Week 2

Solve the separable diff. eqn:

$$\frac{dy}{dx} = \frac{9x}{4y}.$$

$$y(0) = 2$$

Soln. $\int 4y\, dy = \int 9x\, dx + c$

gives $2y^2 = \frac{9x^2}{2} + c.$

$\therefore \quad y^2 = \frac{9}{4}x^2 + \frac{c}{2}$

and $y = \pm\sqrt{\frac{9}{4}x^2 + \frac{c}{2}}$

Q. Which solution do we choose?

A. Use the initial condition! squaring,

$y(0) = 2 \Rightarrow 2 = \pm\sqrt{\frac{c}{2}} \Rightarrow \frac{c}{2} = 4$

$y(x) = \sqrt{\frac{9}{4}x^2 + 4}$ is the required solution.

Last time we saw an example of a DE with a solution that was valid only on a certain part of the domain ($\mathbb{R}$).

This calls for understanding the question of "existence and uniqueness" of solutions, i.e., ① does the D.E. have solutions at all?

② If we know that the D.E. can be solved, how many solutions can it have?

Let us see two examples to further motivate the question:

(*) Consider the IVP: $y' = \frac{1}{x}$; $y(0) = 0$.

This is a separable equation, and on solving it: $\frac{dy}{dx} = \frac{1}{x} \to \int dy = \int \frac{dx}{x} + C$

we get $y(x) = \log x + C$, but plugging in the initial condition $y(0) = 0$ gives us NO solution, since log is not defined at 0.

(B) Consider the IVP: $y' = 2\sqrt{y}$; $y(0) = 0$.

This equation has two solutions:

$$y_1(x) = x^2.$$

$$y_2(x) = 0.$$

Theorem: (Existence and uniqueness of solutions)

Suppose that both the function $f(x,y)$ and its partial derivative $D_y(f(x,y))$ are both continuous on some rectangle R in the xy-plane that contains the point (a,b) in its interior. Then, for some open interval I containing the point a, the IVP:

$$\frac{dy}{dx} = f(x,y) \; ; \quad y(a) = b$$

has one and only one solution defined on the interval I.

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let's go back to the examples Ⓐ and Ⓑ above:

Ⓐ : Since $\frac{1}{x}$ is not continuous at $x=0$, the existence/uniqueness fails.

Ⓑ : $f(x,y) = 2\sqrt{y}$ is continuous everywhere (on $y>0$)

but the partial derivative $\frac{1}{\sqrt{y}}$ fails to be conts. $\forall$ $y=0$, so existence/uniqueness fails.

Recall the example from last class: $y' = y^2, y(0)=1$ had the soln. $y(x) = \frac{1}{1-x}$.

$f(x,y) = y^2$ and $D_y(f(x,y)) = 2y$ are both continuous. Yet, we talked about "two" solutions. Does this contradict uniqueness?

No, because the theorem guarantees "some" interval where the solution is unique.

Now consider the following:

$$x \frac{dy}{dx} = 2y.$$

Here, $f(x,y) = \frac{2y}{x}$ and $\frac{\partial f}{\partial y} = \frac{2}{x}$. From the theorem we conclude that we must have a unique solution near any point (x,y) as long as $x \neq 0$. Let's see what happens when $x = 0$:

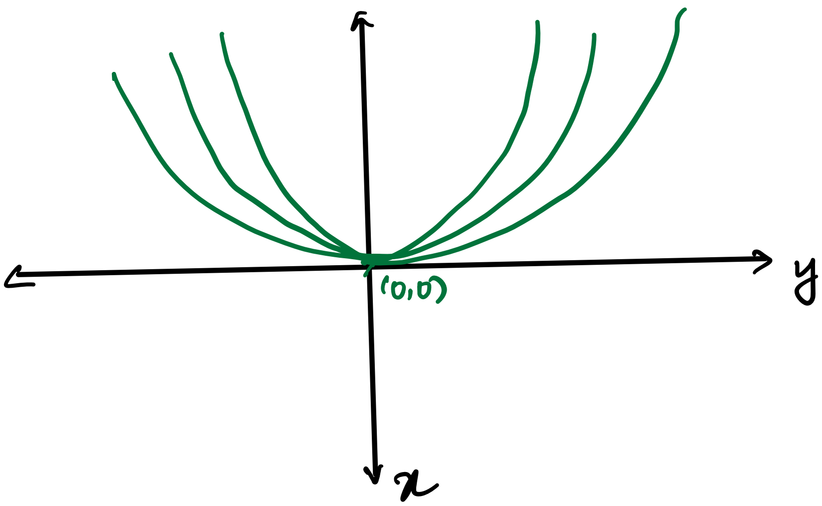
Using separation of variables, we can solve this D.E:

$$y(x) = Cx^2.$$

Suppose we had an initial condition

$$y(0) = b.$$

The graph of $y(x) = Cx^2$ for different values of C is:



So if $x=0$, $y(x)=0$ else there is no solution. On the other hand if $y(x)=0$, uniqueness fails!

Back to separable equations:

<u>Example</u>: Solve: $y' = (x^2-4)(3y+2)$ using the method of separation of variables

<u>Solution</u>: $\frac{dy}{dx} = (x^2-4)(3y+2)$

is equivalent to $\int \frac{dy}{3y+2} = \int (x^2-4)\,dx + C_1$

(provided $y \neq -2/3$!)

$$\frac{1}{3} \log|3y+2| = \frac{x^3}{3} - 4x + C_1.$$

$$\log|3y+2| = x^3 - 12x + 3C_1$$

$$\text{So } |3y+2| = e^{x^3-12x+3C_1}$$

define $C_2 = e^{3C_1}$ so

$$|3y+2| = C_2 e^{x^3-12x}$$

$$\therefore 3y+2 = \pm C_2 e^{x^3-12x}$$

$$\therefore \underset{y(x)}{\overset{}{\underset{\shortparallel}{y}}} = \frac{-2 \pm C_2 e^{x^3-12x}}{3}.$$

The sign +/- can be absorbed into the constant so we let C be this constant $\frac{\pm C_2}{3}$

$\therefore$ A general solution is

$$y(x) = \frac{-2}{3} + C e^{x^3-12x}, \quad C \neq 0.$$

Implicit solutions : We were quite lucky to be able to express the final solution as $y(x) = \cdots$ back in the previous problem. Often, we have to settle for an implicit solution. In the previous example,

$$\log|3y+2| = x^3 - 12x + 3C$$

is an implicit solution, since the "y" is mixed up with other functions.

Example : $$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}.$$

Separating variables, we get

$$y^3 - 5y = 4x - x^2 + C.$$

This is an implicit solution.

Each solution then lies on what is

called a "level curve" of

$$H(x,y) = y^3 - 5y - 4x + x^2.$$

i.e, ~~we~~ given c, the solution is

$$H(x,y) = c.$$

Another example:

$$x + y\frac{dy}{dx} = 0, \quad y(0) = 2$$

$$\int y\,dy = -\int x\,dx + c$$

so $y^2 + x^2 = c'$ $\quad (c' = 2c)$

$y(0) = 2$ gives $4 = c'$.

$\therefore$ An implicit solution is $y^2 = 4 - x^2$

and this time we have 2 explicit solutions

$$y = \sqrt{4-x^2}$$

$$y = -\sqrt{4-x^2}.$$

Note however, that $y(x) = -\sqrt{4-x^2}$ does not solve the IVP, since $y(0) = -2$ in this case.

Therefore, we must be a little careful when dealing with implicit solutions. Not every algebraic manipulation will lead to an actual solution of the D.E.

Problems to try:

I. Which of the following D.E.s are separable?

a) $\frac{dy}{dx} = x + y$

b) $3x^2 \frac{dy}{dx} = \log x . \sin y$

c) $\frac{dy}{dx} = \frac{4+x^2}{x+y^2}$

d) $3x + y^2 \frac{dy}{dx} = 5 + x^2$

II. Solve the foll. diff. eqns.

a) $\frac{dy}{dx} = 2x(y+2)$

b) $\frac{dy}{dx} = 2xy^2 + 3x^2y^2$

$y(1) = \frac{1}{2}$

Solutions

I. b), d).

II. a) $\int \frac{dy}{y+2} = \int 2x\,dx + C_0$

$$\log|y+2| = x^2 + C_0$$

$$|y+2| = C_1 e^{x^2} \qquad C_1 > 0$$

$$y+2 = \pm C_1 e^{x^2}$$

$$y(x) = -2 + C_2 e^{x^2} \qquad C_2 \in \mathbb{R},\ C_2 \neq 0.$$

However $y(x) = -2$ is also a solution, so the general solution is given by

$$y(x) = -2 + C e^{x^2}, \qquad C \in \mathbb{R}.$$

b) $\frac{dy}{dx} = 2xy^2 + 3x^2y^2 = y^2(2x+3x^2)$

$$\int \frac{dy}{y^2} = \int (2x+3x^2)\,dx + C_0 \rightsquigarrow \frac{-1}{y} = x^2 + x^3 + C_0$$

$$y = \frac{-1}{C_0 + x^2 + x^3}.$$

$y(1) = \frac{1}{2}$ gives $c_0 = -4$.

$\therefore$ $y(x) = \dfrac{-1}{x^3+x^2-4}$ is the particular

solution of the given IVP.

Let's verify!

LHS: $\dfrac{dy}{dx} = \dfrac{-1}{(x^3+x^2-4)^2} \cdot (3x^2+2x)(-1)$.

RHS $2xy^2 + 3x^2y^2 = \dfrac{1}{(x^3+x^2-4)^2} \cdot (2x+3x^2)$

LHS = RHS ✓

Applications

The diff. eqn. $\frac{dx}{dt} = kx$, k: constant

serves as a model for many phenomena.

(i) Population growth

$P(t)$: no. of individuals in a population

Assumption: The population has constant birth and death rates β and δ.
(births or deaths per individual per unit of time)

In a short interval Δt of time, approximately $\beta P(t) \Delta t$ births occur, and $\delta P(t) \Delta t$ deaths occur. So, the change in $P(t)$ is given by:

$$\Delta P \approx (\beta - \delta) P(t) \Delta t.$$

$$\therefore \frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = kP$$

where $k = \beta - \delta$.

② **Radioactive decay**: Sample contains $N(t)$ atoms of a ratioactive isotope at time t.

Assumptions: A constant fraction of these atoms decay during each unit of time.

So this is like the population model with zero birth rate, and a constant death rate.

So we get the diff. eqn:

$$\frac{dN}{dt} = -kN$$

(The value of k depends on the isotope.)

Read the textbook for info on "radiocarbon dating", used to find the age of fossils.

③ **Cooling and Heating**:

Newton's Law of Cooling states that the time rate of change of the temperature $T(t)$

of a body immersed in a medium of constant temperature A is proportional to the difference A - T. That is,

$$\frac{dT}{dt} = k(A-T).$$

④ <u>Compound interest</u>

If P is the principal amount, then at the end of t years, we have Pe^{rt} dollars in the account.

Where does this formula come from?

A(t) : amount in \$ in the account at t years.

r : annual interest rate, compounded continuously.

Then, in a short interval of time Δt, The amount of interest accumulated is: $\quad r A(t) \Delta t$.

Interest accumulated in the time period Δt is the change in the amount of money in the account over the time period Δt:

i.e., $\quad \Delta A = r A(t) \Delta t$.

$$\therefore \quad \frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = r A(t).$$

This is a separable D.E., solving it we get $\quad A(t) = C e^{rt}$, where

$C = A(0)$, the principal amount!

Thus, this formula matches with the compound interest formula recalled earlier.

Note: Alternatively, we use a limiting expression to the discrete compounding case:

$A = P(1+r)^n$ ← discrete compounding, n years, r: annual rate.

↓

Amount when interest is compounded (in 1 year)

- half-yearly: $A = P\left(1+\frac{r}{2}\right)^2$
- quarterly: $A = P\left(1+\frac{r}{4}\right)^4$
- compounded m times: $A = P\left(1+\frac{r}{m}\right)^m$

"continuously compounded" ⟶ let $m \to \infty$.

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^m = e^r$$

So for t years, this factor becomes e^{rt}.

$\therefore \; A(t) = Pe^{rt}$.

Prototype for all the above applications:

$$\frac{dy}{dx} = kx.$$

Using separation of variables, we get the solutions

$$y(x) = C e^{kx}$$

where C can be found using an initial condition.

Problems

1. A field has 30 ants. Their population is increasing at a rate of 2 ants per month. How many ants will be there at the end of 4 months?

Soln. The diff. eqn. is $\frac{dP}{dt} = (2-0)\,P(t)$, assuming no deaths in 4 months.

$$\therefore \quad P(t) = C e^{2t}.$$

We are given $P(0) = 30$, so $30 = C \cdot 1 = C$

We need to calculate $P(4)$.

$$P(4) = 30 e^{2 \times 4} = 30 e^{8} \approx 89{,}430.$$

2. A specimen of charcoal found at Stonehenge turns out to contain 63% as much ^{14}C as a sample of present-day charcoal of equal mass. What is the age of the sample?
(Decay constt: $k = 0.0001216$)

Solution: The diff. eqn. is $\frac{dN}{dt} = -kN$.

$$\therefore \quad N(t) = N_0 e^{-kt} \text{ where } N_0 = N(0).$$

We are given: $N_0 (0.63) = N_0 e^{-kt}$

$$\therefore \quad 0.63 = e^{-(0.0001216)t}$$

$$\therefore \quad t = -\frac{\log(0.63)}{0.0001216} \approx 3800y.$$

3. The half-life of a radioactive element is 20 years. Find the time it will take for the element to be a quarter of its original amount.

Ans. $\frac{dN}{dt} = -\lambda N$

$$N_{(t)} = N_0 e^{-\lambda t}$$

given: $\frac{N_0}{2} = N_0 e^{-\lambda \cdot 20}$

$$e^{-20\lambda} = \frac{1}{2}$$

$$\Rightarrow \quad -20\lambda = -\log 2$$

$$\therefore \quad \lambda = \frac{1}{20} \log 2 .$$

For it to become a quarter of its original amount,

$$\frac{N_0}{4} = N_0 \, e^{-\frac{1}{20}\log 2 \cdot t}$$

$$-\log 4 = -t \frac{\log 2}{20}$$

$$\overset{\shortparallel}{-2\log 2} = -t \frac{\log 2}{20} \qquad \therefore \; t = 40 \text{ years.}$$

7. An object has a temperature of 30°C, surrounded by a room of constant temp. of 15°C. It is seen that the temp. of the object falls to 25°C in 10 min. Calculate how much more time will it take to drop to 20°C.

A: ambient temp.

<u>Soln.</u> $$\frac{dT}{dt} = k(A-T)$$ is the diff. eqn.

Solving, $$-\log|A-T| = kt + C_0$$

$$A - T = Ce^{-kt}.$$

Given: ① $T(0) = 30$, so

$$15 - 30 = C \Rightarrow C = -15.$$

② $T(10) = 25$, so

$$15 - 25 = -15\, e^{-k(10)}$$

$$e^{-k(10)} = \frac{2}{3}$$

$$50 - 10k = \log 2/3$$

$$k = \frac{-1}{10} \log(2/3).$$

Solving :

$$T(x) = 20$$

$$15 - 20 = -15\, e^{-\frac{1}{10} \log \frac{2}{3} \cdot t}$$

$$\frac{1}{3} = e^{-\frac{1}{10} \log \frac{2}{3} t}$$

$$\log \frac{1}{3} = -\left(\frac{1}{10} \log \frac{2}{3}\right) t$$

$$t = -10 \frac{\log 1/3}{\log 2/3}$$

$$\approx 27 \text{ min}.$$

$\therefore$ 17 more minutes.