TUTORIAL-1

Problem 1

$$A = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 1 & 1 & 5 & 5 \\ 2 & 1 & 9 & 8 \\ 2 & 0 & 6 & 3 \end{pmatrix}$$

a) Use elementary row operations to find the inverse of A

b) Without performing any further elementary row operations, use your

solution in part a) to solve the sytem of linear equations: $[A^{-1}][\vdots]$

$$\begin{aligned} x_1 \qquad\quad + 3x_3 + x_4 &= 1 \\ x_1 + x_2 + 5x_3 + 5x_4 &= 0 \\ 2x_1 + x_2 + 9x_3 + 8x_4 &= 0 \\ 2x_1 \qquad\quad + 6x_3 + 3x_4 &= 0 \end{aligned}$$

.

Solution

$$A = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 1 & 1 & 5 & 5 \\ 2 & 1 & 9 & 8 \\ 2 & 0 & 6 & 3 \end{pmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 5 & 5 & 0 & 1 & 0 & 0 \\ 2 & 1 & 9 & 8 & 0 & 0 & 1 & 0 \\ 2 & 0 & 6 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \mapsto R_2 - R_1$
$R_3 \mapsto R_3 - 2R_1$:
$R_4 \mapsto R_4 - 2R_1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 6 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \mapsto R_3 - R_2$:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

~~$R_3 \mapsto R_3 - 2R_4$~~

$R_1 \mapsto R_1 - 3R_3$

$R_2 \mapsto R_2 - 2R_3$

$+$

$R_3 \mapsto R_3 - 2R_4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -5 & 4 & 3 & -3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 3 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \mapsto R_1 + 5R_4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -6 & 3 & -3 & 5 \\ 0 & 1 & 0 & 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & 0 & 3 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$A = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 1 & 1 & 5 & 5 \\ 2 & 1 & 9 & 8 \\ 2 & 0 & 6 & 3 \end{pmatrix} \underbrace{\begin{pmatrix} -6 & 3 & -3 & 5 \\ 1 & 3 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ -2 & 0 & 0 & 1 \end{pmatrix}}^{A^{-1}}$$

$= I.$

$$x = A^{-1} b = \begin{pmatrix} -6 & 3 & -3 & 5 \\ 1 & 3 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ -2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \\ 3 \\ -2 \end{pmatrix}.$$

Problem 2

(i) Write down the elementary matrices corresponding to the following row operations:

a) $R_2 \mapsto 3R_2$ $\begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$ (in a 3×3 matrix)

b) $R_1 \mapsto R_1 + R_3$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (in a 3×3 matrix)

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ c) $R_3 \mapsto R_3 + R_4$ (in a 4×4 matrix)

$\begin{bmatrix} 1 & 0 & -5 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ d) $R_1 \mapsto R_1 - 5R_3$ (in a 5×5 matrix)

e) $R_2 \mapsto R_2 - 2R_1$ (in a 4×4 matrix) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(ii) Write down the row operation corresponding to the following elementary matrices:

a) $\begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_1 \mapsto R_1 + \sqrt{2}R_2$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \mapsto R_2 + 3R_1$

c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $R_3 \mapsto R_3 - 3R_1$

d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix}$ $R_4 \mapsto R_4 + \frac{1}{2}R_2$

e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $R_2 \mapsto R_2 - R_3$

f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $R_2 \leftrightarrow R_4$

Problem 3 Write down the product of the following matrices.

a) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

Ans. $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 37 & 38 & 39 \\ 0 & 1 & 0 \\ 40 & 40 & 42 \end{pmatrix}$

$\begin{pmatrix} 37 & 38 & 39 \\ 0 & 1 & 0 \\ 40 & 41 & 42 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 5 & 0 \\ -1 & -5 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 3 & 0 & -5 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Problem 4 Write down the inverse of the following matrices.

a) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Hint: can you write this as a product of elementary matrices?

Ans. a) $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

d) $= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

Problem 5

Solve the following system using LU decomposition

$$2x + 5y = 21$$

$$x + 2y = 8$$

Solution. The coefficient matrix is given by:

$A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$. First we convert it to REF:

$R_2 \mapsto R_2 - \frac{1}{2}R_1$: $\begin{bmatrix} 2 & 5 \\ 0 & -1/2 \end{bmatrix} = U$

i.e.,

$$\underset{L^{-1}}{\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 5 \\ 0 & -1/2 \end{bmatrix}}$$

$$A = \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 5 \\ 0 & -1/2 \end{bmatrix}$$

$$= \underset{L}{\begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 5 \\ 0 & -1/2 \end{bmatrix}}.$$

$$LU\bar{x} = \overset{\bar{b}}{\begin{bmatrix} 21 \\ 8 \end{bmatrix}}$$

First we solve $L\bar{y} = \bar{b}$

i.e., $\begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 21 \\ 8 \end{bmatrix}$

$c_1 = 21 \qquad c_2 = -\frac{5}{2}.$

$U\bar{x} = \bar{y}$:

$$\begin{bmatrix} 2 & 5 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 21 \\ -5/2 \end{bmatrix}$$

$x_2 = 5 \qquad x_1 = \frac{1}{2}(21 - 5.5)$

$= -2.$

$\therefore$ The solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$