Sec. 2.2

Theorem 1: (Principle of superposition)

Let $y_1, y_2, \dots, y_n$ be n solutions of the hom. linear equation

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \dots + P_n(x)\, y = 0 \quad - \boxtimes$$

on the interval I. If $c_1, \dots, c_n$ are constants then the linear combination

$$y(x) = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

is also a solution of Eq. $\boxtimes$ on I.

Theorem 2: Existence and Uniqueness for Linear equations

Suppose that the functions $p_1, p_2, \dots, p_n$ and f are continuous on the open interval I containing the point a. Then, given n numbers $b_0, b_1, \dots, b_{n-1}$, the n^{th} order linear equation

$$y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_{n-1}(x) y' + p_n(x) y = f(x)$$

has a unique solution on the entire interval I that satisfies the n initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}.$$

Wronskian of solutions

Suppose $y_1, y_2, \ldots, y_n$ are n solutions of the hom. linear eqn. $\boxtimes$ on the interval I.

Let $W = W(y_1, \ldots, y_n)$ be the Wronskian.

i.e., $W = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & & \cdots & y_n^{(n-1)} \end{pmatrix}$

a) If $y_1, \ldots, y_n$ are lin. independent, then $W \neq 0$ on I.

b) If $y_1, \dots, y_n$ are lin. dependent then $W \equiv 0$ on I.

Examples

1) Check whether the foll. functions are linearly independent or not, on $\mathbb{R}$:

$y_1(x) = x^2$; $y_2(x) = \sin x$; $y_3(x) = \cos x$

$$W(y_1, y_2, y_3) = \begin{vmatrix} x^2 & \sin x & \cos x \\ 2x & \cos x & -\sin x \\ 2 & -\sin x & -\cos x \end{vmatrix}$$

$= -x^2 - 2$, which is never zero on $\mathbb{R}$.

$\therefore$ The given functions are linearly independent

2) Try: $y_1(x) = x$, $y_2(x) = x\log x$, $y_3(x) = x^2$

<u>Theorem</u>: General solutions of homogeneous equations:

Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0;$$

on an open interval I where $p_1, \ldots, p_n$ are continuous on I. If Y is a solution of the above equation, then it is of the form

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

for some constants $c_1, \ldots, c_n$.

<u>Example</u>: Show that $y_1(x) = x$, $y_2(x) = x^2$, $y_3(x) = x\log x$ are linearly independent

solutions of the D.E.

$$x^3 y^{(3)} - x^2 y^{(2)} + 2xy' - 2y = 0.$$

Find a particular solution on the open interval $x > 0$ satisfying:

$$y(1) = 3 \qquad y'(1) = 2 \qquad y''(1) = 1.$$

<u>Solution</u> (sketch) : Verify that $W(x, x^2, x\log x) \neq 0$ for $x > 0$.

Thus the general solution is given by

$$y(x) = C_1 x + C_2 x \log x + C_3 x^2.$$

Then $y'(x) = C_1 + C_2(\log x + 1) + 2C_3 x$

$$y''(x) = C_2 \cdot \frac{1}{x} + 2C_3$$

Use $y(1) = 3$, $y'(1) = 2$, $y''(1) = 1$ to obtain a linear system of 3 eqns. in 3 variables.

Solve to get $c_1 = 1$, $c_2 = -3$, $c_3 = 2$.

Thus, a particular solution is

$$y(x) = x - 3x\log x + 2x^2$$

<u>Solving homogeneous linear equations with constant coefficients</u> :

- Similar to $n=2$: write down the characteristic equation and find its roots, then identify the linearly independent solutions of the ODE using the nature of these roots (real repeated / distinct real / complex)

It is a hard problem to actually find roots of a polynomial of degree n (>2). But assume we have the roots

$$r_1, r_2, \ldots, r_n$$

of the char. eqn.

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0 .$$

coming from the homogeneous diff. eqn.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 .$$

Idea:

① For each root r_i, we get a solution y_i

② Put them together to get n (lin. ind.) solutions to write a general solution $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$.

Let me elaborate:

Now, a root can be distinct, or repeated, real, or complex.

The multiplicity of a root is the no. of times it appears in the list of ALL roots of an equation.

[A polynomial of degree n has n roots] eg. 1. $(x-2)(x-3)=0$ $\rightsquigarrow$ Two roots : 2, 3

char. eqn

multiplicity : 1 1

The root $r_1 = 2$ corresponds to the solution e^{2x}

Similarly, $r_2 = 3$ corresponds to the solution e^{3x}

Put them together: $y = c_1 e^{2x} + c_2 e^{3x}$ $\leftarrow$ general solution

2. $(x-2)(x+4)^2 = 0$ $\rightsquigarrow$ Two roots : 2, -4

char. eqn

multiplicity : 1 2

The root $r_1 = 2$ again corresponds to the soln. e^{2x}

The root $r_2 = 4$ has multiplicity two, so we need <u>two</u> lin. independent solutions :

$$e^{-4x}, \quad xe^{-4x}.$$

$\therefore$ a general solution is $y = c_1 e^{-4x} + c_2 x e^{-4x} + c_3 e^{2x}$.

3. $(x-4)^3 = 0$ $\rightsquigarrow$ One root : 4

multiplicity : 3.

So we need <u>three</u> lin. independent solutions we know two: e^{4x} and xe^{4x}. The third one is $x^2 e^{4x}$.

In general, the part of the general solution corresponding to a repeated real root α of multiplicity k is given by:

$$c_1 e^{\alpha x} + c_2 x e^{\alpha x} + \dots + c_k x^{k-1} e^{\alpha x},$$

i.e., the k linearly ind. solutions are

$$e^{\alpha x}, \; x e^{\alpha x}, \dots, x^{k-1} e^{\alpha x}.$$

Complex-roots of characteristic eqn

Can try to replicate the real root situation, where $r \rightsquigarrow e^{rx}$

Let $z = a+ib$ be a complex root.

We could try $e^{zx} = e^{(a+ib)x}$.

$$= \underbrace{e^{ax}}_{\text{familiar}} \cdot \underbrace{e^{ibx}}_{\text{unfamiliar!}}$$

We know:

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

$$\therefore \quad e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots$$

Using $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and so on,

$$e^{it} = \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right)$$

Taylor series for $\cos t$

Taylor series for $\sin t$

$\therefore$ We obtain Euler's formula:

$$\boxed{e^{it} = \cos t + i \sin t}, \quad \text{for } t \in \mathbb{R}.$$

<u>Fact</u>: If $a+ib$ is a root of a polynomial, its conjugate: $a-ib$ is also a root. In other words, complex roots <u>always</u> occur in pairs.

Therefore, if $a+ib$ is a root of the characteristic eqn, then so is $a-ib$, so we get two (complex) solutions: $e^{(a+ib)x}$ and $e^{(a-ib)x}$.

However, we want two linearly independent <u>real</u> valued functions as solutions to our diff. eqn. We obtain them in the following manner:

$$* \begin{cases} y(x) = c_1 e^{ax}(\cos bx + i\sin bx) \\ \qquad\qquad + c_2 e^{ax}(\cos bx - i\sin bx) \end{cases}$$

where c_1 and c_2 are allowed to be complex numbers. We make certain choices of c_1 and c_2 in order to extract real-valued solutions:

① Let $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$ in $*$:

$$y_1(x) = e^{ax}\cos bx$$

② Let $c_1 = -\frac{i}{2}$, $c_2 = \frac{i}{2}$ in $*$:

$$y_2(x) = e^{ax}\sin bx.$$

You can verify that these two are linearly independent functions by computing the Wronskian.

Thus, a general (real-valued) solution corresponding to the roots $a+ib$, $a-ib$ is given by $y(x) = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$.

or

$$y(x) = e^{ax}\left[c_1 \cos bx + c_2 \sin bx\right].$$

<u>Repeated complex roots</u>: If the characteristic eqn. has a repeated root that is complex, say $r = a+ib$, with multiplicity k [for example $(x^2+1)^2 = 0$] then the part of the general solution corresponding to this repeated root is given by

$$(c_1 + c_2 x + c_3 x^2 + \dots + c_k x^{k-1})\, e^{ax}\cos bx$$

$$+ (D_1 + D_2 x + D_3 x^2 + \dots + D_k x^{k-1})\, e^{ax}\sin bx.$$

Example: A certain 10-degree polynomial which is the char. eqn. of a 10th order linear hom. diff. eqn with constt. coeff. has the following roots:

$$3, -5, 0, 0, 0, -5, 2 \pm 3i, 2 \pm 3i.$$

Write down the general solution.

Solution:

S.no.	Root	Type	component of general solution
1.	3	Real, distinct	$c_1 e^{3x}$
2.	-5	Real, repeated (2)	$(c_2 + c_3 x) e^{-5x}$
3.	0	Real, repeated (3)	$c_4 + c_5 x + c_6 x^2$
4.	$2+3i$	Complex, repeated (2)	$(c_7 + c_8 x) e^{2x} \cos 3x + (c_9 + c_{10} x) e^{2x} \sin 3x$

$\boxed{\text{Tip:}}$ Number the constants as $c_1, c_2, \dots$ so it is clear whether you have the correct number of linearly independent solutions or not.

Try: The char. eqn of a certain lin. hom. ODE with constt coeff is (partially) factorized as

$$(9r^5 - 6r^4 + r^3)(r^2+1)^2 = 0.$$

Write down the general solution.

One type of linear homogeneous ODE we can solve, which does not have constant coefficients:

[Ex 51. of Sec 2.1]

A second-order Euler equation is one of the form $a x^2 y'' + b x y' + c y = 0$

where a, b, c are constants.

If $x > 0$, the substitution $v = \log x$ transforms the given equation into a constant-coefficients lin. eqn:

Let $v = \log x$, so $v' = \frac{dv}{dx} = \frac{1}{x}$.

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx} = \frac{1}{x} \frac{dy}{dv}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dv}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dv} + \frac{1}{x}\underbrace{\left[\frac{d}{dx}\left(\frac{dy}{dv}\right)\right]}$$

$$\frac{d^2y}{dx\,dv} = \frac{d^2y}{dv^2} \cdot \frac{dv}{dx}$$

$$= \frac{1}{x}\frac{d^2y}{dv^2}.$$

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}.$$

Therefore, the given diff. eqn. becomes

$$a x^2 \cdot \frac{1}{x^2}\left[\frac{d^2y}{dv^2} - \frac{dy}{dv}\right] + bx \cdot \frac{1}{x}\frac{dy}{dv} + cy = 0$$

i.e., $$a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0$$

Constant coefficients!
