LECTURE-3

We will see the row-reduction or Gaussian elimination algorithm now keeping the following example alongside:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

STEP 1:

Begin with the leftmost nonzero column. This is a pivot column, and the pivot position is at the top of the column.

pivot position

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

pivot column

STEP 2 :

Select a non-zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

$R_1 \leftrightarrow R_3$:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

pivot

STEP 3 : Use row replacement operations to create all zeros in all the positions below the pivot.

$R_2 \mapsto R_2 - R_1$:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

STEP 4: Cover the row containing the pivot and all rows (if any) above it. Apply steps 1-3 to the submatrix that remains. Repeat the process until there are no more rows to modify:

$$R_3 \mapsto R_3 - \frac{3}{2} R_2 : \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

STEP 5: Beginning with the rightmost pivot and working upwards and to the left, create zeros in the column above the pivot. Also use the scaling operation to make the pivots equal to 1.

$$\begin{matrix} R_2 \mapsto R_2 - 2R_3 \\ R_1 \mapsto R_1 - 6R_3 \end{matrix} : \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_2 \mapsto R_2/2$:
$R_1 \mapsto R_1/3$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_1 \mapsto R_1 + 3R_2$:

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

We can see how much easier it is to solve linear systems in row echelon form:

consider:

$$\begin{bmatrix} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{bmatrix}.$$

The associated linear system is:

$$\begin{aligned} x_1 + 5x_2 + 2x_3 &= -6 && \text{— 1} \\ 4x_2 - 7x_3 &= 2 && \text{— 2} \\ 5x_3 &= 0 && \text{— 3} \end{aligned}$$

Eqn 3 gives $x_3 = 0$, Substituting this in Eqn 2 gives $4x_2 = 2$, so $x_2 = \frac{1}{2}$. Then Eqn 1 becomes

$$x_1 + 5 \cdot \frac{1}{2} + 2 \cdot 0 = -6$$

$$\text{so } x_1 = -6 - \frac{5}{2} = \frac{-17}{2}.$$

Therefore the solution is

$$x_1 = -\frac{17}{2}, \quad x_2 = \frac{1}{2}, \quad x_3 = 0.$$

As we saw earlier, a linear sytem may have no solution or infinitely many solutions. The row-echelon form can immediately show us if a linear system is consistent or not.

<u>Example</u> :

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

and Gaussian elimination converts this to

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}.$$

Looking at the third row it is clear that there are no solutions. The above matrix is a typical example of the REF of an inconsistent linear system.

The case of infinite solutions is a bit different, this occurs, usually when we have more variables than the number of equations (but not always!)

Let us see two examples:

i) Suppose the augmented matrix of a linear system is converted to RREF and we get:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The associated system of equations is

$$\begin{aligned} x_1 \quad - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

We then have a "free variable", that can be used to describe the values of the remaining variables:

- Set x_3 as a free variable.

  Then $x_2 = 4 - x_3$, $x_1 = 1 + 5x_3$.

Therefore, once we choose a value of x_3, the values of x_1 and x_2 get decided. In this way we get <u>infinitely many solutions</u>.

Let $x_3 = c$, then $x_1 = 1+5c$, $x_2 = 4-c$

for any $c \in \mathbb{R}$.

Note that we could have set x_2 (or x_1) as the free variable instead:

If x_2 is free, then
$$
\begin{aligned}
x_3 &= 4 - x_2 \\
x_1 &= 1 + 5x_3 \\
&= 1 + 5(4 - x_2) \\
&= 21 - 5x_2.
\end{aligned}
$$

Let's do a quick check to see that the solution sets match regardless of which variable was chosen to be free:

$$
u = \begin{bmatrix} 1+5c \\ 4-c \\ c \end{bmatrix} \quad \text{vs.} \quad v = \begin{bmatrix} 21-5d \\ d \\ 4-d \end{bmatrix}
$$

Let $c = 1$. Then $x_2 = 4 - c = 3$, so we must take $d = 3$.

Then $u = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$ $\qquad$ $v = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$.

<u>Problem</u>: Solve:

$$x_1 - 2x_2 + x_3 = 2$$
$$x_1 - x_2 + 2x_3 = 5$$
$$x_1 + x_2 = 3$$

<u>Solution</u>: The augmented matrix corresponding to the given linear system is

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & -1 & 2 & 5 \\ 0 & 1 & 1 & 3 \end{bmatrix}.$$

Performing elementary row operations to convert it to REF:

$R_2 \mapsto R_2 - R_1$: $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

$R_3 \mapsto R_3 - R_2$ $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The corresponding linear system is

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ x_2 + x_3 &= 3 \end{aligned}$$

Set x_3 as a free variable.

Then $x_2 = 3 - x_3$

$$\begin{aligned} x_1 &= 2 + 2x_2 - x_3 \\ &= 2 + 2(3 - x_3) - x_3 \\ &= 8 - 3x_3. \end{aligned}$$

Therefore, the set of solutions is given by $\begin{bmatrix} 8-3c \\ 3-c \\ c \end{bmatrix}$ for $c \in \mathbb{R}$.

<u>Note</u>: It can happen that there is more than one free variable in the set of

solutions.

In general, for a consistent linear system, the number of free variables is no. of variables − no. of equations.

Try: Suppose the REF of the augmented matrix of a given linear system is

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Describe the solution set.

[one answer:

$x_1 = -6x_2 - 3x_4$

x_2

$x_3 = 5 + 4x_4$

x_4

$x_5 = 7$]