TEST 2 Solutions

1. Use the formula for T to get images of basis vectors in $\mathbb{R}^4$:

$T(1,0,0,0) = (1,0,0) = 1.e_1 + 0.e_2 + 0.e_3$

$T(0,1,0,0) = (1,1,0) = e_1 + e_2 + 0.e_3$

$T(0,0,1,0) = (0,1,0) = 0e_1 + e_2 + 0e_3$

$T(0,0,0,1) = (0,0,2). = 0e_1 + 0e_2 + 2e_3$

$\therefore$ The matrix is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

2. $N(T) = \{ v \in V : T(v) = \bar{0} \}$

a) It consists of vectors in V that map to the zero vector in W, under the linear transformation T.

b) Verify that $Av_1 \neq \bar{0}$, while $A\bar{v}_2 = \bar{0}$.

So $v_1 \notin N(A)$, $v_2 \in N(A)$.

3. See this solved in Lecture 17. The given set is an orthogonal basis, so there is a simple formula for the coefficients!

4. a) Check that the properties of W being a subspace are satisfied, by taking arbitrary vectors in W.

eg. to show W is closed under addition,

let $u = (x_1, x_2, x_3) \in W \Rightarrow x_1 = 2x_2$

$v = (y_1, y_2, y_3) \in W \Rightarrow y_1 = 2y_2$

Then, $u+v = (x_1+y_1, x_2+y_2, x_3+y_3)$

Observe : $x_1 + y_1 = 2x_2 + 2y_2$

$= 2(x_2+y_2)$

$\therefore u+v \in W$.

b) Since $x_1 = 2x_2$ for a vector $(x_1, x_2, x_3) \in W$, an arbitrary vector in W is of the form $(2x_2, x_2, x_3) : x_2, x_3 \in \mathbb{R}$.

$= x_2\,(2,1,0) + x_3\,(0,0,1).$

$W = \text{span}\{(2,1,0),\ (0,0,1)\}$

Further, these vectors are linearly independent.

$\therefore$ a basis of W is $\{(2,1,0),\ (0,0,1)\}$.

5. a) You can take any matrix or (finite) set of matrices in $M_2(\mathbb{R})$ and take the span of these.

eg. $\text{span}\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$ or $\text{span}\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}$

or $\text{span}\left\{\begin{pmatrix}1&2\\3&4\end{pmatrix},\ \begin{pmatrix}2&1\\1&1\end{pmatrix}\right\}$

But the word "span" is important.

Many of you wrote one or two matrices in $M_2(\mathbb{R})$ as an example.

This is incorrect! A non-zero subspace has infinitely many vectors, not just one or two.

Other examples are : • trace 0 matrices in $M_2(\mathbb{R})$
• upper triangular matrices in $M_2(\mathbb{R})$
• diagonal matrices in $M_2(\mathbb{R})$

b) Using rank-nullity theorem,

(i) $\dim \text{Range}(T) = 3 - \dim \ker T$
$= 3 - 0$
$= 3.$

(ii) The co-domain is $\mathbb{R}^4$, which has dimension 4. Since Range (T) is not of dimension 4, the map T is not surjective.

c) Many examples can be given.
eg. $(1, 1, -1)$, $(0, -3, 2)$

$\|v\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}.$

d) $\mathbb{R}^2$.