Logistic equation

A logistic eqn. is an ODE whose solution is a logistic function : a function that models "bounded growth".

It is of the form $\frac{dP}{dt} = kP(M-P)$ where r and M are constants.

The standard logistic equation sets $r = M = 1$, giving

$$\frac{dy}{dx} = y(1-y)$$

which is separable. Solving, we get

$$\int \frac{dy}{y(1-y)} = \int dx + C_0$$

$$\frac{A}{y} + \frac{B}{1-y}$$

$$\int \frac{dy}{y} + \int \frac{dy}{1-y} = x + C_0$$

$$\log|y| - \log|1-y| + \log C = x$$

$$\log \left| \frac{cy}{1-y} \right| = x$$

$$\frac{cy}{1-y} = e^x$$

$$y = e^x - y e^x$$

$$y(c + e^x) = e^x$$

$$y = \frac{e^x}{e^x + c}.$$

Other variations: Suppose the birthrate β is a linear decreasing function of the population size, i.e.,

$$\beta(t) = \beta_0 - \beta_1 P(t)$$

for some +ve constants β_0 and β_1.

Further suppose the death rate $\delta = \delta_0$ remains constant.

Then, going by our earlier model of $\frac{dP}{dt} = (\beta(t) - \delta(t))\, P(t)$,

we get $\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0) P$.

$$= \beta_1 \left(\frac{(\beta_0 - \delta_0)}{\beta_1} - P \right) P$$

so this is the logistic equation with $k = \beta_1$ and $M = \frac{\beta_0 - \delta_0}{\beta_1}$

Example

If we take $k = 0.0004$ and $M = 150$, the logistic equation takes the form

$$\frac{dP}{dt} = 0.0004 P (150 - P)$$

This is a separable equation and on solving we get :

$$\int \frac{dP}{P(150-P)} = \int 0.0004\, dt$$

$$\log|P| - \log|150-P| = 0.06t + C$$

$$\therefore \quad \frac{P}{150-P} = \pm e^{C} e^{0.06t}$$

$$\text{or} \quad = B e^{0.06t}$$

If $t = 0$ and $P(0) = P_0$, then (as long as $P \neq 150$)

we get the solution as

$$P(t) = \frac{150 P_0}{P_0 + (150 - P_0) e^{-0.06t}}.$$

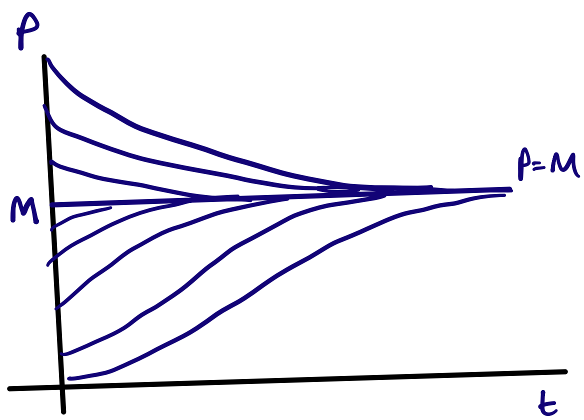
[What happens if $P_0 = 150$? The D.E will tell us that $\frac{dP}{dt} = 0$, i.e., the population remains constant.]

Analysis of the general solution to the logistic equation:

Solving the IVP: $\frac{dP}{dt} = kP(M-P)$; $P(0) = P_0$,

we get $$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}.$$

- If $P_0 = M$, then $P(t) \equiv M$.
- If $P_0 < M$, then $P' > 0$ and $P(t) < M$
- If $P_0 > M$, then $P' < 0$ and $P(t) > M$



M is therefore referred to sometimes as the "carrying capacity" of the environment.

Note: If you see other sources, you might see the logistic equation expressed as

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

where K is the carrying capacity.

Example Suppose in 1885 the population of a certain country was 50 million and was growing at the rate of 750,000 people per year at the time. Suppose in 1940 the population was 100 million and was growing at the rate of 1 million per year. Assume this population satisfies the logistic eqn. Determine the limiting population M and the predicted population for the year 2000.

Solution Using the given data,

$$0.75 = 50K(M-50) \quad , \quad 1.00 = 100K(M-100)$$

$$\frac{50K(M-50)}{100K(M-100)} = 0.75$$

$$\frac{M-50}{M-100} = 1.5$$

$$M - 50 = 1.5(M-100)$$

$$100 = 0.5M \Rightarrow M = 200$$

This leads to $k = 0.0001$.

$\therefore$ The limiting population is 200 million.

Let $t=0$ correspond to the year 1940, so $P_0 = 100$.

Then the year 2000 will be $t=60$, and

$$P(60) = \frac{100 \times 200}{100 + (200-100)e^{-(0.0001)(200)(60)}} \approx 153.705$$

[If $t=0$ corresponds to the year 1885, $P_0 = 50$

and 2000 is 115.

$$P(115) = \frac{200 \times 50}{50 + (200-50)e^{-(0.0001)(200)(115)}}$$

$$\approx 153.7543$$]

What if we didn't know the rate at which the population was growing?

<u>Example</u>: The U.S. population in 1800 was 5.308 million, in 1850 it was 23.192 million and in 1900 it was 76.212 million. Using the logistic model formula, we get two equations in two variables: k, M, but the equations are not linear:

$$\frac{5.308M}{5.308 + (M-5.308)e^{-50kM}} = 23.192 \rightsquigarrow (1850)$$

Use $P_0 = 5.308$ million taking $t=0$ as the year 1800.

$$\frac{5.308M}{5.308 + (M-5.308)e^{-100kM}} = 76.212 \rightsquigarrow (1900)$$

How do we solve this? We can use an algebraic trick here!

eqn. for 1850 gives

$$5.308 + (M-5.308)e^{-50kM} = \frac{5.308M}{23.192}$$

$$\text{so } e^{-50kM} = 5.308\left(\frac{M}{23.192} - 1\right) \times \frac{1}{(M-5.308)} \quad \text{— } ⓐ$$

eqn. for 1900 gives, similarly :

$$e^{-100kM} = 5.308\left(\frac{M}{76.212} - 1\right) \times \frac{1}{(M-5.308)} \quad \text{— (b)}$$

Since $(e^{-50kM})^2 = e^{-100kM}$, we get

$$5.308^{\cancel{2}}\left(\frac{M}{23.192} - 1\right)^2 \frac{1}{(M-5.308)^{\cancel{2}}} = \cancel{5.308}\left(\frac{M}{76.212} - 1\right)\cdot\frac{1}{\cancel{(M-5.308)}}$$

$$\left(\frac{M}{\underset{a}{23.192}} - 1\right)^2 = \left(\frac{M-\overset{b}{5.308}}{\underset{b}{5.308}}\right)\left(\frac{M}{\underset{c}{76.212}} - 1\right)$$

$$\frac{(M-a)^2}{a^2} = \frac{(M-b)}{b}\frac{(M-c)}{c}$$

$$M^2 - 2aM + a^2 = (M^2 - (b+c)M + bc)\frac{a^2}{bc}$$

$$M^2\left(1 - \frac{a^2}{bc}\right) + M\left(-2a + \frac{(b+c)a^2}{bc}\right) = 0$$

$$M\left(M\left(1-\frac{a^2}{bc}\right) - 2a + \left(\frac{1}{b}+\frac{1}{c}\right)a^2\right) = 0$$

$$M = \left(1 - \frac{a^2}{bc}\right)^{-1}\left(2a - a^2\left(\frac{1}{b}+\frac{1}{c}\right)\right)$$

$$= \left(\frac{bc}{bc-a^2}\right)\left(\frac{2abc-a^2(b+c)}{bc}\right)$$

$$M = \frac{2abc - a^2(b+c)}{bc-a^2}$$

Plugging in the values we get $M \approx 188.1208.$

$$\approx 188.121.$$

Using this in:

$$e^{-50kM} = 5.308\left(\frac{M}{23.192} - 1\right) \times \frac{1}{(M-5.308)}.$$

$$e^{-50k(188.121)} = 0.206482207$$

$$k = \frac{-1}{50 \times 188.121} \log(0.206482207)$$

$$= 0.000167716$$

Finally, we get

$$P(t) = \frac{998.546}{5.308 + (182.813)e^{-0.031551t}} \quad \text{— } (L)$$

Compare with the exponential model:

$$P(t) = P_0 e^{rt}$$

$P(0) = 5.308$, $P(100) = 76.212$ gives
$\downarrow$
year 1800

$$r = \frac{1}{100} \log \frac{76.212}{5.308} \approx 0.026643.$$

So, $P(t) = 5.308\, e^{(0.026643)t}$. —— (N)

We can use (L) and (N) to get the population numbers that the logistic eqn and natural growth eqn. models give and tabulate these values.

The error (RMSE): square root of average of the squares of the individual errors

are: 3.162 for exponential / natural growth model
0.452 for logistic eqn model

for data in years 1800 - 1900.

Logistic model is good for a while and then underestimates. Exponential model overestimates rather quickly.

Acceleration - Velocity models

Neglecting air resistance, Newton's Law ($F = ma$) can be expressed as the differential equation:

$$m \frac{dv}{dt} = F_G$$

Force due to gravity

where $F_G = -mg$.

we will follow the convention: $\uparrow$ +ve $\bullet$ $\downarrow$ -ve for direction of forces.

Notice that we have ignored air-resistance. Now we incorporate that into the model:

$$m \frac{dv}{dt} = F_G + F_R$$

Newton showed that under some physical assumptions, F_R is prop. to v^2.

In practice: $F_R \sim kv^p \quad 1 \leq p \leq 2$.

We consider 2 cases: when $p = 1$ and $p = 2$.

I. Resistance proportional to velocity:

Consider an object falling downwards.
We assume that the total force acting on the body is given by $F = F_R + F_G = -kv - mg$

$k(-v)$ — since velocity is downwards

$-(mg)$ — since gravity acts downwards.

Newton's Law then gives

$$m\frac{dv}{dt} = -kv - mg$$

and we rewrite this as

$$\frac{dv}{dt} = -\rho v - g, \quad \text{where } \rho = \frac{k}{m} > 0.$$

This is separable, and the solution is

$$v(t) = \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}. \quad [v(0) = v_0]$$

Observe: $\lim\limits_{t\to\infty} v(t) = -\frac{g}{\rho} := v_\tau$

$\therefore$ The speed does not increase indefinitely but approaches a finite "limiting" speed or "terminal" speed: $|v_\tau| = \frac{g}{\rho} = \frac{mg}{k}$

Velocity at time t is therefore

$$v(t) = (v_0 - v_\tau)e^{-\rho t} + v_\tau$$

and the height of the object at time t can be obtained by integrating this equation to give

$$y(t) = y_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t}).$$

ρ is called the "drag coefficient": the constant such that the acceleration due to air resistance is $-\rho v$.

For a person descending with a parachute, ρ is typically 1.5, so terminal speed is $\approx$ 14.5 mi/hr

$\downarrow$

? km/hr

Let us compare the solutions of a problem, one worked out ignoring air-resistance and the other incorporating it.

Example: Suppose an arrow is shot straight upward from the ground ($y_0 = 0$) with initial velocity $v_0 = 49 m/s$. What is the maximum height achieved?

Solution (Ignoring air resistance)

Here, $v(t) = v_0 - 9.8t = 49 - 9.8t$

So the position at time t is

$$y(t) = \int (49 - 9.8t)\,dt = 49t - 4.9t^2 + \underbrace{y_0}_{=0}$$

$$= 49t - 4.9t^2.$$

At its maximum height, $v(t) = 0$, so the time taken to reach this height is $\frac{49}{9.8} = 5s.$

The height at $t = 5$ is $49(5) - 4.9(25) = 122.5m.$

Solution (Taking into account air resistance) with $\rho = 0.04$

$$v(t) = \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}$$

$$= \left(49 + \frac{9.8}{0.04}\right)e^{-0.04t} - \frac{9.8}{0.04}$$

$$= 294 e^{-0.04t} - 245$$

and $y(t) = y_0 + v_t t + \frac{1}{\rho}(v_0 - v_t)(1 - e^{-\rho t})$

$$= -245t + \frac{1}{0.04}(49 + 245)(1 - e^{-0.04t})$$

$$= 7350 - 245t - 7350 e^{-0.04t}$$

$v(t) = 0$ at $t = \frac{-1}{0.04} \log\left(\frac{245}{294}\right)$

$$\approx 4.558 s.$$

$y(4.558) \approx 108.28 m.$