# Orthogonal complements

Let $W$ be a subspace of $\mathbb{R}^n$, and suppose $z$ is a vector that is orthogonal to <u>every</u> vector in $W$. Then $z$ is said to be orthogonal to $W$.

The set of all such vectors $z$ that are orthogonal to $W$ is called the orthogonal complement of W, and is denoted by $W^{\perp}$.

(pronounced W "perp") $W^{\perp} := \{ z \in \mathbb{R}^n \mid z \cdot w = 0 \ \forall w \in W \}$.

<u>Examples</u> 1. In $\mathbb{R}^2$, what is $W^{\perp}$, if

$W = \{ (x,0) : x \in \mathbb{R} \}$?

we want $(a,b) \in \mathbb{R}^2$ such that

$(a,b) \cdot (x,0) = 0$ for all $x \in \mathbb{R}$.

i.e, $ax = 0$ for all $x \in \mathbb{R}$

$\Rightarrow a = 0$. $\therefore$ $W^{\perp} = \{ (0,b) : b \in \mathbb{R}^2 \}$

2. Compute $W^{\perp}$ where

$$W = \text{span} \left\{ \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

<u>Soln.</u> We want $(x,y,z)$ such that

$(x,y,z) \cdot (1,7,2) = 0$ — ①

<u>AND</u>

$(x,y,z) \cdot (-2,3,1) = 0$ — ②

Observe that if the above conditions are satisfied, then $(x,y,z)$ is orthogonal to every vecctor in W, which would be of the form $\alpha(1,7,2) + \beta(-2,3,1)$.

① gives $x + 7y + 2z = 0$

② gives $-2x + 3y + z = 0$

We get a system of linear equations!

$$\left(\begin{array}{ccc|c} 1 & 7 & 2 & 0 \\ -2 & 3 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 7 & 2 & 0 \\ 0 & 17 & 5 & 0 \end{array}\right)$$

$$17y + 5z = 0 \rightarrow y = -\frac{5}{17}z$$

$$x + 7y + 2z = 0$$

$$\begin{aligned} x &= -7y - 2z \\ &= \frac{35}{17}z - 2z \\ &= \frac{1}{17}z. \end{aligned}$$

So the solutions are $\left\{ c \begin{pmatrix} \frac{1}{17} \\ -5/17 \\ 1 \end{pmatrix} \mid c \in \mathbb{R} \right\}$

or $\operatorname{span}\left\{ \begin{pmatrix} 1 \\ -5 \\ 17 \end{pmatrix} \right\}$

$$\therefore \quad W^{\perp} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -5 \\ 17 \end{pmatrix} \right\}.$$

**Defn.** A set of vectors $\{u_1, \dots, u_p\}$ in $\mathbb{R}^n$ is said to be an <u>orthogonal set</u> if each pair of distinct vectors from the set is orthogonal, i.e., if $u_i \cdot u_j = 0$ whenever $i \neq j$.

Example: Let $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$

Then,
$$u_1 \cdot u_2 = 3(-1) + 1(2) + 1(1) = 0$$
$$u_2 \cdot u_3 = -1(-1/2) + 2(-2) + 1(7/2) = 0$$
$$u_1 \cdot u_3 = 3(-1/2) + 1(-2) + 1(7/2) = 0$$

so the set $\{u_1, u_2, u_3\}$ is orthogonal.

<u>Note</u>: If we plot these points in $\mathbb{R}^3$ and connect them to the origin $(0,0,0)$, the line segments will be

mutually perpendicular.

<u>Theorem</u> If $S = \{u_1, \dots, u_p\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, then $S$ is linearly independent.

<u>Proof</u> If $0 = c_1 u_1 + \cdots + c_p u_p$

for some scalars $c_1, c_2, \dots, c_p$, then

$$
\begin{aligned}
0 &= 0 \cdot u_1 \\
&= (c_1 u_1 + \cdots + c_p u_p) \cdot u_1 \\
&= c_1 (u_1 \cdot u_1) + c_2 (u_2 \cdot u_1) \\
&\quad + \cdots + c_p (u_p \cdot u_1)
\end{aligned}
$$

$$0 = c_1 (u_1 \cdot u_1)$$

(since $u_i$'s form an orthogonal set, $u_i \cdot u_j = 0$ for $i \neq j$)

Now, $u_1 \neq 0$ and we have the property that $v \cdot v = 0 \Leftrightarrow v = 0$

Thus, $u_1 \cdot u_1 \neq 0$.

This means that $c_1 (u_1 \cdot u_1) = 0$ forces $c_1$ to be zero.

The same arguement can be repeated starting with $0 = 0 \cdot u_i$ to show that $c_i = 0$ for $i = 2, 3, \ldots, p$.

Thus, $c_1 = c_2 = \cdots = c_p = 0$ and we conclude that $u_1, \ldots, u_p$ are linearly independent.

__Defn.__ An orthogonal basis for a subspace $W$ of $\mathbb{R}^n$ is a basis for $W$ that is also an orthogonal set.

Why are we interested in orthogonal bases? The following theorem shows that, with an orthogonal basis, the scalars in a linear combination can be computed very easily:

__Theorem__ Let $\{u_1, \ldots, u_p\}$ be an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$. For each $y \in W$, the scalars in the linear combination

$$y = c_1 u_1 + \cdots + c_p u_p$$

are given by:

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j} \quad j = 1, \dots, p.$$

<u>Proof</u>: If

$$y = c_1 u_1 + \cdots + c_p u_p$$

then

$$y \cdot u_i = (c_1 u_1 + \cdots + c_p u_p) \cdot u_i$$

$$= c_i (u_i \cdot u_i)$$

and as before, we know that $u_i \cdot u_i \neq 0$,

so

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i} ; \quad i = 1, \dots, p.$$

Example: We saw earlier that

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

are orthogonal. Now we also know that $\{u_1, u_2, u_3\}$ are linearly independent in $\mathbb{R}^3$, and hence form a basis of $\mathbb{R}^3$.

Side remarks: ① If $V$ is a vector space of dimension $n$ and $\{v_1, \ldots, v_n\}$ is ANY linearly ind. set of $n$ vectors then the set $\{v_1, \ldots, v_n\}$ is automatically a basis of $V$.

② The ONLY vector space of dimension $0$ is the zero vector space: $\{0\}$

③ If $U$ is a subspace of $V$ then $\dim U \leq \dim V$, and $\dim U = \dim V$ if and only if $U = V$.

Consider the vector $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$. If we wanted to express this vector as a linear combination of $u_1$, $u_2$ and $u_3$, that is

$$y = c_1 u_1 + c_2 u_2 + c_3 u_3,$$

instead of solving linear equations we can use inner products!

$$c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1}$$

$$= \frac{6(3) + 1(1) - 8(1)}{3(3) + 1(1) + 1(1)}$$

$$= \frac{11}{11} = 1$$

$$c_2 = \frac{y \cdot u_2}{u_2 \cdot u_2}$$

$$= \frac{6(-1) + 1(2) - 8(1)}{(-1)(-1) + 2(2) + 1(1)}$$

$$= \frac{-12}{6} = -2$$

$$c_3 = \frac{y \cdot u_3}{u_3 \cdot u_3} = \frac{-33}{33/2} = -2.$$

$$\therefore \; y = u_1 - 2u_2 - 2u_3.$$

**Example 2**

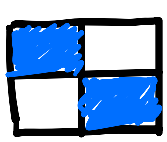
Let $u_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and

$$x = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

Verify that $\{u_1, u_2, u_3\}$ is an orthogonal set. This then becomes an orthogonal basis for $\mathbb{R}^3$. Express $x$ as a linear combination of $\mathbb{R}^3$.

## Application to pattern recognition:

In order to input a $2 \times 2$ coloured block (say, with blue and white) into a computer, we first convert it into a $4 \times 1$ vector by assigning a 1 to each block that is blue and a 0 to each block that is white.



$$\rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \quad v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

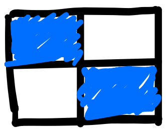
Notice that

$$Bv = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $B\bar{0} = \bar{0}$:

$$B\bar{0} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Moreover, if $\bar{w}$ is any vector that is not $v$ or $\bar{0}$, then $B\bar{w} \neq \bar{0}$.

Thus, if the computer receives a $2 \times 2$ coloured block and has to verify if it is  or not, taking the $4 \times 1$ vector $x$ corresponding to the given block, it computes $B\bar{x}$

- If $B\bar{x} \neq \bar{0}$ then the given block does NOT match the pattern.

- If $B\bar{x} = 0$, then $x = v$ or $x = \bar{0}$. We can then distinguish between the two by checking if $x = \bar{0}$ or not.

Two points remain to be explained:

1) How do we form the matrix $B$?
2) How are we guaranteed that $B\bar{x}$ will be non-zero for every vector $x \neq v$, $x \neq \bar{0}$?

We will answer these next class!