# Reducible Second-order equations

The general form is

$$F(x, y, y', y'') = 0.$$

If either $y$ is missing or $x$ is missing, we can do a substitution to convert it to a first order D.E.

① Dependent variable ($y$) missing:

So the eqn. is of the form $F(x, y', y'') = 0$.

Then, we substitute $p = y' = \frac{dy}{dx}$

so $p' = y'' = \frac{dp}{dx}$.

This converts the D.E. into:

$$F(x, p, p') = 0.$$

We solve this for $p = \frac{dy}{dx}$, then solve again to get $y$.

<u>Exercise</u>: Perform the substitution $p = y'$ to reduce the following 2nd order equations to first order:

a) $y'' + x^2 y' = 3x^2$ $\rightsquigarrow$ $p' + x^2 p = 3x^2$

b) $y'' + \frac{2}{x} y' = 6$ $\rightsquigarrow$ $p' + \frac{2}{x} p = 6$

c) $y'' = -4x$ $\rightsquigarrow$ $p' = -4x$.

<u>Examples</u>

1. Solve: $xy'' + y' = 4x$.

<u>Solution</u>: This is a 2nd order D.E. with $y$ missing. Let $p = y'$ so $p' = y''$ to get

$$xp' + p = 4x,$$ which is linear. (1st order)

The integrating factor is $e^{\int \frac{1}{x} dx} = x$.

solve to get $\quad xp = 2x^2 + C_1$

so $\quad p = \dfrac{dy}{dx} = 2x + \dfrac{C_1}{x}$

$$\therefore\ y(x) = \int 2x + \frac{C_1}{x}\, dx$$

$$= x^2 + C_1 \log|x| + C_2.$$

---

Note: Very quickly the reduced eqn. can be tricky to solve! We see an example below:

2. Solve : $\quad y'' + x^2 y' = 3x^2.$

Solution: This is a 2nd order D.E with $y$ missing. Setting $p = \dfrac{dy}{dx}$, $p' = y''$

we get $\quad p' + x^2 p = 3x$, which is linear.

The integrating factor here is $\quad e^{\int x^2 dx}$

$$= e^{x^3/3}.$$

$$e^{x^3/3}(p' + x^2 p) = 3x^2 \cdot e^{x^3/3}$$

$$\therefore \quad p \cdot e^{x^3/3} = \int 3x^2 \cdot e^{x^3/3}\, dx$$

set $\frac{x^3}{3} = t$. $x^2 dx = dt$.

So the integral on the RHS becomes

$$3\int e^t\, dt = 3e^t + C_1 = 3e^{x^3/3} + C_1$$

$$\therefore \quad p(x) = 3 + C_1 e^{-x^3/3}.$$

$$\overset{\shortparallel}{\frac{dy}{dx}}$$

$$\therefore \quad y(x) = \int (3 + C_1 e^{-x^3/3})\, dx$$

$$= 3x + C_1 \int e^{-x^3/3}\, dx + C_2$$

Another kind of reducible 2nd order D.E. is when $x$ is missing, so the equation is of the form $F(y, y', y'') = 0$

Here we substitute $p = y'$

and $y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx}$

$= p\frac{dp}{dy}$

So $p$ now becomes the dependent variable and $y$ behaves like an independent one.

<u>Example 3</u>: Solve: $yy'' = 3(y')^2$.

Solution: Substitute $y' = p$.

Then $y'' = p\frac{dp}{dy}$

So the diff. eqn. becomes $yp\frac{dp}{dy} = 3p^2$

or, $y \dfrac{dp}{dy} = 3p$, which is separable.

Solving: $$\int \frac{dp}{p} = 3\int \frac{dy}{y}$$

$$\log p = 3\log y + \log C_1$$

$$\therefore \underset{\overset{\shortparallel}{\frac{dy}{dx}}}{p} = C_1 y^3.$$

This is again a separable D.E.

Solving, $$\int \frac{dy}{y^3} = C_1 \int dx$$

$$-\frac{1}{2} y^{-2} = C_1 x + C_2 ,$$

$$\text{or } y^{-2} = C_3 x + C_4 \qquad (C_3 = -2C_1,\ C_4 = -2C_2)$$

Explicit: $y = (C_3 x + C_4)^{-1/2}$.

Another kind of reducible 2nd order D.Es is one where we know one solution:

<u>Example</u>: Find another solution to

$$2t^2y'' + ty' - 3y = 0; \quad t > 0$$

given that $y_1(t) = t^{-1}$ is a solution.

(Note that this eqn has $t, y, y'$ and $y''$)

<u>Soln</u>. We assume that the second solution $y_2(t)$ is of the form $v(t)y_1(t)$, plug this into the diff. eqn, obtain another D.E. that we solve. In the given problem,

$$y_2(t) = \frac{v(t)}{t}, \quad \text{so } y_2'(t) = -\frac{1}{t^2}v + \frac{v'}{t},$$

$$y_2''(t) = \frac{2}{t^3}v - \frac{2}{t^2}v' + \frac{1}{t}v''$$

$\therefore$ The D.E. becomes

$$2t^2\left(2t^{-3}v - 2t^{-2}v' + t^{-1}v''\right) + t\left(t^{-2}v + t^{-1}v'\right) - 3\left(t^{-1}v\right) = 0$$

$$4t^{-1}v - 4v' + 2tv'' + t^{-1}v + v' - 3t^{-1}v = 0$$

$$2tv'' - 3v' = 0.$$

Now let $v' = p$ so $v'' = p'$ and we get

$$2tp' - 3p = 0$$

$$\int \frac{dp}{3p} = \int \frac{dt}{2t} \rightsquigarrow \frac{1}{3}\log p = \frac{1}{2}\log t + \log C$$

$$\Rightarrow p^2 = Ct^3$$

or $p = Ct^{3/2}$,

so $v' = Ct^{3/2}$.

$$\therefore \quad v = \frac{2}{5}C\,t^{5/2} + C_2$$

$\therefore$ The most general form of $y_2(t)$

is $y_2(t) = \left(C_1 t^{5/2} + C_2\right)\cdot\frac{1}{t}$.

Exercise: Find another solution to $t^2y'' + 2ty' - 2y = 0$, given that $y_1(t) = t$ is a solution.

Soln: $y_2(t) = C_1 t^{-2}$

Now we move on to Chapter 2!

Second order linear equations :

**General form :**

$$\boxed{A(x)y'' + B(x)y' + C(x)y = F(x).} \quad - \boxed{\blacktriangle}$$

So the diff. eqn is linear in $y$, $y'$ and $y''$ but $A(x)$, $B(x)$, $C(x)$ and $F(x)$ need not be linear.

**Homogeneous & Non-homogeneous lin. eqns :**

If the function $F(x)$ in eqn $\boxed{\blacktriangle}$ is the zero function, then the diff. eqn is said to be a homogeneous 2nd order lin. eqn.

If $F(x) \neq 0$ then it is non-homogeneous.

Example : $y'' + xy' + \frac{2}{x}y = \cos x$ is non-homogeneous

$\downarrow$ (1)

but $y'' + xy' + \frac{2}{x} y = 0$ is homogeneous.

(2)

Eqn. (2) is called the homogeneous equation associated to the non-homogeneous equation (1).

In general, the homogeneous eqn associated to the non-homogeneous eqn $\boxed{\blacktriangle}$ is

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

For simplicity, we keep the coefficient of $y''$ as 1 and consider the general homogeous 2nd order eqn as

$$y'' + p(x)y' + q(x)y = 0,$$

which we study first.

**Exercise:** Consider: $y'' + y' - 2y = 0$.

Let $y(x) = e^{rx}$. Find the value(s) of $r$ such that $y(x)$ is a solution to the given differential equations.

**Solution**

$$y = e^{rx} \qquad y' = r e^{rx} \qquad y'' = r^2 e^{rx}.$$

$\therefore$ $y'' + y' - 2y = 0$ with $y = e^{rx}$ gives

$$(r^2 + r - 2) e^{rx} = 0.$$

$e^{rx} \neq 0$ for any $r, x$ so $r^2 + r - 2$ must be zero.

Solving this quadratic equation we get $r = 1$ or $r = -2$.

$\therefore$ $y_1 = e^{x}$ and $y_2 = e^{-2x}$ are two solutions of the given D.E.

**Theorem 1** : Principle of superposition for homogeneous equations

Let $y_1$ and $y_2$ be two solutions of the homogeneous linear equation:

$$y'' + p(x)\,y' + q(x)\,y = 0 \quad — (1)$$

If $c_1, c_2 \in \mathbb{R}$, then the linear combination

$y = c_1 y_1 + c_2 y_2$ is also a solution!

**Proof** : We plug in $y = c_1 y_1 + c_2 y_2$ in the LHS of (1), given that $y_1'' + p(x) y_1' + q(x) y_1 = 0$

and $y_2'' + p(x) y_2' + q(x) y_2 = 0$

$$\text{LHS} = (c_1 y_1'' + c_2 y_2'') + p(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2)$$

$$= c_1 (y_1'' + p(x) y_1' + q(x) y_1) + c_2 (y_2'' + p(x) y_2' + q(x) y_2)$$

$$= c_1 \cdot 0 + c_2 \cdot 0 = 0 = \text{RHS}.$$

Theorem 2 (Existence and Uniqueness for Linear equations):

Suppose that the functions $p, q$ and $f$ are continuous on the open interval $I$ containing the point $a$. Then, given any two numbers $b_0$ and $b_1$, the equation

$$y'' + p(x)y' + q(x)y = f(x)$$

has a unique solution on the entire interval $I$ that satisfies the initial conditions

$$y(a) = b_0 \; ; \; y'(a) = b_1 .$$

Remarks: Observe the nature of the initial conditions!

- For a given point $(a, b_0)$ there are infinitely many solution curves passing through it! (one for each initial slope $: b_1$ at $x=a$)
- Similarly, for a given slope $b_1$ at $a$, there are infinitely many solutions having $y'(a) = b_1$.

<u>Example</u>: Consider the D.E: $y'' + y = 0$.

It is easy to verify that $y_1 = \cos x$

and $y_2 = \sin x$

are both solutions. This D.E. is homogeneous.

Theorem 1 tells us that $y = c_1 y_1 + c_2 y_2$ will also be a solution for any choice of constants $c_1$ and $c_2$.

Suppose we had the initial conditions

$$y(0) = 2 \;, \quad y'(0) = 5 .$$

Then $2 = c_1 \cos 0 + c_2 \sin 0 = c_1$

$5 = -c_1 \sin 0 + c_2 \cos 0 = c_2$

$\therefore$ The unique solution for the initial value problem $y'' + y = 0$; $y(0) = 2$, $y'(0) = 5$

is $y(x) = 2\cos x + 5\sin x$.

Theoretically speaking if $y_1(x)$ and $y_2(x)$ are "different" solutions of the D.E

$$y'' + p(x)\, y' + q(x)\, y = 0,$$

then to solve for $c_1$, and $c_2$ such that $y = c_1 y_1 + c_2 y_2$ is the unique solution satisfying the initial conditions

$$y(a) = b_0 \quad , \quad y(a) = b_1 ,$$

we need to solve the linear system:

$$c_1\, y_1(a) + c_2\, y_2(a) = b_0$$
$$c_1\, y_1'(a) + c_2\, y_2'(a) = b_1 .$$

Observe that this is just

$$\begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

which will have a unique solution (for each $b_0, b_1$) if and only if $\begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix}$ is invertible.

Remark: Theorem 2 also tells us that we will not find three "different" solutions $y_1, y_2, y_3$ to a 2nd order lin. homogeneous D.E. because then, using Theorem 1, $y = c_1 y_1 + c_2 y_2 + c_3 y_3$ would be a general solution. Plugging in the initial conditions $y(a) = b_0$, $y'(a) = b_1$ gives us

$$\begin{bmatrix} y_1(a) & y_2(a) & y_3(a) \\ y_1'(a) & y_2'(a) & y_3'(a) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

which doesn't have a unique solution for every $b_0, b_1$. This contradicts Theorem 2. Thus, such a $y_3$ does not exist.

Q. How can we ensure

$$\begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \text{ is invertible?}$$

Ans: It works iff $y_1, y_2$ are linearly independent functions!

**Fact from linear algebra:**

n vectors in $\mathbb{R}^n$ are lin. independent if and only if the $n \times n$ matrix formed by these n vectors as columns of a matrix has non-zero determinant.

Using this for $n=2$ in our question, what we want is for $\begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix}$

to have non-zero determinant for all $a \in I$.