We now apply the translation theorem to solve differential eqns. where we have a step-function appearing:

1. Solve: $x'' + 4x = g(t)$; $x(0) = 0$, $x'(0) = 0$

$$\text{where } g(t) = \begin{cases} 0 & \text{if } t < 1 \\ t-1 & \text{if } t \geq 1 \end{cases}$$

<u>Solution</u>: Take Laplace transforms on both sides:

$$\mathcal{L}\{x''\} + 4\,\mathcal{L}\{x\} = \mathcal{L}\{g(t)\} \quad \text{— } (*)$$

$$\mathcal{L}\{x''\} = s^2 X(s) - s\,x(0) - x'(0) = s^2 X(s)$$

$$\mathcal{L}\{x\} = X(s)$$

To find $\mathcal{L}\{g(t)\}$ we need to express it as $u(t-a)f(t-a)$, i.e., we want $a, f(t-a)$ such that

$$g(t) = \begin{cases} 0 & t < 1 \\ t-1 & t \geq 1 \end{cases} = u(t-a)f(t-a) = \begin{cases} 0 & 0 \leq t < a \\ f(t-a) & t \geq a \end{cases}$$

$\therefore \quad a = 1, \quad f(t-1) = t-1$, so $f(t) = t$.

$$\Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s^2} = F(s)$$

Translation thm 2 gives $\mathcal{L}\{g(t)\} = e^{-as}F(s)$

$$= \frac{e^{-s}}{s^2}.$$

Substituting all the transforms in (*):

$$s^2 X(s) + 4X(s) = \frac{e^{-s}}{s^2}.$$

$$\Rightarrow X(s) = \frac{e^{-s}}{s^2(s^2+4)}$$

We obtain the solution to the DE by taking inverse transforms on both sides.

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s^2+4)}\right\} = ?$$

This is of the form $e^{-as}F(s)$ with $a=1$

$$F(s) = \frac{1}{s^2(s^2+4)}.$$

By translation theorem 2 we know

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,f(t-a)$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$

We know $a = 1$.

So let us find $f(t)$:

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

Solve to get $A=0$, $B=\frac{1}{4}$, $C=0$, $D=\frac{-1}{4}$.

$$\therefore F(s) = \frac{1}{4s^2} - \frac{1}{4(s^2+4)}$$

$$\therefore \mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{4}t - \frac{1}{4}\cdot\frac{1}{2}\sin 2t$$

$$\therefore \quad x(t) = u(t-a)\, f(t-a)$$

$$= u(t-1)\, f(t-1)$$

$$= \begin{cases} 0 & t<1 \\ \frac{t-1}{4} - \frac{1}{8}\sin 2(t-1) & t \geq 1 \end{cases}$$

Linear Systems of Differential Equations

We now learn how to solve a system of first order linear differential equations. Solving a general system of n linear diff. eqns involves finding

$$(x_1(t), x_2(t), \ldots, x_n(t))$$

(functions) that solve:

$\boxed{A}$:

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t)$$

$$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t)$$

$$\vdots$$

$$x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t)$$

which can be expressed in matrix-vector notation:

Write $P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & & & \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{bmatrix}$

$\bar{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $\dfrac{d\bar{x}}{dt} = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$, $\bar{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$

Then, the system $\boxed{A}$ can be expressed as

$$\frac{d\bar{x}}{dt} = \bar{P}(t)\,\bar{x}(t) + \bar{f}(t)$$

A <u>solution</u> of such a system is a column vector of functions $\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ such that the component functions $x_1(t), x_2(t), \ldots, x_n(t)$ satisfy all the equations in $\boxed{A}$ simult-aneously.

<u>Example</u>: The first order system

$$x_1' = 4x_1 - 3x_2$$
$$x_2' = 6x_1 - 7x_2$$

can be written as:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

It is easily verified that

$$y_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad y_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

are solutions of the given linear system.

① Checking that $\bar{x} = y_1$ is a solution:

LHS: $$\frac{d\bar{x}}{dt} = \begin{bmatrix} \frac{d}{dt}(3e^{2t}) \\ \frac{d}{dt}(2e^{2t}) \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}.$$

RHS: $$\begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$$

Since LHS = RHS, y_1 is verified to be a solution.

② Checking that $\bar{x} = y_2$ is a solution:

LHS: $$\frac{d\bar{x}}{dt} = \begin{bmatrix} \frac{d}{dt}(e^{-5t}) \\ \frac{d}{dt}(3e^{-5t}) \end{bmatrix} = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix}$$

R.H.S: $$\begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix}$$

How many solutions can this linear system have?

Observe that this system is homogeneous (i.e., the $\bar{f}(t)$ vector is $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$). As we had in the case of one DE, here too we have:

Change to y_i

<u>Theorem</u>: Principle of superposition

If $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ are solutions of the homogeneous linear system $\bar{x}' = \bar{P}(t)\bar{x}$, then, the linear combination $\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n$

is also a solution.

and

<u>Theorem</u>: General solution of a homogeneous linear system:

If $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ are linearly independent solutions of the linear system $\bar{x}' = \bar{P}(t)\bar{x}$ on an interval I where $\bar{P}(t)$ is continuous, then <u>every</u> solution of the linear system is of the form

$$\bar{x}(t) = c_1\bar{x}_1(t) + c_2\bar{x}_2(t) + \cdots + c_n\bar{x}_n(t),$$

for constants $c_1, c_2, \ldots, c_n$.

How can we tell if our solutions are lin. independent? Using Wronskians!

If $\bar{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$, ..., $\bar{x}_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$ are

vectors (whose components are functions of t)

then $$W(\bar{x}_1, \dots, \bar{x}_n) = \det \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}.$$

If $W(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \neq 0$ on I, then $\bar{x}_1, \dots, \bar{x}_n$ are linearly independent.

If $W(\bar{x}_1, \dots, \bar{x}_n) \equiv 0$ on I, then $\bar{x}_1, \dots, \bar{x}_n$ are linearly dependent.

Symbol for "identically zero", i.e., zero everywhere

Consider the example from earlier:

$$\bar{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \bar{x},$$

$$\bar{y}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \qquad \bar{y}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

$$W(\bar{y}_1, \bar{y}_2) = \det \begin{bmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{bmatrix}$$

$$= 7e^{-3t} \neq 0 \text{ on } \mathbb{R},$$

so $\bar{y}_1$ and $\bar{y}_2$ are linearly independent solutions, and a general solution is of the form

$$\bar{x} = c_1 \bar{y}_1 + c_2 \bar{y}_2$$

$$= c_1 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}.$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Explicitly, the solution vector is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3c_1 e^{2t} + c_2 e^{-5t} \\ 2c_1 e^{2t} + 3c_2 e^{-5t} \end{bmatrix}$$

What about a particular solution?

If we were given an initial condition

$\bar{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, we use this to find c_1, c_2.

Observe that this <u>one</u> condition is sufficient:

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3c_1 + c_2 \\ 2c_1 + 3c_2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 3c_1 + c_2 &= 0 \\ 2c_1 + 3c_2 &= 2 \end{aligned}$$

Solve to get

$c_1 = -\frac{2}{7}$, $c_2 = \frac{6}{7}$

Therefore, a particular solution of the linear system is

$$\bar{x}(t) = -\frac{2}{7}\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + \frac{6}{7}\begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}.$$

$$= -\frac{2}{7}e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{6}{7}e^{-5t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

<u>3x3 example</u>:

Verify that $\bar{x}_1 = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}$, $\bar{x}_2 = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}$, $\bar{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$

are linearly independent solutions of the system:

$$x' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\bar{x}.$$

Write out the general solution and then find a particular solution satisfying the initial condition: $\bar{x}(0) = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$.

<u>Solution</u>. Verify on your own that $\bar{x}_1' = A\bar{x}_1$, $\bar{x}_2' = A\bar{x}_2$ and $\bar{x}_3' = A\bar{x}_3$ where A is the matrix $\begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}$.

These solutions are linearly independent because their Wronskian:

$$W = \begin{vmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & -e^{3t} & e^{5t} \end{vmatrix}$$

$$= 2e^t\left(0 - 2e^{8t}\right) - 2e^{3t}\left(2e^{6t} + 2e^{6t}\right) + 2e^{5t}\left(-2e^{4t}\right)$$

$$= e^{9t}(-4 - 8 - 4)$$

$$= -16e^{9t} \neq 0 \text{ on } \mathbb{R}.$$

Thus, the general solution is given by

$$\bar{x}(t) = c_1 \begin{pmatrix} 2e^t \\ 2e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{pmatrix} + c_3 \begin{pmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{pmatrix}.$$

To find c_1, c_2, c_3 we use the initial condition: $\bar{x}(0) = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$

$$\bar{x}(0) = c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$$

This gives us a linear system of equations:

$$\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= 0 \\ 2c_1 + 0 \cdot c_2 - 2c_3 &= 2 \\ c_1 - c_2 + c_3 &= 6 \end{aligned}$$

Can use row operations on the augmented matrix to solve this system!

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 0 & -2 & 2 \\ 1 & -1 & 1 & 6 \end{array}\right] \xrightarrow{R_1 \mapsto R_1/2,\ R_2 \mapsto R_2/2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & 6 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto R_2 - R_1,\ R_3 \mapsto R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & -2 & 0 & 6 \end{array}\right]$$

This leads to:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_2 - 2c_3 &= 1 \\ -2c_2 &= 6 \end{aligned}$$

giving $c_2 = -3$, $c_3 = 1$ and $c_1 = 2$.

Thus, a particular solution is given by

$$\bar{x}(t) = 2\begin{bmatrix} 2e^{t} \\ 2e^{t} \\ e^{t} \end{bmatrix} - 3\begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix} + \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}.$$

Eigenvalue Method for Solving Linear Systems of Differential Equations:

Consider

$$\bar{x}' = A\bar{x} \quad \text{—— } (\star)$$

Suppose $\bar{x}(t) = \bar{v}e^{\lambda t}$ where $\bar{v}$ is a vector with constants / scalars as components.

If $\bar{x}(t)$ were to be a solution of $(\star)$,

we would have:

$$\bar{x}' = \bar{v}\lambda e^{\lambda t}$$

i.e., $$A\bar{x} = \bar{v}\lambda e^{\lambda t} = \lambda\bar{v}e^{\lambda t} \quad \text{— } (1)$$

But $A\bar{x}$ is $A(\bar{v}e^{\lambda t}) = A\bar{v}e^{\lambda t}$ — (2)

(1) = (2), so we must have

$$A\bar{v}e^{\lambda t} = \lambda\bar{v}e^{\lambda t}$$

Since $e^{\lambda t} \neq 0$ for any $t \in \mathbb{R}$, we can cancel it on both sides to obtain

$$\boxed{A\bar{v} = \lambda\bar{v}}.$$

This means $\bar{v}$ is an eigenvector corresponding to the eigenvalue λ of the matrix A!

Recall :

- Eigenvalues are calculated by solving the equation $\det(A-\lambda I)=0$.
  ↳ solve for λ
- For each eigenvalue λ, we find the corresponding eigenvectors by solving
  $$(A-\lambda I)\bar{v} = 0.$$
  ↳ solve for $\bar{v}$

Fact:

- Eigenvectors corresponding to distinct eigenvalues are linearly independent.

So if A is an $n \times n$ matrix, and we obtain n linearly independent eigenvectors, then we can write down the general solution to $\bar{x}' = A\bar{x}$ easily:

<u>Step 1</u> : Solve the eqn. $\det(A - \lambda I) = 0$ to get $\lambda_1, \lambda_2, \ldots, \lambda_n$ (can be repeated eigenvalues!)

<u>Step 2</u> : Find n-linearly independent eigenvectors (if possible):

$\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$

<u>Step 3</u> : Write down the general solution:

$$\bar{x}(t) = c_1 \bar{v}_1 e^{\lambda_1 t} + c_2 \bar{v}_2 e^{\lambda_2 t} + \cdots + c_n \bar{v}_n e^{\lambda_n t}.$$

Case of distinct eigenvalues (real)

Eg! Solve: $x_1' = x_1 + 3x_2$

$x_2' = x_1 - x_2$

Soln. The above system can be expressed as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightsquigarrow \bar{x}' = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \bar{x}$$

Step 1: Finding eigenvalues of A:

We need to solve $\det(A - \lambda I) = 0$:

i.e., $\begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)(-1-\lambda) - 3 = 0$

$\lambda^2 = 4$

$\therefore \lambda_1 = 2, \lambda_2 = -2$ are the two eigenvalues.

Step 2: Finding eigenvectors:

(i) For $\lambda_1 = 2$: Solve $(A - \lambda_1 I)\bar{v} = \bar{0}$:

$$\begin{bmatrix} 1-2 & 3 \\ 1 & -1-2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

leads to the equations $\quad -a+3b = 0$

$$a - 3b = 0$$

$\therefore \quad a = 3b$, b is a free variable.

So an eigenvector for $\lambda_1 = 2$ is $\bar{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(ii) For $\lambda_2 = -2$: $\quad \begin{bmatrix} 1+2 & 3 \\ 1 & -1+2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

leads to $a + b = 0$,

So $a = -b$, b is free.

An eigenvector for $\lambda_2 = -2$ is $\bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Note: Recall from Linear Algebra that if $\bar{v}$ is an eigenvector for λ then $c\bar{v}$ for any $c \in \mathbb{R}$ is also an eigenvector for λ. So the choice of eigenvectors is not unique. We could have taken

$\bar{v}_1 = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$ and $\bar{v}_2 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$. However we choose scalar multiples in such a way that the eigenvectors are simple.

<u>Step 3</u>: The general solution is therefore

$$\bar{x}(t) = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}.$$

---

2. Solve:

$$x_1' = x_1 + 2x_2 + x_3$$
$$x_2' = 6x_1 - x_2$$
$$x_3' = -x_1 - 2x_2 - x_3$$

<u>Solution</u>: The given system is

$$\bar{x}' = A\bar{x} \quad \text{where} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}.$$

Step1 : Eigenvalues:

$$\det(A-\lambda I)=0$$

$$(1-\lambda)\left[(-1-\lambda)^2-0.(-2)\right]-2\left[6(-1-\lambda)-(-1)(0)\right]$$
$$+1\left[6(-2)-(-1)(-1-\lambda)\right]=0$$

$$(1+\lambda)^2(1-\lambda)+2\left(6(1+\lambda)\right)-12-(1+\lambda)=0$$

$$(1-\lambda)(1+\lambda)^2+12+12\lambda-12-(1+\lambda)=0$$

$$(1+\lambda)(1-\lambda^2-1)+12\lambda=0$$

$$-\lambda^3-\lambda^2+12\lambda=0$$

$$\Rightarrow\quad -\lambda(\lambda+4)(\lambda-3)=0$$

So the eigenvalues are $\lambda_1=0$
$\lambda_2=-4$
$\lambda_3=3$.

Step2 : Standard way to find eigenvectors is by solving $(A-\lambda I)\bar{v}=\bar{0}$.

There is a quicker way! (Works if the

multiplicity of λ is one)

Steps:

1. Write down the matrix $A - \lambda I$.

2. Since λ is such that $\det(A - \lambda I) = 0$, this means $A - \lambda I$ has linearly dependent rows. Choose two rows of $A - \lambda I$ that are linearly independent (i.e., two rows that are not scalar multiples of each other.) Call them r_1 and r_2

3. The required eigenvector will be the cross product of r_1 and r_2!

Let us try this in the case $\lambda_1 = 0$ and A as above:

- $A - \lambda I = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$.

- Pick row 1 and row 2,

- Compute their cross product:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 6 & -1 & 0 \end{vmatrix} = \hat{i}(1) - \hat{j}(-6) + \hat{k}(-13)$$

Thus, $v_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}$.

<u>Note</u>: Remember that eigenvectors are not unique, and cv_1 for any $c \in \mathbb{R}$ is also an eigenvector for λ_1. Thus, the cross-product method will give <u>one</u> eigenvector, which can be scaled if necessary to match with the solution obtained from

$$(A - \lambda_1 I) v_1 = 0.$$

Let us continue to find the remaining eigenvectors using the cross-product idea:

ii) Eigenvector for $\lambda_2 = -4$:

- $A - \lambda_2 I = \begin{bmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{bmatrix}$

- Choose Rows 2 & 3 this time

- Compute cross-product:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{vmatrix}$$

$$= \hat{i}(9) - \hat{j}(18) + \hat{k}(-9)$$ $-12+3$

$\therefore$ $v_2 = \begin{pmatrix} 9 \\ -18 \\ -9 \end{pmatrix}$, which we can rescale to get $v_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

(iii) Eigenvector for $\lambda_3 = 3$:

- $A - \lambda_3 I = \begin{bmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{bmatrix}$

- Choose Row 1 and Row 3

- Compute cross product :

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 1 \\ -1 & -2 & -4 \end{vmatrix}$$

$$= \hat{i}\,(-6) - \hat{j}\,(9) + \hat{k}\,(6)$$

$$\therefore v_3 = \begin{pmatrix} -6 \\ -9 \\ 6 \end{pmatrix} \quad \text{or} \quad v_3 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}.$$

The general solution therefore is

$$\bar{x}(t) = c_1 \bar{v}_1 e^{\lambda_1 t} + c_2 \bar{v}_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t}$$

$$= c_1 \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} e^{-4t} + c_3 \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} e^{3t}.$$

It could very well happen that we do not have distinct eigenvalues.

In this case, there is still a possibility that we will have enough eigenvectors.

Examples :

1. Consider the system $\bar{x}' = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \bar{x}$.

Find the general solution.

Soln. $A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}$$

$= (\lambda-1)(-\lambda^2+11\lambda-10)$

$= -(\lambda-1)(\lambda-1)(\lambda-10).$

Therefore, the eigenvalues are 1, 1, 10.

Finding eigenvectors:

(i) $\lambda = 10$: This is not a repeated eigenvalue so we can compute the eigenvector using the cross-product trick: $A-\lambda I = \begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix}$

Pick Row 2 and Row 3:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{vmatrix} = \hat{i}(40-4) - \hat{j}(-32-4) + \hat{k}(8-(-10))$$

$$= 36\hat{i} + 36\hat{j} + 18\hat{k}.$$

$$\therefore \quad \bar{v}_1 = \begin{bmatrix} 36 \\ 36 \\ 18 \end{bmatrix}$$

which we simplify to $\bar{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

ii) $\lambda = 1$: This is a repeated eigenvalue so we proceed using the usual method:

$$(A - \lambda I)\bar{v}_2 = \bar{0} \rightsquigarrow \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

leads to <u>ONE</u> equation:

$$2a + 2b + c = 0$$

We have two free variables, so choosing appropriate values we get two linearly independent eigenvectors.

i) Pick $c = 0$, $b = 1$. Then $a = -1$

ii) Pick $c = 1$, $b = 0$. Then $a = -\frac{1}{2}$

This gives $\bar{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$, or $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

by scaling by 2.

Finally we write the general solution

$$\bar{x}(t) = c_1 \bar{v}_1 e^{10t} + c_2 v_2 e^{t} + c_3 v_3 e^{t}$$

$$= c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{10t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} e^{t}.$$

Let's do another one :

Solve : $$\bar{x}' = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix} \bar{x}.$$

Solution :

$$\bar{x}(t) = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{9t} + c_2 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{can vary depending on choice of free variables.}} e^{2t}$$