Determinants

The determinant of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is the number } (ad - bc).$$

For a general $n \times n$ matrix, it is defined recursively using determinants of certain $(n-1) \times (n-1)$ submatrices. Let us do 3×3 matrices. We introduce some terminology to make it easy to give the formula for the determinant of any square matrix.

For a matrix A, let A_{ij} denote the sub-matrix formed by deleting the i^{th} row and j^{th} column of A.

eg. if $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \\ 3 & 1 & 1 \end{bmatrix}$, delete row 1

then $A_{11} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}$

$i=1$
$j=1$

delete column 1

$A_{23} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$

$i=2$
$j=3$

delete row 2

delete column 3

let's do one more :

$A_{31} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$

$i=3$
$j=1$

delete row 3

delete column 1

<u>Definition:</u> The determinant of an $n \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & & \vdots \\ \vdots & & \ddots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

is the sum $\sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$

$$= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} - \cdots + (-1)^{1+n} a_{1n} \det A_{1n}.$$

<u>Example</u> Compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

Solution. Here $n = 3$, so applying the formula above,

$$\det A = 1 \cdot \det \begin{pmatrix} 4 & -1 \\ -2 & 0 \end{pmatrix} - 5 \cdot \det \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \det \begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}.$$

$$= 1(-2) - 5(0) + 0(-4)$$

$$= -2.$$

<u>Example 2</u> : Calculate the determinant

of $A = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 4 & -1 \\ 0 & 3 & 1 \end{bmatrix}$

<u>Solution</u>.

$$\det A = 1 \cdot \det \begin{bmatrix} 4 & -1 \\ 3 & 1 \end{bmatrix} - 5 \det \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

$$+ 2 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

$$= 1(7) - 5(2) + 2(6)$$

$$= 9.$$

The definition provided above is not the only way to calculate the determinant. We used $a_{11}, a_{12}, \dots, a_{1n}$, i.e., elements of the first row, but we can actually take any other row (or column!). We show what this means :

Defn. given $A = [a_{ij}]$, the (i,j)-cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

Then, using this notation

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}.$$

The point we are trying to make is that we can expand along any row, to get

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}.$$

so in Example 1, using row 2, i.e., $i=2$ in the above formula

$$\det\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= 2\cdot(-1)^{2+1}\det\begin{pmatrix} 5 & 0 \\ -2 & 0 \end{pmatrix} + 4\cdot(-1)^{2+2}\det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ (-1)\cdot(-1)^{2+3}\det\begin{pmatrix} 1 & 5 \\ 0 & -2 \end{pmatrix}$$

$$= -2\times 0 + 4\times 0 + 1\times(-2)$$

$$= -2.$$

We can also expand along any column:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

So in example 1, if we expand along column 3,

$$\det\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$$

$$= 0\cdot(-1)^{1+3}\det\begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix} + (-1)\cdot(-1)^{2+3}\det\begin{pmatrix} 1 & 5 \\ 0 & -2 \end{pmatrix}$$

$$+ 0\cdot(-1)^{3+3}\det\begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}$$

$$= 0 + 1(-2) + 0$$

$$= -2.$$

The fact that the determinant can be computed by expanding along any row or column allows us to choose a way that simplifies calculations, especially if a certain row or column has many zeros.

Example: If $A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$

then we can expand along column 1

to get

$$\det A = 3 \cdot (-1)^{1+1} \cdot \det \begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\underbrace{+0+0+0+0}_{\text{coming from remaining cofactors.}}$$

Expand again along column 1 in this submatrix:

$$= 3 \cdot \left(2 \cdot (-1)^{1+1} \cdot \det \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \right)$$

same matrix as in Example 1

$$= 3 \cdot 2 \cdot (-2)$$

$$= -12.$$

either upper triangular or lower triangular.

Theorem: If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

So $\det \begin{pmatrix} 2 & -3 & 4 & 5 \\ 0 & 5 & 3 & -1 \\ 0 & 0 & -2 & 7 \\ 0 & 0 & 0 & 4 \end{pmatrix} = 2 \times 5 \times -2 \times 4 = -80.$

Properties of determinants

Theorem: Let A be a square matrix.

1. If the matrix is transformed to B by adding a multiple of one row to another row of A, then
$$\det B = \det A.$$

2. If two rows of A are interchanged and the resulting matrix is B, then
$$\det B = (-1) \det A$$

3. If one row of A is scaled by multiplying the row by k, and the resulting matrix is B, then
$$\det B = k \cdot \det A.$$

This can help us compute the determinant with fewer calculations:

<u>Example</u> Compute det A where

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$

<u>Solution</u>: Scale the first row:

$$\det A = 2 \det \underbrace{\begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}}_{B}$$

Bring B into REF using row replacements:

$$= 2 . \det \begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= 2 \times (1 \times 3 \times -6 \times 1)$$

$$= -36 .$$

Some important properties of determinants

1) A square matrix A is invertible if and only if $\det A \neq 0$.

2) If A is an $n \times n$ matrix, then $\det A^T = \det A$.

3) If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

4) There is another way to compute inverses of matrices, using determinants:
Recall C_{ij}, the cofactors in the formula for the determinant.
Consider the matrix where the $(i,j)^{th}$ entry is C_{ji}, i.e.,
$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T.$$

This matrix is called the adjugate of the original matrix A, denoted by adj A.

Then,

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A.$$

<u>Example</u> Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

<u>Solution</u> The cofactors are:

$C_{11} = \det\begin{bmatrix} -1 & 1 \\ 4 & -2 \end{bmatrix} \cdot (-1)^{1+1} = -2 \qquad C_{21} = 14 \qquad C_{31} = 4$

$C_{12} = \det\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \cdot (-1)^{1+2} = 3 \qquad C_{22} = -7 \qquad C_{32} = 1$

$C_{13} = \det\begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix} \cdot (-1)^{1+3} = 5 \qquad C_{23} = -7 \qquad C_{33} = -3$

$\therefore$ The adjugate of A is $\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$

The determinant of A is calculated next:

$$\det A = 2(2-4) - 1(-2-1) + 3(4+1)$$

$$= 2(-2) + 1(3) + 3(5)$$

$$= 14.$$

$$\therefore A^{-1} = \frac{1}{14}\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}.$$