LECTURE-12

Review: basis and dimension.

<u>Exercise</u>: Consider the set of 3×3 upper triangular matrices:

$$U = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} : a,b,c,d,e,f \in \mathbb{R} \right\}$$

(1) Show that U is a subspace of $M_3(\mathbb{R})$

(2) Find a basis of U.

(3) What is the dimension of U?

Usefulness of basis: Given a basis $\{v_1, \dots, v_n\}$ of an n-dimensional vector space V, every vector in V can be <u>uniquely</u> expressed as a linear combination of $v_1, \dots, v_n$.

<u>Example</u>: Consider $V = \mathbb{R}^3$, and two bases:

$$B_1 = \{ (1,0,0), (0,1,0), (0,0,1) \}$$

$$B_2 = \{ (1,0,0), (1,1,0), (1,1,1) \}.$$

Express the vector $(1,2,3)$ using basis $\mathcal{B}_1$ as well as basis $\mathcal{B}_2$:

<u>Solution</u>: Using $\mathcal{B}_1$ as basis:

$$(1,2,3) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1).$$

Comparing component-wise on both sides,

$$c_1 = 1, \quad c_2 = 2, \quad c_3 = 3.$$

So $(1,2,3) = 1\cdot(1,0,0) + 2(0,1,0) + 3(0,0,1)$.

Using $\mathcal{B}_2$ as basis:

$$(1,2,3) = c_1(1,0,0) + c_2(1,1,0) + c_3(1,1,1)$$

$$\Rightarrow \quad \begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_2 + c_3 &= 2 \\ c_3 &= 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_3 &= 3, \\ c_2 &= -1, \\ c_1 &= -1. \end{aligned}$$

So $(1,2,3) = -1\cdot(1,0,0) - 1\cdot(1,1,0) + 3(1,1,1)$.

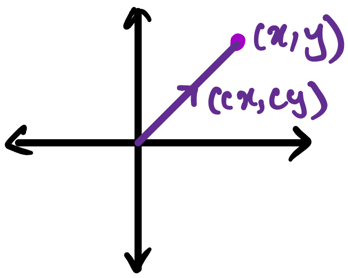
Linear Transformations

Suppose $x \in \mathbb{R}^n$. When we multiply x on the left by A : Ax, it transforms the vector x. Further, this happens for every point $x \in \mathbb{R}^n$, so we have a mapping $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, defined by $T(x) = Ax$.

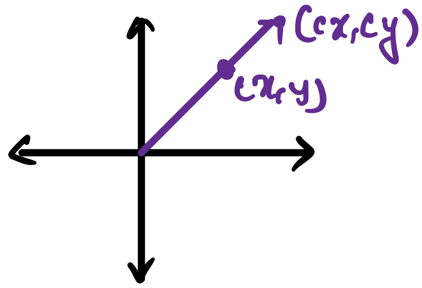
Let us see four such transformations (on $\mathbb{R}^2$):

1. $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

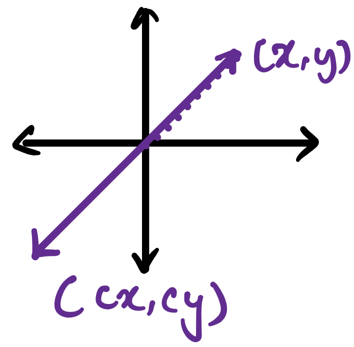
   Then $A = cI$ <u>stretches</u> every vector by the same factor c.
   The whole of $\mathbb{R}^2$ expands or contracts (or goes through the origin and out the opposite side)



$0 < c < 1$



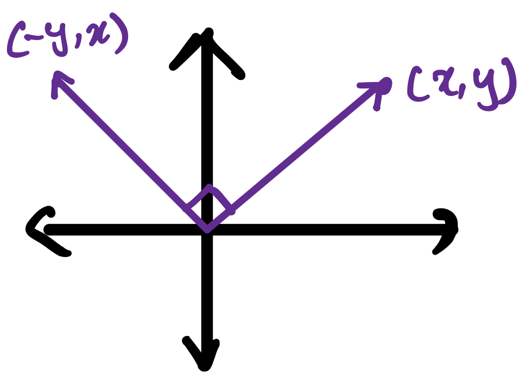
$c > 1$



$c < 0$.

2. A rotation matrix turns the whole space around the origin.

eg. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates every vector anti-clockwise by $90°$:



3. A <u>reflection</u> matrix transforms every vector into its mirror image along a certain line.

eg. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

reflects a vector (x,y) along the line $y = x$.

4. A <u>projection</u> matrix collapses the whole space onto a lower-dimensional subspace.

eg. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ collapses the y-coordinate to zero for any vector (x,y).

i.e., $(x,y) \mapsto (x,0)$.

so $\mathbb{R}^2$ is transformed to the x-axis.

$\boxed{QQ}$ Can every transformation of the plane $\mathbb{R}^2$ be encoded using a matrix?

What about the transformation that takes $(x,y) \mapsto (x+a, y+a)$?

(This is a translation in of the vector (x,y) by (a,a).)

Answer: No, not every transformation can be defined using a matrix, eg. translations. The ones that can be are "linear" transformations.

Def. A linear transformation T from a vector space V into a vector space W is a map that assigns to each vector $x \in V$ a unique vector $T(x) \in W$ such that

(i) $T(u+v) = T(u) + T(v) \quad \forall u, v \in V$

(ii) $T(cu) = c \cdot T(u) \quad \forall u \in V, c \in \mathbb{R}$.

Examples

(1) $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (y, x)$

Let us check that this is a lin. transformation

Let $u = (x_1, y_1)$
$v = (x_2, y_2)$ be any two vectors in $\mathbb{R}^2$.

Checking (i): $T(u+v) = T(u) + T(v)$:

$$\begin{aligned} LHS = T(u+v) &= T((x_1,y_1)+(x_2,y_2)) \\ &= T((x_1+x_2,\ y_1+y_2)) \\ &= (y_1+y_2,\ x_1+x_2) \\ &= (y_1,x_1) + (y_2,x_2) \\ &= T(u) + T(v) = RHS. \end{aligned}$$

Checking (ii):

$$\begin{aligned} \underset{LHS}{T(cu)} &= T(c(x_1,y_1)) \\ &= T((cx_1, cy_1)) \\ &= (cy_1, cx_1) \\ &= c(y_1, x_1) \\ &= c\,T(u) = RHS. \end{aligned}$$

Therefore, the given map is a linear transformation. More examples:

(B) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$T(x,y) = (x+2y,\ 3x-y)$ is a lin. trans.

(c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$T(x, y, z) = (x, y+z, z+x)$ is a lin. trans.

(d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$T(x, y) = (x+1, y+2)$ is <u>NOT</u> a lin trans.

because $T(\bar{u} + \bar{v}) = T((x_1+x_2, y_1+y_2))$

$= (x_1+x_2+1, \; y_1+y_2+2)$

$T(\bar{u}) + T(\bar{v}) = (x_1+1, y_1+2) + (x_2+1, y_2+2)$

$= (x_1+x_2+2, \; y_1+y_2+4)$

So $T(\bar{u}+\bar{v}) \neq T(\bar{u}) + T(\bar{v})$.

→ Check that $T(c\bar{u}) \neq cT(\bar{u})$ as well.

<u>An important class of linear transformations:</u>

Any matrix leads immediately to a linear transformation. Let $A \in M_{m\times n}(\mathbb{R})$

Then we have : $T: \mathbb{R}^n \to \mathbb{R}^m$

$x \mapsto Ax$.

and T can be checked to be a linear

transformation:

Let $\bar{u}, \bar{v} \in \mathbb{R}^n$. Then $T\bar{u} \in \mathbb{R}^m$, $T\bar{v} \in \mathbb{R}^m$

since $T\bar{u} = A\bar{u}$

$T\bar{v} = A\bar{v}$.

So $T(\bar{u}+\bar{v}) = A(\bar{u}+\bar{v}) = A\bar{u}+A\bar{v} = T\bar{u}+T\bar{v}$.

Therefore, property (i) is satisfied.

Further, $T(c\bar{u}) = A(c\bar{u}) = cA\bar{u} = cT(\bar{u})$

So property (ii) is satisfied.

let us see this with an example:

let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(x_1, x_2, x_3) = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1+2x_2+3x_3 \\ 4x_1+5x_2+6x_3 \end{bmatrix}$$

Then, for any $c \in \mathbb{R}$, and $\bar{u} = (x_1, x_2, x_3)$,

$c\bar{u} = (cx_1, cx_2, cx_3)$. So $T(c\bar{u}) = A(c\bar{u})$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix} = \begin{bmatrix} cx_1+2cx_2+3cx_3 \\ 4cx_1+5cx_2+6cx_3 \end{bmatrix}$$

$$= c \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \end{bmatrix} = c\,T(\bar{u}).$$

$$\therefore \quad T(c\bar{u}) = c\,T(\bar{u}).$$

Therefore, this is a linear transformation.

In fact, we can do the opposite: starting with a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, can find a $m \times n$ matrix A such that T is given by $Tx = Ax$!

Examples (same as before :)

(A) $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \leftrightarrow \quad \underset{\downarrow}{\overset{T(\bar{x})}{T(x,y)}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{=A} \overset{A\bar{x}}{\overbrace{\begin{bmatrix} x \\ y \end{bmatrix}}}$

$T(x,y) = (y, x)$

(B) $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \leftrightarrow \quad T(x,y) =$

$T(x,y) = (x + 2y, 3x - y)$ $\quad \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}}_{=A} \begin{bmatrix} x \\ y \end{bmatrix}$

(c)

$T : \mathbb{R}^3 \to \mathbb{R}^2$ $\iff$ $T(x, y, z) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$

$T(x, y, z) = (x+z, y+z)$

$$\underset{A}{\overset{\parallel}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}}$$

Q. How do we find the matrix A given a transformation T?

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