Lecture 1

What are differential equations?

It is an equation where the unknown is a function, and the function and its derivatives appear in the equation.

Some examples:

- $$\frac{dx}{dt} = x^2 + t^2$$

- $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 7y = 3.$$

Q. Why use differential equations?

A. Algebraic equations do not take into account a "change" aspect of an unknown quantity. However, derivatives can, and so it is natural to use derivatives to "model" a process.

You have seen differential equations before!

Examples

a) Newton's Law (of motion): $F = ma$

can be expressed as:

$$m \frac{d^2x}{dt^2} = f\left(t, x(t), \frac{dx}{dt}(t)\right)$$

Here the unknown is the function $x(t)$: the position of the particle in space at the time t.

The force may depend on time, on the position of the particle and the velocity.

b) Toricelli's Law: The time rate of change of the volume V of water in a draining tank is proportional to the square root of the depth y of water in the tank:

$$\frac{dV}{dt} = k\sqrt{y}$$

c) Population model: The rate of change of a population with constant birth and death rates is proportional to the size of the population:

$$\frac{dP}{dt} = kP.$$

let us discuss this further.

$$P(t) = Ce^{kt} \quad \text{for any constant } C \in \mathbb{R}$$

is a solution of the above differential

equation, and we can verify this:

$$\text{LHS: } \frac{dP}{dt} = Cke^{kt} = k(Ce^{kt}) = kP(t).$$
$$\overset{\shortparallel}{\text{RHS}}.$$

So how many solutions does the diff. eqn. $\frac{dP}{dt} = kP$ have? Infinitely many! (one for each $C \in \mathbb{R}$).

What more can we do with diff. eqns?

<u>Example</u>: Suppose $P(t) = Ce^{kt}$ is the population of a colony of bacteria at time t, that the pop at time $t = 0$ hours was 1000 and that the population doubled after 1 hour. Can you predict the no. of bacteria after 1.5 hours?

Solution: We are given:

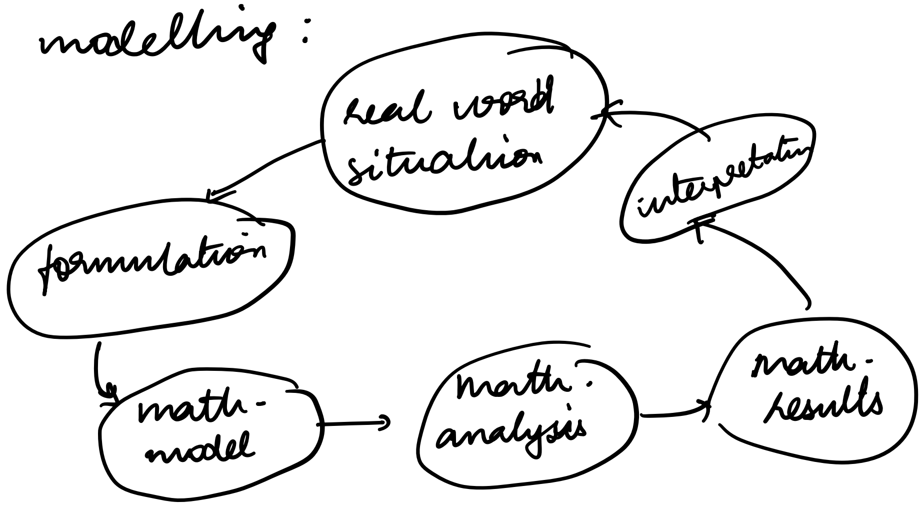
$$P(0) = 1000 = Ce^{k\cdot 0} = C$$
$$P(1) = 2000 = Ce^{k}$$

$\therefore C = 1000$, and using this we get $e^k = 2$, so $k = \log 2$ (or $\ln 2$)

Now we can calculate $P(1.5)$ easily:

$$P(1.5) = Ce^{k(1.5)}$$
$$= 1000\, e^{\frac{3}{2}\log 2}$$
$$= 1000\, 2^{3/2}$$
$$\approx 2828.$$

This describes a baby instance of "mathematical modelling:



- Remarks about math modeling:

It must be sufficiently detailed to represent the real-world situation with decent accuracy. On the other hand it must be sufficiently simple so that one can actually carry out the mathematical analysis.

In this course we begin to only scratch the surface, and our differential equations will be rather simple. But, it will still provide us with enough material to discuss the general theory, the different approaches we could take etc.

<u>Book:</u> "Elementary Differential Equations" by C. Henry Edwards and David E. Penney.

Course Outline:

Chapter 1 : First-Order differential equations

Sec 1.1 – 1.7

- Integrals as general and particular solutions
- Separable equations and applications
- Linear first order equations
- substitution methods and exact equations
- Population models
- Slope fields and solution curves.

Chapter 2 Linear equations of Higher Order

Sec 2.1 – 2.5 , selected topics from 2.6 – 2.8.

- General solutions
- Homogeneous eqns with constant coefficients, mechanical vibrations

- Inhomogeneous eqns and method of undetermined coefficients

Chapter 4 Laplace Transform Methods

Sec 4.1 - 4.3

- Transforms & Inverse transforms
- Using Laplace transforms for IVPs
- Translation and partial fractions

Chapter 5 Linear Systems of differential equations

Sec 5.1

Time permitting:

Either Numerical Methods or Power Series Methods

$\downarrow$ Chapter 6 $\qquad\qquad$ $\downarrow$ Chapter 3

Assessments : Homework : 40%

$4 \times 10\%$

Due dates : HW1 Tue 5 Mar

HW2 Mon 25 Mar

HW3 Tue 16 Apr

HW4 Mon 6 May

In-class tests : 30%

$2 \times 15\%$

Test dates : Test 1 Tue 26 Mar

Test 2 Tue 7 May

We start with precise definitions:

Defn: The order of a differential equation is the order of the highest derivative that appears in it.

Examples

(i) $y' = -2t + 4t^2$ is a first order ODE

(ii) $\frac{d^2x}{dt^2} + 3xt^2 = 0$ is a 2nd order ODE

(iii) $(y')^2 + y' + y = 0$ is a first order ODE

Defn: A first order ODE is an equation of the form

$$y' = f(t, y(t)),$$

where f is a given function of t and y and $y' = \frac{dy}{dt}$.

The most general form of an nth order ODE is

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

where x is an independent variable,

y is an unknown function depending on x

F is a specific real valued function of $n+2$ variables.

<u>Definition</u> A differential eqn.

$$G(x, y, y', y'', \dots, y^{(n)}) = 0$$

is said to be linear if G is a linear function in the variable y and its derivatives $y', y'', \dots, y^{(n)}$.

<u>Examples</u>

(i) $y' + 2y + 3 = 0$ is a linear 1st order DE.

(ii) $y'' + xy + x^2 = 0$ is a linear 2nd order D.E.

(iii) $y' = -\frac{2}{y} + 3t$ is a non-linear 1st order D.E.

<u>Definition</u> A continuous function $u = u(x)$ is a solution of the differential equation

$$F(x, y, y', \dots, y^{(n)}) = 0$$

on the interval $I \subset \mathbb{R}$ provided that the derivatives $u', u'', \dots, u^{(n)}$ exist on I and

$$F(x, u, u', \dots, u^{(n)}) = 0$$

for all $x \in I$.

<u>Examples</u>

(i) The function $y(t) = e^{2t} - \frac{3}{2}$ is a solution of the equation

$$y' = 2y + 3.$$

(ii) Consider the differential eqn.

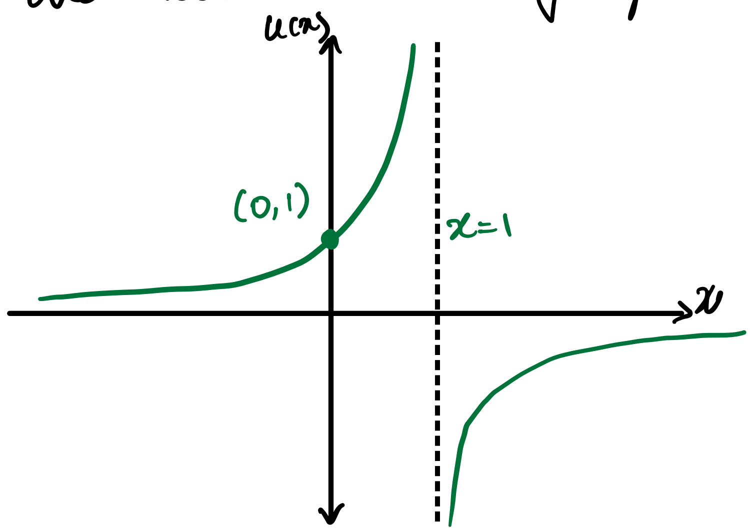
$$y' = y^2.$$

and the function $u(x) = \frac{1}{1-x}$.

Clearly, $u'(x) = \frac{-1}{(1-x)^2} \cdot (-1) = \frac{1}{(1-x)^2} = u(x)^2$.

However, $u(x)$ is not continuous at $x=1$.

If we look at the graph:



we see two "branches" of the function $u(x) = \frac{1}{1-x}$, one in the interval

$x \in (-\infty, 1)$

and the other in the interval $x \in (1, \infty)$.

Thus, we actually have two solutions given by the function $u(x) = \frac{1}{1-x}$, each valid on different domains.

For a while we will focus on solving first order differential equations, i.e., eqns of the form

$$\frac{dy}{dx} = f(x, y).$$

A typical application will have some information about the initial condition that $y(x)$ has to satisfy, and given as

"$y(x_0) = y_0$,"

where x_0 need not be 0.

Then, the differential eqn. along with an initial condition is called as an "initial value problem" which we will often abbreviate as IVP.

Problems

I. Verify by substitution that each given function is a solution of the given differential eqn.

1. $y' = 3x^2$; $y = x^3 + 7$

2. $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$

3. $y' + 2xy^2 = 0$; $y = \frac{1}{1+x^2}$

II. In the foll. problems, verify that $y(x)$ satisfies the given D.E. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition.

i) $y' = y + 1$; $y(x) = Ce^{x} - 1$; $y(0) = 5$

ii) $y' + 3x^2 y = 0$; $y(x) = Ce^{-x^3}$, $y(0) = 7$

iii) $xy' + 3y = 2x^5$; $y(x) = \frac{x^5}{4} + Cx^{-3}$, $y(2) = 1$

Now let's start learning how to solve (1st order) differential equations!

If the equation is of the form

$$\frac{dy}{dx} = y' = f(x),$$

i.e., the RHS involves no y-term, then we just integrate both sides.

<u>Example:</u>

Solve: $y' = 3x^2$

<u>Solution</u>: $\frac{dy}{dx} = 3x^2.$

Integrating both sides,

$$y(x) = x^3 + c, \quad c \in \mathbb{R}$$

This form of solution is called a "general solution".

If the problem was presented as an IVP,

eg. solve $y' = 3x^2$, $y(2) = 3$, then we get a "particular" solution, i.e., a specific value for the constant 'C':

$$y(2) = 2^3 + C = 3$$

$$\text{so } C = 3 - 8 = -5.$$

$\therefore$ $y(x) = x^3 - 5$ is a solution of the given IVP.

Using direct integration to solve D.E.s:

Newton's 2nd law of motion: If a force $F(t)$ acts on the particle and is directed along its line of motion, then

$$F(t) = m \times \underbrace{a(t)}_{x''(t)}.$$

If the force $F(t)$ is known, then the position function $x(t)$ can be obtained by integrating twice:

$$\frac{d^2x}{dt^2} = \frac{F(t)}{m}.$$

If we have constant acceleration,

then writing the eqn as $\frac{dv}{dt} = a$

v: velocity, $\frac{dx}{dt}$

we integrate to get $v(t) = at + C_1$

If we are given the initial velocity

$v(0) = v_0$, then $C_1 = v_0$ so

$$v(t) = at + v_0.$$

Integrate again to get

$$x(t) = \int v(t)\,dt = a\frac{t^2}{2} + v_0 t + C_2.$$

If we are given the initial position

$x(0) = x_0$, then $C_2 = x_0$ so

$$x(t) = \frac{at^2}{2} + v_0 t + x_0.$$

Problem: A lunar lander is falling freely towards the moon's surface at a speed of 450 m/s. Its retrorockets, when fired, provide a constant deceleration of $2.5\, m/s^2$. At what height above the lunar surface should the retrorockets be activated to ensure a "soft touchdown", i.e., $v = 0$ on impact?

Solution We use

$$v(t) = at + v_0 \qquad \text{— (i)}$$

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0 . \qquad \text{— (ii)}$$

(i) is $v(t) = 2.5t - 450$

(because velocity of rocket is ↓, acceleration is ↑, so opposite signs)

This will be zero at $t = \frac{450}{2.5} = 180\,s$

i.e., 3 min after retrorockets are fired

The position of the lander is given by (ii):

$$0 = \frac{1}{2} 2.5(180)^2 - 450(180) + x_0$$

after 3 min, lander is on the moon

height of lander from the moon at $t=0$, when rockets are fired.

$$\therefore x_0 = 450(180) - 1.25(180)^2$$

$$= 40,500.$$

$\therefore$ The rockets must be fired when the lander is 40.5 km above the moon's surface, and it takes 3 min for the soft landing to happen.

Separable Equations

The 1st order D.E.

$$\frac{dy}{dx} = H(x,y)$$

is called <u>separable</u> provided that $H(x,y)$ can be written as the product of a function of x and a function of y.

i.e., $$\frac{dy}{dx} = g(x)\,h(y)$$

Let $f(y) = \frac{1}{h(y)}$.

Then $$f(y)\frac{dy}{dx} = g(x) \quad \text{— (1)}$$

<u>Claim</u>: This is equivalent to

$$\int f(y)\,dy = \int g(x)\,dx + C. \quad \text{— (2)}$$

Proof. Let $F(y) = \int f(y)\,dy$,

$G(x) = \int g(x)\,dx$.

Then $\dfrac{d}{dx}(F(y(x))) = F'(y(x))\cdot y'(x)$

$= f(y).y'$

OR $f(y(x)).\dfrac{dy}{dx}$

From (*), this is equal to $g(x)$

$= \dfrac{d}{dx}G(x)$.

We have shown:

$$\frac{d}{dx}F(y(x)) = \frac{d}{dx}G(x)$$

$$\Rightarrow F(y(x)) = G(x) + C,$$

which is eqn (2).

<u>Note</u>: In doing problems, we informally write $f(y)\frac{dy}{dx} = g(x)$ as

$$\text{"} f(y)\,dy = g(x)\,dx \text{"},$$

as if multiplying both sides by dx, but while this gives the same equation as (2), you should remember that this is only like a shortcut, and one cannot cancel "dx" or multiply/divide an equation by it.

<u>Examples</u>

1. Solve: $y' = 6y^2x$, $y(1) = \frac{1}{25}$.

<u>Soln</u>: This is equivalent to

$$\int \frac{dy}{y^2} = \int 6x\,dx + C$$

$$\frac{-1}{y} = 3x^2 + C_1$$

$\therefore$ $y(x) = \dfrac{1}{C - 3x^2}$ is a general solution for the differential equation.

Using the initial condition $y(1) = \frac{1}{25}$, we get $\dfrac{1}{25} = \dfrac{1}{C-3}$, so $C = 28$.

$\therefore$ $y(x) = \dfrac{1}{28 - 3x^2}$ is a particular solution of the given IVP.

Remark: Where is this solution valid?