Eigenvectors:

1) For $\lambda = 10$, $v_1 = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

Solve

2) For $\lambda = 1$,

$(A - \lambda I) v = 0$:

$$\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get $2x_1 + 2x_2 + x_3 = 0$.

We have TWO free variables, so

$$x_1 = \frac{1}{2}(-2x_2 - x_3).$$

$$\therefore \quad v = \begin{bmatrix} -x_2 - \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$v = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

Thus, we get <u>TWO</u> linearly independent eigenvectors:

$$v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

for the same eigenvalue $\lambda = 1$.

Not everytime will we get lucky to have n eigenvectors (lin. independent) for an $n \times n$ matrix.

We see a simple example of this:

Let $A = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix}$.

The char. poly. of this matrix

is $(1-\lambda)(9-\lambda)+16 = (\lambda-5)^2$.

So A has repeated eigenvalues $\lambda = 5, 5$.

Eigenvector: $(A-5I)v = 0$

$$\Updownarrow$$

$$\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot c$ is the <u>only</u> eigenvector

up to scalar multiples.

Two things I didn't mention last class:

① The polynomial we get on expanding $\det(A-\lambda I)$ for a square matrix A is called the "characteristic polynomial" of A.

(2) The eigenvalue - eigenvector relation can be seen as $\boxed{Av = \lambda v}$:

i.e., for a given matrix A, if there exists a real (or complex) number λ and a vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$) such that $Av = \lambda v$, then such a v is called an eigenvector and λ an eigenvalue of A.

<u>Defn</u>: A and B are called similar matrices if there is an invertible matrix P such that $B = P^{-1}AP$.

Similar matrices have the same eigenvalues. This is shown as follows:

Suppose $Bv = \lambda v$, i.e., λ is an eigenvalue of B with eigenvector v.

Then, $(P^{-1}AP)v = \lambda v$

so $APv = \lambda Pv$

i.e., Pv is an eigenvector of A with the same eigenvalue λ.

$\det(B - \lambda I) = \det(P^{-1}AP - \lambda I)$

Now, $P^{-1}AP - \lambda I = P^{-1}(A - \lambda I)P$

So $\det(P^{-1}AP - \lambda I) = \det P^{-1} \det(A - \lambda I) \cdot \det P$.

$= \det(A - \lambda I)$.

Therefore, the characteristic polynomials of the matrices $P^{-1}AP$ and A are the same, which means their eigenvalues are the same!

Diagonalizability of matrices

Working with diagonal matrices is easy! We can check for invertibility, find the determinant, find its eigenvalues etc in a much simpler way.

If our matrix A is not diagonal, it can be useful to know if it is similar to a diagonal matrix.

This is especially useful in one application:

Powers of matrices.

Suppose we want to find A^k for a square matrix A. What kind of matrices are easiest to take powers of?

Diagonal matrices!

$$\begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}^k = \begin{pmatrix} a_{11}^k & & & \\ & a_{22}^k & & \\ & & \ddots & \\ & & & a_{nn}^k \end{pmatrix}$$

What about the matrix

$$A = \begin{pmatrix} 17 & -10 & -5 \\ 45 & -28 & -15 \\ -30 & 20 & 12 \end{pmatrix} ?$$

Suppose we wanted to get hold of A^{98}.

Let $P = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 3 \\ -2 & 3 & 0 \end{pmatrix}$. Then one

can check that $P^{-1} = \begin{pmatrix} -3 & 2 & 1 \\ -2 & 4/3 & 1 \\ 3 & -5/3 & -1 \end{pmatrix}$

Further, if $D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, then

it can be checked that $\boxed{A = PDP^{-1}}$.

So $A^{98} = \underbrace{(PDP^{-1}) \cdots (PDP^{-1})}_{98 \text{ times}} = PD^{98}P^{-1}$

$$= P \begin{pmatrix} 3^{98} & & \\ & 2^{98} & \\ & & 2^{98} \end{pmatrix} P^{-1},$$

which is much, much easier to calculate. How do we find this special matrix P?

Answer: Using eigenvectors of A.

<u>Proposition</u>: Suppose the $n \times n$ matrix A has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix P, then $P^{-1}AP$ is a diagonal matrix D. The eigenvalues are on the diagonal of D.

<u>Proof</u>: Let $x_1, x_2, \ldots, x_n$ be the n lin. independent eigenvectors of A. Form the matrix P by setting x_i to be the i^{th} column of P.

Then,

$$AP = A \cdot [x_1 \cdots x_n]$$

column vectors

$$= [Ax_1 \cdots Ax_n]$$

$$= [\lambda_1 x_1 \cdots \lambda_n x_n]$$

Now, we can express the above $n \times n$ matrix as

$$\begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

(Recall: this is just multiplying the matrix P to the <u>left</u> by elementary matrices, which result in elementary operations to the <u>columns</u> of A)

Therefore, $AP = PD$.

$$\Rightarrow P^{-1}AP = D.$$

Remarks:

1. Not all matrices are diagonalizable. The above proof tells us that if there is a matrix P such that $P^{-1}AP$ is D, then the diagonal elements of D have to be eigenvalues and columns of P have to be the corresponding eigenvectors.

2. If a matrix has distinct eigenvalues then it is diagonalizable. Thus, diagonalization can fail if and only if there are repeated eigenvalues.

3. The diagonalizing matrix is not unique, but up to ordering of columns and multiplication of

the columns by scalars, it is unique

Using QR-factorization to compute eigenvalues (not to be tested in final exam)

Suppose A is an $n \times n$ matrix. Let

$$A = Q_0 R_0$$

be a QR factorization of A, and set

$A_1 = R_0 Q_0$.

Observe: $Q_0 A_1 Q_0^T$

$$= Q_0 R_0 Q_0 Q_0^T$$

$$= Q_0 R_0 \quad \text{(since } Q_0 \text{ is an orthogonal matrix so } Q_0 Q_0^T = I\text{)}$$

$$= A.$$

i.e., $A = Q_0 A_1 Q_0^T$.

Now, let $A_1 = Q_1 R_1$ be the QR factorization of A_1. Set $A_2 = R_1 Q_1$.

By the same logic as before,

$$A_1 = Q_1 A_2 Q_1^T$$

$$\text{So } A = Q_0 (Q_1 A_2 Q_1^T) Q_0^T$$

$$= (Q_0 Q_1) A_2 (Q_0 Q_1)^T$$

Continue this process: Once A_m has been created, let $A_m = Q_m R_m$ be the QR factorization of A_m, and create $A_{m+1} = R_m Q_m$.

Stop when A_m seems to look upper triangular (or if it seems like A_m's are not converging to an upper triangular matrix.)

Final Exam

6 Dec 2023 9:30 - 12:30 40 marks.

Syllabus:
- Vector spaces and subspaces, basis and dimension, matrix of a linear transformation.
- Matrices for computer graphics.
- Orthogonality, pattern recognition application,
- Orthogonal projection, Gram-Schmidt process, QR factorization, least squares
- Determinants, eigenvalues, eigenvectors.