Week 4

Some other techniques for solving 1st order equation: Substitution.

I. Linear Substitutions:

If the given D.E is of the form

$$\frac{dy}{dx} = f(Ax + By + C)$$

where A, B, and C are known constants, then a (possible) substitution comes from letting $u = Ax + By + C$.

Examples: 1. Solve: $\frac{dy}{dx} = \frac{1}{2x - 4y + 7}$.

Soln. Set $u = 2x - 4y + 7$

Then $y = \frac{1}{4}(2x + 7 - u)$

$$\therefore \frac{dy}{dx} = \frac{1}{2} - \frac{1}{4}\frac{du}{dx}.$$

Therefore, $\frac{1}{2} - \frac{1}{4}\frac{du}{dx} = \frac{1}{u}$, which rearranges

to $\frac{du}{dx} = \frac{2u-4}{u}$, which is separable.

Solving, we get $u + 2\log|u-2| = 2x + C$.

Substituting $u = 2x - 4y + 7$:

$$2x - 4y + 7 + 2\log|2x - 4y + 5| = 2x + C$$

$\Rightarrow \log(2x - 4y + 5)^2 = 4y + C$, an implicit solution.

Note: In general:

$$\frac{dy}{dx} = f(Ax + By + C)$$

and the substitution $u = Ax + By + C$

leads to $\frac{du}{dx} = A + B\frac{dy}{dx}$

$\therefore \frac{du}{dx} = A + Bf(u)$, which is

always separable. But $\int \frac{du}{A+Bf(u)}$ can get

complicated to evaluate.

Another kind of substitution:

Solve: $2xe^{2y}\frac{dy}{dx} = 3x^4 + e^{2y}$

Here 'y' appears as just as e^{2y}, so we can try: $v = e^{2y}$ Then $\frac{dv}{dx} = 2e^{2y}\frac{dy}{dx}$

$= 2v\frac{dy}{dx}$

transforming the diff. eqn. to

$$2xv\left(\frac{1}{2v}\frac{dv}{dx}\right) = 3x^4 + v$$

i.e., $\displaystyle \frac{dv}{dx} - \frac{1}{x}v = 3x^3$, which is linear.

Now we study more systematic substitutions.

II. Homogeneous Equations

This is when the DE is of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

Then, we make the substitution $v = \dfrac{y}{x}$

i.e., $y = vx$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

and we get the separable eqn. $x\frac{dv}{dx} = f(v) - v$.

Examples

· Express the following diff. eqns. as $y' = f\left(\frac{y}{x}\right)$:

a) $x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}$ $\qquad$ $\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \frac{y^2}{x^2}}$

b) $x^2 y' - y^2 = xy$ $\qquad$ $y' = \frac{y^2}{x^2} + \frac{y}{x}$

c) $xy^2 \frac{dy}{dx} = x^3 + y^3$ $\qquad$ $\frac{dy}{dx} = \frac{1 + \left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2}$

Solution of the D.E. in c) :

Take $y = vx$. Then $y' = v + xv'$

and we get $\quad y' = v + xv' = \dfrac{1+v^3}{v^2}$

$$\therefore \quad xv' = \frac{1}{v^2},$$

which is a separable equation.

$$\int v^2\,dv = \int \frac{dx}{x} + C$$

$$\frac{v^3}{3} = \log|x| + C$$

$$\therefore \quad v = (3\log|x| + 3C)^{1/3}$$

$$\| $$

$$\frac{y}{x}$$

So $y(x) = x\,(3\log|x| + C_1)^{1/3}.$

ii) Solve: $x^2 y' = xy + x^2 e^{y/x}$

Solution: $y(x) = -x \log\left|-\log|x| - C\right|.$

III. Bernoulli Equations

In 1696, Jacob Bernoulli solved what is now known as the Bernoulli differential eqn:

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$$

Here, n is any real number.

This is a first order non-linear D.E for $n \neq 0, 1$.
The following year, Leibniz solved this equation by transforming it into a linear equation, and we use this method as well.

We use the substitution

$$v = y^{1-n}.$$

Then, $y = v^{\frac{1}{1-n}}$

so $$\frac{dy}{dx} = \frac{1}{1-n} v^{\frac{n}{1-n}} \frac{dv}{dx}.$$

Therefore the Bernoulli equation transforms to:

$$\frac{1}{1-n} v^{\frac{n}{1-n}} \frac{dv}{dx} + P(x) v^{\frac{1}{1-n}} = Q(x) v^{\frac{n}{1-n}}$$

Dividing the equation on both sides by $\frac{1}{1-n} v^{\frac{n}{1-n}}$, we get

$$\frac{dv}{dx} + (1-n) P(x) v = (1-n) Q(x),$$ which

is a linear D.E.

Example:

Solve: $$y^2 \frac{dy}{dx} + 2x y^3 = 6x$$

Solution: Dividing both sides by y^2:

$$y' + 2xy = 6xy^{-2},$$

which is a Bernoulli eqn with $n = -2$.

Making the substitution $v = y^{1-n} = y^3$,

$$y = v^{1/3} \quad ; \quad \frac{dy}{dx} = \frac{1}{3} v^{-2/3} \frac{dv}{dx}$$

So the DE becomes

$$\frac{dv}{dx} + 6xv = 18x \rightarrow \text{linear (also separable)}$$

The integrating factor is e^{3x^2}

$$\therefore \quad ve^{3x^2} = 18 \int x e^{3x^2} dx$$

$$= 3e^{3x^2} + C$$

$$\therefore \underset{\text{''} y^3}{v(x)} = 3 + Ce^{-3x^2}.$$

A combination of substitutions:

Example. Solve: $y' = \dfrac{t(y+1) + (y+1)^2}{t^2}$

Solution: This is very close to being homogeneous, if $y+1 = u$. So we make this substitution. Then,

$$u' = \frac{tu + u^2}{t^2} = \frac{u}{t} + \frac{u^2}{t^2}$$

Set $v = \dfrac{u}{t}$ so $u = vt$

$$\frac{du}{dt} = v + t\frac{dv}{dt}.$$

$\therefore$ $v + t\dfrac{dv}{dt} = v + v^2$, which is separable.

$$\therefore \int \frac{dv}{v^2} = \int \frac{dt}{t} + c$$

$$-\frac{1}{v} = \log|t| + c.$$

$$\therefore \underset{\frac{y+1}{t}}{\overset{\shortparallel}{v}} = \frac{-1}{\log|t|+c}$$

$$\therefore y(t) = -1 - \frac{t}{\log|t|+c}.$$

Exact Differential Equations

Recall: when we have an implicit solution to a diff. eqn., we verify the solution using implicit differentiation, i.e., if

$$F(x,y) = c$$

is a solution curve, then we recover the D.E. it solves by differentiating it w.r.t x on both sides:

$$\underbrace{\frac{\partial F}{\partial x}}_{M(x,y)} + \underbrace{\frac{\partial F}{\partial y}}_{N(x,y)} \cdot \frac{dy}{dx} = 0.$$

We express this in a symmetric form as:

$$M(x,y)\,dx + N(x,y)\,dy = 0, \quad \text{---} (*)$$

called its "differential form".

Now, we turn this around (we are reverse-engineering this process to obtain $F(x,y)=C$ starting from an equation of the form $(*)$);

Suppose the functions $M(x,y)$ and $N(x,y)$ in the given D.E are such that there exists a function $F(x,y)$ satisfying:

$$\frac{\partial F}{\partial x} = M(x,y), \quad \frac{\partial F}{\partial y} = N(x,y)$$

Then the implicit solution of the diff. eqn would be $F(x,y)=C$.

If such an F exists, the diff. eqn $(*)$ is said to be an <u>exact differential equation</u>.

How do we know a given D.E. is exact?

Fact from multivariable calc:

If the mixed partial derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \text{ are continuous}$$

$\frac{\partial}{\partial x}N.$ $\frac{\partial}{\partial y}M$

on an open set in the xy-plane, then they are equal: $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$.

$\therefore$ If such an F exists, we must necessarily have: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. — $(**)$

This

What about the converse? i.e., if we have a differential form $M(x,y)dx + N(x,y)dy = 0$ such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, is it true that the diff. eqn. is exact? Turns out the ans. is YES, so $(**)$ is called the <u>criterion for exactness</u> of a differential equation:

Theorem (Criterion for exactness)

Suppose that the functions $M(x,y)$ and $N(x,y)$ are continuous and have conts. first order partial derivatives in the open rectangle R : $a<x<b$, $c<y<d$. Then, the diff. equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact in R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ at each point of R.

Proof: $\Rightarrow$) seen earlier

$\Leftarrow$) We need to show that if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then there is a function $F(x,y)$ so that $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$. Observe that, for any function $g(y)$, the function

$$F(x,y) = \int M(x,y)\,dx + g(y)$$

satisfies the condition $\frac{\partial F}{\partial x} = M$, since $g(y)$ is merely a function of y, so is treated as a constant when integrating with respect to x.

Now, we must choose $g(y)$ in such a way that $\frac{\partial f}{\partial y} = N$.

i.e., $\frac{\partial}{\partial y}\left(\int M(x,y)\,dx + g(y) \right) = N$

or, $\frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + g'(y) = N$.

Therefore we need $g'(y) = N - \frac{\partial}{\partial y}\int M(x,y)\,dx$ — G

But how do we get hold of $g(y)$? What happens is that our hypothesis $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ will ensure that $N - \frac{\partial}{\partial y}\int M(x,y)\,dx$ is a function of y alone! So we can integrate eqn. G to obtain $g(y)$.

Let's verify this claim: $\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int M(x,y)\,dx\right)$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial x \partial y} \int M(x,y)\,dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M(x,y)\,dx\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Thus, the RHS of eqn. (6) is a function of y alone, since the partial derivative w.r.t x is zero.

Therefore we find a suitable $g(y)$ such that $\frac{\partial F}{\partial y} = N$ and this completes the proof.

<u>Steps for solving an exact equation:</u>

[Note: We do this <u>after</u> first verifying that the D.E. is exact, i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.]

<u>Step 1</u> : Use the fact that $\frac{\partial F}{\partial x} = M(x,y)$ to find

$F(x,y)$ by integrating with respect to x:

$$F(x,y) = \int M(x,y)\,dx + \underset{\substack{\uparrow \\ \text{"constant" of} \\ \text{integration}}}{g(y)}$$

Step 2: Use the fact that $\dfrac{\partial F}{\partial y} = N(x,y)$ to find $g'(y)$ by differentiating the equation obtained in Step 1, w.r.t y.

Step 3: Find $g(y)$ by integrating the expression obtained for $g'(y)$ in Step 2.

Step 4: Finally, substitute for $g(y)$ in the expression for $F(x,y)$ obtained in Step 1 to arrive at the final answer.

We look at some solved examples:

Verify if the following differential equations are exact or not. If they are, then solve it.

1. $y^3 dx + 3xy^2 dy = 0$

<u>Soln</u>. Here, $M(x,y) = y^3$, $N(x,y) = 3xy^2$.

$$\frac{\partial M}{\partial y} = 3y^2$$

$$\frac{\partial N}{\partial x} = 3y^2.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given D.E is exact.

We now perform the steps to get a solution:

$$F(x,y) = \int M(x,y)dx + g(y)$$

$$= \int y^3 dx + g(y)$$

$$F(x,y) = xy^3 + g(y) \text{ —— (A)}.$$

To find $g(y)$: $\frac{\partial F}{\partial y} = N(x,y)$, so $\quad 3xy^2 + g'(y) = 3xy^2$

$$\Rightarrow g'(y) = 0$$

So $g(y) = c$, a constant.

$\therefore \; F(x,y) = c$

where $F(x,y) = xy^3$ (plugging $g(y)=c$ in eqn Ⓐ).

is the general solution to the given D.E.

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2. $(\cos x + \log y)\,dx + \left(\frac{x}{y} + e^y\right) dy = 0$

<u>Soln</u>. Here, $M(x,y) = \cos x + \log y$

$$N(x,y) = \frac{x}{y} + e^y.$$

Check for exactness: $\frac{\partial M}{\partial y} = \frac{1}{y} = \frac{\partial N}{\partial x}$, so the

given eqn. is exact.

$$F(x,y) = \int (\cos x + \log y)\,dx + g(y)$$

$$= \sin x + x \log y + g(y).$$

$$\frac{\partial F}{\partial y} = N(x,y) \Rightarrow \frac{x}{y} + g'(y) = \frac{x}{y} + e^y$$

So $g'(y) = e^y$

$$\Rightarrow g(y) = e^y + c.$$

Thus $F(x,y) = C_1$ where $F(x,y) = \sin x + x \log y + e^y$ the general solution.

Exercises: Solve, if the DE is exact:

1. $2xy^2 + 3x^2 + (2x^2y + 4y^3)\frac{dy}{dx} = 0.$

   Soln: $F(x,y) = x^2y^2 + x^3 + y^4$

2. $\frac{dy}{dx} = \frac{-2x - y}{x-1}$; $y(0) = 1$

   Soln: $F(x,y) = x^2 + xy - y$

   particular soln: $y = \frac{1+x^2}{1-x}.$

3. $2x + y + xy\frac{dy}{dx} = 0$

   Soln: This is not exact.