LECTURE-15

Kernel and Range

Consider a linear transformation

$$T: \mathbb{R}^n \to \mathbb{R}^m.$$

matrix computed using standard basis for $\mathbb{R}^n, \mathbb{R}^m$.

Suppose A is the standard matrix for T. Then, the range (or image) of the lin. trans'n T is given by

$$\mathrm{Im}(T) \text{ or } R(T) = \{T(x) : x \in \mathbb{R}^n\}$$

$$= \{Ax : x \in \mathbb{R}^n\}.$$

It is not hard to see that $R(T)$ is the set of all linear combinations of the columns of the matrix A, so $R(T)$ is the same as the column space of A.

On the other hand, the set

$$\{ x \in \mathbb{R}^n : T(x) = 0 \}$$

is the kernel of the linear transformation T and is the same as the set

$$\{ x \in \mathbb{R}^n : Ax = 0 \},$$

which is the nullspace of A.

However, we will talk about kernel and range without bringing up the matrix of the linear transformation from now on, because we want to extend the idea to arbitrary real vector spaces V and W:

<u>Def</u>. Suppose that $T: V \rightarrow W$ is a linear transformation from a real vector

space V into a real vector space W.

Then the set

$$R(T) := \{T(u) : u \in V\}$$

is called the range of T and the set

$$\ker(T) := \{u \in V : T(u) = 0\}$$

is called the kernel of T.

Examples

1.
$$\begin{aligned} T : V &\to W \\ v &\mapsto 0 \quad \text{for all } v \in V. \end{aligned}$$

Then. $\ker(T) = V$
$R(T) = \{0\}$

2. (Identity)
$T(v) = v.$
$$\begin{aligned} T : V &\to V \\ v &\mapsto v \quad \text{for all } v \in V. \end{aligned}$$

Then, $R(T) = V$, $\ker(T) = \{0\}$

3. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, projection onto x-axis

$T(x,y) = (x,0)$

Then, $R(T) = \{(x,0) \mid x \in \mathbb{R}\}$ (x-axis)

$\ker(T) = \{(0,y) \mid y \in \mathbb{R}\}$ (y-axis)

4. $T: P_3 \longrightarrow P_2$, where $T(p(x)) = p'(x)$,

the derivative of $p(x)$.

What is the kernel of this map? $\{a_0 \mid a_0 \in \mathbb{R}\} = \mathbb{R}$

range of this map? P_2

$$P_3 = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

Proposition: Suppose $T: V \to W$ is a linear transformation. Then $\ker(T)$ is a subspace of V, while $R(T)$ is a subspace of W.

Proof: Exercise?

Theorem: Suppose that $T: V \to W$ is a linear transformation from an n-dimensional real vector space V into a real vector space W. Then

"Rank-Nullity Theorem" $\rightarrow$ $\boxed{\dim \ker(T) + \dim R(T) = n = \dim V}$

Problems

1. Consider:

$T: \mathbb{R}^3 \to \mathbb{R}^3$ with (standard basis) matrix $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{pmatrix}$

Find Ker(T), Range(T), and its dimensions.
Verify the rank-nullity theorem.

2. Verify the rank-nullity theorem in the examples presented earlier in this lecture

Solutions

1. $T(\bar{x}) = A(\bar{x})$, so from page 1 of this lecture, $\ker T$ = Null space of $A = N(A)$

$N(A)$ is calculated by solving

$$\begin{pmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Augmented matrix: $\begin{pmatrix} 1 & -1 & 3 & 0 \\ 5 & 6 & -4 & 0 \\ 7 & 4 & 2 & 0 \end{pmatrix}$.

$R_2 \mapsto R_2 - 5R_1$, $R_3 \mapsto R_3 - 7R_1$ gives $\begin{pmatrix} 1 & -1 & 3 & 0 \\ 0 & 11 & -19 & 0 \\ 0 & 11 & -19 & 0 \end{pmatrix}$

$R_3 \mapsto R_3 - R_2$ gives $\begin{pmatrix} 1 & -1 & 3 & 0 \\ 0 & 11 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Solving,

$$x_1 - x_2 + 3x_3 = 0$$

$$11x_2 - 19x_3 = 0 \qquad \therefore \; x_2 = \frac{19}{11}x_3$$

$$x_1 = x_2 - 3x_3$$
$$= -\frac{14}{11}x_3$$

so the nullspace is given by

$$N(A) = \text{span}\left\{\begin{pmatrix} -14/11 \\ 19/11 \\ 1 \end{pmatrix}\right\} \text{ or span}\left\{\begin{pmatrix} 14 \\ 19 \\ 11 \end{pmatrix}\right\}.$$

$\therefore$ dim ker T = dim $N(A) = 1$, since $\{(-14, 19, 11)\}$ is a basis of $N(A)$.

Range (T) = $C(A)$ = column space of A.

dim Range (T) = no. of lin. ind. columns of A

= rank (A)

= no. of non-zero rows in REF of A

$= 2$ $\left[\text{REF is } \begin{pmatrix} 1 & -1 & 3 \\ 0 & 11 & -19 \\ 0 & 0 & 0 \end{pmatrix}\right]$

T is a lin. transformation from $\mathbb{R}^3 \to \mathbb{R}^3$, and dim $\mathbb{R}^3 = 3$.

Now, $\dim \ker T + \dim \text{Range}\, T = 1 + 2 = 3$
$= \dim \mathbb{R}^3$

So Rank-Nullity theorem is verified.

Examples from earlier:

2\. 1.

$T : V \to W$

$v \mapsto 0$ for all $v \in V$.

Then. $\ker(T) = V \to \dim \ker(T) = \dim V$

$R(T) = \{0\} \to \dim R(T) = 0$.

So $\dim \ker(T) + \dim R(T) = \dim V + 0 = \dim V$.

$\therefore$ Rank-nullity theorem is verified.

2\.

(Identity) $T : V \to V$, $T(v) = v$ for all $v \in V$.

Then, $R(T) = V$, $\ker(T) = \{0\}$

So $\dim \ker(T) + \dim R(T) = 0 + \dim V = \dim V$

$\therefore$ Rank-nullity theorem is verified.

3. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$$T(x,y) = (x,0)$$

Then, $R(T) = \{(x,0) \mid x \in \mathbb{R}\}$ (x-axis)

$\dim R(T) = 1$

$\ker(T) = \{(0,y) \mid y \in \mathbb{R}\}$ (y-axis)

$\dim \ker(T) = 1$

so $\dim \ker T + \dim R(T) = 1+1 = 2 = \dim \mathbb{R}^2$

$\therefore$ Rank-nullity theorem is verified.

4. $T : P_3 \longrightarrow P_2$, where $T(p(x)) = p'(x)$,

the derivative of $p(x)$.

$R(T) = P_2$, $\ker T = \{a_0 \mid a_0 \in \mathbb{R}\} = \mathbb{R}$

$\therefore \dim R(T) = 3$, $\dim \ker T = 1$

$\dim \ker T + \dim R(T) = 1 + 3 = 4 = \dim P_3$.

$\therefore$ Rank-nullity theorem is verified.

Observation :

Let T be a linear transformation from $V \to W$

Can you show that $T(0) = 0$?

(Notice I have not defined T. So this means the above property is true for ALL linear transformations!)

Injective, Surjective map review.

- What does $T: V \to W$, T: surjective tell us about $\dim R(T)$?

Ans: T Surjective $\Rightarrow R(T) = W$.

So $\dim R(T) = \dim W$.

<u>Recall</u>: $f: X \to Y$ is injective if the foll. property is satisfied: for any x_1, x_2 such that $f(x_1) = f(x_2)$, we must have $x_1 = x_2$.

<u>Lemma</u>: If V and W are vector spaces

,

then $T: V \to W$ is injective $\iff \ker T = \{0\}$

<u>Proof</u>: If T is injective, then clearly $\ker T = \{0\}$ since every linear transformation maps 0 to 0.

If $\ker T = \{0\}$, and v_1 and v_2 are vectors in V such that

$$T(v_1) = T(v_2),$$

then $T(v_1 - v_2) = 0$

$\Rightarrow v_1 - v_2 = 0$

$\Rightarrow v_1 = v_2$,

so T is injective.

$T(v_1) - T(v_2) = 0$

$\Updownarrow$

$T(v_1 - v_2)$ is because T is linear