Solutions to practice problem set II

1. a) $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. (i) $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

(ii) The equation $x_1 + x_2 - x_3 = 0$ is the same as $x_3 = x_1 + x_2$.

So the subspace consists of the vectors

$$(x_1, x_2, x_1+x_2) = x_1(1,0,1) + x_2(0,1,1).$$

This shows that the vectors $(1,0,1)$ and $(0,1,1)$ span the subspace.

Moreover, $(1,0,1)$ and $(0,1,1)$ are lin. independent

Thus, a basis is

$$\{(1,0,1), (0,1,1)\}.$$

3.

(a) Yes.

Basis : $\{x, x^2\}$

(b) No. This set is not closed under scalar multiplication : take $\alpha = -1$, $v = (1,-2)$

Then $v \in W$ but $\alpha v = (-1, 2)$ does not lie in W.

(c) No. This set is not closed under scalar multiplication : take $\alpha = -1$, $v = (1,2)$

Then $v \in W$ but $\alpha v = (-1,-2)$ does not lie in W.

(d) Yes.

Basis: $\{(0,1,0), (0,0,1)\}$

4. Find the determinant:

a) $A = \begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}$

Ans: -2

b) $A = \begin{bmatrix} -1 & 5 & 2 \\ 3 & 1 & 6 \\ -5 & 2 & 4 \end{bmatrix}$

Ans: -180

c) $A = \begin{bmatrix} 12 & 13 & 25 \\ -5 & 6 & 1 \\ 3 & -22 & -19 \end{bmatrix}$

Ans: 0

Notice $C_1 + C_2 = C_3$

C_i : i^{th} column.

so the columns are lin. dependent,

$\Rightarrow$ determinant $= 0$.

d) $A = \begin{bmatrix} 11 & 22 & 5 \\ 5 & 10 & 3 \\ 6 & 12 & -45 \end{bmatrix}$

Ans: 0.

Notice: $C_1 \times 2 = C_2$

So the columns are lin. dependent.

$\Rightarrow$ determinant $= 0$.

5. i) : $\begin{bmatrix} 4 & -3 \\ 4 & -2 \end{bmatrix}$.

ii) $T(1,1) = (1,2) = a_1(1,1) + a_2(0,-1)$

$T(0,-1) = (3,2) = b_1(1,1) + b_2(0,-1)$

Solve to get $a_1 = 1$, $a_2 = -1$, $b_1 = 3$, $b_2 = 1$

$\therefore$ matrix is $\begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$.

6.

Soln. $Tv_1 = (0,1,1) = a_1(1,3,1) + a_2(0,1,1) + a_3(1,1,0)$

$Tv_2 = (2,1,3) = b_1(1,3,1) + b_2(0,1,1) + b_3(1,1,0)$

$Tv_3 = (0,0,1) = c_1(1,3,1) + c_2(0,1,1) + c_3(1,1,0)$.

$\rightsquigarrow$ $a_1 + a_3 = 0$ $\quad 3a_1 + a_2 + a_3 = 1$

$a_1 + a_2 = 1$

$\circledast$ $2a_1 + a_2 + a_3 = 1 \Rightarrow a_1 = 0$

$a_3 = 0$

$a_2 = 1$

$$b_1 + b_3 = 2$$

$\leadsto$ $3b_1 + b_2 + b_3 = 1$ $\Rightarrow$ $b_1 + 3 + 2 = 1$

$b_1 + b_2 = 3$ $\Rightarrow$ $b_1 = -4$ $\quad b_2 = 7$

$b_3 = 6$

$\leadsto$ $c_1 + c_3 = 0$

$3c_1 + c_2 + c_3 = 0$ $\quad c_1 + 1 + 0 = 0$

$c_1 + c_2 = 1$ $\Rightarrow$ $c_1 = -1$, $\quad c_2 = 2$

$c_3 = 1$.

$\therefore$ matrix is $\begin{pmatrix} 0 & -4 & -1 \\ 1 & 7 & 2 \\ 0 & 6 & 1 \end{pmatrix}$.

7. First check that $u_1 \cdot u_2 = 0$. Since this is true, $\{u_1, u_2\}$ form an orthogonal basis for W. We can use the foll. formula for

$$\hat{v} = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \frac{4}{6}(1,2,1) + \frac{-2}{2}(1,0,-1)$$

$$= \frac{2}{3}(1,2,1) - (1,0,-1)$$

$$= \left(\frac{-1}{3}, \frac{4}{3}, \frac{5}{3}\right).$$

8. $A = \begin{bmatrix} 1 & -2 \\ 3 & -2 \\ 2 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the least-squares solution vector. This satisfies

$$A^T A x = A^T b$$

Solving:

$$\underbrace{\begin{bmatrix} 1 & 3 & 2 \\ -2 & -2 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 & -2 \\ 3 & -2 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 & 3 & 2 \\ -2 & -2 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{b}$$

$$\begin{bmatrix} 14 & -6 \\ -6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$7x_1 - 3x_2 = 3$$
$$-2x_1 + 3x_2 = -1$$

$$5x_1 = 2 \qquad \therefore x_1 = \frac{2}{5}$$

$$x_2 = \left(-1 + \frac{4}{5}\right)\frac{1}{3} = \frac{-1}{15}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$ is a least-squares solution.

9. a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$

ans. eigenvalues are $3, -2$

eigenvectors: $\alpha \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $A = \begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix}$

Eigenvalues	eigenvectors
3	$\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$
-5	$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$
6	$\begin{pmatrix} 1 \\ 6 \\ 16 \end{pmatrix}$

Q.10 $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -2 \end{pmatrix}$. Compute the QR-factorization of A.

Soln.

Let $x_1 = (2, 1, 0)$

$x_2 = (-1, 3, 1)$

$x_3 = (1, -2, -2)$

Apply Gram-Schmidt :

$v_1 = (2, 1, 0)$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= (-1, 3, 1) - \frac{1}{5}(2, 1, 0)$$

$$= \left(-\frac{7}{5}, \frac{14}{5}, 1\right)$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (1, -2, -2) - 0 \cdot v_1 - \frac{-9.5}{54}\left(-\frac{7}{5}, \frac{14}{5}, 1\right)$$

$$= (1, -2, -2) + \frac{5}{6}\left(-\frac{7}{5}, \frac{14}{5}, 1\right)$$

$$= \left(\frac{-1}{6}, \frac{1}{3}, \frac{-7}{6} \right)$$

Scaling the vectors to clear denominators,

we can take

$v_1 = (2, 1, 0)$

$v_2 = (-7, 14, 5)$

$v_3 = (-1, 2, -7)$.

Note: this is just for simplification. The method will work even if you don't do this step

$\therefore$ dividing by norms to get unit vectors, and placing them as columns, we get

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-7}{3\sqrt{30}} & \frac{-1}{3\sqrt{6}} \\ 1/\sqrt{5} & \frac{14}{3\sqrt{30}} & \frac{2}{3\sqrt{6}} \\ 0 & \frac{5}{3\sqrt{30}} & \frac{-7}{3\sqrt{6}} \end{bmatrix}$$

$R = Q^T A =$

$$\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -7/3\sqrt{30} & 14/3\sqrt{30} & 5/3\sqrt{30} \\ -1/3\sqrt{6} & 2/3\sqrt{6} & -7/3\sqrt{6} \end{bmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -2 \end{pmatrix}$$

$$= \begin{bmatrix} 5/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 18/\sqrt{30} & \frac{-15}{\sqrt{30}} \\ 0 & 0 & 3/\sqrt{6} \end{bmatrix}$$

$\therefore$ $A =$

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-7}{3\sqrt{30}} & \frac{-1}{3\sqrt{6}} \\ 1/\sqrt{5} & \frac{14}{3\sqrt{30}} & \frac{2}{3\sqrt{6}} \\ 0 & \frac{5}{3\sqrt{30}} & \frac{-7}{3\sqrt{6}} \end{bmatrix} \begin{bmatrix} 5/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 18/\sqrt{30} & \frac{-15}{\sqrt{30}} \\ 0 & 0 & 3/\sqrt{6} \end{bmatrix}$$

is a QR factorization of A.