Extra step for Reduced row echelon form:

<u>STEP5</u>: Beginning with the right-most pivot and working upwards and to the left, create zeros in the column above the pivot. Also use the scaling operation to make the pivots equal to 1.

Example: Convert the following matrix into RREF.

$$A = \begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & -3 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$
$$\begin{bmatrix} 1 & -1 & 1 & -3 \\ 3 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

$R_2 \mapsto R_2 - 3R_1$
$$\begin{bmatrix} 1 & -1 & 1 & -3 \\ 0 & 4 & -4 & 10 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

$R_3 \mapsto R_3 - 2R_1$
$$\begin{bmatrix} 1 & -1 & 1 & -3 \\ 0 & 4 & -4 & 10 \\ 0 & 3 & -1 & 6 \end{bmatrix}$$

$R_2 \mapsto R_2/4$:
$$\begin{bmatrix} 1 & -1 & 1 & -3 \\ 0 & 1 & -1 & 5/2 \\ 0 & 3 & -1 & 6 \end{bmatrix}$$

$R_3 \mapsto R_3 - 3R_2$
$$\begin{bmatrix} 1 & -1 & 1 & -3 \\ 0 & 1 & -1 & 5/2 \\ 0 & 0 & 2 & -3/2 \end{bmatrix}$$

$6 - \frac{15}{2}$

$R_3 \mapsto R_3/2$
$$\begin{bmatrix} 1 & -1 & 1 & -3 \\ 0 & 1 & -1 & 5/2 \\ 0 & 0 & 1 & -3/4 \end{bmatrix}$$

$R_1 \mapsto R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & -1 & 5/2 \\ 0 & 0 & 1 & -3/4 \end{bmatrix}$$

$\frac{5}{2} - 3$

$R_2 \mapsto R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 7/4 \\ 0 & 0 & 1 & -3/4 \end{bmatrix}$$

$\frac{5}{2} - \frac{3}{4}$

Note: The row-echelon form of a matrix can look varied, i.e., depending on the row operations, one can get two different matrices, both in echelon form, starting from the same matrix.

However, the <u>reduced</u> echelon form of a matrix is <u>unique</u>.

Application to network flow :

A network consists of a set of points, called the nodes, and directed lines connecting some or all of the nodes. The flow is indicated by a number or a variable.

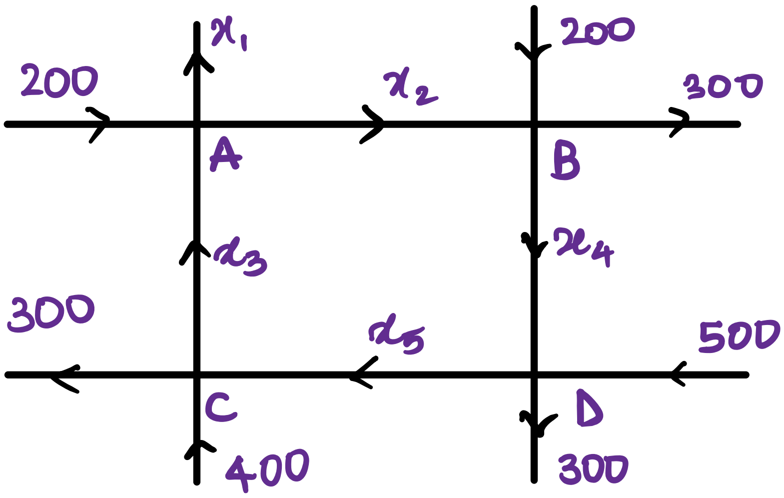
Two assumptions are followed:

1) The total flow into a node is equal to the total flow out of a node.

2) The total flow into the network is equal to the total flow out of a network.

<u>Example</u>: The picture below represents a system of one way roads in a part of a

city and the traffic flow along the roads between the junctions:



Find x_1, x_2, x_3, x_4 and x_5.

<u>Solution</u>: We first equate the total flow into each node with the total flow out of the same node:

Node A: $200 + x_3 = x_1 + x_2$

Node B: $200 + x_2 = 300 + x_4$

Node C: $400 + x_5 = 300 + x_3$

Node D: $500 + x_4 = 300 + x_5$.

We then equate the total flow in and out of the network:

$$400 + 200 + 200 + 500 = 300 + 300 + 300 + x_1$$

We get:

$$\begin{aligned}
x_1 + x_2 - x_3 \quad &= 200 \\
x_2 \quad - x_4 \quad &= 100 \\
x_3 \quad - x_5 &= 100 \\
x_4 - x_5 &= -200 \\
x_1 \quad &= 400
\end{aligned}$$

gives the foll. augmented matrix:

$$\begin{bmatrix}
1 & 1 & -1 & 0 & 0 & 200 \\
0 & 1 & 0 & -1 & 0 & 100 \\
0 & 0 & 1 & 0 & -1 & 100 \\
0 & 0 & 0 & 1 & -1 & -200 \\
1 & 0 & 0 & 0 & 0 & 400
\end{bmatrix}$$

$R_1 \leftrightarrow R_5$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 400 \\ 0 & 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 1 & 1 & -1 & 0 & 0 & 200 \end{bmatrix}$$

$R_5 \mapsto R_5 - R_1$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 400 \\ 0 & 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 1 & -1 & 0 & 0 & -200 \end{bmatrix}$$

$R_5 \mapsto R_5 - R_2$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 400 \\ 0 & 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & -1 & 1 & 0 & -300 \end{bmatrix}$$

$R_5 \mapsto R_5 + R_3$ $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 400 \\ 0 & 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & 0 & 1 & -1 & -200 \end{bmatrix}$$

$R_5 \mapsto R_5 + R_4$: $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 400 \\ 0 & 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_4 - x_5 = -200$

$x_5 = t$. $\quad x_4 = t - 200$

$x_3 = 100 + t$

$$x_2 = t - 100$$

$$x_1 = 400 .$$

Thus, for any $t \in \mathbb{N}$

$$(x_1, x_2, \dots, x_5) = (400, t-100, t+100, t-200, t).$$

Since no. of vehicles cannot be negative,

$$t \geq 200 .$$

Application to economics :

An economy is divided into sectors.
We know the total output for each sector as well as how outputs are exchanged among the sectors. The value of the total output of a given sector is known as the price of the output.

Example : An economy consists of 3 sectors : A, B, C which purchase from each other according to the table below:

	Proportion of output from sector:		
	A	B	C
purchased by sector A	0.2	0.6	0.1
purchased by sector B	0.4	0.1	0.5
purchased by sector C	0.4	0.3	0.4

If possible,
Find the value of the output of each sector so that the income matches the expenditure.

(Observe that the rows give the expenditure of each sector.)

Solution: Let p_A, p_B and p_C be the value of the total outputs i.e., income of sectors A, B and C respectively. Then the table can be seen as:

Purchased by: \ Output of:	A	B	C
A	$0.2p_A$	$0.6p_B$	$0.1p_C$
B	$0.4p_A$	$0.1p_B$	$0.5p_C$
C	$0.4p_A$	$0.3p_B$	$0.4p_C$

incomes

expenditures

$$p_A = 0.2p_A + 0.6p_B + 0.1p_C$$

$$p_B = 0.4p_A + 0.1p_B + 0.5p_C$$

$$p_C = 0.4p_A + 0.3p_B + 0.4p_C$$

leading to the (homogeneous) linear equations:

$$-0.8P_A + 0.6P_B + 0.1P_C = 0 \rightarrow \text{eqn1}$$

$$0.4P_A - 0.9P_B + 0.5P_C = 0$$

$$0.4P_A + 0.3P_B - 0.6P_C = 0$$

Note: The variables are P_A, P_B and P_C, so for eg. in eqn1:

$P_A = 0.2P_A + 0.6P_B + 0.1P_C \rightsquigarrow 0 = (0.2-1)P_A + 0.6P_B + 0.1P_C$

so $-0.8P_A + 0.6P_B + 0.1P_C = 0$.

Augmented matrix:

$$\begin{bmatrix} -8 & 6 & 1 & 0 \\ 4 & -9 & 5 & 0 \\ 4 & 3 & -6 & 0 \end{bmatrix}$$

row operations

$$\begin{bmatrix} 16 & 0 & -13 & 0 \\ 0 & 12 & -11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore (P_A, P_B, P_C) = \left(\frac{13}{16}t, \frac{11}{12}t, t\right).$$