LECTURE-13.

Finding the matrix of a linear transformation

Given : $T : V \longrightarrow W$

$\dim V = n$, $\dim W = m$

Step 0 : Fix a basis of V : $\{v_1, v_2, \ldots, v_n\}$

and a basis of W : $\{w_1, w_2, \ldots, w_m\}$

Step 1 : Calculate Tv_1, and express it as a linear combination of $w_1, \ldots w_m$:

i.e., find scalars $a_{11}, a_{21}, \ldots, a_{m1} \in \mathbb{R}$

so that

$$Tv_1 = a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m .$$

Similarly for $Tv_2, \ldots Tv_n$:

$$Tv_2 = a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m$$

$$\vdots$$

$$Tv_n = a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m$$

<u>Step 3</u>: Write the scalars column-wise:

$$\begin{array}{cccc} \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & & \uparrow \\ \text{Scalars in } Tv_1 & \text{Scalars in } Tv_2 & & \text{Scalars in } Tv_n \end{matrix} \end{array}$$

<u>Examples</u>.

1. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by
$$T(x,y) = (3x-y, y).$$
Find the matrix of this linear transformation using the standard basis $\{e_1, e_2\}$ i.e., $B = \{(1,0), (0,1)\}$.

<u>Solution</u>: We start by finding Te_1 and Te_2 and expressing them as linear combinations of e_1 & e_2:

$$Te_1 = T(1,0) = (3\cdot 1 - 0, 0) = (3,0)$$
$$Te_2 = T(0,1) = (3\cdot 0 - 1, 1) = (-1,1).$$

Now, $(3,0) = 3(1,0) + 0\cdot(0,1)$

$(-1,1) = -1(1,0) + 1\cdot(0,1)$

So the matrix is given by:

$$\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}.$$

Sometimes, the transformation T might be defined in an alternate way, like the following example. However, the method remains the same.

<u>Example 2</u>

. $V = \mathbb{R}^2$, $W = \mathbb{R}^2$, T is the transformation that rotates a vector anti-clockwise by 90°. Find the matrix of this lin. transformation.

using the standard basis e_1, e_2.

<u>Solution</u>:

The standard basis for $\mathbb{R}^2$: $(1,0)$ and $(0,1)$.

Then $T(1,0) = (0,1)$

$T(0,1) = (-1,0)$

Next, we write the images of the basis elements as linear combination of the basis elements of W $(= \mathbb{R}^2)$.

$$T(1,0) = (0,1) = 0\cdot(1,0) + 1\cdot(0,1)$$

$$T(0,1) = (-1,0) = -1\cdot(1,0) + 0\cdot(0,1)$$

Thus, the matrix of this linear transformation is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Example 3

Let us do an example where V and W are different vector spaces:

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$

$$T(x,y,z) = (x+y, y+z)$$

Step 0: Choose bases: $\mathbb{R}^3$: $(1,0,0), (0,1,0), (0,0,1)$

$\mathbb{R}^2$: $(1,0), (0,1)$

Step 1: Find the images of basis elements of $\mathbb{R}^3$, and express them using the chosen basis of $\mathbb{R}^2$:

$$T(1,0,0) = (1,0) = 1\,(1,0) + 0\,(0,1)$$

$$T(0,1,0) = (1,1) = 1\,(1,0) + 1\,(0,1)$$

$$T(0,0,1) = (0,1) = 0\,(1,0) + 1\,(0,1)$$

Step 2: Write the matrix with the coefficients corresponding to $T(v_i)$ as the i^{th} column:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \leftarrow$$ This is the required matrix.

Example 4

Let P_3 be the vector space of polynomials in $\mathbb{R}[x]$ of degree at most 3.

$$D : P_3 \longrightarrow P_3$$

$$D(f) = \frac{df}{dx}$$

using the Basis $\{1, t, t^2, t^3\}$ for P_3.

Solution :

$$D(1) = 0 = 0.1 + 0.t + 0.t^2 + 0.t^3$$

$$D(t) = 1 = 1.1 + 0.t + 0.t^2 + 0.t^3$$

$$D(t^2) = 2t = 0.1 + 2.t + 0.t^2 + 0.t^3$$

$$D(t^3) = 3t^2 = 0.1 + 0.t + 3.t^2 + 0.t^3.$$

So the required matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let us do an example where we use a non-standard basis:

<u>Example:</u> Compute the matrix of the following linear transformation using the basis $\{(1,0,0), (1,1,0), (1,1,1)\}$ of $\mathbb{R}^3$:

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x,y,z) \longmapsto (2x, 2x+3y, x+z) \quad — (*)$$

Solution: First find the image of basis vectors; then express it as a lin. comb. of this basis.

$T(1,0,0) = (2,2,1)$, calculated using $(*)$.

We want to find c_1, c_2, c_3 such that

$$(2,2,1) = c_1(1,0,0) + c_2(1,1,0) + c_3(1,1,1)$$

This leads us to solve:

$$c_1 + c_2 + c_3 = 2$$
$$c_2 + c_3 = 2$$
$$c_3 = 1$$

which gives $c_3 = 1, \quad c_2 = 1, \quad c_1 = 0.$

i.e., $T(1,0,0) = 0\,(1,0,0) + 1\,(1,1,0)$
$+ 1\,(1,1,1)$

Do the same for $T(1,1,0)$ and $T(1,1,1)$:

$$T(1,1,0) = (2,5,1).$$

$$= d_1(1,0,0) + d_2(1,1,0) + d_3(1,1,1)$$

so

$$\begin{aligned} d_1 + d_2 + d_3 &= 2 \\ d_2 + d_3 &= 5 \\ d_3 &= 1 \quad \Rightarrow d_2 = 4 \Rightarrow d_1 = -3. \end{aligned}$$

So $T(1,1,0) = -3(1,0,0) + 4(1,1,0) + 1\,(1,1,1)$

Similarly, $T(1,1,1) = (2,5,2)$

$$= -3\,(1,0,0) + 3\,(1,1,0) + 2\,(1,1,1)$$

Therefore, the matrix using this basis is given by

$$A = \begin{bmatrix} 0 & -3 & -3 \\ 1 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

<u>Remark!</u> If we used the standard bases $\{e_1, e_2, e_3\}$ for $\mathbb{R}^3$, then the matrix would look like:

$$E = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Observe a key difference:

$$E\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2x+3y \\ x+z \end{bmatrix} \quad \text{but} \quad A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3y-3z \\ x+4y+3z \\ x+y+2z \end{bmatrix}$$

These don't match!

So asking for a matrix B that defines the linear transformation, i.e., B such that $T(\bar{x}) = B\bar{x}$, is accomplished ONLY using standard basis for $\mathbb{R}^n$.

(But we might have another purpose for the matrix of a given linear transf. which will benefit from an alternate basis. So it is not a useless exercise to change the basis!)

Towards an application to computer graphics:

Let us review some linear transformations of the plane $\mathbb{R}^2$, and add some new ones.

1. **Reflections:**

a) $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ satisfies $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$

so it represents reflection across the x-axis.

b) $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ satisfies $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$

so it represents reflection across the y-axis.

c) $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ satisfies $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$

so it represents reflection across the origin, on the line containing (x,y), $(0,0)$.

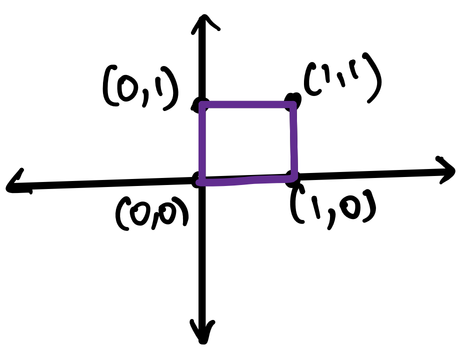
d) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

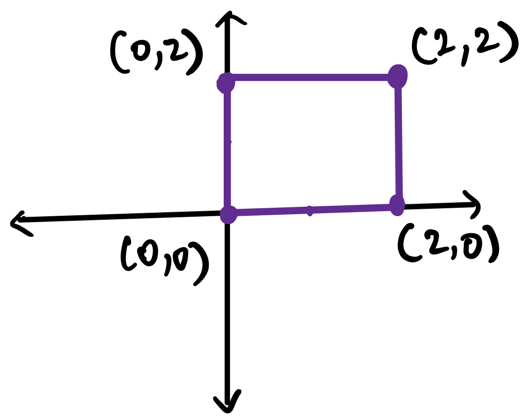
so it represents reflection across the line $y = x$.

2. Stretch Let $k \in \mathbb{R}$, $k > 0$.

a) $A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ satisfies $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$

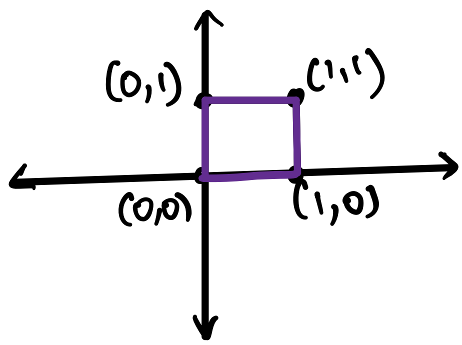
so it represents an expansion if $k > 1$
and " " compression if $0 < k < 1$.

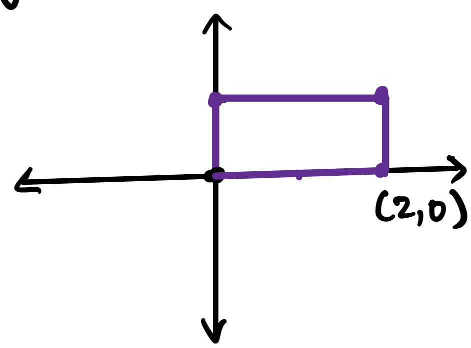




b) $A = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ satisfies $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ y \end{pmatrix}$

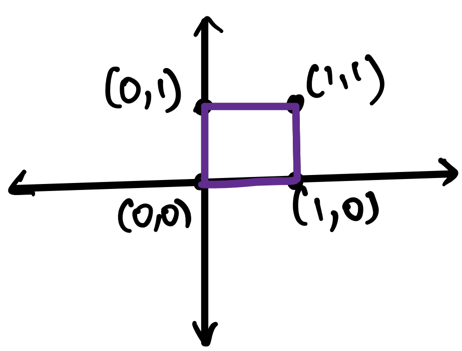
so it represents an expansion/compression in the x-direction only.

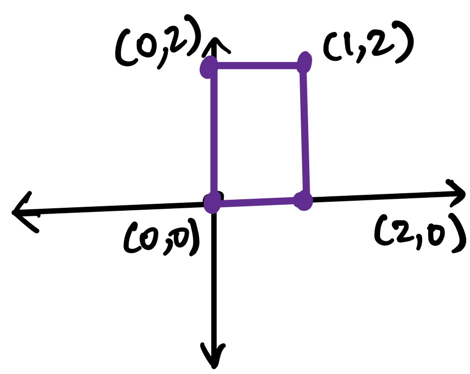




c) $A = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ satisfies $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ ky \end{pmatrix}$

so it represents an expansion/compression in the y-direction only.

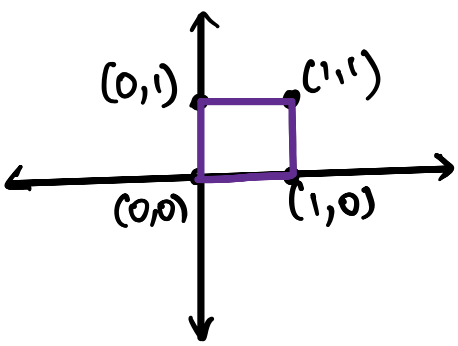


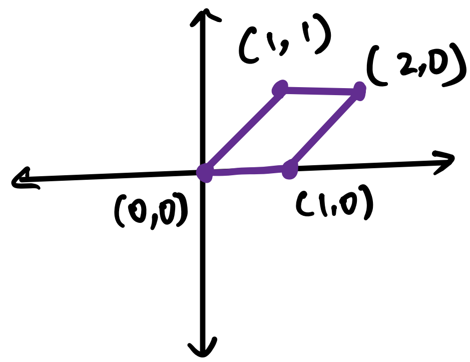


d) <u>Shear</u>

$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ satisfies $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ky \\ y \end{pmatrix}$

represents a "shear" in the x-direction.

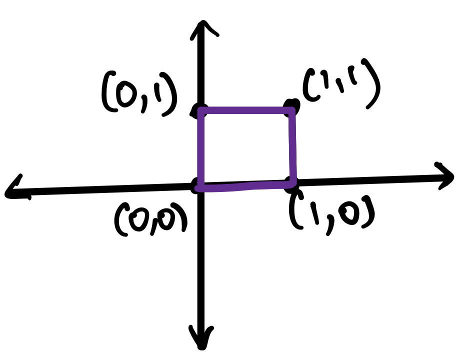


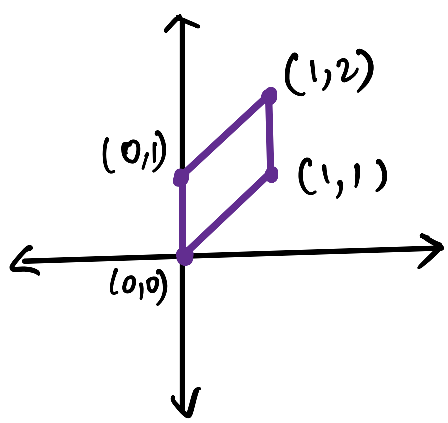


$k=1$

e) $A = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ satisfies $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ kx+y \end{pmatrix}$

represents a "shear" in the y-direction.





(Note that for $k<0$, the shearing will take place in the opposite direction).

3. <u>Rotations</u>

Anticlockwise rotation by an angle θ is given by

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$