# Laplace Transforms of derivatives: (Sec 4.2)

piecewise conts., and differentiable on [a,b] except at finitely many points.

**Theorem 1** Suppose that the function $f(t)$ is continuous and piecewise smooth for $t \geq 0$ and is of exponential order as $t \to +\infty$, so $\exists$ non-negative constants $M, c, T$ such that

$$|f(t)| \leq M e^{ct} \quad \text{for } t \geq T.$$

Then, $\mathcal{L}\{f'(t)\}$ exists for $s > c$ and

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$
$$= sF(s) - f(0).$$

Note: We show the proof for the case that $f$ is smooth. (not just piecewise smooth) and you can read the proof for the general case detailed in the textbook at the end of Sec 4.2.

Using integration by parts, we get

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-st} f'(t)\,dt$$

$$= \lim_{b\to\infty} \left( \left[ e^{-st} f(t) \right]_0^b + s\int_0^b e^{-st} f(t)\,dt \right) - (\dagger)$$

Consider the first summand: $\left[ e^{-st} f(t) \right]_0^b$.

upper limit: When $s > c$, $|e^{-st} f(t)| \le e^{-(s-c)b} \to 0$ as $b \to \infty$.

lower limit: $f(0)$.

The second summand with $\lim_{b\to\infty}$ is just $s\mathcal{L}\{f(t)\}$

$\therefore$ $(\dagger)$ is just $-f(0) + s\mathcal{L}\{f(t)\}$

so $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0)$.

How do we use this in solving differential equations?

Consider a linear differential eqn. with constant coefficients:

$$a x''(t) + b x'(t) + c x(t) = f(t).$$

with initial conditions $x(0) = x_0,$

$x'(0) = x_1.$

Using Linearity of transforms,

$$a \mathcal{L}\{x''(t)\} + b \mathcal{L}\{x'(t)\} + c \mathcal{L}\{x(t)\} = \mathcal{L}\{f(t)\}.$$

Let $x'(t) = g(t).$

$$\mathcal{L}\{x''(t)\} = \mathcal{L}\{g'(t)\}$$

$$= s \mathcal{L}\{g(t)\} - g(0)$$

$$= s \mathcal{L}\{x'(t)\} - x'(0)$$

$$= s \left[ s \mathcal{L}\{x(t)\} - x(0) \right] - x'(0)$$

$$= s^2 X(s) - s x_0 - x_1.$$

$\mathcal{L}\{x'(t)\} = sX(s) - x_0.$

$\mathcal{L}\{x(t)\} = X(s)$, and $\mathcal{L}\{f(t)\} = F(s)$.

So we get an algebraic equation in terms of $X(s)$. Then, taking inverse transforms, we get $x(t)$!

## Examples

1. Solve: $x'' - 3x' + 2x = 0$; $x(0) = 3$, $x'(0) = 4$

Solution: Apply $\mathcal{L}$ to both sides:

$$\mathcal{L}\{x''(t)\} - 3\mathcal{L}\{x'(t)\} + 2\mathcal{L}\{x(t)\} = 0$$

i.e., $$s^2 X(s) - s x(0) - x'(0)$$
$$- 3(sX(s) - x(0)) + 2X(s) = 0$$

So $$X(s)(s^2 - 3s + 2) - 3s - 4 + 9 = 0$$

$$\Rightarrow \quad X(s) = \frac{3s-5}{s^2-3s+2} = \frac{3s-5}{(s-1)(s-2)}$$

Using the method of partial fractions,

$$X(s) = \frac{3s-5}{(s-1)(s-2)} = \frac{2}{s-1} + \frac{1}{s-2}$$

Taking inverse transforms,

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\text{so} \quad x(t) = 2e^{t} + e^{2t}.$$

<u>Example 2</u> : Solve : $x' - 2x = \sin 2t$, $x(0)=0$, $x'(0)=0$.

<u>Solution</u> : Taking Laplace transforms:

$$sX(s) - 2X(s) = \frac{2}{s^2+4}.$$

$$\Rightarrow \quad X(s) = \frac{2}{(s-2)(s^2+4)}.$$

$$\frac{2}{(s-2)(s^2+4)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+4}$$

Solve to get $A = \frac{1}{4}$, $B = \frac{-1}{4}$, $C = \frac{-1}{2}$.

$$\therefore X(s) = \frac{1}{4} \cdot \frac{1}{s-2} - \frac{1}{4} \cdot \frac{s}{s^2+4} - \frac{1}{2} \cdot \frac{1}{s^2+4}.$$

Taking inverse transforms,

$$\mathcal{L}^{-1}\{X(s)\} = \frac{1}{4} e^{2t} - \frac{1}{4} \cos(2t) - \frac{1}{4} \sin(2t).$$

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Q. How do we find inverse transforms if we have a quadratic term like $\frac{1}{(s+1)^2}$?

We know what to do if we had $\frac{1}{s^2}$, and $\frac{1}{s+1}$

$\frac{1}{s^2} \xrightarrow{\mathcal{L}^{-1}} t$, $\quad \frac{1}{s+1} \xrightarrow{\mathcal{L}^{-1}} e^{-t}$

what about the composition?

This brings us to: Translation Theorem-1

If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > c$, then $\mathcal{L}\{e^{at} f(t)\}$ exists for $s > a + c$ and we have:

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

Equivalently,

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t).$$

Using this, we obtain:

$$\mathcal{L}\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{e^{at} \sin(kt)\} = \frac{k}{(s-a)^2 + k^2}$$

$$\mathcal{L}\{e^{at} \cos(kt)\} = \frac{(s-a)}{(s-a)^2 + k^2}$$

<u>Examples</u>

1) Find $\mathcal{L}\{e^{-3t} \sin(4t)\}$

<u>Ans</u>. Here, $f(t) = \sin 4t$, $a = -3$

and $F(s) = \dfrac{4}{s^2+16}$, so $F(s-a)$

$$= F(s-(-3))$$

$$= \frac{4}{(s+3)^2+16}.$$

2) Find $\mathcal{L}\{5te^{-t} + e^{2t}\cos(2t)\}$

<u>Soln.</u> By linearity, this is equal to

$$5\,\mathcal{L}\{te^{-t}\} \quad + \quad \mathcal{L}\{e^{2t}\cos(2t)\}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

using Translation Thm 1

$$5 \cdot \frac{1}{(s+1)^2} \qquad\qquad \frac{(s-2)}{(s-2)^2+4}$$

3) Find $\mathcal{L}^{-1}\left\{\dfrac{5}{(s-9)^3}\right\}$

<u>Soln</u> This is of the form $\mathcal{L}^{-1}\{F(s-a)\}$

where $F(s) = \dfrac{5}{s^3} \rightsquigarrow \dfrac{5}{2} \cdot \dfrac{2}{s^3}$, and $(s-a) = (s-9)$

$\therefore$ Translation theorem $\Rightarrow \mathcal{L}^{-1}\left(\dfrac{5}{(s-9)^3}\right) = \dfrac{5}{2} e^{9t} \cdot t^2$

4. Find $\mathcal{L}^{-1}\left\{\dfrac{5s}{2(s-1)^2+4}\right\}$

Soln. We want to express

$\dfrac{5s}{2(s-1)^2+4}$ as $F(s-a)$ for some $a$.

From the $(s-1)$ term in the denominator it gives us the idea that $a=1$. However the numerator has $5s$, not of the form $s-1$. So we add and subtract appropriately to bring the expression to the form $F(s-a)$:

$$\frac{5(s-1+1)}{2(s-1)^2+4} = \frac{5(s-1)}{2(s-1)^2+4} + \frac{5}{2(s-1)^2+4}$$

$$= \frac{5}{2}\left[\frac{(s-1)}{(s-1)^2+2}\right] + \frac{5}{2}\left[\frac{1}{(s-1)^2+2}\right]$$

$$\downarrow$$

$$\frac{5}{2\sqrt{2}}\left[\frac{\sqrt{2}}{(s-1)^2+\sqrt{2}^2}\right]$$

$$\therefore \mathcal{L}^{-1}\left(\frac{5s}{2(s-1)^2+4}\right) = \frac{5}{2} e^t \cos(\sqrt{2}t) + \frac{5}{2\sqrt{2}} e^t \sin(\sqrt{2}t).$$

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Suppose we were asked to calculate

$$\mathcal{L}^{-1}\left(\frac{5s}{2s^2-4s+6}\right)$$

how do we proceed?

Notice: $2(s-1)^2+4 = 2s^2-4s+6$, so this is actually the same problem! But how to go from an expression of the form

$as^2+bs+c$ to $\alpha\left((s+A)^2+B\right)$, which we know how to take inverse transforms of?

Ans: Completion of squares!

$$\boxed{s^2 + as + b = \left(s + \frac{a}{2}\right)^2 + b - \left(\frac{a}{2}\right)^2.}$$

Don't memorize it, work it out on your own! The idea is simple:

Recall: $(A+B)^2 = A^2 + 2AB + B^2$.

Now, in $s^2 + as + b$, we want "$as$" to be the "$2AB$" term.

If it were really so, then we would "complete the square" by writing the $B^2$ term as:

$$s^2 + as + \left(\frac{a}{2}\right)^2.$$

However we have $b$, which is not always the perfect square $\left(\frac{a}{2}\right)^2$, so we add and subtract it:

$$s^2 + as + b = \underbrace{s^2 + as + \left(\frac{a}{2}\right)^2} - \left(\frac{a}{2}\right)^2 + b$$

$$\therefore \quad s^2 + as + b = \left[\left(s + \frac{a}{2}\right)^2\right] + \left[b - \left(\frac{a}{2}\right)^2\right].$$

Examples :

1) $s^2 + 4s + 10 = (s+2)^2 + 10 - 2^2$

$= (s+2)^2 + 6$

2) $y^2 + 3y + 1 = \left(y + \frac{3}{2}\right)^2 + 1 - \left(\frac{3}{2}\right)^2$

$= \left(y + \frac{3}{2}\right)^2 - \frac{5}{4}$

3) $x^2 - 6x - 2 = (x-3)^2 - 2 - (-3)^2$

$= (x-3)^2 - 11$

Let us also review Partial Fraction Decomposition:

Rule 1: Linear factor with power 'n'.

In decomposing $\frac{P(s)}{Q(s)}$ into partial fractions, if $Q(s)$ has a term $(s+a)^n$ appearing, then the corresponding term in the decomposition is

$$\frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \cdots + \frac{A_n}{(s+a)^n}.$$

Rule 2: Quadratic factor of power 'n'.

If $Q(s)$ has a term $((s+a)^2+b)^n$ appearing, then the corresponding term in the decomposition is

$$\frac{A_1 s + B_1}{(s+a)^2+b} + \frac{A_2 s + B_2}{((s+a)^2+b)^2} + \cdots + \frac{A_n s + B_n}{((s+a)^2+b)^n}.$$

Use Laplace Transforms to solve the following differential equations:

1) $x'' - 6x' + 9x = 18$ ; $x(0) = 0$, $x'(0) = 0$.

Solution : Taking Laplace Transforms on both sides:

$$\mathcal{L}\{x''\} - 6\mathcal{L}\{x'\} + 9\mathcal{L}\{x\} = \mathcal{L}\{18\}$$

$$\Rightarrow \quad s^2 X(s) - s x(0) - x'(0)$$
$$- 6\left[ s X(s) - x(0) \right] + 9X(s) = \frac{18}{s}.$$

Simplifying,

$$X(s)\left(s^2 - 6s + 9\right) = \frac{18}{s}$$

$$X(s) = \frac{18}{s(s^2 - 6s + 9)} = \frac{18}{s(s-3)^2}.$$

Use partial fraction decomposition:

$$\frac{18}{s(s-3)^2} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{(s-3)^2}$$

So $18 = A(s^2-6s+9) + B(s-3)s + Cs$

i.e., $18 = s^2(A+B) + s(-6A-3B+C) + 9A$

$9A = 18$, $\Rightarrow A = 2$

$A + B = 0$ $\Rightarrow B = -2$

$-6A - 3B + C = 0$ $\Rightarrow C = 3B + 6A = 6$.

$$\therefore X(s) = \frac{2}{s} - \frac{2}{s-3} + \frac{6}{(s-3)^2}.$$

Therefore, $x(t) = \mathcal{L}^{-1}\{X(s)\}$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 6\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

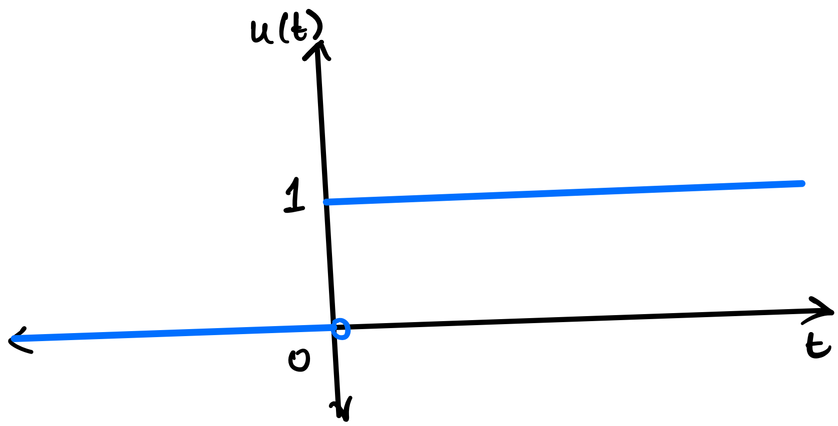
$= 2 - 2e^{3t} + 6te^{3t}$, using translation theorem 1.

# Translation theorem - 2

So far we have been computing transforms and inverse transforms of continuous functions, we now consider piecewise cont's ones.

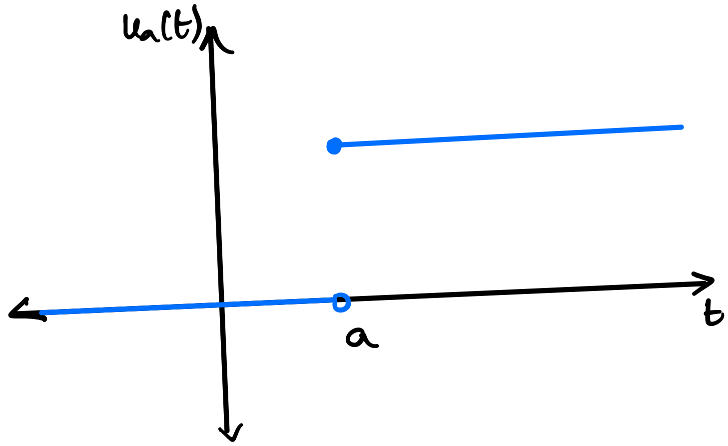
The most basic piecewise continuous function is the unit step function :

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$




More generally, the jump can happen at any point on the real line :

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$




Observe :

$$u(t-a) = \begin{cases} 0 & \text{if } t-a < 0 \\ 1 & \text{if } t-a \geq 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

$$= u_a(t)$$

Laplace Transform of $u_a(t)$:

$\mathcal{L}\{u_a(t)\}$ by definition is

$$\int_0^\infty e^{-st}\, u_a(t)\, dt.$$

$$= \int_0^a e^{-st} \cdot 0 \, dt + \int_a^\infty e^{-st} \cdot 1 \, dt$$

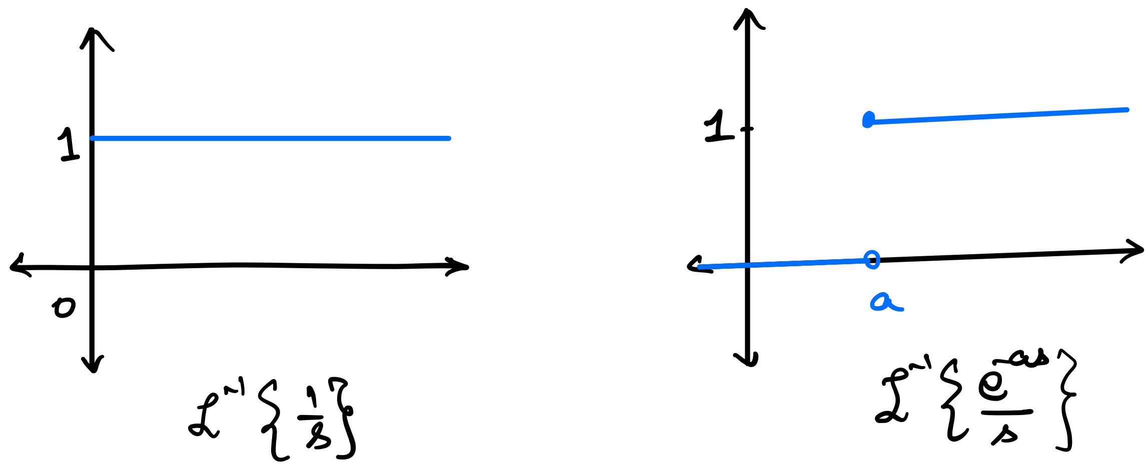
$$= 0 + \lim_{b \to \infty} \left[ \frac{e^{-st}}{-s} \right]_a^b = \frac{e^{-as}}{s}.$$

$$\therefore \quad \mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s} \quad \text{and} \quad \mathcal{L}^{-1}\left\{ \frac{e^{-as}}{s} \right\} = u(t-a)$$

or $u_a(t)$

In particular:

$$\mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} = 1 \quad \text{and} \quad \mathcal{L}^{-1}\left\{ \frac{e^{-as}}{s} \right\} = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$




So there is a translation by 'a' units when $\frac{1}{s}$ is multiplied by $e^{-as}$.

This pattern goes through for other functions too, and we state it as:

## Translation Theorem - 2

If $\mathcal{L}\{f(t)\}$ exists for $s > c$, then

$$\mathcal{L}\{u(t-a)\,f(t-a)\} = e^{-as}F(s)$$

and $$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,f(t-a).$$

for $s > c+a$.

NOTE :

$$u(t-a)\,f(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t \geq a . \end{cases}$$

## Examples

1. Find $\mathcal{L}^{-1}\left\{ \dfrac{e^{-2s}}{s^3} \right\}$

Here, $a = 2$, $F(s) = \dfrac{1}{s^3}$.

Trans. thm. 2 $\Rightarrow$ $\mathcal{L}^{-1}\left\{ \dfrac{e^{-2s}}{s^3} \right\} = u(t-2)\,f(t-2)$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s^3} \right\} = \frac{1}{2} t^2 .$$

$$\therefore\ u(t-2)\,f(t-2) = \begin{cases} 0 & \text{if } t < 2 \\ \frac{1}{2}(t-2)^2 & \text{if } t \geq 2 \end{cases} .$$

2. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s+2}\right\}$

Here, $a = 3$, $F(s) = \dfrac{1}{s+2}$.

Trans. thm 2 $\Rightarrow$ $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s+2}\right\} = u(t-3)f(t-3)$

where $f(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s+2}\right\} = e^{-2t}$

$\therefore \quad u(t-3)f(t-3) = u(t-3)\,e^{-2(t-3)}$

$$= \begin{cases} 0 & \text{if } t<3 \\ e^{6-2t} & \text{if } t\geq 3 \end{cases}.$$

3. Find $\mathcal{L}\{g(t)\}$ where $g(t) = \begin{cases} 0 & \text{if } t<2 \\ e^{2t-4} & \text{if } t\geq 2. \end{cases}$

Soln. The Translation theorem-2 can be used to calculate transforms of piecewise conts. fns if it is of the form

$$u(t-a)\,f(t-a).$$

i.e., $\begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t \geq a \end{cases}$

for some 'a' and function f.

Therefore, we first express g(t) as $u(t-a)f(t-a)$ for the appropriate 'a' and function f.

Clearly, $a = 2$. Observe: $e^{2t-4}$

$= e^{2(t-2)}$

$= f(t-2)$ for $f(t) = e^{2t}$.

$\therefore \mathcal{L}\{g(t)\} = e^{-as}F(s)$.

$$F(s) = \mathcal{L}\{e^{2t}\} = \frac{1}{s-2}.$$

Finally, we obtain $\mathcal{L}\{g(t)\} = \dfrac{e^{-3s}}{s-2}$.

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Sometimes the function will need some manipulation in order to express it as $f(t-a)$:

Example: Find $\mathcal{L}\{g(t)\}$ where

$$g(t) = \begin{cases} 0 & \text{if } t < 4 \\ t^2 & \text{if } t \geq 4 \end{cases}$$

We want $t^2$ to be the form $f(t-4)$.

$$t^2 = \underbrace{((t-4)+4)^2}_{= f(t-4)}$$

So take $f(t) = (t+4)^2$.

$$\Rightarrow F(s) = \mathcal{L}\{t^2 + 8t + 16\}$$

$$= \frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}.$$

$$\therefore \mathcal{L}\{g(t)\} = e^{-as} F(s) \quad \text{with } a = 4$$

$$= e^{-4s}\left(\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right).$$

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One last example:

Find the Laplace transform of

$$g(t) = \begin{cases} 0 & \text{if } t < 2 \\ \sin 3t & \text{if } t \geq 2 \end{cases}$$

<u>Soln</u>. We want to express $g(t)$

as $u(t-a)\, f(t-a)$.

Easy to see $a = 2$. So we want

$$\sin 3t = f(t-2)$$

$$\sin 3t = \sin(3(t-2+2))$$

$$= \sin(3(t-2) + 6)$$

$$\therefore \quad f(t) = \sin(3t+6)$$

$$= \sin 3t \cos 6 + \cos 3t \sin 6 .$$

constants!

So $F(s) = \mathcal{L}\{f(t)\}$

$$= \cos 6 \cdot \frac{3}{s^2+9} + \sin 6 \cdot \frac{s}{s^2+9} .$$

Finally, $\mathcal{L}\{g(t)\} = e^{-2s} F(s)$

$$= e^{-2s} \left( \cos 6 \cdot \frac{3}{s^2+9} + \sin 6 \cdot \frac{s}{s^2+9} \right) .$$

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What if the step function is given as

$$g(t) = \begin{cases} h(t) & 0 \le t < a \\ 0 & t \ge a \end{cases} \quad ?$$

In this case we add and subtract $h(t)$ :

$$g(t) = \begin{cases} h(t) & 0 \le t < a \\ h(t) - h(t) & t \ge a \end{cases}$$

$$= h(t) - \begin{cases} 0 & 0 \le t < a \\ h(t) & t \ge a \end{cases}$$

and proceed.

<u>Example</u>: Find $\mathcal{L}\{g(t)\}$ where

$$g(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

$$g(t) = t^2 - \underbrace{\begin{cases} 0 & 0 \le t < 2 \\ t^2 & t > 2 \end{cases}}_{g_1(t)}$$

$$\mathcal{L}\{g_1(t)\} = u(t-2)f(t-2)$$

where $f(t) = (t+2)^2$

So $F(s) = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$.

$$\therefore \mathcal{L}\{g_1(t)\} = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$$

and $\mathcal{L}\{t^2\} = \frac{2}{s^3}$.

So $\mathcal{L}\{g(t)\} = \frac{2}{s^3} - e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$.