TEST-1 SOLUTIONS

1. a) $e^{\int \frac{3}{t} dt} = e^{3\log t} = t^3$.

b) Linear, second-order.

c) Here $M = x^3 + \frac{y}{x}$, $N = y + ?$

Criterion for exactness is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Since $\frac{\partial M}{\partial y} = \frac{1}{x}$, N can be a function of the form $\log x + f(y)$.

d) $\frac{dT}{dt} = k(70 - T)$; $T(0) = 210$.

2. $y' = ye^x$; $y(0) = 2e$

This is a separable equation.

$$\int \frac{dy}{y} = \int e^x + C$$

$$* \quad \log|y| = e^x + C$$

$$\therefore \quad y = e^{e^x + C}$$

Now, $y(0) = 2e$ gives $2e = e^{1+C}$

$$\therefore \quad C + 1 = \log(2e)$$

$$= 1 + \log 2$$

$$\therefore \quad C = \log 2.$$

$$\therefore \quad y(x) = e^{e^x + \log 2}$$

$$= 2e^{e^x}.$$

(OR) $*$ $\rightsquigarrow$ $\log|y| + \log C = e^x$

$$Cy = e^{e^x}$$

$y(0) = 2e$ gives $2e \cdot C = e$ $\therefore$ $C = \frac{1}{2}$.

and $y(x) = 2e^{e^x}$.

3. $xy' = y + 2\sqrt{xy}$

we can rearrange to get

$$y' = \frac{y}{x} + 2\sqrt{\frac{y}{x}},$$

which is a homogeneous equation.

Substituting $y = vx$

$y' = v + xv'$

we get

$$v + 2\sqrt{v} = v + xv'$$

so $v' = \frac{2\sqrt{v}}{x}$, which is separable.

$$\int \frac{dv}{2\sqrt{v}} = \int \frac{dx}{x} + C$$

$$v^{1/2} = \log x + C.$$

$$\therefore y(x) = x(\log x + C)^2$$

4. Here, $F(x,y) = \dfrac{xy}{\cos x}$ and $(a,b) = (0,0)$.

$$\frac{\partial}{\partial y} F(x,y) = \frac{x}{\cos x}.$$

Since $\cos(0) \neq 0$, there is an interval around $x = 0$ for which $\cos x \neq 0$, say $[-\frac{\pi}{4}, \frac{\pi}{4}]$. Then, $F(x,y)$ and $\frac{\partial}{\partial y} F(x,y)$ are both continuous on $[-\frac{\pi}{4}, \frac{\pi}{4}] \times [-1,1]$.

Thus, the hypothesis of the existence-uniqueness theorem is valid in this rectangle, and so the IVP has a unique solution in some interval around $x = 0$.

5. $$(2x+y)\,dx + (x-1)\,dy = 0$$

Here, $M(x,y) = 2x+y$

$N(x,y) = x-1$.

Let $F(x,y)=C$ be the general solution.

Since the equation is exact, $\frac{\partial F}{\partial x} = M$.

i.e., $F(x,y) = \int (2x+y)\,dx + g(y)$ for some g.

$\therefore$ $F(x,y) = x^2+xy+g(y)$.

$\frac{\partial F}{\partial y} = N \rightarrow$ $x+g'(y) = x-1$

$\therefore$ $g'(y) = -1$

and so $g(y) = -y+c$.

$\therefore$ The general solution is given by

$$x^2+xy-y = C.$$

Plugging in the initial condition $y(0)=1$,

we get $C=-1$.

so the particular solution to the IVP is

$$y(x-1) + x^2 = -1$$

$$\text{i.e.,} \quad y = \frac{-1-x^2}{x-1}$$

$$y(x) = \frac{1+x^2}{1-x}.$$

6. This was done in class. See Week 3 notes, example 2.