Week 3

Strange things happen when one studies non-linear differential equations!

Example: Solve: $y' = \sqrt{x+y}$

Let $x+y=t$

$$1+y'=t' \quad \text{so} \quad \underset{\sqrt{t}}{\underset{\parallel}{y'}} = t'-1.$$

So we get $t' = 1+\sqrt{t}$, a separable equation

$$\int \frac{dt}{1+\sqrt{t}} = \int dx + C$$

Take $1+\sqrt{t} = u$

$$t = (u-1)^2$$

$$dt = 2(u-1)du$$

The integral becomes

$$\int \frac{2(u-1)du}{u} = \int dx + C$$

So $\quad 2u - 2\log u = x + C$

i.e., $\quad 1+\sqrt{t} - \log(1+\sqrt{t}) = \frac{x}{2} + C$

Substitute $t = x+y$ to get the (implicit) solution

$$1+\sqrt{x+y} - \log|1+\sqrt{x+y}| = \frac{x}{2} + C.$$

Let's verify that this is indeed a solution:

We do this by "implicit" differentiation:

$$\frac{1}{2\sqrt{x+y}}(1+y') - \frac{1}{1+\sqrt{x+y}} \cdot \frac{1}{2\sqrt{x+y}}(1+y') = \frac{1}{2}$$

Simplifying to:

$$\frac{1+y'}{\sqrt{x+y}}\left(1 - \frac{1}{1+\sqrt{x+y}}\right) = 1$$

or $1+y' = 1+\sqrt{x+y}$, so $y' = \sqrt{x+y}$,

the original D.E!

Now try the same process for the IVP:

$$\frac{dy}{dx} = \sqrt{x-y} \quad ; \quad y(2) = 1.$$

Let $x - y = t$. $\quad 1 - y' = t'$, or $\underset{\sqrt{t}}{\overset{}{y'}} = 1 - t'$

So we have the separable equation

$$t' = 1 - \sqrt{t}.$$

Solving $\displaystyle\int \frac{dt}{1-\sqrt{t}} = \int dx + C$ — *

$1 - \sqrt{t} = u$, $\quad$ so $t = (1-u)^2$

$\sqrt{t} = 1 - u$ $\quad\quad dt = -2(1-u)\,du$

$\therefore$ The integral becomes

$$\int \frac{-2(1-u)\,du}{u} = \int dx + C$$

so $2u - 2\log u = x + C$

i.e., $2(1-\sqrt{t}) - 2\log|1-\sqrt{t}| = x + C$

In terms of x & y:

$$2(1-\sqrt{x-y}) - 2\log|1-\sqrt{x-y}| = x + C$$

But there seems to be a problem if we substitute the initial condition $y(2)=1$:

$1-\sqrt{x-y}$ is then 0, so this is <u>not</u> a valid solution to the IVP.

Observe however that, in eqn *, when we divided by $1-\sqrt{t}$, the assumption (although unsaid) was that $\sqrt{t} \neq 1$. So this general solution holds only when $1-\sqrt{t} \neq 0$ i.e., $y(a)=b$ whenever $a = b+1$.

So what about these cases?

Check that the function $y = x-1$ is then a solution to the D.E!

$$\sqrt{x-y} \rightsquigarrow 1 \qquad \frac{dy}{dx} = 1, \text{ so } \frac{dy}{dx} = \sqrt{x-1}$$

is satisfied!

(and the existence-uniqueness theorem will then tell us that this is the only one!)

Linear first-order equations

- Borrow the idea of separable equations and see if we can modify it to solve other equations.

linear diff. eqns.

Def. A linear first-order differential equation is of the form

$$\frac{dy}{dx} + P(x)\, y = Q(x).$$

Idea: Multiply both sides with a suitable function so that one side can be expressed as a derivative. Then, integrate both sides to solve the equation.

"Integrating Factor" : $\rho(x) = e^{\int P(x)\,dx}$.

Multiplying this on both sides:

$$\underbrace{e^{\int P(x)\,dx}\frac{dy}{dx} + P(x)\,e^{\int P(x)\,dx}\,y}\ = \underbrace{Q(x)e^{\int P(x)\,dx}}.$$

$$\frac{d}{dx}\left[y(x)\cdot e^{\int P(x)\,dx}\right] = Q(x)\,e^{\int P(x)\,dx}.$$

Then, $$y(x)\cdot e^{\int P(x)\,dx} = \int \left(Q(x)\,e^{\int P(x)\,dx}\right)dx + C$$

So $$y(x) = e^{-\int P(x)\,dx}\left[\int \left(Q(x)\,e^{\int P(x)\,dx}\right)dx + C\right].$$

<u>Remark</u>: There is no need to add a constt of integration while computing $\int P(x)\,dx$. Why? Because this constant gets cancelled on both sides, making its value irrelevant.

Solved examples

1. Solve: $\quad y' + 3x^2y = 6x^2$

<u>Solution</u>:

The integrating factor is $e^{\int 3x^2 dx}$

$$= e^{x^3}.$$

Multiplying both sides by this factor:

$$\underbrace{e^{x^2}y' + 3x^2e^{x^3}y} = 6x^2e^{x^3}.$$

$$\frac{d}{dx}\left(y(x) \cdot e^{x^3}\right) = 6x^2e^{x^3}$$

$$\therefore\ y(x)\, e^{x^3} = \int 6x^2e^{x^3}dx$$

$$= 2\int e^u\, du$$

$u = x^3$
$du = 3x^2 dx$

$$= 2e^{x^3} + C$$

$$\therefore \quad y(x) = 2 + Ce^{-x^3}.$$

2. Solve the IVP:

$$y' + \frac{3y}{t} = t^2 \; ; \; y(1) = \frac{1}{2}.$$

for $t > 0$.

Solution

The integrating factor is $e^{\int \frac{3}{t} dt}$

$$= e^{3 \log t} = t^3.$$

Multiplying the D.E. by t^3 on both sides,

$$\underbrace{t^3 y' + 3t^2 y}_{\frac{d}{dt}(y(t)\, t^3)} = t^5$$

$$\frac{d}{dt}(y(t)\, t^3) = t^5$$

Integrating,

$$y(t) \cdot t^3 = \int t^5 dt + C$$

$$= \frac{t^6}{6} + C$$

Therefore, the general solution is

$$y(t) = \frac{t^3}{6} + Ct^{-3}.$$

Using the initial condition $y(1) = \frac{1}{2}$:

$$\frac{1}{2} = \frac{1}{6} + C, \qquad C = \frac{1}{3}.$$

$\therefore$ The solution is

$$y(t) = \frac{t^3}{6} + \frac{1}{3t^3}.$$

Theorem : Existence-uniqueness for linear 1st order ODEs :

If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the IVP $\dfrac{dy}{dx} + P(x)y = Q(x)$; $y(x_0) = y_0$

has a unique solution $y(x)$ on I.

<u>Exercises</u>: Solve the following linear first order equations:

a) $y' + 4y = 8$, $y(0) = 1$

$[y(t) = 2 - e^{-4t}]$

b) $y' - y = t^2$, $y(0) = 4$

$[y(t) = 6e^{t} - (t^2 + 2t + 2)]$

c) $y' + ty = 5t$, $y(2) = 1$

$[y(t) = 5 - 4e^{2 - (1/2)t^2}]$

Mixture Problems

Consider a tank containing a solution: a mixture of solute and solvent.
There is an inflow and outflow, and the task is to compute the amount of solute in the tank at time t.

What info. do we have?

1. $x(0) = x_0$, the amount of solute at time 0.
2. The solution flowing into the tank has a concentration of c_i grams/Litre and flows in at rate r_i litres/sec (constant)
3. The outflow is a constant rate r_o L/sec

How do we set up the differential eqn?

We estimate the change Δx in a brief interval Δt of time.

$\Delta x = \underbrace{\text{grams input}}_{\text{in } \Delta t} - \underbrace{\text{grams output}}_{\text{in } \Delta t}$

$\qquad\quad \| \qquad\qquad\qquad\qquad \|$

$\qquad r_i c_i \Delta t \qquad\qquad\qquad r_o c_o \Delta t.$

$\hookrightarrow$ concentration of outflow at time t.

so $\dfrac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o(t)$

where $c_o(t) = \dfrac{x(t)}{V(t)}$ $\leftarrow$ volume of solvent at time t

$\hookrightarrow V(0) + (r_i - r_o)t.$

$V(0)$: initial volume

$\therefore$ The differential eqn. is

$$\frac{dx}{dt} = r_i c_i - \frac{r_o x}{V(t)},$$

which is a linear equation.

Example :

① A 50 litre tank of pure water has a brine mixture with concentration of 2 grams per litre entering at a rate of 5 liters per minute. At the same time, the well-mixed contents drain out at the rate 5 litres per minute. Find the amount of salt at time t.

Solution

inflow rate $= r_i c_i = 5\frac{L}{min} \times 2\frac{g}{L} = 10$ grams per minute.

outflow rate $= r_o c_o = 5 \cdot \frac{x(t)}{V(t)}$ g. per minute

$$V(t) = V(0) + (r_i - r_o)t$$
$$= 50$$

So the diff. eqn. is $\frac{dx}{dt} = 10 - \frac{5}{50}x(t)$

or, $$\frac{dx}{dt} + \frac{1}{10}x = 10.$$

$$P(t) = \frac{1}{10}, \quad Q(t) = 10$$

Integrating factor is $e^{\int P(t)dt} = e^{\int \frac{dt}{10}} = e^{t/10}$.

$$e^{t/10}\frac{dx}{dt} + \frac{1}{10}e^{t/10}x = e^{t/10}.10$$

$$x e^{t/10} = \int 10 e^{t/10} dt$$

$$x e^{t/10} = 100 e^{t/10} + C$$

"pure water"

$\therefore \quad x(t) = 100 + Ce^{-t/10}$.

$x(0) = 0$, so $C = -100 \therefore x(t) = 100 - 100e^{-t/10}$.

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2. A tank contains 1500L of water and 20kg of dissolved salt. Fresh water is entering at 15L/min and the solution drains at the rate 10L/min. How much salt is in the tank at t minutes and at 10 minutes?

Soln. $r_i = 15\,L/min$ $\quad c_i = 0$, since fresh water flows in.

$r_0 = 10\,L/min.$ $\quad C_0 = \dfrac{x(t)}{1500 + (15-10)t}$

$$\therefore \quad \frac{dx}{dt} = r_i c_i - r_0 c_0 = 0 - \frac{x(t)}{1500+5t} \cdot 10$$

This equation is actually separable as well.

$$\int \frac{dx}{x} = -\int \frac{10\,dt}{1500+5t} = -\int \frac{2\,dt}{300+t}$$

$$\therefore \quad \log|x| = -2\log|300+t| + c_1$$

c_1 can write as $\log C$

$$\log|x| = \log\left(|300+t|^{-2} C\right)$$

$$x = C(300+t)^{-2}$$

$x(0) = 20$, so $20 = C(300)^{-2}$

$$\therefore \quad C = 18{,}00{,}000.$$

$$x(10) = 1800000\,(310)^{-2}$$

$$\approx 18.73.$$

So, after 10 min, there is 18.73 kg of salt in the tank.

<u>Example 3</u> A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and has a concentration of $\frac{1}{5}(1+\cos t)$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?

<u>Solution</u>: Proceed as in previous examples to get the diff. eqn. as

$$x(t) = \frac{9}{5}(1+\cos t) - \frac{2x(t)}{200+t} \;;\quad x(0) = 5.$$

When does the tank overflow?

The tank overflows when $V(t) = 1500$

$$= 600 + 3t$$

$$\therefore \; t = 300.$$

The question therefore asks us to calculate $x(300)$.

We find $x(t)$ first :

$$(200+t)^2 x(t) = \frac{9}{5}\left(\frac{1}{3}(200+t)^3 + \sin t\,(200+t)^2 + 2(200+t)\cos t - 2\sin t + c\right) + c$$

$$\therefore \; x(t) = \frac{9}{5}\left(\frac{1}{3}(200+t) + \sin t + \frac{2\cos t}{200+t} - \frac{2\sin t}{(200+t)^2}\right) + \frac{c}{(200+t)^2}$$

<u>Note</u> :

$$\int (200+t)^2 \cos t$$

$$= \sin t\,(200+t)^2 - \int 2(200+t)\sin t$$

$$- \left[2(200+t) \cdot -\cos t + \int 2\cos t \, dt \right]$$

$$= \sin t \, (200+t)^2 + 2(200+t)\cos t - 2\sin t + C$$

Find c by using $x(0) = 5$: $c = -4600720$.

$\therefore \; x(300) = 279.797$ lbs.