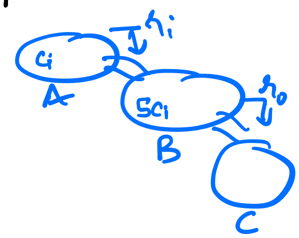
HW2 - SOLUTIONS

1. Given data: $V(0) = 1640\ km^3$. $r_i = r_o = 410\ km^3/yr.$

$V(t) = V(0) + (r_i - r_o)t$ gives $V(t) = 1640\ km^3$
(constant volume)

Let c_i be the pollutant concentration of Lake A



Then, we are given: $\dfrac{x(0)}{V(0)} = 5c_i$

$\therefore\ x(0) = 5c_i\,(1640)$.

We need to find t such that $x(t) = 2c_i(1640)$.

$$\frac{dx}{dt} = 410\,c_i - \frac{410}{1640}\,x \rightsquigarrow \frac{dx}{dt} + \frac{1}{4}x = 410\,c_i$$

This is a linear equation with integrating factor $e^{t/4}$.

Solve to get $x(t) = 1640\,c_i + Ce^{-t/4}$.

Use $x(0) = 5c_i\,(1640)$ to get $C = 4c_i\,(1640)$.

$\therefore$ t for which $x(t) = 2c_i\,(1640)$ is given by solving $2c_i(1640) = 1640\,c_i\,[1 + 4e^{-t/4}]$

$$t = 4\log 4 \approx 5.5452\ \text{years}.$$

2.

i) $xy\dfrac{dy}{dx} = y^2 + x\sqrt{4x^2+y^2}$.

Rearranging, $\dfrac{dy}{dx} = \dfrac{y}{x} + \sqrt{\dfrac{4x^2}{y^2}+1}$

Letting $v = y/x$ or $y = vx$ we get

$y' = v + xv'$

and the given differential equation becomes

$v + xv' = v + \dfrac{\sqrt{4+v^2}}{v}$ ⇝ separable.

$\therefore \displaystyle\int \frac{v\,dv}{\sqrt{4+v^2}} = \int \frac{dx}{x}$

$\left.\begin{array}{l} 4+v^2 = u \\ 2v\,dv = du \end{array}\right\} \rightarrow \displaystyle\int \frac{du}{2\sqrt{u}} = \int \frac{dx}{x}$

$u^{1/2} = \log x + C$

so $\sqrt{4+\dfrac{y^2}{x^2}} = \log x + C \Rightarrow 4 + \dfrac{y^2}{x^2} = (\log x + C)^2$

$\therefore y^2 = x^2(\log x + C)^2 - 4x^2$

$\therefore y(x) = \sqrt{x^2(\log x + C)^2 - 4x^2}$.

(ii) $x^2 y' + 2xy = 5y^4$

Dividing by x^2 throughout, the given D.E. takes the form of a Bernoulli equation:

$$y' + \frac{2}{x} y = \frac{5}{x^2} y^4 .$$

with $n = 4$. ∴ we substitute $v = y^{-3}$, or $y = v^{-1/3}$

and $y' = -\frac{v^{-4/3}}{3} \frac{dv}{dx}$.

The D.E. then becomes

$$-\frac{v^{-4/3}}{3} \frac{dv}{dx} + \frac{2}{x} v^{-1/3} = \frac{5}{x^2} v^{-4/3}, \text{ which}$$

simplifies to: $\frac{dv}{dx} - \frac{6}{x} v = -\frac{15}{x^2}$

This is a linear D.E, with integrating factor $\frac{1}{x^6}$.

Solve to get $v = \frac{15}{7x} + Cx^6 = y^{-3}$

$$\therefore y^3(x) = \frac{7x}{15 + 7Cx^7} .$$

(iii) $y' = x^2+2xy+y^2 = (x+y)^2$

Soln. Using the substitution $v = x+y$

$$\frac{dv}{dx} = 1+\frac{dy}{dx}$$

we get

$$\frac{dv}{dx} - 1 = v^2$$

i.e., $\frac{dv}{dx} = v^2+1$ a separable equation.

Solving,

$$\int \frac{dv}{v^2+1} = \int dx + C$$

$$\tan^{-1} v = x + c$$

or, $v = \tan(x+C)$

i.e., $y(x) = \tan(x+C) - x$.

3.

$$\underbrace{\left(x^3 + \frac{y}{x}\right)}_{M(x,y)} dx + \underbrace{\left(y^2 + \log x\right)}_{N(x,y)} dy = 0 \text{ .}$$

$$\frac{\partial M}{\partial y} = \frac{1}{x} \qquad \frac{\partial N}{\partial x} = \frac{1}{x}, \quad \text{so the eqn. is exact.}$$

Let the solution be given by $F(x,y) = C$.

$\frac{\partial F}{\partial x} = M$

$\therefore F(x,y) = \int M\,dx + g(y)$

Then, $F(x,y) = \int \left(x^3 + \frac{y}{x}\right) dx + g(y)$

$$= \underbrace{\frac{x^4}{4} + y \log x + g(y)}$$

$\frac{\partial F}{\partial y} = N$

$$\frac{\partial F}{\partial y} = N \rightsquigarrow \log x + g'(y) = y^2 + \log x \text{ .}$$

$$\therefore \quad g'(y) = y^2$$

$$\text{So} \quad g(y) = \frac{y^3}{3} + c_0 \text{ .}$$

$\therefore$ The general solution is given by

$$\frac{x^4}{4} + y \log x + \frac{y^3}{3} = C \text{ .}$$

4. Show that the substitution $v = \log y$ transforms the differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)\, y \log y$$

into the linear equation

$$\frac{dv}{dx} + P(x) = Q(x)v(x).$$

Use this idea to solve:

$$x\frac{dy}{dx} - 4x^2y + 2y\log y = 0.$$

Solution: If $v = \log y$, then $\frac{dv}{dx} = \frac{1}{y}\frac{dy}{dx}$

Thus, dividing the given D.E. on both sides by y, and using the substitutions $v = \log y$, $v' = \frac{y'}{y}$,

we get

$$v' + P(x) = Q(x)v.$$

Now, $x\dfrac{dy}{dx} - 4x^2y + 2y\log y = 0$

is equivalent to

$$\frac{dy}{dx} - 4xy = -\frac{2}{x}\, y\log y$$

So here $P(x) = -4x$, $Q(x) = -\dfrac{2}{x}$.

The substitution $v = \log y$ transforms the equation to

$$\frac{dv}{dx} - 4x = -\frac{2}{x}v \rightsquigarrow \frac{dv}{dx} + \frac{2}{x}v = 4x$$

The integrating factor here is $e^{\int \frac{2}{x}dx}$ is x^2.

$$\underbrace{x^2\frac{dv}{dx} + 2xv}_{\frac{d}{dx}(v\cdot x^2)} = 4x^3$$

$$\frac{d}{dx}(v\cdot x^2) = 4x^3 \quad \therefore\ vx^2 = x^4 + C$$

and so $\underset{\log y}{\underset{\shortparallel}{v(x)}} = x^2 + Cx^{-2}$

$\therefore\ y(x) = e^{\left(x^2 + \frac{C}{x^2}\right)}$ is the general solution.

5.

Solve:

$$\frac{dP}{dt} = kP(M-P) \quad ; \quad P(0) = P_0,$$

<u>Solution</u>: Since k and M are constants, we see that this is a separable equation. Using separation of variables:

$$\int \frac{dP}{P(M-P)} = k\int dt + C$$

$$\frac{A}{x} + \frac{B}{M-x}$$

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$ gives the equation

$$1 = A(M-P) + BP = AM + (B-A)P$$

$AM = 1$

$\Rightarrow A = \frac{1}{M}$

$B - A = 0 \Rightarrow A = B$

$$\therefore \; A = \frac{1}{M}, \quad B = \frac{1}{M}.$$

So $$\int \frac{dP}{P(M-P)} = \frac{1}{M}\log|P| - \frac{1}{M}\log|M-P| + \log C$$

$$= kt$$

$$\therefore \quad \log\left|\frac{Pc}{M-P}\right|^{1/M} = kt$$

$$\frac{Pc}{M-P} = (e^{kt})^M$$

$$Pc = (M-P)(e^{kt})^M$$

$$P(c + e^{ktM}) = M e^{ktM}$$

$$P = \frac{M e^{kMt}}{c + e^{kMt}} = \frac{M}{ce^{-kMt} + 1}$$

Using the initial condition : $P(0) = P_0$,

$$P_0 = \frac{M}{c+1} \qquad \text{So, } c = \frac{M - P_0}{P_0}$$

Plugging this value into the general solution :

$$\therefore \; P(t) = \frac{M P_0}{(M - P_0)e^{-kMt} + P_0}.$$