LECTURE - 8

Cost of elimination

Qn: How many arithmetic operations does Gaussian elimination require, for n equations in n variables?

We first focus on the row echelon form of the $n \times n$ matrix A (and not the augmented matrix.)

Observe that every step involves finding what multiple of the pivot should be subtracted from the row we are applying the row operation to.

For example,
$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 3 & 2 & 5 \end{bmatrix}.$$

To make $a_{21} = 0$, we use

$$R_2 \mapsto R_2 - \frac{1}{2} R_1$$

The "$\frac{1}{2}$" is just $\frac{a_{21}}{a_{11}}$.

Similarly, to make $a_{31} = 0$, we use

$$R_3 \mapsto R_3 - \frac{3}{2} R_1,$$

and $\frac{3}{2} = \frac{a_{31}}{a_{11}}$.

So every row operation to get to row echelon form involves a "multiply and subtract" combination → call this one operation

To make $a_{21} = 0$, we have the following:

1. One division $\frac{a_{21}}{a_{11}}$
2. $(n-1)$ subtractions $a_{2j} - \frac{a_{21}}{a_{11}} \cdot a_{1j}$

$2 \leq j \leq n$

($\because$ we don't need to perform it for $j=1$, since we know it is going to become zero.)

So a total of n operations.

Similarly for $a_{31}, a_{41}, \ldots, a_{n1}$.

Thus, to make the first column into row echelon form, we need

$$n(n-1) \text{ operations.}$$

$$\overset{\shortparallel}{n^2-n}$$

The next stage we do the same thing for $(n-1) \times (n-1)$ sized submatrix, and so we perform

$$(n-1)^2 - (n-1)$$

number of operations.

Counting them up we have:

$$1^2 + 2^2 + \cdots + n^2 - (1 + 2 + \cdots + n)$$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

$$= \frac{n^3 - n}{3}.$$

Now if we had the augmented matrix we then have additionally:

$$\underbrace{(n-1)}_{\substack{\text{changes} \\ \text{while make} \\ \text{1st column} \\ \text{into echelon} \\ \text{form}}} + \underbrace{(n-2)}_{\substack{\text{2nd} \\ \text{colum.}}} + \cdots + \underbrace{1}_{\substack{n-1^{\text{st}} \\ \text{column.}}} \quad \text{operations}$$

Back-substitutions : | No. of operations

		No. of operations
Row n	: one unknown, so one division	1
Row $n-1$	: one subtraction, one division	2
Row $n-2$	: two subtractions, one division	3
$\vdots$		
Row 1	: $n-1$ subtractions, one division	n

So totally : $((n-1)+\cdots+1)+[1+2+\cdots+n]$

$\frac{(n-1)(n-2)}{2}+n(n$

$= n^2$

So totally, $\frac{n^3-n}{3}+n^2$ steps to solve $Ax=\bar{b}$.

In contrast, suppose we have the decomposition $A = LU$.

Then solving $Ax = \bar{b}$ involves first solving $L\bar{c} = \bar{b}$ and then $U\bar{x} = \bar{c}$.

L and U are triangular, so the back / forward substitution takes $\frac{n(n+1)}{2}$ steps each, so totally

$$n^2 + n.$$

Of course, to get the L & U matrices we need $\frac{n^3 - n}{3}$ steps, but the advantage comes when we want to solve $Ax = \bar{b}$ for various choices of $\bar{b}$. In using the augmented matrix method, we need to go through $\frac{n^3-n}{3}$ $+n^2$ steps <u>for every</u> $\bar{b}$. So if we wish to solve

$A x = \bar{b}_1$, $A x = \bar{b}_2$, ..., $A x = \bar{b}_k$

we perform $k\left(\frac{n^3-n}{3} + n^2\right)$ operations.

But the $A = LU$ method, we do this ONCE, taking $\frac{n^3-n}{3}$ steps, and the two triangular systems $Lc = b$, $Ux = c$ are solved in $\frac{n^2+n}{2}$ steps each, so a total of

$$\frac{n^3-n}{3} + k\cdot\left(n^2+n\right) \text{ steps.}$$

Vector spaces

A (real) vector space is a set V on which are defined two operations:

1) addition, denoted by $+$

2) scalar multiplication, denoted by $\cdot$ (or omitted)

subject to the following rules:

Given u, v, w in V and scalars $c, d \in \mathbb{R}$:

1) The sum of u and v, i.e., $u+v \in V$ (closure under addition)

2) $u+v = v+u$ (commutativity of addition)

3) $(u+v)+w = u+(v+w)$ (associativity of addition)

4) There is an element $0 \in V$ such that

$0 + u = u$ for all $u \in V$. (additive identity)

5) For each $u \in V$, there is a vector $-u$ in V such that $u + (-u) = 0$ (additive inverse)

6) For every $c \in \mathbb{R}$, $u \in V$, the product $c \cdot u \in V$ (closed under scalar mult'n)

7) $c(u+v) = cu + cv$ (distributivity of scalar mult. over vector addition)

8) $(c+d)u = cu + du$

9) $c(du) = (cd)u$ (Associativity of scalar mult.)

10) $1 . u = u$. (Multiplicative identity)

Examples

1) $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^n$, $M_2(\mathbb{R})$

2) The set $\mathbb{P}$ of polynomials with coefficients in $\mathbb{R}$.

3) The set of all functions $f : [0,1] \to \mathbb{R}$

4) Set of all sequences $\{ (a_n)_{n \geq 1} \mid a_n \in \mathbb{R} \}$

NOTE: Elements of a vector space are called "vectors". This does not always mean we are talking about elements of $\mathbb{R}^2$ or $\mathbb{R}^n$. So, if $V = \mathbb{P}$, the set of polynomials with coefficients in $\mathbb{R}$, vector space

then $u = 1 + x$, $v = x^2 + \sqrt{2}x + \frac{3}{4}$ are examples of vectors.