LECTURE 6

Recall:

Theorem: Suppose A is an $n \times n$ matrix and suppose that B is obtained from A by an elementary row operation. Suppose E is an elementary matrix obtained from I_n by the same elementary row operation. Then $B = EA$.

the $n \times n$ identity matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}_{n \times n}$$

Getting comfortable with elementary matrices:

→ Taking inverses of elementary matrices is very easy (we will see this soon).

→ Multiplying EA where E is an elementary matrix and A is any matrix (of the same size as E) is very easy!

As we saw, there are 3 kinds of elementary matrices, obtained from the 3 kinds of elementary row operations:

Note: We focus on 3×3 matrices for examples, but these can be applied to any $n\times n$ matrix.

1. Row replacement:

eg. $R_1 \mapsto R_1 + 2R_2$: $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_3 \mapsto R_3 - R_1$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

2. Row exchange / interchange

eg $R_1 \leftrightarrow R_2$: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \leftrightarrow R_3$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

3. Scaling :

eg. $R_1 \mapsto 3R_1$ $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \mapsto \frac{-1}{2} R_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Can you identify the row operations corresponding to the following elementary matrices?

1). $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Ans. $R_1 \mapsto R_1 - 2R_3$ $R_3 \mapsto -R_3$

3) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 \mapsto R_2 - 3R_3$

4) $\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_1 \mapsto \sqrt{2} R_1$

NOTE: It is important that you do only **ONE** operation → either row replacement, row interchange or scaling.

So the following are not considered to be elementary matrices:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

, which are obtained by performing more than one elementary row operation to the identity matrix.

Multiplying A by an elementary matrix (on the left)

Let us do some examples to see how much easier it is to multiply two matrices if the matrix on the left is an elementary matrix:

1. $$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

Identify that this corresponds to $R_2 \mapsto -R_2$ $\rightsquigarrow$ apply it to A $\rightsquigarrow$ write the resulting matrix

2. $$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 4 \\ 5 & 6 & 6 \end{bmatrix}}_{A} = \begin{bmatrix} 7 & 7 & 10 \\ 3 & 3 & 4 \\ 5 & 6 & 6 \end{bmatrix}$$

Identify that this corresponds to $R_1 \mapsto R_1 + 2R_2$ $\rightsquigarrow$ apply it to A $\longrightarrow$ write the resulting matrix.

What about:

3. $$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = ?$$

Observe, these are both elementary matrices, but the rule does not change, and it shouldn't confuse you!

As before, we focus on the matrix on the <u>left</u>, identify the corresponding row operation and apply it to the matrix on the right.

So the answer to the above question is $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, because the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ corresponds to $R_1 \leftrightarrow R_2$.

Try some more, and more than two:

1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

2) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3) $\underset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_2}{\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$:

We start from the right, two at a time.

i.e., Do : $E_1 (\underbrace{E_2 E_3}_{= A})$ then compute $E_1 A$.

If we start from the left : $(\underbrace{E_1 E_2}_{= B}) E_3$

we cannot use the trick because E_3 ← elementary matrix is placed to the <u>right</u> of the matrix B.

Also, B is not an elementary matrix.

So we have :

$$E_1(\underset{R_1 \mapsto R_1 - 3R_2}{E_2}E_3) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{R_2 \mapsto R_2 + 2R_3} \begin{bmatrix} 1 & -15 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -15 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Try another one:

Q. $$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}} = ?$$

A:

$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Finding A^{-1}:

Consider an $n \times n$ matrix A, and suppose we know that it is possible to reduce the matrix A by a sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ of elementary row operations to the identity matrix I_n.

If $E_1, E_2, \ldots, E_k$ are respectively the elementary $n \times n$ matrices obtained from I_n by performing the same operations $\alpha_1, \ldots, \alpha_k$, then:

$$I_n = E_k \cdots E_3 E_2 E_1 A.$$

We therefore must have

$$A^{-1} = E_k \cdots E_2 E_1 A A^{-1}$$

$$= E_k \cdots E_1 I_n.$$

Algorithm to find A^{-1} for an $n \times n$ matrix A:

1. Arrange the matrix A & I_n side by side to get the rectangular matrix $[A \vdots I_n]$.

2. Perform row operations to convert A to I_n, and apply these operations to I_n as well.

3. If A can be reduced to I_n, then what turns up on the right (by performing the same operations to I_n) is the inverse of A. Else, A^{-1} does NOT exist.

$$[A \vdots I_n] \rightsquigarrow [I_n \vdots A^{-1}].$$

<u>Example 1</u>: Find the inverse of $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ if it exists.

<u>Solution</u>:

$$\left(\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$R_2 \mapsto R_2 + 3R_1$

$R_3 \mapsto R_3 - 2R_1$

$$\left(\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right)$$

$R_3 \mapsto R_3 + 3R_2$ $\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right)$

$R_1 \mapsto R_1 + R_3$
$R_2 \mapsto R_2 + R_3$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right)$

$R_3 \mapsto R_3/2$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right)$

Therefore the inverse of $\begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}$

is the matrix $\begin{pmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{pmatrix}$.

Example 2: Find the inverse of

$$B = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$ if it exists.

Solution:

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 6 & 2 & 0 & 1 \end{array}\right]$$

$R_2 \mapsto R_2 - 2R_1$: $$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

There is no way the second row can be made $[0\ 1\ *\ *]$, so $\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ cannot be reduced to I_2, therefore the inverse does not exist.

Try as an exercise: Find the inverse of

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix},$$ if it exists.

Note: Since inverses are unique, whatever you calculate as the inverse matrix, (let's call it B) **MUST** satisfy $BA = AB = I$.

So you should multiply your final answer by A and verify that you indeed get the identity matrix. If not, you have made a mistake somewhere!