form satisfies the following additional properties, then it is in <u>reduced row echelon form.</u> (RREF)

4. The leading entry in each nonzero row is 1
5. Each leading 1 is the only non-zero entry in its column.

Example :

$$\begin{bmatrix} 0 & 11 & 3 & 1 & 1 & 2 & 4 & 6 \\ 0 & 0 & 7 & 2 & 6 & 3 & 5 & 6 \\ 0 & 0 & 0 & 3 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 6 \end{bmatrix}$$

(Row-echelon form)

Reduced row-echelon form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 0 & 6 & 3 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

Every matrix can be brought to REF or RREF. How? Using "elementary row operations".

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.

   eg. $R_2 \mapsto R_2 + 3R_1$

   $R_3 \mapsto R_3 - \frac{1}{2}R_2$

2. (Interchange) Interchange two rows.

   eg. $R_1 \leftrightarrow R_3$

3. (Scaling) Multiply all entries in a row by a non-zero constant.

When we solve linear systems, what is the matrix we are trying to bring to row-echelon form? It has a special name:

Augmented matrix: It is the matrix formed by writing down the cofficients of each variable as columns and then inserting one more column formed by the entries on the RHS of each eqn:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n \end{aligned} \quad \rightsquigarrow \quad \begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{bmatrix}$$

Upshot: We perform elementary row operations on the augmented matrix to convert it to row echelon form, and then use back-substitution to solve the (equivalent) linear system obtained.

We would like a systematic way to convert a given matrix into REF (or RREF). This is given by the "Gauss-Jordan elimination" algorithm. We proceed to describe this.

<u>Def</u>. A <u>pivot position</u> in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A <u>pivot column</u> is a column of A that contains a pivot position.

(Roughly, pivot positions are the entries that make up a new "step" in the step pattern of an echelon form of a matrix.)