Laplace Transforms

Defn. Given a function $f(t)$ defined for all $t \geq 0$, the Laplace transform of f is the function F defined as follows:

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt,}$$

defined for all values of s for which the integral converges.

What does it mean for an integral to converge? Since one of the limits of integration is ∞, we have to be careful.

For any integrable function g, we define

$$\int_0^\infty g(t)\,dt = \lim_{b\to\infty} \int_0^b g(t)\,dt.$$

If this limit exists, then the integral on the LHS converges. If the limit does not exist, then the limit diverges.

We now find the Laplace transforms of some functions using the definition we just encountered:

1) $f(t) = 3$ (constant function)

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot 3\,dt = 3 \int_0^\infty e^{-st}\,dt$$

$$= 3 \times \lim_{b \to \infty} \int_0^b e^{-st}\,dt$$

$$= 3 \times \lim_{b \to \infty} \left[\frac{-e^{-st}}{s} \right]_0^b$$

$$= 3 \lim_{b \to \infty} \left[\frac{-e^{-sb}}{s} + \frac{1}{s} \right]$$

$$= 3 \times 0 + \frac{3}{s} = \frac{3}{s}, \quad s > 0.$$

<u>Recall</u>: $\lim_{x \to \infty} e^{-x} = 1$, but $\lim_{x \to \infty} e^{x}$ does not exist.

In general for constant functions $f(t) = a$, the same calculation above shows that

$$\mathcal{L}\{a\} = \frac{a}{s}, \quad s > 0.$$

2) $f(t) = e^{at}, \quad t > 0.$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \lim_{b \to \infty} \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^b$$

$$= \lim_{b \to \infty} \left[\frac{e^{-(s-a)b}}{-(s-a)} + \frac{1}{s-a} \right]$$

$$= \frac{1}{s-a}, \quad s > a.$$

<u>Note</u>: The above formula holds for complex 'a' as well: If $a = \alpha + i\beta$, then

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > \alpha.$$

Proof omitted.

3. $\mathcal{L}\{t^a\}$

To compute the integral $\int_0^\infty e^{-st} t^a \, dt$, it helps to use the Gamma function:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \quad , \quad x > 0 .$$

This function has properties

$$\Gamma(1) = 1, \qquad \Gamma(x+1) = x\,\Gamma(x).$$

So if x is a natural number, say n,

then
$$\begin{aligned} \Gamma(n+1) &= n\,\Gamma(n) \\ &= n(n-1)\,\Gamma(n-1) \\ &= n(n-1)(n-2)\,\Gamma(n-2) \\ &\;\;\vdots \\ &= n(n-1)(n-2)\cdots\Gamma(1) \\ &= n! \quad , \text{ since } \Gamma(1) = 1. \end{aligned}$$

So Γ somewhat generalises the factorial function on natural numbers, to any real number. It is not easy to evaluate $\Gamma(x)$ where x is not a natural number, but one useful identity you should memorize is:

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

which helps us evaluate Laplace transforms of $t^{1/2}$, $t^{3/2}$, $t^{5/2}$ etc.

Now we compute $\mathcal{L}\{t^a\}$: $\int_0^\infty e^{-st} t^a \, dt$

Make the substitution $u = st$, so $du = s\,dt$.

$\Rightarrow t = \frac{u}{s}$ so $dt = \frac{du}{s}$.

$$\therefore \mathcal{L}\{t^a\} = \int_0^\infty e^{-u} \frac{u^a}{s^a} \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a \, du$$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^{(a+1)-1} \, du$$

$$= \frac{\Gamma(a+1)}{s^{a+1}}$$

- If $a \in \mathbb{N}$, say $a = 5$, then $\mathcal{L}\{t^5\} = \frac{5!}{s^6}$.

- If a is a "half-integer", say $a = \frac{1}{2}$

then $\Gamma(a+1) = \Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)$

(using $\Gamma(x+1) = x\Gamma(x)$)

$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{\sqrt{\pi}}{2}$, using the identity mentioned above.

$$\therefore \mathcal{L}\{t^{1/2}\} = \frac{\sqrt{\pi}}{2s^{3/2}}.$$

Linearity of Laplace Transforms.

If $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exist for $s > r_1$, $s > r_2$ respectively, then

$$\mathcal{L}\{af(t) + bg(t)\} = a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$$

for $s > \max\{r_1, r_2\}$.

Linearity helps us compute the Laplace transforms of many functions without needing to start from the definitions:

1) $f(t) = \cos kt$

Use $\cos kt = \dfrac{e^{ikt} + e^{-ikt}}{2}$

so $$\mathcal{L}\{\cos(kt)\} = \frac{1}{2}\left(\frac{1}{s-ik} + \frac{1}{s+ik}\right)$$

$$= \frac{1}{2}\left(\frac{2s}{s^2+k^2}\right)$$

$$= \frac{s}{s^2+k^2} \cdot \quad , \quad s>0.$$

Similarly, $\mathcal{L}\{\sin(kt)\} = \dfrac{k}{s^2+k^2}$, $s>0$.

use $\sinh(kt) = \dfrac{e^{kt} - e^{-kt}}{2}$ $\leftarrow$ $\mathcal{L}\{\sinh(kt)\} = \dfrac{k}{s^2-k^2}$, $s>|k|$

use $\cosh(kt) = \dfrac{e^{kt}+e^{-kt}}{2}$ $\leftarrow$ $\mathcal{L}\{\cosh(kt)\} = \dfrac{s}{s^2-k^2}$, $s>|k|$

2) $f(t) = (e^t + e^{-t})^2$

$$= e^{2t} + 2 + e^{-2t}$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s-2} + \frac{2}{s} + \frac{1}{s+2}, \quad s>2.$$

3) $f(t) = \sin(3t)\cos(2t)$ $\quad \left[\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right) \right]$

$$= \frac{1}{2}\left(\sin 5t + \sin t\right)$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{2}\,\frac{5}{s^2+25} + \frac{1}{2}\cdot\frac{1}{s^2+1}.$$

Qn Does the Laplace transform exist for any function $f(t)$, $t \geq 0$?

Ans. In general no, since we need the improper integral $\int_0^\infty e^{-st} f(t)\,dt$ to converge.

Without going into details, we record two conditions that are necessary for the Laplace Transform of f to exist:

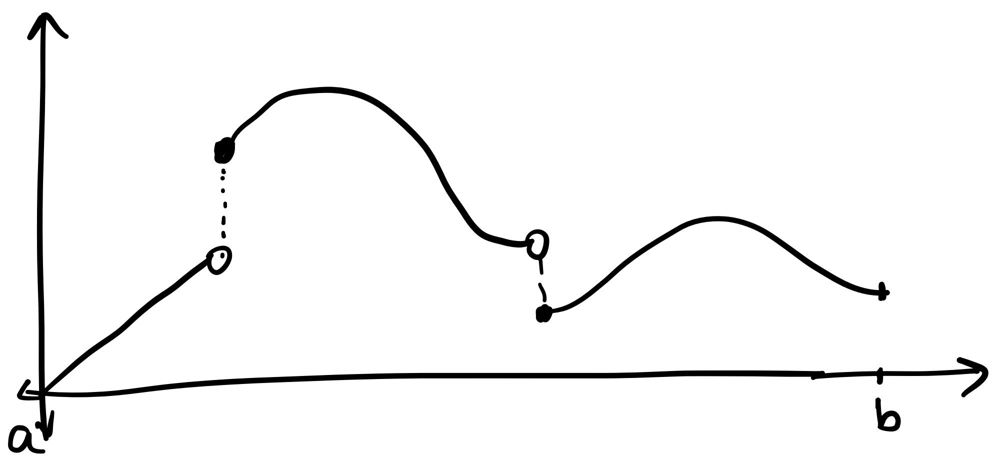
1) $f(t)$, $t \geq 0$ should be piecewise-continuous
2) $f(t)$ should be of exponential order

Piecewise continuous: "continuous in pieces"

Defn. A function f is said to be piecewise continuous on the interval $[a, b]$ if $[a, b]$ can be divided into finitely many subintervals so that

1) f is continuous on each subinterval

2) $f(t)$ has a finite limit as t approaches any point of discontinuity in the interval $[a,b]$.

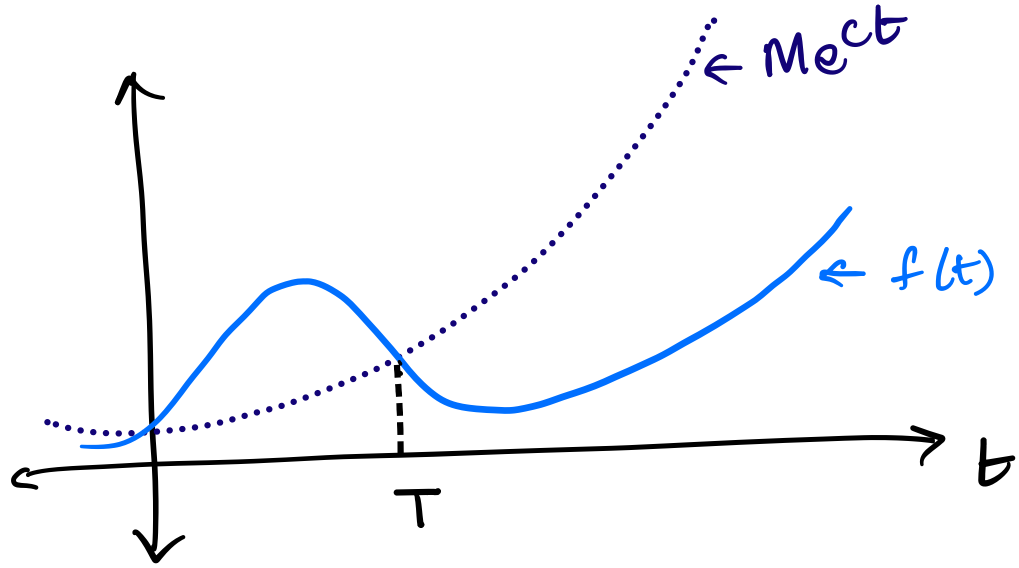


Exponential order:

A function f is said to be of exponential order as $t \to \infty$ if there exist non-negative constants M, c, T such that

$$|f(t)| \leq M e^{ct} \quad \text{for } t \geq T.$$

Loosely speaking, this means the graph of f_n eventually lies under the graph of Me^{ct} for some M, c:



An example of a function that is not of exponential order: e^{t^2}.

Inverse Transforms

If $F(s) = \mathcal{L}\{f(t)\}$, then the inverse transform of $F(s)$ is $f(t)$.

i.e., $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Examples :

1) $F(s) = \dfrac{1}{s+5}$, then $\mathcal{L}^{-1}\{F(s)\}$

$$= e^{-5t}$$

2) $F(s) = \dfrac{21}{s^3}$, then $\mathcal{L}^{-1}\{F(s)\}$

$$= \mathcal{L}^{-1}\left\{\frac{21}{2} \cdot \frac{2}{s^3}\right\}$$

$$= \frac{21}{2} \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}$$

$$= \frac{21}{2} t^2 .$$

(inverse transforms are linear as well:

$$\mathcal{L}^{-1}\{aF(s)+bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}.)$$

3) $F(s) = \dfrac{9+s}{4-s^2}$

$$\frac{9+s}{4-s^2} = \frac{9}{4-s^2} + \frac{s}{4-s^2}$$

$$= -\frac{9}{2}\cdot\frac{2}{s^2-4} - \frac{s}{s^2-4}$$

$$= -\frac{9}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\}$$

$$= -\frac{9}{2}\sinh(2t) - \cosh(2t)$$

4) $$F(s) = \frac{s}{(s+3)(s+5)}$$

Use partial fraction decomposition:

$$\frac{s}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5}$$

Solve to get $A = -\frac{3}{2}$, $B = \frac{5}{2}$

$$\therefore \frac{s}{(s+3)(s+5)} = -\frac{3}{2} \cdot \frac{1}{s+3} + \frac{5}{2} \cdot \frac{1}{s+5}$$

$$\therefore \mathcal{L}^{-1}\left(\quad , \quad \right) = -\frac{3}{2} e^{-3t} + \frac{5}{2} e^{-5t}$$

5) Find $\mathcal{L}^{-1}\left\{\frac{1}{s^3+5s^2}\right\}$

Ans. $s^3+5s^2 = s^2(s+5)$

$$\therefore \frac{1}{s^3+5s^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+5}$$

$$1 = As(s+5) + B(s+5) + Cs^2$$

$$= s^2(A+C) + s(B+5A) + 5B.$$

$$\Rightarrow \left.\begin{aligned} A+C &= 0 \\ B+5A &= 0 \\ 5B &= 1 \end{aligned}\right\} \Rightarrow$$ solve to get $B=\frac{1}{5}$, $A=\frac{-1}{25}$, $C=\frac{1}{25}$.

$$\therefore \frac{1}{s^2(s+5)} = \frac{-1}{25} + \frac{1}{5s^2} + \frac{1}{25}\cdot\frac{1}{s+5}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = \frac{-1}{25} + \frac{t}{5} + \frac{1}{25}e^{-5t}.$$