LECTURE-10

Review

Null space of a matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

then
$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives $u = c$, $v = c$, $w = -c$

for any $c \in \mathbb{R}$.

i.e., $N(A) = \left\{ c \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} : c \in \mathbb{R} \right\}$

Exercise: Describe the column space and null space of the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 5 \end{bmatrix}.$$

From the definition of column space,

$$C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

(can you see that this simplifies to :

$$\left\{ c \begin{bmatrix} 1 \\ 0 \end{bmatrix} : c \in \mathbb{R} \right\} ?$$

Notice that $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = (c_1 - c_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

and c_1, c_2 are just scalars, so we can denote it using 'c'.)

$$C(B) = \left\{ c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 3 \end{bmatrix} : c_1, c_2, c_3 \in \mathbb{R} \right\}$$

(what does this simplify to ?)

$$C(D) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 5 \end{bmatrix} : c_1, c_2, c_3 \in \mathbb{R} \right\}$$

(Does this set simplify further ?)

The "simplification" we are doing is just removing extra/redundant elements.

We would like to study subsets that span a vector space / subspace as "efficiently" as possible. In order to do so, we need the concept of "Linear Independence":

<u>Definition</u>: A set of vectors $\{v_1, v_2, \dots, v_p\}$ in a vector space V is said to be linearly independent if the vector equation

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = \bar{0} \quad — 1$$

has only the trivial solution $\begin{matrix} c_1 = 0 \\ c_2 = 0 \\ \vdots \\ c_p = 0 \end{matrix}$.

If not, that is, suppose there is a

solution to eqn (1) with some c_i's non-zero,

for $c_1, \dots, c_p$

then the vectors $v_1, \dots, v_p$ are said to be

linearly dependent.

Example

1) $V = \mathbb{R}^2$

The set $\{v_1 = (1,2), v_2 = (2,0), v_3 = (0,1)\}$

is linearly dependent because

$$\underbrace{1 \cdot (1,2)}_{\substack{c_1 v_1 \\ (c_1 = 1)}} - \underbrace{\frac{1}{2}(2,0)}_{\substack{c_2 v_2 \\ (c_2 = -\frac{1}{2})}} - \underbrace{2(0,1)}_{\substack{c_3 v_3 \\ (c_3 = -2)}} = \underbrace{(0,0)}_{\bar{0}}.$$

On the other hand, the set $\{w_1 = (2,0), w_2 = (0,1)\}$

is linearly independent because

if we try to solve:

$$c_1 w_1 + c_2 w_2 = \bar{0}$$

i.e., $c_1 (2,0) + c_2 (0,1) = (0,0)$

we get the equations

$c_1 \cdot 2 + c_2 \cdot 0 = 0$ → equating 1st component on both sides

$c_1 \cdot 0 + c_2 \cdot 1 = 0$

↳ equating 2nd components on both sides

and we see $2c_1 = 0$, $c_2 = 0$ $\Rightarrow c_1 = c_2 = 0$.

So $c_1 = c_2 = 0$ is the <u>ONLY</u> solution,

thus, the set $\{(2,0), (0,1)\}$ is **linearly independent**.

2). $V = \mathbb{P}$ (set of polynomials with coefficients in $\mathbb{R}$)

$v_1 = 1$ (constant polynomial)

$v_2 = t$ $\quad v_3 = 4 - t$

Then $\{v_1, v_2, v_3\}$ is linearly dependent because $4v_1 + (-1)v_2 - v_3 = 0$.

How to we check if a given set of vectors are linearly dependent or independent?

<u>Example</u> : Determine whether the vectors

$$v_1 = (1,2) \qquad v_2 = (2,1) \qquad v_3 = (3,0)$$

are linearly independent or not.

<u>Solution</u> : We want to see whether there are scalars $c_1, c_2, c_3 \in \mathbb{R}$ not all zero, so that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \bar{0}.$$

i.e., $c_1(1,2) + c_2(2,1) + c_3(3,0) = (0,0)$

This leads to the linear system :

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + c_2 + 0c_3 &= 0 \end{aligned}$$

This gives $2c_1 = -c_2$ and $c_1 - 4c_1 + 3c_3 = 0$

so $c_3 = c_1$.

$\therefore$ This has infinitely many solutions.

In particular, taking $c_1 = 1$, $c_2 = -2$, $c_3 = 1$,

$$(1,2) - 2(2,1) + (3,0) = (0,0).$$

Therefore, the vectors v_1, v_2, v_3 are linearly dependent.

<u>NOTE</u>: The intuitive idea for linear dependence is to see if one vector is "dependent" on the remaining, i.e., given $\{v_1, \dots, v_m\}$, whether v_1 (or any other vector in the set) can be expressed as a linear combination of the remaining vectors.

Some properties that hold for any vector space:

① If a set contains only ONE vector. Is it always linearly independent?

<u>Ans</u>: We want to see if our set is $\{v\}$, then $c \cdot v = 0 \Rightarrow c = 0$?

Notice: $c \cdot v = 0$ forces $c = 0$ only if v is not the zerovector.

Therefore we infer:

1) $\{v\}$ where $v \neq \bar{0}$ is ALWAYS linearly independent.

2) $\{\bar{0}\}$ is ALWAYS linearly dependent.

↳ in fact, any set with $\bar{0}$ in it is always linearly dependent!

(2) Consider a set of two vectors : $\{v_1, v_2\}$.

If these are linearly dependent, this would mean

$$c_1 v_1 + c_2 v_2 = 0$$

with one (and hence both) constants c_1 & c_2 to be non-zero.

Therefore, $c_1 v_1 = -c_2 v_2$.

$$v_1 = -\frac{c_2}{c_1} v_2.$$

In other words, v_1 is a (scalar) multiple of v_2!

Similarly, $v_2 = -\frac{c_1}{c_2} v_1$ so v_2 is a scalar multiple of v_1.

If either c_1 or c_2 is zero, it forces both c_1 & c_2 to be zero, so we conclude:

$\{v_1, v_2\}$ is lin. dependent if and only if v_1 and v_2 are scalar multiples of each other.

Example 4) The columns of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

are linearly dependent, since $C_2 = 3C_1$.

Alternatively,

$$-3 \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 3 \\ 6 \\ -3 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} = 0$$

is a linear combination involving non-zero scalars that is equal to zero.

The rows are also linearly dependent:

$$2R_2 - 5R_1 = R_3$$

Example 5) The columns of $\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

are linearly independent.

Pf. We need to show that if

$$c_1 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} = 0$$

Then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Let us express the above equation as a system of linear equations.

$$3c_1 + 4c_2 + 2c_3 = 0 \quad - ⓐ$$
$$c_2 + 5c_3 = 0 \quad - ⓑ$$
$$2c_3 = 0 \quad - ⓒ$$

ⓒ $\Rightarrow c_3 = 0$, then ⓑ $\Rightarrow c_2 = 0$ and using

$c_2 = c_3 = 0$ in ⓐ we get $c_1 = 0$.

Observe that we have shown that the system $A\bar{c} = 0$ has the only solution $\bar{c} = \bar{0}$, But solutions to $A\bar{c} = \bar{0}$ as we saw earlier form the <u>null space</u> of A.

This indicates (and the above reasoning can be shown to hold for any matrix) that the following is true:

The columns of A are linearly independent if and only if we have $N(A) = \{0\}$.

Example: The non-zero rows of a matrix in echelon form are linearly independent.

Further, if we pick out the columns that contain the pivots, they are also linearly independent.

eg if $U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the REF of $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$

pivot columns are C_1 and C_3

then $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix} \right\}$ is a linearly independent set.

<u>Def</u> The <u>rank of a matrix A</u> is the no. of linearly independent rows (or columns) of the matrix.

It can be found by counting the no. of non-zero rows in the row-echelon form of the matrix.

<u>Example</u> : $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$

has the row echelon form $U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Then, the rank of A is 2, since the REF has two non-zero rows.

(We will see more about rank of a matrix later in the course)

We now check our understanding of some key concepts covered in the following example:

Example (★)

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

1. What is $C(A)$?
2. What is $N(A)$?
3. Determine whether the vectors $(1,2,1)$, $(1,3,1)$ and $(0,1,0)$ are linearly independent in $\mathbb{R}^3$.
4. What is the rank of A?