Solutions

Total score: 48 for this assignment

Problem 1 (12 marks) We find the inverse by starting with the augmented matrix $[A \mid I]$ and row-reducing A to I_n:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 1 & 5 & 3 & 0 & 1 & 0 \\ 2 & 6 & -1 & 0 & 0 & 1 \end{array}\right)$$

$R_2 \mapsto R_2 - R_1$

$R_3 \mapsto R_3 - 2R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 3 & 5 & -1 & 1 & 0 \\ 0 & 2 & 3 & -2 & 0 & 1 \end{array}\right)$$

$R_3 \mapsto R_3 - \frac{2}{3}R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 3 & 5 & -1 & 1 & 0 \\ 0 & 0 & \frac{-1}{3} & \frac{-4}{3} & \frac{-2}{3} & 1 \end{array}\right)$$

$R_3 \mapsto -3R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 3 & 5 & -1 & 1 & 0 \\ 0 & 0 & 1 & 4 & 2 & -3 \end{array}\right)$$

$R_2 \mapsto R_2 - 5R_3$
$R_1 \mapsto R_1 + 2R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 9 & 4 & -6 \\ 0 & 3 & 0 & -21 & -9 & 15 \\ 0 & 0 & 1 & 4 & 2 & -3 \end{array}\right)$$

$R_1 \mapsto R_1 - \frac{2}{3} R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 23 & 10 & -16 \\ 0 & 3 & 0 & -21 & -9 & 15 \\ 0 & 0 & 1 & 4 & 2 & -3 \end{array}\right)$$

$R_3 \mapsto R_3/3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 23 & 10 & -16 \\ 0 & 1 & 0 & -7 & -3 & 5 \\ 0 & 0 & 1 & 4 & 2 & -3 \end{array}\right)$$

Therefore the inverse matrix is given by

$$A^{-1} = \begin{pmatrix} 23 & 10 & -16 \\ -7 & -3 & 5 \\ 4 & 2 & -3 \end{pmatrix}.$$

b) We obtain the solution by computing $A^{-1}b$ where $b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$:

$$\begin{pmatrix} 23 & 10 & -16 \\ -7 & -3 & 5 \\ 4 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 1 \end{pmatrix}$$

(12 marks)

Problem 2 (Total 24 marks)

a) (in brief) Using the following sequence of row operations:

1) $R_2 \mapsto R_2 + 2R_1$

2) $R_3 \mapsto R_3 + 3R_1$

3) $R_3 \mapsto R_3 - R_2$

The given matrix is brought to the foll. REF:

$$\begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

This is U. That is,

$$\overbrace{\underset{3}{\begin{pmatrix}1&0&0\\0&1&0\\0&-1&1\end{pmatrix}}\underset{2}{\begin{pmatrix}1&0&0\\0&1&0\\3&0&1\end{pmatrix}}\underset{1}{\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}}}^{L^{-1}}\underset{A}{\begin{pmatrix}1&4&3\\-2&-6&0\\-3&-10&0\end{pmatrix}}=\underset{U}{\begin{pmatrix}1&4&3\\0&2&6\\0&0&3\end{pmatrix}}$$

$$\therefore L=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0\\0&1&0\\3&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0\\0&1&0\\0&-1&1\end{pmatrix}^{-1}=\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-3&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0\\-2&1&0\\-3&1&1\end{pmatrix}$$

→ Your L & U will look different if you scaled R_2 by 2 and/or R_3 by 3. The factorization is fine as long as L is lower triangular, U is upper triangular and $A = LU$.

$\therefore$ The LU decomposition is given by

$$\begin{pmatrix}1&4&3\\-2&-6&0\\-3&-10&0\end{pmatrix}=\begin{pmatrix}1&0&0\\-2&1&0\\-3&1&1\end{pmatrix}\begin{pmatrix}1&4&3\\0&2&6\\0&0&3\end{pmatrix}$$

(8 marks)

b) To solve the system we first solve:

$$L\bar{y} = \bar{b}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

i.e., $y_1 = 1$

$y_2 = -1 + 2y_1 = 1$

$y_3 = 2 + 3y_1 - y_2 = 4.$

Then solve $U\bar{x} = \bar{y}$:

$$\begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$3x_3 = 4$

$2x_2 + 6x_3 = 1$

$x_1 + 4x_2 + 3x_3 = 1$

$\rightsquigarrow$

$x_3 = 4/3$

$x_2 = \frac{1}{2}\left(1 - 6 \cdot \frac{4}{3}\right) = -\frac{7}{2}$

$x_1 = \left(1 - \frac{4.3}{3} + 4 \cdot \frac{7}{2}\right)$

$= 11$

Thus, the solution to the original system

is $x_1 = 11$, $x_2 = -\frac{7}{2}$, $x_3 = \frac{4}{3}$.

(4 marks)

c) L^{-1} = ③②① from part a)

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Problem 3 (12 marks)

a)

(i) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\rightsquigarrow$ $R_1 \longleftrightarrow R_4$ $\rightsquigarrow$ $R_4 \mapsto R_4 - 5R_1$

b) (i) 5×5 matrix, $R_5 \mapsto R_5 + 3R_2$ $\rightsquigarrow$ $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 1 \end{pmatrix}$

(ii) 4×4 matrix, $R_2 \longleftrightarrow R_3$ $\rightsquigarrow$ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(iii) 3×3 matrix,

$R_3 \mapsto \sqrt{2}R_3$ $\rightsquigarrow$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$

(iv) 2×2 matrix

$R_2 \mapsto R_2 - 2R_1$ $\rightsquigarrow$ $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.