Linear independence of functions:

Let $f(t)$ and $g(t)$ be functions on $\mathbb{R}$. Then, they are called linearly dependent if there are non-zero constants c_1 and c_2 with

$$c_1 f(t) + c_2 g(t) = 0$$

for all t. Otherwise, they are lin. independent.

Note: "all" t is important. The functions could have $f(x_0) = g(x_0)$ at some points x_0 but not all, and this is enough for them to be classified as linearly independent.

eg. $y_1 = \sin x$, $y_2 = \cos x$. These are linearly independent, even though $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$ (and this repeats periodically, so inf. many points!)

Def Given two functions f and g, the **Wronskian** of f and g is the determinant

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g .$$

→ Do examples.

Theorem 3 Suppose that y_1 and y_2 are two solutions of the homogeneous 2nd order linear equation

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval I on which p and q are continuous.

(a) If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) \equiv 0$ on I.

(b) If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ for any point on I.

↳ note how this is a "strong" converse.

Proof a) If y_1 and y_2 are linearly dependent there is a constant $k \neq 0$ such that if $y_2 = k y_1$, i.e., $y_2(x) = k y_1(x)$ for all x in I.

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Then, $$W(y_1, y_2) = \begin{vmatrix} y_1(x) & k y_1(x) \\ y_1'(x) & k y_1'(x) \end{vmatrix}$$

$$= k y_1'(x) y_1(x) - k y_1'(x) y_1(x)$$

$$= 0, \text{ for any } x.$$

b) We show that if $W(y_1, y_2) = 0$ for some point in I, then y_1 and y_2 are linearly dependent:

Suppose x_0 is in I such that

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0.$$

$\therefore$

$$y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0) = 0$$

But these are coefficients of the linear system

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0$$

This means the homogeneous linear system

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has a non-trivial solution: $(c_1, c_2) \neq (0,0)$.

[This follows from the general fact that $A\bar{x} = \bar{0}$ has a non-trivial solution (i.e., $\bar{x} \neq \bar{0}$) if and only if the columns of A are linearly dependent. An equivalent condition for this is $\det A = 0$.]

Use these c_1 and c_2 to form

$$Y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Then, $Y(a) = 0, \quad Y'(a) = 0$.

Thus, Y has to be the function that is identically 0 by the theorem on existence-uniqueness

$$\Rightarrow \quad c_1 y_1(x) + c_2 y_2(x) = 0 \quad \forall x \in I,$$

with $(c_1, c_2) \neq (0,0)$.

$\therefore$ y_1 and y_2 are linearly dependent.

Theorem 4 General Solutions of Homogeneous Equations:

Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation

$$y'' + p(x) y' + q(x) y = 0,$$

with p and q cont's on an open interval I.

If $Y(x)$ is any solution of this eqn. on I, then there exist numbers c_1 and c_2 such that

$$Y(x) = c_1 y_1 + c_2 y_2.$$

Conclusion: For homogeneous 2nd order lin. eqns, We only need to find two linearly independent solutions, then these two describe all possible solutions!

Remark: We are actually considering the vector space of all solutions of the hom. eqn. and finding a basis for it. Once we have a basis, every solution can be expressed as a lin. combination of the basis solutions.

Example Let $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$.

These are solutions of the diff. eqn:

$$y'' - 4y = 0.$$

But $y_3(x) = \cosh(2x)$ and $y_4(x) = \sinh(2x)$ are also solutions:

Using: $\frac{d}{dx}(\sinh(kx)) = k\cosh(kx)$ for $k \in \mathbb{R}$,

$$\frac{d}{dx}(\cosh(kx)) = k\sinh(kx)$$

This can be easily verified.

So are we getting four lin. ind. solutions?

No, because $\sinh(kx) = \dfrac{e^{kx} - e^{-kx}}{2}$

$$\cosh(kx) = \frac{e^{kx} + e^{-kx}}{2}$$

So $y_3(x) = \frac{1}{2} y_1(x) + \frac{1}{2} y_2(x)$

and $y_4(x) = \frac{1}{2} y_1(x) - \frac{1}{2} y_2(x)$.

Examples.

Given a homogeneous eqn, two solutions y_1 and y_2 and a pair of initial conditions, find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the initial conditions:

1. $y'' - 3y' = 0$; $y_1 = 1$, $y_2 = e^{3x}$; $y(0) = 4$, $y'(0) = -2$

Soln. (Verify that y_1 and y_2 solve the D.E.)

The wronskian $W(1, e^{3x}) = \begin{vmatrix} 1 & e^{3x} \\ 0 & 3e^{3x} \end{vmatrix}$

$= 3e^{3x}$, which does not vanish on $\mathbb{R}$.

$\therefore$ y_1 and y_2 are lin. independent solutions.

Thus, a general solution of the diff. eqn. is of the form $y(x) = c_1 + c_2 e^{3x}$.

$y'(x) = 3c_2 e^{3x}$

$y(0) = 4 \rightsquigarrow 4 = c_1 + c_2$

$y'(0) = -2 \rightsquigarrow -2 = 3c_2$

Solving, we get $c_2 = -\frac{2}{3}$, so $c_1 = 4 + \frac{2}{3} = \frac{14}{3}$

$\therefore$ A particular solution is given by

$$y(x) = \frac{14}{3} - \frac{2}{3} e^{3x}.$$

Try on your own : 2. $y'' + 2y' + y = 0$; $y_1 = e^{-x}$, $y_2 = xe^{-x}$

$y(0) = 2$; $y'(0) = -1$.

Solution : $y(x) = 2e^{-x} + xe^{-x}$.

3. Show that $y_1(x) = x^2$ and $y_2(x) = x^3$ are two different solutions of $x^2 y'' - 4xy' + 6y = 0$, both satisfying the initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = 0.$$

Why do these facts not contradict Theorem 2 with respect to uniqueness?

Soln. Observe that the theorem applies to diff. eqns of the form $y'' + p(x)y' + q(x)y = f(x)$, i.e., those 2nd order linear eqns with coeff. of $y'' = 1$. The given D.E. in this form would be

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = 0$$

and $-\frac{4}{x}$, $\frac{6}{x^2}$ are not continuous at $x=0$.

Thus, the hypothesis of the theorem doesn't hold, so we cannot apply Theorem 2.

Linear, 2nd order, homogeneous eqns. with constant coefficients:

The general form is

$$ay'' + by' + cy = 0,$$

where a, b, c are constants.

Substituting e^{rx} in place of y, we see that $y(x) = e^{rx}$ solves this D.E. whenever

$$ar^2 + br + c = 0 :$$

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}.$$

$$\therefore \quad ay'' + by' + cy = 0 \iff ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$\text{i.e., } (ar^2 + br + c)e^{rx} = 0$$

$$\iff ar^2 + br + c = 0.$$

This quadratic eqn. associated to the diff. eqn $ay'' + by' + cy = 0$ is known as the "<u>characteristic equation</u>" of the homogeneous lin. diff. eqn.

Now, the quadratic can have:

(i) Two (distinct) real roots.

(ii) A repeated real root.

(iii) complex roots.

We consider each case, and find two lin. independent solutions, which will then allow us to write a general solution to the hom. lin. diff. eqn.

<u>Case 1</u> : Distinct Real Roots.

→ If r_1 and r_2 are distinct roots of the characteristic eqn. $ar^2+br+c=0$, then $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are two linearly independent solutions.

Lin. ind. because the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix}$$

$$= (r_2 - r_1) e^{(r_1+r_2)x}$$

$\neq 0$ as long as $r_2 \neq r_1$.

Therefore, a general solution of

$$ay'' + by' + cy = 0$$

if $ar^2 + br + C = 0$ has distinct, real roots r_1 and r_2 is given by:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

Case 2: Suppose the char. eqn. $ar^2 + br + C = 0$ has a real, repeated root.

i.e., the quadratic is of the form $(r-\alpha)^2 = 0$.

$$\shortparallel$$
$$r^2 - 2\alpha r + \alpha^2$$

Then, the D.E must be of the form

$$y'' - 2\alpha y' + \alpha^2 y = 0 \quad \text{—} \quad (*)$$

Now, $e^{\alpha x}$ is a root. $v(x)$ not a constt. fn.

Let $y_2(x) = v(x)e^{\alpha x}$, be the other root.

Note that it makes sense to choose y_2 as this, because

$$W(y_1, y_2) = v'(x) \neq 0.$$

Plugging in $y_2(x)$ in $(*)$ and simplifying, we get the eqn. $v''(x) = 0$.

$\therefore v(x) = ax + b$, for some constants a, b, $a \neq 0$.

Therefore, a general solution is

$$y(x) = c_1 e^{rx} + (c_2 x + b) e^{rx}$$

combining b into the constant c_1, we write

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}$$
$$= (c_1 + c_2 x) e^{rx}.$$

[Alternatively, we could just verify that $y_2(x) = x e^{rx}$ also solves the D.E. I wanted to shed some light into <u>why</u> we consider this function as y_2, which is missing in the ~~textbook~~.]

Examples

Solve the following ODEs:

1. $2y'' - 7y' + 3y = 0$ $\quad : y(x) = c_1 e^{x/2} + c_2 e^{3x}, \quad c_1, c_2 \in \mathbb{R}$

2. $3y'' + 6y' + y = 0$ $\quad y(x) = c_1 e^{\left(-1 + \frac{\sqrt{24}}{6}\right)x} + c_2 e^{\left(-1 - \frac{\sqrt{24}}{6}\right)x}$

3. $y'' + 2y' + y = 0$ $\quad y(x) = (c_1 + c_2 x) e^{-x}$

4. $25y'' - 10y' + y = 0$ $\quad y(x) = (c_1 + c_2 x) e^{x/5}$.

Before going to the complex roots part, we generalize the results for the case of real distinct and repeated roots, to homogeneous linear n^{th} order diff. eqns.