Solutions to Assignment-1

I. (3 marks)

Verify that the given function satisfies the differential equation and find a value of C that satisfies the initial condition:

$$y' + y\tan x = \cos x \; ; \; y(x) = (x+C)\cos x \; ; \; y(\pi) = 0$$

Solution: $y(x) = (x+C)\cos x.$

$$y'(x) = -(x+C)\sin x + \cos x$$

LHS: $y' + y\tan x =$

$$-(x+C)\sin x + \cos x + (x+C)\cos x \cdot \frac{\sin x}{\cos x}$$

$$= \cos x$$

$$= \text{RHS}.$$

The function is verified to be a solution.

$$y(\pi) = (\pi + C)\cos\pi = -(\pi + C)$$
$$= 0$$
$$\therefore C = -\pi.$$

II. Solve :

(a) $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2+9}}$; $y(4) = 2$

(b) $\dfrac{dy}{dx} = y^2 + 2$

(3 marks)

Soln. a) $\displaystyle\int dy = \int \frac{x\,dx}{\sqrt{x^2+9}} + C$

↳ use substitution :
$u = x^2+9$
$du = 2x\,dx$

$$y(x) = \frac{1}{2}\int \frac{du}{\sqrt{u}} + C$$

$$\therefore y(x) = \sqrt{x^2+9} + C$$

$$2 = \sqrt{25} + C$$

$$\therefore C = -3.$$

$\therefore y(x) = \sqrt{x^2+9} - 3$ is the required solution

(3 marks)

b) $\int \frac{dy}{y^2+2} = \int dx + C$

writing $2 = \sqrt{2}^2$ and using $\int \frac{dy}{y^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{y}{a}\right)$

$$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) = x + C$$

$$\frac{y}{\sqrt{2}} = \tan(\sqrt{2}(x+c))$$

$$\therefore y(x) = \sqrt{2}\tan(\sqrt{2}(x+c)).$$

III. Use the existence-uniqueness theorem to deduce whether the following IVPs have a unique solution or not:

(a) $\frac{dy}{dx} = \sqrt{x-y}$; $y(2) = 2$ (3 marks)

(b) $\frac{dy}{dx} = \sqrt{x-y}$; $y(2) = 1$ (3 marks)

Soln. Here, $f(x,y)$ (from the Theorem)

$= \sqrt{x-y}$, which is continuous on $y, x \geq 0$, $x \geq y$.

$D_x(f(x,y)) = \dfrac{1}{2\sqrt{x-y}}$; which is continuous on $y, x \geq 0$, $x > y$.

, a) No rectangle containing $(2,2)$ works since $D_y f(x,y)$ is not continuous at $x=y$. Therefore existence/uniqueness fails.

b) Consider the rectangle

$$\left\{ (x,y) \in \mathbb{R}^2 : \begin{matrix} 1.5 < x < 2.5 \\ 0 < y < 1.5 \end{matrix} \right\}$$

Then, both f and $D_x f$ are continuous here; so the hypothesis of the theorem holds and there exists a unique solution in some interval $I \subseteq \mathbb{R}$ containing $x = 2$.

IV The solution of the D.E. $\frac{dN}{dt} = -kN$ is:

(4 marks) $N(t) = N_0 e^{-kt}$

$$\frac{\cancel{N_0}}{2} = \cancel{N_0}\, e^{-k\,5.27}$$ ⇝ using info about half-life

$$-k(5.27) = -\log 2$$

$$k = \frac{\log 2}{5.27}.$$

$$\frac{\cancel{N_0}}{100} = \cancel{N_0}\, e^{-\frac{\log 2}{5.27}t}$$ ⇝ since we need to know t for which $N(t) = \frac{N_0}{100}$.

$$\log \frac{1}{100} = -\frac{\log 2}{5.27} t$$

$$t = 5.27 \times \frac{\log 1/100}{\log 1/2}$$

$$\approx 35 \text{ yrs.}$$

V. $\frac{dN}{dt} = kN(P-N)$, where k is the constant of proportionality.

[1 mark]