Let V be a vector space and $W \subseteq V$ be a subspace.

Recall: $W^{\perp} = \{ v \in V : v \cdot w = 0 \text{ for every } w \in W \}$,

the orthogonal complement of W.

FACT 1: $W^{\perp}$ is a subspace of V too!

Proof: * $0 \in W^{\perp}$, since $0 \cdot w = 0$ for every $w \in W$.

* If $v_1, v_2 \in W^{\perp}$, by definition this means $\left.\begin{array}{l} v_1 \cdot w = 0 \\ v_2 \cdot w = 0 \end{array}\right\}$ for each $w \in W$.

Therefore $(v_1 + v_2) \cdot w = v_1 \cdot w + v_2 \cdot w$
$= 0$ for each $w \in W$

showing that $v_1 + v_2 \in W^{\perp}$.

* If $\alpha \in \mathbb{R}$, $v_1 \in W^{\perp}$ then for each $w \in W$,

$\alpha v_1 \cdot w = \alpha (v_1 \cdot w) = \alpha \cdot 0 = 0$,

since $v_1 \cdot w = 0$ for every $w \in W$.

FACT 2: $(W^{\perp})^{\perp} = W$.

Now we are ready to use the properties of inner products and orthogonality to find the matrix that will identify a grid pattern of 2 colours! We will use the following:

(i) If $W \subseteq V$ is a subspace, then $W^\perp$ is also a subspace of V, so it will have some basis.

(ii) $(W^\perp)^\perp = W$

(iii) Matrix multiplication uses the standard inner product on $\mathbb{R}^n$:

Let $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k$ be k vectors in $\mathbb{R}^n$, so each $\bar{a}_i$ is an n-tuple.

Let $\bar{b}$ be another vector in $\mathbb{R}^n$,

Then

$$\underset{k \times n}{\begin{bmatrix} \text{——} & \bar{a}_1 & \text{——} \\ \text{——} & \bar{a}_2 & \text{——} \\ & \vdots & \\ \text{——} & \bar{a}_k & \text{——} \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} | \\ \bar{b} \\ | \end{bmatrix}} = \begin{bmatrix} \bar{a}_1 \cdot \bar{b} \\ \bar{a}_2 \cdot \bar{b} \\ \vdots \\ \bar{a}_k \cdot \bar{b} \end{bmatrix}$$

Let ω be the $n\times 1$ vector generated from a pattern of blue and white squares.

Let $W = \text{span}\{\omega\}$.

Choose a basis $\{v_1, \ldots, v_{n-1}\}$ for $W^\perp$.

Create the matrix $B = \begin{bmatrix} v_1^T \\ \vdots \\ v_{n-1}^T \end{bmatrix}$. → Here we are thinking of $v_1, \ldots, v_{n-1}$ as column vectors, so $v_1^T, \ldots, v_{n-1}^T$ are row vectors

Notice that if $\bar{u}$ is any $n\times 1$ vector,

$$B\bar{u} = \bar{0} \rightsquigarrow \underbrace{\begin{bmatrix} \text{---} & v_1^T & \text{---} \\ \text{---} & v_2^T & \text{---} \\ & \vdots & \\ \text{---} & v_{n-1}^T & \text{---} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} | \\ \bar{u} \\ | \end{bmatrix}}_{\bar{u}} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}_{\bar{0}}$$

From (iii) above, $B\bar{u} = \begin{bmatrix} v_1^T \cdot \bar{u} \\ v_2^T \cdot \bar{u} \\ \vdots \\ v_{n-1}^T \cdot \bar{u} \end{bmatrix}$

Therefore $B\bar{u} = \bar{0} \iff v_1^T \cdot \bar{u} = 0,\ v_2^T \cdot u = 0, \ldots,\ v_{n-1}^T \cdot u = 0$

In other words, $Bu = \bar{0}$ iff u is orthogonal to $v_1, \ldots, v_{n-1}$, i.e., iff u is orthogonal to every vector in $W^\perp \iff u \in (W^\perp)^\perp$, so $u \in W$.

Summarizing: $\boxed{Bu = \bar{0} \iff u \in W}$.

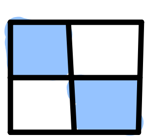
Now, $W = \text{span}\{w\} = \{\alpha w \mid \alpha \in \mathbb{R}\}$ ← pattern vector.

We are assuming only 2 colours are used, represented by 1 and 0. So there are only 2 vectors in W that will be of relevance to us:

1. $\alpha = 1 \rightsquigarrow 1w = w$
2. $\alpha = 0 \rightsquigarrow 0w = \bar{0}$.

For any other vector u, $Bu \neq \bar{0}$.

This key fact distinguishes the given pattern from any other pattern.

Let us work out the previous example again:  $\rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = w$

Let $W = \text{span}\{w\} = \text{span}\left\{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}$.

$W^\perp = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_4 = 0\}$

$= \text{span}\{\underbrace{(1,0,0,-1)}_{v_1^T}, \underbrace{(0,1,0,0)}_{v_2^T}, \underbrace{(0,0,1,0)}_{v_3^T}\}$.

Form the matrix B by inserting the basis vectors of $W^\perp$ as rows:

$$B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Check : $B\bar{w} = \bar{0}$? $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ ✓

Choose a random $\bar{x} \neq \bar{w}$, consisting of 1's & 0's:

say $\bar{x} = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}$. Then $B\bar{x} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

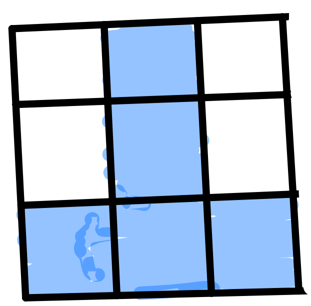
$\neq \begin{bmatrix}0\\0\\0\end{bmatrix}$.

Some final remarks:

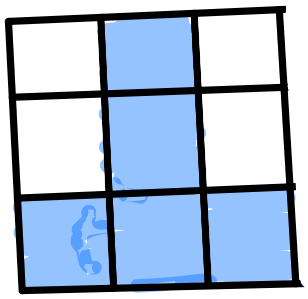
① It doesn't matter what basis you choose for $W^\perp$, as long as it is a valid basis, it will work. B will look different depending on your basis, but that's ok!

② If your pattern is a $n \times n$ grid, $W^\perp$ will have $n-1$ vectors in its basis, each of length n, so B is an $n-1 \times n$ matrix.

Example 2: Find a matrix B that can be used to identify the pattern:



Solution: First we convert this into a 9×1 vector:

 $\rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \omega$

Next, set $W = \text{span}\{\omega\}$. We need to find a basis for $W^{\perp}$. To do this, we solve $x \cdot \omega = 0$:

$$[x_1 \; x_2 \; \cdots \; x_9] \cdot (0\;0\;1\;1\;1\;1\;0\;0\;1) = 0$$

$$\Rightarrow x_3 + x_4 + x_5 + x_6 + x_9 = 0 .$$

i.e., $W = \left\{ (x_1, \ldots, x_9) \in \mathbb{R}^9 \;\middle|\; \begin{array}{l} x_3 + x_4 + x_5 \\ + x_6 + x_9 \end{array} = 0 \right\}$

$\therefore$ a basis is :

$\{(1,0,0,0,0,0,0,0,0), (0,1,0,0,0,0,0,0,0),$

$(0,0,0,0,0,0,1,0,0), (0,0,0,0,0,0,0,1,0),$

$(0,0,-1,1,0,0,0,0,0), (0,0,-1,0,1,0,0,0,0),$

$(0,0,-1,0,0,1,0,0,0), (0,0,-1,0,0,0,0,0,1)\}$

(OR) using the notation for standard basis of $\mathbb{R}^n$ described earlier, the above set is

$\{e_1, e_2, e_7, e_8, e_4-e_3, e_5-e_3, e_6-e_3, e_9-e_3\}.$

Now, $B = \begin{bmatrix} - v_1 - \\ \vdots \\ - v_{n-1} - \end{bmatrix}$

$$\therefore B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ is a matrix that works.