Non-homogeneous equations.

general form:

$$\text{(I)} - \quad y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_{n-1}(x) y' + p_n(x) y = f(x)$$

Suppose we had one particular solution

y_p of eqn. (I), and Y is a general solution of (I). That is,

$$y_p^{(n)} + p_1(x) y_p^{(n-1)} + \cdots + p_{n-1}(x) y_p' + p_n(x) y_p = f(x)$$

and $Y^{(n)} + p_1(x) Y^{(n-1)} + \cdots + p_{n-1}(x) Y' + p_n(x) Y = f(x).$

Observe that the difference $Y - y_p$ satisfies the homogeneous eqn. associated to (I):

$$(Y-y_p)^{n} + p_1(x)(Y-y_p)^{(n-1)} + \cdots + p_n(x)(Y-y_p) = f(x)-f(x)$$
$$= 0.$$

and we have learned how to solve homogeneous equations (with constt coeff.)

$\therefore$ $Y - y_p$ is a general solution to the homogeneous eqn associated to (I), so it is of the form $y_c = c_1 y_1 + \cdots + c_n y_n$ where $y_1, \ldots, y_n$ are lin. ind. solutions.

We call y_c the 'complementary function', the general solution of the hom. eqn associated to the given non-hom. eqn.

Examples Find a general solution :

1. $y'' - 2y' + 2y = 2x, \quad y_p = x + 1.$

1. $y'' - 4y = 12 \; ; \quad y_p = -3.$

Q: How to get hold of y_p?

A. In general, hard, but if $f(x)$ is relatively "simple", we can use the "method of undetermined coefficients" to get a particular solution y_p.

	Simple forms of $f(x)$	"Trial" y_p
1.	x (or any deg 1 polynomial)	$Ax + B$
2.	x^2 (or any deg 2 polynomial)	$Ax^2 + Bx + C$.
3.	deg m polynomial	$A_m x^m + A_{m-1} x^{m-1} + \cdots + A_0$
	given in problem.	variables
4.	$a \cos kx + b \sin kx$	$A \cos kx + B \sin kx$.
		same k as in $f(x)$.
5.	e^{kx}	Ae^{kx}

Then, substitute the trial y_p into the diff. eqn. and solve for the constants $A, B, \ldots$

Examples : 1. Find a particular solution of $y'' - 4y = 2e^{3x}$.

Solution : $f(x) = 2e^{3x}$.

Trial $y_p = Ae^{3x}$.

$y_p' = 3Ae^{3x}$, $y_p'' = 9Ae^{3x}$.

Plugging this into the given D.E :

$$y_p'' - 4y_p = 2e^{3x}$$

$$9Ae^{3x} - 4Ae^{3x} = 2e^{3x}$$

$$5Ae^{3x} = 2e^{3x}$$

$$\Rightarrow A = \frac{2}{5}.$$

$\therefore$ A particular solution is $y_p = \frac{2}{5}e^{3x}$.

2. Solve: $y'' + 3y' + 4y = 3x + 2$.

<u>Solution</u>: First solve the associated homogeneous eqn: $y'' + 3y' + 4y = 0$

char. eqn: $r^2 + 3r + 4 = 0$

roots $\rightsquigarrow$ $-\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$

so $y_c = e^{-3/2x}\left(c_1 \cos \frac{\sqrt{7}}{2}x + C_2 \sin\frac{\sqrt{7}}{2}x\right)$

Now for y_p:

RHS: $3x + 2$; so trial $y_p = Ax + B$.

$y_p' = Ax$, $y_p'' = A$.

Substitute: $y_p'' + 3y_p' + 4y_p = 3x + 2$ and solve for A and B by equating coefficients of like powers of x on both sides to get: $A = \frac{3}{4}$, $B = -\frac{1}{16}$.

$\therefore$ $y_p = \frac{3}{4}x - \frac{1}{16}$, and the general solution of the given diff. eqn is

$$Y = y_c + y_p = e^{-3/2\,x}\left(C_1 \cos\frac{\sqrt{7}}{2}x + C_2 \sin\frac{\sqrt{7}}{2}x\right) + \frac{3}{4}x - \frac{1}{16}.$$

Try: Solve $y'' - 4y = 2e^{3x}$. Soln:

$$Y = C_1 e^{2x} + C_2 e^{-2x} + \frac{2}{5}e^{3x}$$

Suppose the D.E. was $y'' - 4y = e^{2x}$.

Trial solution Ae^{2x} will <u>NOT</u> work!

This is because e^{2x} solves the homogeneous ODE.

However, Axe^{2x} would work.

We outline the method of undetermined coefficients below, to solve:

$$ay'' + by' + cy = f(x)$$

This works as long as $f(x)$ is a linear combination of finite products of functions of the following types:

1. A polynomial in x
2. An exponential function e^{rx}
3. $\sin kx$ or $\cos kx$.

First, obtain the complimentary function y_c.

<u>Rule 1</u> If no term appearing in $f(x)$ or any of its derivatives satisfies the associated homogeneous equation, then take as a trial solution y_p, a linear combination of all linearly independent such terms and its derivatives.

If Rule 1 is violated, we proceed to apply:

Rule 2 (Case of duplication): If the candidate y_p attempted in Rule 1 has some terms in common with some terms in y_c, then multiply that part of y_p with x^s where s is the smallest number so that the modified y_p and y_c now have no terms in common.

Let's do lots of examples to get the idea:

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① $y_c : c_1 e^{2x} + c_2 e^{5x}$, $f(x) = e^{-x}$

Rule 1 applies! Take $y_p = A e^{-x}$.

② $y_c = c_1 + c_2 x$, $f(x) = 3x+2$

Rule 2 applies, since y_c & $f(x)$ have terms in common, so $Ax+B$ will not work.

Try : $x(Ax+B) = Ax^2 + Bx$ → common term with $c_2 x$ in y_c

Try: $x^2(Ax+B) = Ax^3+Bx^2$ ✓ no terms in common with y_c!

to multiply ↑ make sure the whole term $(Ax+B)$ by x^2, not just Bx.

③ $y_c = c_1 e^x \sin 2x + c_2 e^x \cos 2x$,

$f(x) = x \sin 2x$

Rule 1 applies! Note that $x \sin 2x$ and $e^x \sin 2x$ are not considered duplicates.

Set $y_p = (Ax+B)\sin 2x + (Cx+D)\cos 2x$.

④ $y_c = c_1 e^{2x} + c_2 \cos x + c_3 \sin x$

$f(x) = \frac{1}{2}e^{2x} + x$

Rule 2 applies, since $c_1 e^{2x}$ and $\frac{1}{2}e^{2x}$ are duplicates. But "x" is not a problem so we set

$$y_p = \underbrace{x e^{2x}}_{\text{using } \frac{1}{2}e^{2x}} + \underbrace{Ax+B}_{\text{using } x}$$

Example 1: Solve $y'' - 5y' + 6y = 2e^{3x} + x$

Solution: Finding y_c:

We solve the homogeneous equation

$$y'' - 5y' + 6y = 0$$

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0 \quad \therefore r_1 = 2, r_2 = 3$$

$$\therefore y_c = C_1 e^{2x} + C_2 e^{3x}.$$

$f(x)$ from the given D.E. is $2e^{3x} + x$.

A trial y_p would have been $Ae^{3x} + Bx + C$

However, Ae^{3x} duplicates the $C_2 e^{3x}$ term in y_c, therefore we replace it with Axe^{3x}, to obtain the modified trial y_p as

$$y_p = Axe^{3x} + Bx + C.$$

Now we proceed to find A, B, C:

want: $y_p'' - 5y_p' + 6y_p = 2e^{3x} + x$

$y_p' = A(e^{3x} + 3xe^{3x}) + B.$

$y_p'' = 3A(2e^{3x} + 3xe^{3x})$

$\therefore$ we want

$(6Ae^{3x} + 9Axe^{3x}) - 5(Ae^{3x} + 3Axe^{3x} + B)$
$+ 6(Axe^{3x} + Bx + C) = 2e^{3x} + x.$

Equate coefficients of like terms:

e^{3x}: $6A - 5A = 2 \Rightarrow A = 2$

x: $6B = 1 \Rightarrow B = \frac{1}{6}$

xe^{3x}: $9A - 15A + 6A = 0$

1: $-5\cancel{B} + 6C = 0 \Rightarrow C = \frac{5}{36}.$

A particular solution is given by:

$$y_p = 2e^{3x} + \frac{1}{6}x + \frac{5}{36},$$

and a general solution of the given ODE

is $\quad Y(x) = y_c + y_p$

$$= (c_1 e^{2x} + c_2 e^{3x}) + 2e^{3x} + \frac{x}{6} + \frac{5}{36}.$$

<u>Example 2</u> : Solve: $\quad y'' - 4y' + 4y = \cos 2x$

Solution : $\quad (c_1 + c_2 x) e^{2x} - \frac{1}{8} \sin 2x$

<u>Example 3</u> : Solve : $\quad y'' - 6y' + 9y = 2e^{3x}$

Solution : $\quad (c_1 + c_2 x) e^{3x} + x^2 e^{3x}$

Variation of Parameters

There is an all-purpose formula to find y_p, but it involves integrals.

The idea is to set up a particular solution (say for 2nd order) as

$$y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are lin. ind. solutions appearing in y_c: $y_c = c_1 y_1(x) + c_2 y_2(x)$.

Then, one can derive:

$$y_p = -y_1(x) \int \frac{y_2(x) f(x)}{W}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W}\, dx$$

where $W = W(y_1, y_2)$ is the Wronskian of y_1 and y_2.

<u>Example</u> : Solve: $y'' + y = \tan x$

<u>Solution</u> : The associated hom. eqn. is $y'' + y = 0$.

Solve to get $y_c = A\cos x + B\sin x$.

Now, we use the method of variation of parameters to get y_p:

Let $y_1 = \cos x$, $y_2 = \sin x$.

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$(\ddagger) \therefore \ y_p = -\cos x \int \sin x \tan x \, dx + \sin x \int \cos x \tan x \, dx$$

- $\displaystyle\int \sin x \tan x \, dx = \int \frac{\sin^2 x}{\cos x} \, dx$

$$= \int \frac{1 - \cos^2 x}{\cos x} \, dx$$

$$= \int (\sec x - \cos x)\,dx$$

$$= \log|\sec x + \tan x| - \sin x$$

- $\int \cos x \tan x\,dx = \int \sin x\,dx = -\cos x.$

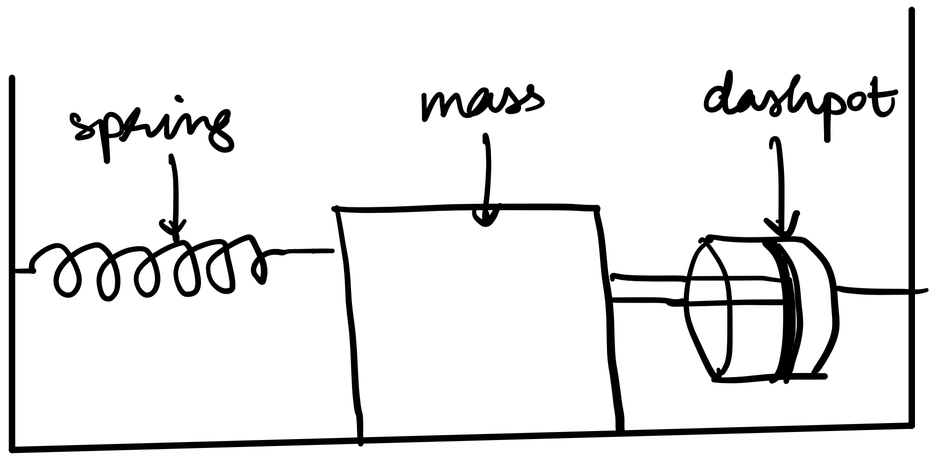
$$\therefore y_p = -\cos x\,(\log|\sec x + \tan x| - \sin x) + \sin x\,(-\cos x).$$

$$= -\cos x \log|\sec x + \tan x|.$$

Finally, a general solution is given by

$$\boxed{y(x) = A\cos x + B\sin x - \cos x \log|\sec x + \tan x|}.$$

Application: Mechanical Vibrations



Setup of the differential equation:

We consider a mass-spring-dashpot system where a mass m is attached to a spring and (may be) connected to a dashpot (shock-absorber).

We now set up a D.E. that describes various properties of the system:

We take $x > 0$ when the spring is stretched

$x < 0$ when the spring is compressed

$x = 0$ at equilibrium.

Hooke's Law: The restorative force exerted by the spring on the object acts in the opposite direction of motion and is proportional to it:

$$(A) \qquad \underbrace{F_s}_{\text{Restorative force of spring}} = -\overbrace{k}^{\text{positive constant called "spring constant"}} \underbrace{x}_{\text{displacement}}$$

Note: F_s and x are of opposite signs.

The dashpot provides a force directed opposite to the instantaneous direction of motion of the mass / object. It is designed so that it is proportional to the velocity of the object.

This gives:

(B) $F_R = -c\dfrac{dx}{dt} = -cx'$

Here 'c' is the damping constant, which is also always positive.

F_R and x' are in opposite directions, so they will have values with opposite signs.

In addition, we may have an external force acting on the system:

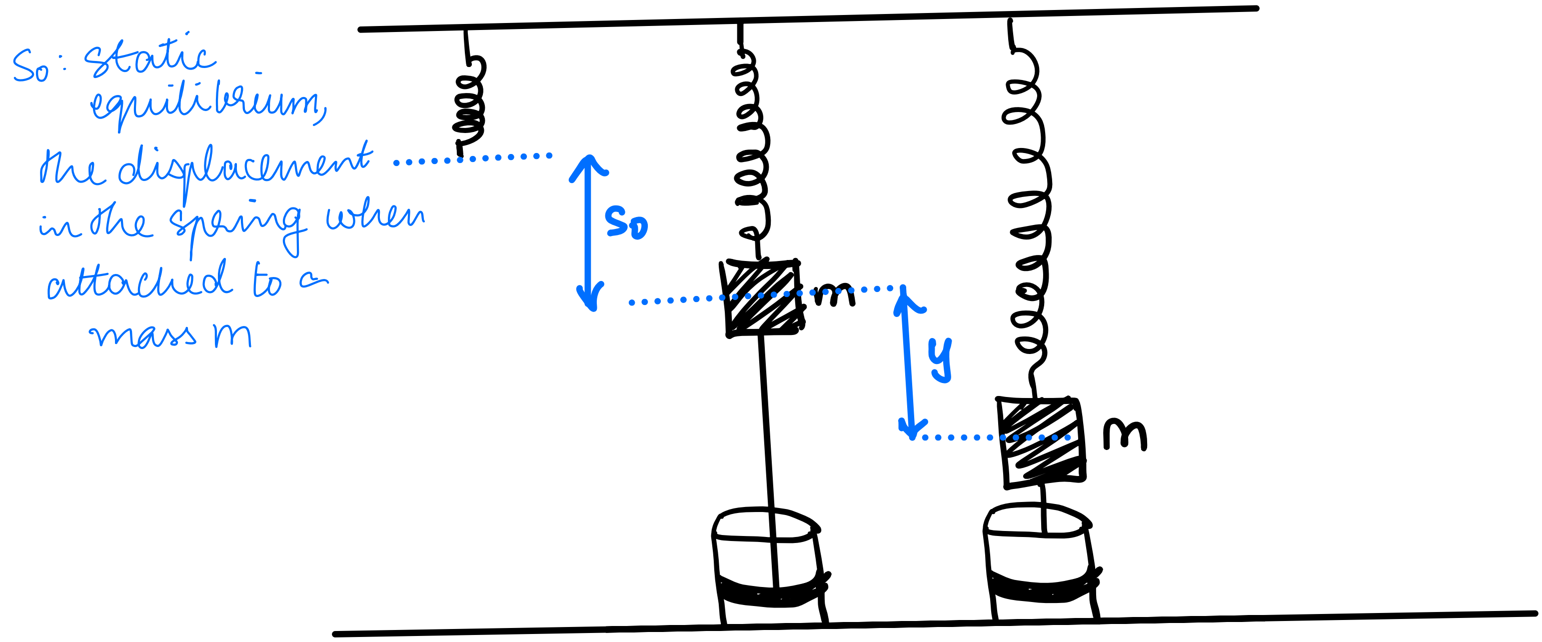
(C). $F_E = f(t)$

The total force is given by Newton's Law: $F_S + F_R + F_E = F = m\dfrac{d^2x}{dt^2}$.

and we arrive at our diff. eqn:

$$\boxed{mx'' + cx' + kx = F(t)}$$

Alternatively, we could have the situation where the object is suspended vertically while attached to a spring:



The restorative force in this case is

$$F_s = -W = -mg.$$

We have: $F_s = -ks_0 \Rightarrow mg = ks_0$

$$\therefore s_0 = \frac{mg}{k}.$$

Here too, we may or may not have a dashpot. The diff. eqn. is the same:

$$my'' + cy' + ky = F(t)$$

where $y(t)$ is the displacement in the vertical direction at time t.

<u>Terminology</u>: If there is no dashpot, $c = 0$ and the motion is said to be "<u>undamped</u>".

Else, if $c > 0$, the motion is said to be "<u>damped</u>".

the D.E. is homogeneous

If there is no external force, then $F(t) = 0$ and the motion is "<u>free</u>", else if $F(t) \neq 0$, it is called "forced".

non-homogeneous.

I. Free, Undamped motion

In this case, $\overset{\text{undamped}}{\overset{\Downarrow}{c=0}}$, $\overset{\text{Free}}{\overset{\Downarrow}{F(t)=0}}$, so the D.E. is

$$mx'' + kx = 0.$$

Let $\omega_0 = \sqrt{\frac{k}{m}}$, the circular frequency.

Then the above D.E. becomes

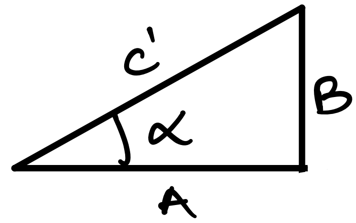
$$x'' + \omega_0^2 x = 0.$$

char eqn:
$r^2 + \omega_0^2 = 0$
$\therefore r = \pm i\omega_0$
$\therefore$ 2 lin. ind. solutions are:
$\cos\omega_0 t, \sin\omega_0 t$

A general solution to this homogeneous 2nd order lin. eqn. is

$$\boxed{x(t) = A\cos\omega_0 t + B\sin\omega_0 t}. \quad -(\mathrm{I})$$

Let us analyze the motion described by this solution:



Let $c' = \sqrt{A^2+B^2}$, $\cos\alpha = \dfrac{A}{c'}$ and $\sin\alpha = \dfrac{B}{c'}$.

Multiply & divide the RHS of (I) by c':

$$x(t) = c'\left(\frac{A}{c'}\cos(\omega_0 t) + \frac{B}{c'}\sin(\omega_0 t)\right).$$

Using the trig formula: $\cos(\theta_1 \pm \theta_2)$
$= \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$

we obtain:

$$x(t) = c' \cos(\omega_0 t - \alpha).$$

Here, c' : amplitude

ω_0 : circular frequency

α : phase angle.

∴ The mass oscillates to and fro about its equilibrium position. Such motion is called "simple harmonic motion".

Some terminology:

T : period (of oscillation $= \frac{2\pi}{\omega_0}$) → time taken to complete one full oscillation.

$\nu = \frac{1}{T}$ is the frequency → no. of complete cycles per second.

If we know the initial position $x(t)$ and initial velocity $x'(t)$ of the mass, then we can calculate the constants A and B of eqn (I), then find the phase angle α.

<u>Note</u>: Although $\tan\alpha = \frac{B}{A}$, the angle α in our derivation is between 0 and 2π, but our calculators take the principle branch of tan to be between $-\pi/2$ and $\pi/2$. So in effect:

$$\alpha = \begin{cases} \tan^{-1}\left(\frac{B}{A}\right) & \text{if } A>0, B>0 \text{ (1st quad)} \\ \pi + \tan^{-1}\left(\frac{B}{A}\right) & \text{if } A<0 \text{ (2nd or 3rd quad)} \\ 2\pi + \tan^{-1}\left(\frac{B}{A}\right) & \text{if } A>0, B<0. \text{ (4th quad)} \end{cases}$$

where $\tan^{-1}\left(\frac{B}{A}\right)$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ given by a calculator/computer.

Free, damped motion

In this case $c > 0$, and we get

$$m x'' + c x' + k x = 0 .$$

Again, let $\omega_0 = \sqrt{\frac{k}{m}}$. Set $p = \frac{c}{2m}$ so the above D.E. becomes

$$x'' + 2p x' + \omega_0^2 x = 0 .$$

The roots of the associated char. eqn are

$$r_{1,2} = -p \pm \sqrt{p^2 - \omega_0^2} .$$

Now, depending on whether $p^2 - \omega_0^2$ is $<, =$ or > 0, we have 3 cases:

Case 1: $\boxed{p^2 > \omega_0^2}$ $\Rightarrow \frac{c^2}{4m^2} > \frac{k}{m}$, i.e., $c^2 > 4km$.

In this situation, we say the motion is "overdamped".

The roots are real and distinct, so the position function is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Note that as $t \to \infty$, $x(t) \to 0$:

$$e^{rt} = e^{t(-p \pm \sqrt{p^2 - \omega_0^2})} \to 0 \text{ as } t \to \infty.$$

(since $\sqrt{p^2 - \omega_0^2} < p$, and $p > 0$, r_1, r_2 are both < 0.)

$\therefore$ The mass settles to its equilibrium position without any oscillations

Case 2: $p^2 = \omega_0^2$, or $c^2 = 4km$: This is called the "critically damped" case.

The char. eqn. has repeated, real roots so

$$x(t) = e^{-pt}(c_1 + c_2 t).$$

Again, as $t \to \infty$, $x(t) \to 0$.

<u>Case 3</u> : $p^2 < \omega_0^2$, or $c^2 < 4km$: This is called the "underdamped" case.

The char. roots have two complex conjugate roots : $-p \pm i\sqrt{\omega_0^2 - p^2}$, and the general solution is

$$x(t) = e^{-pt}\left(A\cos\omega_1 t + B\sin\omega_1 t\right)$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2} = \dfrac{\sqrt{4km - c^2}}{2m}$.

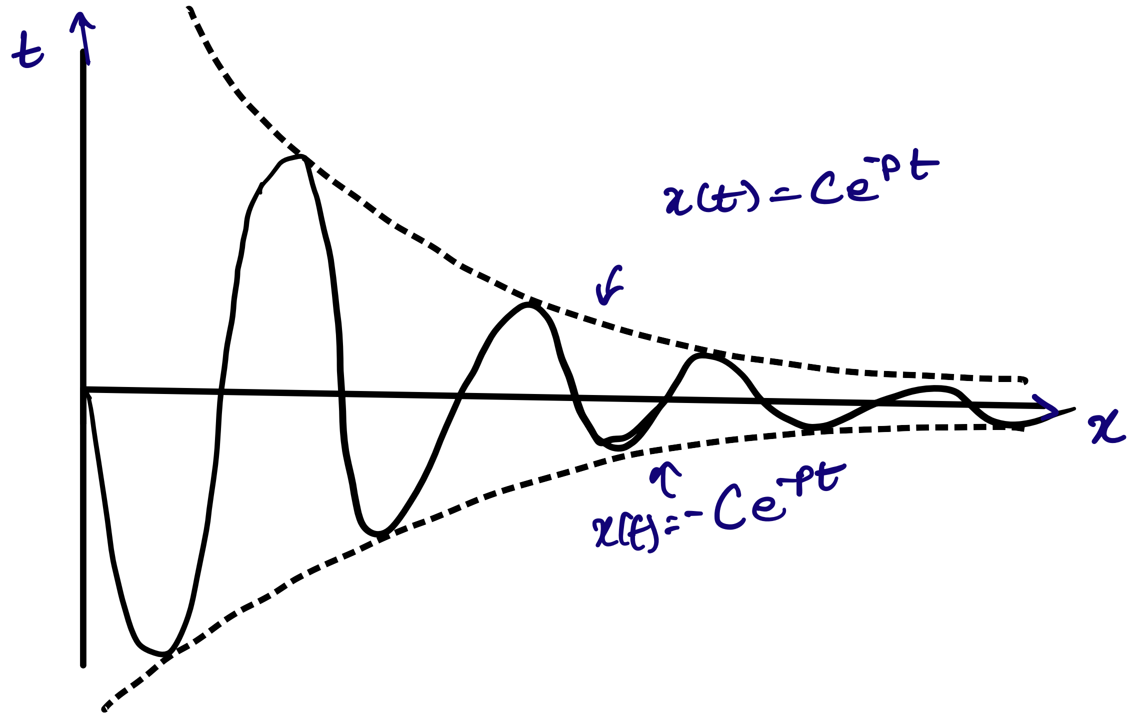
As before, letting $C = \sqrt{A^2 + B^2}$, $\cos\alpha = \frac{A}{C}$, $\sin\alpha = \frac{B}{C}$ and using the addition formula of cosines,

$$x(t) = C e^{-pt} \cos(\omega_1 t - \alpha)$$

The graph of this motion is given below.

The motion is not actually periodic,

so



ω_1 : "pseudofrequency"

$T_1 = \frac{2\pi}{\omega_1}$: pseudoperiod.

Ce^{-pt} : time-varying amplitude.

$\omega_1 < \omega_0$, so $T_1 > T$. i.e., there is a time lag

$\therefore$ The oscillations are damped exponentially, in accordance with the time-varying amplitude.

Forced, undamped motion :

An external force $F_E = f(t)$ is usually incorporated into the system via rotations in most mechanical systems, so it is given as $F_0 \cos(\omega t)$ or $F_0 \sin(\omega t)$.

We assume $\omega \neq \omega_0$.

$\therefore$ we solve the DE: $mx'' + kx = F_0 \cos \omega t$

$$x_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

Trial x_p : $A \cos \omega t + B \sin \omega t$.

If $\omega \neq \omega_0$, there is no duplication and we solve to get $A = \dfrac{F_0}{k - m\omega^2}$, $B = 0$.

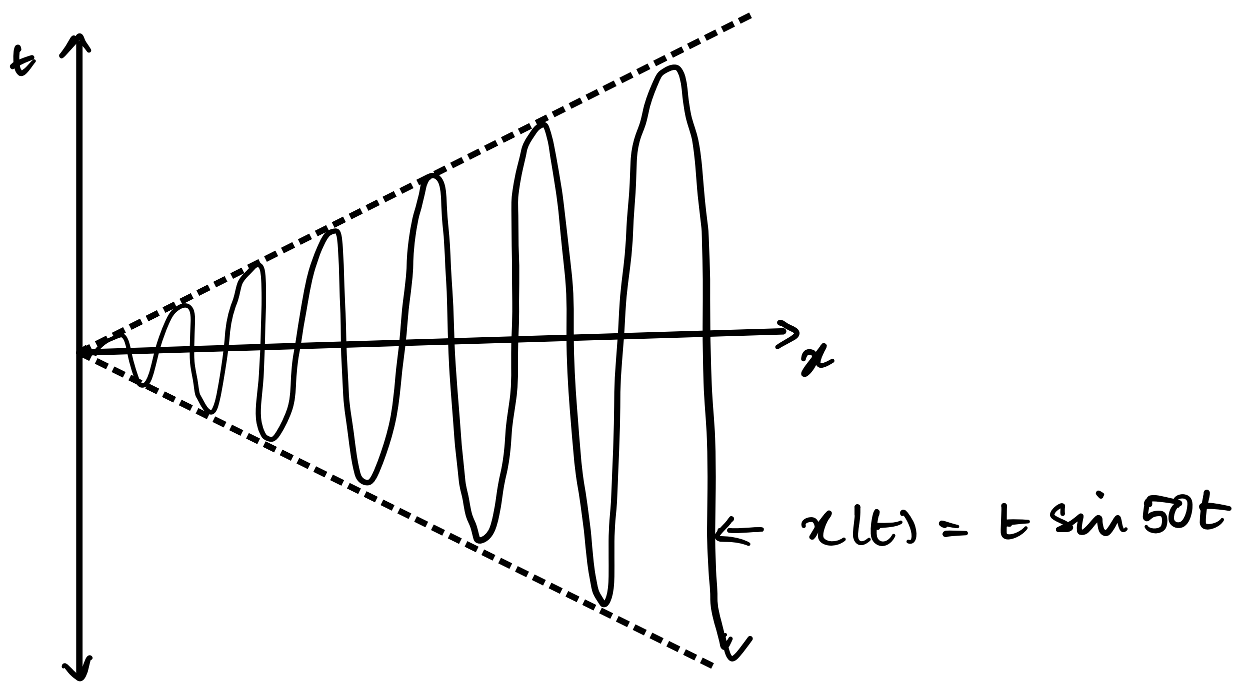
$$\therefore x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{\underbrace{k - m\omega^2}} \cos \omega t.$$

$$= \frac{F_0/m}{\omega_0^2 - \omega^2}.$$

If ω_0 and ω are very close, then $x_p(t)$ becomes very large, this is called "resonance": when freq. of the system and freq. of the external force become very close to each other, causing amplitude to be very large.

If $\omega_0 = \omega$, then method of undetermined coefficients leads to $x_p(t) = \dfrac{F_0}{2m\omega_0} t \sin \omega_0 t$.

we then get a situation of "pure resonance" and the graph of $x_p(t)$ looks like:



Damped, forced oscillations

In this case the D.E. is

$$m x'' + c x' + kx = F_0 \cos \omega t$$

Here, $x_c(t)$ depends on the nature of damping.

In all the 3 cases, $x_c(t) \to 0$ as $t \to \infty$.

Thus, the motion is dominated by oscillations coming from the external force. Due to this, in the case of forced oscillations the $x_c(t)$ term is often referred to as "transient motion".

Let $x_p(t) = A \cos \omega t + B \sin \omega t$.

We find the values of A and B to be:

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}.$$

Observe that there is no danger of unbounded amplitude here, since the denominators are always positive and never going to 0.

$x_p(t)$ is referred to as 'steady periodic motion'

Problems

1. A mass of 3kg is attached to the end of a spring that is stretched 20 cm by a force of 15N.
It is set in motion with initial position $x_0 = 0$ and initial velocity $v_0 = -10\,m/s$
Find $x(t)$, amplitude, period and frequency of the resulting motion

Soln. Since the spring is stretched by 20 cm when acted on by a force of 15 N, so the spring exerts an equal but opposite restorative force F_s of $-15N$:

$$F_s = -kx \Rightarrow -15 = -k(0.2)$$

$$\therefore k = \frac{15}{0.2} = 75\,N/m$$

There is no external force or damping, so the motion is free, undamped. Therefore the equation of the system is

$$mx'' + kx = 0.$$

i.e., $$3x'' + 75x = 0.$$

Solve to get $x(t) = -2\sin 5t$.

Amplitude $= \sqrt{(-2)^2 + 0^2} = 2m$

Period $= T = \frac{2\pi}{\omega_0}$, $\omega_0 = \sqrt{\frac{k}{m}} = 5$

$\therefore$ $T = \frac{2\pi}{5}$ seconds. Frequency $= \frac{1}{T} = \frac{5}{2\pi}\ sec^{-1}$.

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2. A mass of 3kg is attached to the end of a spring that has a spring constant $k = 63 N/m$. There is a dashpot attached to the mass that provides a force of 30N per m/s of velocity. The mass is set in motion at $x_0 = 2$ with initial velocity of 2m/s. Determine the position function and nature of damping.

<u>Soln</u>. Given: $m = 3kg$, $k = 63 N/m$, $c = 30 N/m/s$

$x(0) = 2 \quad , \quad x'(0) = 2.$

The motion is free and damped, so the equation governing the motion is:

$$mx'' + cx' + kx = 0.$$

i.e., $3x'' + 30x' + 63x = 0$

$$\downarrow$$

$$x'' + 10x' + 21x = 0$$

Solve to get $x(t) = c_1 e^{-3t} + c_2 e^{-7t}$.

Then, $x'(t) = -3c_1 e^{-3t} - 7c_2 e^{-7t}$

Use the initial conditions:

$$\left.\begin{aligned} x(0) = 2 &\Rightarrow c_1 + c_2 = 2 \\ x'(0) = 2 &\Rightarrow -3c_1 - 7c_2 = 2 \end{aligned}\right\} \rightarrow \text{solve to get } \begin{aligned} c_1 &= 4 \\ c_2 &= -2 \end{aligned}$$

The position function is

$\therefore \quad x(t) = 4e^{-3t} - 2e^{-7t}.$

The D.E. has real, distinct roots so the motion is <u>overdamped</u>.

3. A 19.6 N weight is attached to a vertically suspended spring that stretches 49 cm. The weight is pulled down 1 m below its static equilibrium position and released from rest at time $t = 0$. Find the position function, amplitude and period.

Solution :

$$\left.\begin{array}{l} W = 19.6\,N \\ \overset{\shortparallel}{mg} = m \times 9.8 \end{array}\right\} \Rightarrow m = 2\,kg$$

$$s_0 = 49\,cm = 0.49\,m.$$

$$mg = ks_0 \Rightarrow k = \frac{mg}{s_0} = \frac{19.6}{0.49} = 40\,N/m.$$

Initial conditions : $y(0) = 1$, $y'(0) = 0$.

$\downarrow$ "released from rest"

The motion is free, undamped. So the D.E is

$$my'' + ky = 0$$

i.e., $2y'' + 40y = 0$

Solve to get $y(t) = A\cos\sqrt{20}\,t + B\sin\sqrt{20}\,t$

$$y'(t) = -\sqrt{20}\,A\sin\sqrt{20}\,t$$
$$+\sqrt{20}\,B\sin\sqrt{20}\,t$$

$y(0) = 1 \Rightarrow A = 1$

$y'(0) = 0 \Rightarrow \sqrt{20}\,B = 0 \Rightarrow B = 0.$

$\therefore y(t) = \cos(\sqrt{20}t)$ is the position function.

- amplitude $= \sqrt{A^2+B^2} = 1$
- period $= \dfrac{2\pi}{\omega_0} = \dfrac{2\pi}{\sqrt{k/m}} = \dfrac{2\pi}{\sqrt{20}}$ sec.

4. Consider a mass-spring system with $m = 1\text{kg}$, $k = 4\,N/m$ and an external force $F_E = 5\sin 3t$. Find $x(t)$ given $x(0) = 0$, $x'(0) = 1$.

Soln. No damping, so the eqn is

$$mx'' + kx = F_E$$

i.e., $x'' + 4x = 5\sin 3t$.

Solve to get

$$x_c(t) = c_1 \cos 2t + c_2 \sin 2t$$

$f(t) = 5\sin 3t$, so no duplication.

Trial $x_p(t) = A\cos 3t + B\sin 3t$.

Solve to find A, B. Get $A = 0$, $B = -1$.

$$\therefore \quad x_p(t) = -\sin 3t.$$

$$\therefore \quad x(t) = c_1 \cos 2t + c_2 \sin 2t - \sin 3t$$

Use initial conditions $x'(0) = 1$, $x(0) = 0$

to get $c_1 = 0$, $c_2 = 2$.

Finally, $x(t) = 2\sin 2t - \sin 3t$.

5. Find the position function $x(t)$ if $m = 1\,kg$, $k = 4\,N/m$, $c = 4\,N/m/s$ and an external force of $10\cos 2t$ is applied.

<u>Soln</u>. The D.E. is

$$mx'' + cx' + kx = F_E,$$

i.e., $$x'' + 4x' + 4x = 10\cos 2t$$

Obtain x_c as $(c_1 + c_2 t)e^{-2t}$

$f(t) = 10\cos 2t$. $\therefore$ Trial $x_p = A\cos 2t + B\sin 2t$.

No duplication, so Rule 1 applies.

Solve to get $B = \frac{5}{4}$, $A = 0$.

$$\therefore x_p(t) = \frac{5}{4}\sin 2t$$

and $$x(t) = x_c(t) + x_p(t)$$

$$= (c_1 + c_2 t)e^{-2t} + \frac{5}{4}\sin 2t.$$