LECTURE-9

Subspaces

A subspace of a vector space V is a subset W of V that has the foll. properties:

a) The zero vector of V is in H, i.e., $0 \in H$

b) H is closed under addition, i.e., for every $u, v \in H$, $u + v \in H$

c) H is closed under scalar multiplication. i.e., for every $c \in \mathbb{R}$, $v \in H$, $c \cdot v \in H$.

Examples:

1) The "trivial" subspace: $\{0\}$ is a subspace of every vector space. [So the empty set is not a subspace or a vector space]

2) The set of polynomials of degree up to n, denoted by P_n is a subspace of P.
↳ what about polynomials of a fixed degree?

3) The set of lower triangular 3x3 matrices is a subspace of $M_3(\mathbb{R})$.

4) The line $x = y$ is a subspace of $\mathbb{R}^2$.

Non-examples

1) $S = \{(x,y) \mid x, y \in \mathbb{R},\ x, y \geq 0\}$ is not a subspace of $\mathbb{R}^2$. Why?

2) The set of invertible 2x2 matrices is not a subspace of $M_2(\mathbb{R})$ Why?

Def. A <u>linear combination</u> of finitely many vectors $v_1, v_2, \ldots, v_m \in V$ is an element of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_m$$

where $\alpha_i \in \mathbb{R}$, $1 \le i \le m$

<u>Examples</u>

1) In $\mathbb{R}^2$,

$$(3, 5) = 3(1,0) + 5(0,1)$$

so the vector $(3,5)$ is a linear combination of the vectors $(1,0)$ and $(0,1)$.

<u>Note:</u>

$$(3,5) = 2(\cancel{1},1) + (0,3)$$
$$= (1,2) + (2,1) + 2(0,1)$$

So $(3,5)$ can be a linear combination of other vectors in more than one way.

2) In $V = \mathbb{P}_3$,

$$1 + 2x + 3x^3 = 1 \cdot 1 + 2 \cdot (x + x^3) + 1 \cdot x^3$$

So here we have expressed the polynomial $1 + 2x + 3x^3$ as a linear combination of the vectors 1, $x + x^3$ and x^3.

3) $V = M_2(\mathbb{R})$.

$$\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}$$

We have expressed $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix}$.

4) The matrix equation $A\bar{x} = \bar{b}$ can be expressed using linear combinations

example:

$$x_1 + 2x_2 + x_3 = 1$$
$$2x_1 - x_2 = 2$$
$$x_1 + x_2 + 5x_3 = 0$$

$$\Updownarrow$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

That is, we have expressed the vector $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ as a linear combination of the columns $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$ but the coefficients x_1, x_2, x_3 are unknown.

Thus, solving the linear system $A\bar{x}=\bar{b}$ is the same as trying to see whether we can write the vector $\bar{b}$ as a linear combination of the columns of A!

Span of a set of vectors

The set of all linear combinations of a fixed set of vectors $v_1, \dots, v_m$ is a subspace of the vector space V and is denoted by $\mathrm{span}\{v_1, \dots, v_m\}$. (Exercise: Prove that it is a subspace)

i.e., If we let $\mathrm{span}\{v_1, \dots, v_m\}$ be denoted by W, then

$$W = \left\{ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m \;:\; \alpha_1, \dots, \alpha_m \in \mathbb{R} \right\}$$

is a subspace of V.

i.e., we have "created" a vector space using linear combinations.

<u>Fact</u>: Span$\{v_1, \ldots, v_n\}$ is a subspace of V.

<u>Examples</u>

1) Let $V = \mathbb{R}^2$
$W = \text{span}\{(0,1)\}$

What is W? It is the set $\{\alpha(0,1) : \alpha \in \mathbb{R}\}$

so it consists of elements of the form $(0,\alpha)$ for each $\alpha \in \mathbb{R}$. This is the y-axis!

→ what about span$\{(1,0)\}$? x-axis

→ " " span$\{(1,1)\}$? line $x=y$

Do you see a pattern? Can you formulate geometrically what span$\{(a,b)\}$ is for any (fixed) a and b in $\mathbb{R}$?

<u>Ans</u>
It is the line through the origin that passes through the point (a,b).

Two important subspaces of $\mathbb{R}^n$:

1) Let A be an $m \times n$ matrix. The <u>column space</u> of A is the set of all linear combinations of the columns of A.

eg. Let $A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}$.

Then $C(A) := \left\{ c_1 \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}$

<u>Ex</u> Show that $C(A)$ is a subspace of $\mathbb{R}^{m}$.

important to understand why 'm' and not 'n'!

Therefore we can say that a linear system $A\bar{x} = \bar{b}$ is solvable only if $\bar{b}$ is in $C(A)$.

2. The nullspace of A

The <u>nullspace</u> of a matrix consists of all vectors x s.t $A\bar{x}=\bar{0}$, and it is denoted by $N(A)$. If A is an $m \times n$ matrix, then $N(A)$ is a subspace of $\mathbb{R}^{n}$. ← important to understand why 'n' and not 'm'!

<u>Proof</u> (that $N(A)$ is a subspace of $\mathbb{R}^n$).
A has n columns, so $N(A)$ is a subset of $\mathbb{R}^n$.
We need to check 3 properties:

1) Does $\bar{0}$ belong to $N(A)$?
2) If $\bar{a}, \bar{b} \in N(A)$, does $\bar{a}+\bar{b} \in N(A)$?
3) If $\bar{a} \in N(A)$ does $c\bar{a} \in N(A)$ for $c \in \mathbb{R}$?

(1) : Here $\bar{0}$ is $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$. Clearly, $A\bar{0}=\bar{0}$ since any matrix multiplied by the zero vector results in the zero vector.

(2) If $\bar{a}, \bar{b} \in N(A)$, by definition this means
$A\bar{a} = \bar{0}$, $A\bar{b} = \bar{0}$.

Observe: $A(\bar{a}+\bar{b}) = A\bar{a} + A\bar{b} = \bar{0} + \bar{0} = \bar{0}$.

so $\bar{a}+\bar{b}$ lies in $N(A)$.

(3) If $\bar{a} \in N(A)$, then $A\bar{a} = \bar{0}$.

$A(c\bar{a}) = \underbrace{(cA)}_{\text{every entry in the matrix gets multiplied by } c.}\bar{a} = c(A\bar{a}) = c \cdot \bar{0} = \bar{0}$.

This shows that $N(A)$ is a subspace of $\mathbb{R}^n$.

<u>Examples</u> (i) If $A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}$, find the set $N(A)$.

<u>Solution</u>: We want to find the set of all $\begin{bmatrix} u \\ v \end{bmatrix} = \bar{y}$ satisfying $A\bar{y} = \bar{0}$. That is,

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} u + 0 \cdot v &= 0 \rightsquigarrow \Rightarrow u = 0, \\ 5u + 4v &= 0 \qquad \text{so } v = 0. \\ 2u + 4v &= 0 \end{aligned}$$

$$\therefore N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$