We have learned what the span of vectors $v_1, \ldots, v_n$ is, and the definition of linear independence of vectors $v_1, v_2, \ldots, v_n$ is.

These two concepts come together in the following important definition:

Def. A basis for V is a set of vectors having the following properties.

1. The vectors in V are linearly independent

2. The vectors in the set span the space V.

A basis B can be thought of as a generating set for V: Since it spans V, it means that every vector $v \in V$ can be written as a linear combination of vectors in B.

The fact that vectors in B are linearly independent actually gives us more: It tells us that this linear combination is unique:

If $\mathbb{B} = \{v_1, \dots, v_k\}$ is a basis of V, and

$w = a_1 v_1 + \cdots + a_k v_k$ and also

$w = b_1 v_1 + \cdots + b_k v_k$

Then subtracting the two eqns gives

$$(a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \cdots + (a_k - b_k) v_k = 0$$

But $v_1, \ldots, v_k$ are linearly independent,

so $a_1 - b_1 = 0, \quad a_2 - b_2 = 0, \quad \ldots, \quad a_k - b_k = 0$

i.e., $a_1 = b_1, \quad a_2 = b_2, \quad \ldots, \quad a_k = b_k$.

Examples of basis of a vector space:

1. $V = \mathbb{R}^3$.

consider:
$e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

* These are linearly independent because if we consider the linear combination

$$c_1 e_1 + c_2 e_2 + c_3 e_3 = \bar{0}$$

i.e., $c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$

then we get the equations

$$1 \cdot c_1 + 0 \cdot c_2 + 0 \cdot c_3 = 0 \rightarrow$$ comparing 1^{st} components on both sides

$$0 \cdot c_1 + 1 \cdot c_2 + 0 \cdot c_3 = 0$$

$$0 \cdot c_1 + 0 \cdot c_2 + 1 \cdot c_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0.$$

* The set $\{e_1, e_2, e_3\}$ is a spanning set for $\mathbb{R}^3$, because an arbitrary vector $(a, b, c) \in \mathbb{R}^3$ can be written as

$$(a, b, c) = a e_1 + b e_2 + c e_3.$$

In general, for $V = \mathbb{R}^n$, the set of vectors $\{e_1, \ldots, e_n\}$ where e_i is a tuple that has 0 in all coordinates except the i^{th} coordinate, which is equal to 1, forms a basis for $\mathbb{R}^n$. This is often denoted as the <u>standard basis</u>

for $\mathbb{R}^n$.

2. $V = M_{2\times 2}(\mathbb{R})$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

forms a basis.

3. $V = P_n$, the space of polynomials in $\mathbb{R}[x]$ with degree at most n has

$$\{ 1, x, x^2, \ldots, x^k \}$$

as a basis.

Does V have a unique basis? No.

eg. $\mathbb{R}^2$: $(1,0), (1,1)$ is a basis for $\mathbb{R}^2$,
$(1,2), (0,3)$ is a basis for $\mathbb{R}^2$.

Recall: In the case of two vectors $\{v_1, v_2\}$ we saw that these are linearly dependent if and only if they are scalar multiples of each other

i.e., $v_1 = c v_2$ for some $c \in \mathbb{R}$.

Thus, in $\mathbb{R}^2$, if $v_1 = (a_1, b_1)$ $v_2 = (a_2, b_2)$, then $\{v_1, v_2\}$ is <u>guaranteed</u> to be linearly independent if $(a_1, b_1) \neq c(a_2, b_2)$ for any $c \in \mathbb{R}$.

same as saying v_1 and v_2 are NOT multiples of each other.

But how do we know if this is a spanning set for $\mathbb{R}^2$? i.e., we want <u>every</u> $(p, q) \in \mathbb{R}^2$ to be expressed as a linear combination of (a_1, b_1) and (a_2, b_2). This amounts to finding $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 a_1 + c_2 a_2 = p$$
$$c_1 b_1 + c_2 b_2 = q$$

Let us see first with an example:

Let $(a_1, b_1) = (1, 2)$ $(a_2, b_2) = (3, 4)$.

So we want to solve

$$\begin{aligned} c_1 + 3c_2 &= p \\ 2c_1 + 4c_2 &= q \end{aligned}$$

(OR) $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Augmented matrix: $\begin{pmatrix} 1 & 3 & p \\ 2 & 4 & q \end{pmatrix} \underset{R_2 \mapsto R_2 - 2R_1}{\rightsquigarrow} \begin{pmatrix} 1 & 3 & p \\ 0 & -2 & q-2p \end{pmatrix}$

which can be solved to get a unique solution for c_1, c_2.

Observe that in general we get

$$\begin{pmatrix} a_1 & b_1 & p \\ a_2 & b_2 & q \end{pmatrix} \underset{R_2 \mapsto R_2 - \frac{a_2}{a_1} R_1}{\rightsquigarrow} \begin{pmatrix} a_1 & b_1 & p \\ 0 & b_2 - \frac{a_2 b_1}{a_1} & q - \frac{a_2}{a_1} p \end{pmatrix}$$

This would have a unique solution as long as $b_2 \neq \frac{a_2 b_1}{a_1}$, i.e., $\frac{b_2}{b_1} \neq \frac{a_2}{a_1}$

which is equivalent to saying $(a_1, b_1) \neq c(a_2, b_2)$ for any $c \in \mathbb{R}$.

Dimension of a vector space

A vector space can have infinitely many distinct bases, but the number of basis vectors remains constant.

Def. The number of elements in any basis of a vector space V is called the dimension of V.

eg. 1) The dimension of $\mathbb{R}^n$ is n.

2) The dimension of $M_n(\mathbb{R})$ is n^2.

3) The dimension of $\mathbb{P}_n$ is $n+1$.

Two important results that we will state without proof:

1) Any linearly independent set in V can be extended to a basis, by

adding more vectors if necessary.

2) Any spanning set can be reduced to a basis, by removing vectors if necessary.

In other words, a basis is a <u>maximal independent set</u>, and <u>minimal spanning set</u>.

<u>Examples</u> :

1. Find the dimension of $C(A)$ where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

<u>soln</u>. $C(A) = \text{span}\left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}} \right\}$

we need to check if this is a linearly independent set:

Set $c_1 v_1 + c_2 v_2 + c_3 v_3 = \bar{0}$.

i.e., $c_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

or

$$c_1 + c_2 = 0 \quad -① $$
$$c_1 + 3c_2 + c_3 = 0 \quad -② $$
$$3c_1 + c_2 - c_3 = 0 \quad -③ $$

Solve: ~~use~~ ① to get $c_2 = -c_1$ and plug it ②, ③:

$$-2c_2 + c_3 = 0$$
$$2c_2 - c_3 = 0 \quad \Rightarrow \quad c_3 = 2c_2$$

$\therefore$ The set is lin. dependent.

Set $c_1 = 1$. Then $c_2 = -1$, $c_3 = -2$.

$$\therefore \; v_1 = v_2 + 2v_3.$$

$\therefore$ Span$\{v_1, v_2, v_3\}$ = Span$\{v_2, v_3\}$.

$\{v_2, v_3\}$ is a linearly independent set since v_2 & v_3 are not multiples of each other.

Thus, a basis for $C(A)$ is $\{v_2, v_3\}$ and the dimension = 2.

2. Find a basis for the null space $N(A)$.

We want to solve : $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Observe, this is the same system as before.

The solution set is given by $c \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$.

$\therefore \quad N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}$.

Basis for $N(A) = \left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}$ $\rightarrow$ Observe the difference

$N(A) = \left\{ c\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} : c \in \mathbb{R} \right\}$

infinite set

Basis is <u>one</u> vector : $\left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}$

In some cases we can find a basis by looking at the min. no. of vectors needed to span the vector space and come up with a guess for a basis.

Example: Find a basis for the foll. subspaces of $\mathbb{R}^3$ and compute its dimension.

a) $W_1 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = x_3 \}$

b) $W_2 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \}$

a) basis for W_1 : $\{ (1,1,1) \}$

b) basis for W_2 : $\{ (1,0,-1), (0,1,-1) \}$

Definition:

- The rank of a matrix A is the dimension of $C(A)$, its column space.
- The nullity of a matrix A is the dimension of $N(A)$, its null space.