If we know that $T: V \to W$ is a lin. trans. of vector spaces of equal dimension, then we have a bonus result, which is

T is injective $\Longleftrightarrow$ T is surjective!

This follows from the fact that

T is injective $\Leftrightarrow$ $\ker T = \{0\}$

and by rank-nullity theorem this means $R(T)$ has dimension n, and so $R(T) = W$, implying T is surjective.

Therefore, in such a case we can talk about the inverse linear map: $T^{-1}: W \to V$, which is the unique map such that

$$T^{-1} \circ T = \text{identity map on } V$$

$$\begin{tikzcd} V \arrow[r, "T"'] \arrow[rr, bend right, dashed, "\text{identity}"'] & W \arrow[r, "T^{-1}"'] & V \end{tikzcd}$$

<u>Exercise</u>

For each of the following, determine whether the linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ is injective. If yes, find also the inverse linear map $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$.

a) $T(x_1, x_2, \dots, x_n) = (x_2, x_1, x_3, \dots, x_n)$

b) $T(x_1, x_2, \dots, x_n) = (x_2, x_3, \dots, x_n, x_1)$

c) $T(x_1, x_2, \dots, x_n) = (x_2, x_2, x_3, \dots, x_n)$

The Euclidean spaces $\mathbb{R}^n$ are not just vector spaces, they have more structure to them. For example, there is a notion of distance between two vectors, and "angle" between two vectors.

These two notions are easier to work with and build on, if we view $\mathbb{R}^n$ as a special case of a general class of spaces called <u>inner product spaces</u>.

<u>Definition</u> An <u>inner product</u> is an operation on a vector space V:

$$\cdot : V \times V \longrightarrow \mathbb{R}$$
$$(u, v) \mapsto u \cdot v$$

that takes two <u>vectors as input</u> and gives a <u>scalar as the output</u>.

It satisfies the following properties:

For vectors $u, w, w \in V$ and scalar $c \in \mathbb{R}$,

a) $u \cdot v = v \cdot u$

b) $(u+v) \cdot w = u \cdot w + v \cdot w$

c) $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$

d) $u \cdot u \geq 0$ and $u \cdot u = 0 \Leftrightarrow u = 0$.

The (standard) inner product on $\mathbb{R}^n$ is what you would have learned earlier as "dot product":

if $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$

then $u \cdot v = u_1 v_1 + \cdots + u_n v_n$.

<u>Example</u> If $u = (2, -5, -1)$ and $v = (3, 2, -3)$ are vectors in $\mathbb{R}^3$, then

$$u \cdot v = (2)(3) + (-5)(2) + (-1)(-3)$$

$$= -1.$$

<u>Note</u>

1) If we view vectors in $\mathbb{R}^n$ as column vectors or $n \times 1$ matrices, then we can define the standard inner product on $\mathbb{R}^n$ as

$$\underline{u \cdot v = u^T v}.$$

2) Matrix multiplication can be expressed in terms of inner product:

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \\ a_{n1} & \cdots & a_{np} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \\ b_{p1} & \cdots & b_{pm} \end{bmatrix}}_{B}$$

Let $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ denote the rows of A, and $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_m$ denote the columns of B.

Then $AB = \begin{bmatrix} \bar{a}_1 \cdot \bar{b}_1 & \bar{a}_1 \cdot \bar{b}_2 & \cdots & \bar{a}_1 \cdot \bar{b}_m \\ \bar{a}_2 \cdot \bar{b}_1 & \bar{a}_2 \cdot \bar{b}_2 & \cdots & \bar{a}_2 \cdot \bar{b}_m \\ \vdots & & & \\ \bar{a}_n \cdot \bar{b}_1 & \bar{a}_n \cdot \bar{b}_2 & \cdots & \bar{a}_n \cdot \bar{b}_m \end{bmatrix}$

Any vector space that can be equipped with an inner product is called an inner product space.

<u>Example</u> of an inner product space that is different from $\mathbb{R}^n$:

$V = C[a,b]$, the collection of real valued functions on $[a,b] \subseteq \mathbb{R}$:

$$C[a,b] := \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous} \}$$

We have already seen that this is a real vector space. $C[a,b]$ is in fact also an inner product space, with the inner product defined as follows:

$$\langle f, g \rangle := \int_a^b f(x) g(x)\, dx .$$

angle brackets are another notation for the inner product.

Whenever we have an inner product, as a consequence we obtain the notion of a 'length' of a vector:

<u>Defn</u>. The length (or norm) of a vector v is the non-negative scalar defined by

$$\|v\| := \sqrt{v \cdot v} ,$$

and for $V = \mathbb{R}^n$, $v = (v_1, v_2, \ldots, v_n)$ we have the formula

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \quad - (*)$$

$\boxed{\text{Qn}}$ What is the std. inner product on $\mathbb{R}$?

The norm can be thought of a function

$$\|\cdot\| : V \longrightarrow \mathbb{R}_{\geq 0}$$
$$v \longmapsto \|v\|$$

and it satisfies the foll. properties.
Suppose $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

a) $\|u\| \geq 0$ and $\|u\| = 0 \iff u = 0$.

absolute value of c.

b) $\|cu\| = |c| \, \|u\|$

c) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality)

- A vector whose length is 1 is called a unit vector.

- For any vector v, the vector $\frac{v}{\|v\|}$ is a unit vector, because

$$\left\| \frac{v}{\|v\|} \right\| = \frac{1}{\|v\|} \cdot \|v\| = 1.$$

Distance in $\mathbb{R}^n$: With the norm at our disposal, we can now define distance between two vectors in $\mathbb{R}^n$:

$$d : V \times V \longrightarrow \mathbb{R}_{\geq 0}$$

$$d(u,v) = \| u - v \| .$$

If u and v are nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, then there is a nice connection between their inner product and the angle θ between the two line segments from the origin to the points identified with u and v. The formula is

$$u \cdot v = \|u\| \, \|v\| \cos\theta .$$

We are going to be interested in vectors that are perpendicular to each other.

<u>Defn</u>. Two vectors u and v in $\mathbb{R}^n$ are orthogonal to each other if $u \cdot v = 0$.

<u>Pythagorean theorem in n-dimensions:</u>

Two vectors u and v are orthogonal if and only if

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

<u>Proof</u>.

L.H.S: $\|u+v\|^2 = (u+v)\cdot(u+v)$

$$= u\cdot(u+v) + v\cdot(u+v)$$

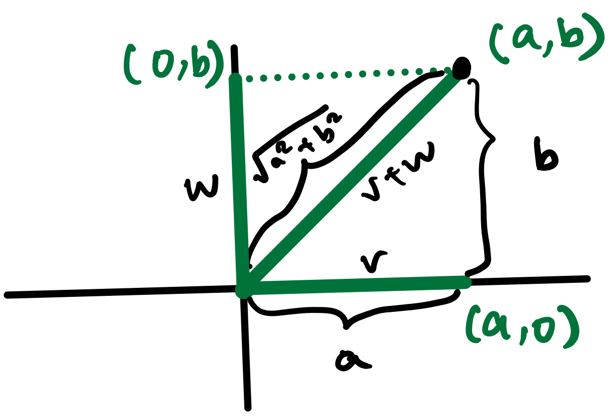
$$= u \cdot u + u \cdot v + v \cdot u + v \cdot v$$

$$= u \cdot u + v \cdot v \iff u \cdot v = v \cdot u = 0$$

i.e., u and v are orthogonal

$$\| u \|^2 + \| v \|^2$$

For $n = 2$,



$v = (a,0) \quad w = (0,b) \quad v+w = (a,b).$

Since $\|v\|^2 + \|w\|^2 = a^2 + b^2 = \|\underbrace{(a,b)}_{v+w}\|^2$,

we know that v and w are orthogonal.

Examples

1. The zero vector is orthogonal to every vector in $\mathbb{R}^n$ because $0 \cdot v = 0$.

2. The vectors $(3, -2, 1, -3)$ and $(1, 1, -1, 0)$ are orthogonal in $\mathbb{R}^4$

Exercises

1. Find the inner product of $\left(\frac{1}{2}, \frac{1}{4}, -3, \frac{1}{2}, \frac{1}{5}\right)$ and $(1, -1, 5, 1, 0)$.

2. What is the norm of $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$?

3. Let $v = (1, 2)$. Find any two vectors w_1 and w_2 in $\mathbb{R}^2$ such that $\|v - w_1\| = \|v - w_2\|$.

4. Find x, y such that $(1, 2, x, 3, 1)$ and $(y, 1, 3, \frac{1}{3}, 0)$ are orthogonal vectors.