ODE-Assignment 4 Solutions

1. Determine the period and frequency of the simple harmonic motion of a body of mass $0.75kg$ at the end of a spring with spring constant $48N/m$. (Note: Simple harmonic motion is the same as free, undamped motion.)
   Solution: From the data given, $m = 0.75kg$ and $k = 48N/m$.
   The circular frequency $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{48}{0.75}} = \sqrt{64} = 8rad/s$. The period of oscillation is given by $\frac{2\pi}{\omega_0} = \frac{\pi}{4}$ seconds and the frequency is $\frac{1}{T} = \frac{4}{\pi}$Hz.

2. A body of mass $250g$ is attached to the end of a spring that is stretched $25cm$ by a force of $9N$. At time $t = 0$, the body is pulled $1m$ to the right, stretching the spring and set in motion with an initial velocity of $5m/s$ to the left. Find the position function, amplitude and period of motion of the body.
   Solution: The DE describing the motion is

$$mx'' + kx = 0.$$

The spring constant can be calcuated by observing that the spring exerts an equal and opposite restorative force $F_S = -9N$ when displaced by $25cm = 0.25m$. Hooke's Law tells us $F_s = -kx$ i.e., $-9 = -k(0.25)$ so we have $k = 36N/m$. The circular frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{36}{0.25}} = 12rad/s.$$

From this we calculate the period:

$$T = \frac{2\pi}{\omega} = \frac{\pi}{6}s$$

and the frequency is $\frac{1}{T} = \frac{6}{\pi}$Hz. The differential equation describing the position of the mass with time is

$$0.25x'' + 36x = 0,$$

where we write $250g = 0.25kg$. Dividing the equation by 0.25, we have

$$x'' + 144x = 0.$$

We solve this now. The characteristic function is $r^2 + 144 = 0$ having roots $\pm 12i$, so the solution is given by

$$x(t) = A\cos 12t + B\sin 12t.$$

The initial conditions given in the problem are: $x(0) = 1m, x'(0) = -5m/s$ (since the motion is in the opposite direction as the direction of displacement of the body) Now, $x'(t) = -12A\sin 12t + 12B\cos 12t$.
$x(0) = 1$ implies $A = 1$ and $x'(0) = -5$ gives $B = \frac{-5}{12}$. Thus, the position function is given by

$$x(t) = \cos 12t - \frac{5}{12}\sin 12t.$$

The amplitude is

$$\sqrt{1^2 + (5/12)^2} = \frac{13}{12}.$$

3. Consider a mass-spring system where mass $m = 1$, damping constant $c = 4$, spring constant $k = 4$ and external force $F_E = 10\cos(3t)$, all in our standard units. Determine the position of the mass at any time.
Solution: Let the position of the mass be denoted by $x = x(t)$ as usual, then the usual DE $mx'' + cx' + kx = F_E$ takes the form
$$x'' + 4x' + 4x = 10\cos(3t). \tag{1}$$
This is a nonhomogeneous linear second order DE with constant coefficients, so we need to find the general solution in two steps. First, we find the complimentary function y_p by solving the associated homogeneous equation $x'' + 4x + 4x = 0$. This should be very easy by now, and the answer should be $x_c = Ae^{-2t} + Bxe^{-2t}$, where A and B are constants (make sure you know how to get this!).

Second, we need to find a particular solution to our original DE. Our trial solution should be a linear combination of all terms resulting from all possible differentiations of the right hand side $10\cos(3t)$, i.e., a linear combinations of $\cos(3t)$ and $\sin(3t)$ (since differentiatng each one of these two functions always results in a multiple of the other). Now suppose $x_p = C\cos(3t) + D\sin(3t)$. Then after straightforward computations of x_p' and x_p'', we see that Equation (1) amounts to
$$(-5D - 12C)\cos(3t) + (-5C + 12D)\sin(3t) = 10\cos(3t).$$
Equatingcoefficients, this forces
$$-5D - 12C = 0, -5C + 12D = 10,$$
from which we find $C = -50/169, D = 120/169$.

Finally, it follows that general solution for x, i.e., the position of the mass (in meters), is
$$x(t) = Ae^{-2t} + Bxe^{-2t} - \frac{50}{169}\cos(3t) + \frac{120}{169}\sin(3t).$$

4. A block of mass of 0.1 kg stretches a spring 0.05 m. Assume there is no damping and the gravitational constant is $g = 9.8$ m/sec^2.

   (a) Suppose the mass is set in motion from its equilibrium position with a downward velocity of 0.1 m/sec, determine the position of the mass at any time.

   (b) What's the amplitude of the motion of the mass?

   (c) When does the mass first return to its equilibrium position?

   Solution: Set the downward direction as the positive direction. Note that Hooke's constant is $k = 0.1 * 9.8/0.05 = 19.6N/m$.

   (a) We are interested in the displacement $y = y(t)$ of the mass from its static equilibrium position, and our DE and initial conditions are
   $$0.1y'' + 19.6y = 0, y(0) = 0, y'(0) = 0.1.$$
   The DE is homogeneous and has charatecteristic equation $0.1r^2 + 19.6 = 0$, which has solutions $r = \pm 16i$, therefore its general solution is $y = A\cos(14t) + B\sin(14t)$. Using the initial conditions, we could easily find $A = 0, B = 1/140$ (do it), therefore the position of the mass (in meters) is
   $$y(t) = \frac{1}{140}\sin(14t).$$

(b) From the formula in (a), the amplitude is simply $1/140$ meters.

(c) The moment the mass first returns to its equilibrium position is the smallest positive t such that $y(t) = 0$, so that t satisfies $14t = \pi$, i.e., the moment is $\pi/14$ seconds after the motion starts.

5. Find the Laplace transform of the following functions:

(a) $f(t) = (t-2)(t^2-2)$
Solution: $F(s) = \dfrac{6}{s^4} - \dfrac{2}{s^2} - \dfrac{4}{s^3} + \dfrac{4}{s}$.

(b) $f(t) = 3\cos^2 t$
Solution: $F(s) = \dfrac{3}{2s} + \dfrac{3}{2}\dfrac{s}{s^2+4}$.

(c) $f(t) = e^{(3t+\frac{1}{2})}e^{4t}$
Solution: $f(t) = e^{7t}e^{1/2}$, so $F(s) = \dfrac{\sqrt{e}}{s-7}$.

(d) $f(t) = \cos(2t + \frac{\pi}{6})$
Solution: $f(t) = \cos(2t)\cos\frac{\pi}{6} - \sin(2t)\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\cos(2t) - \frac{1}{2}\sin(2t)$.
Therefore,

$$F(s) = \frac{\sqrt{3}}{2}\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4}.$$

6. Find the inverse transform of the following functions:

(a) $F(s) = \dfrac{s-2}{s^2-25}$
Solution: Writing $F(s) = \dfrac{s}{s^2-5^2} - \dfrac{2}{s^2-5^2}$, use the table to write the inverse transform:

$$f(t) = \cosh(5t) - \frac{2}{5}\sinh(5t).$$

(b) $F(s) = \dfrac{1}{s+10} - \dfrac{5}{s^4}$
Solution: Writing $F(s) = \dfrac{1}{s+10} - \dfrac{5}{3!}\dfrac{3!}{s^4}$, the inverse tranform is

$$f(t) = e^{-10t} - \frac{5}{6}t^3.$$

(c) $F(s) = \dfrac{s+2}{s^3+2s}$
Solution: Write

$$F(s) = \frac{s+2}{s(s^2+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2}$$

using partial fractions. The constants are calculated to be: $A = 1, B = -1, C = 1$. Thus,

$$F(s) = \frac{1}{s} - \frac{s}{s^2+2} + \frac{1}{s^2+2}.$$

Thus, the inverse transform is

$$f(t) = 1 - \cos(\sqrt{2}t) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}t).$$

Note: In the step above we used:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} = \frac{1}{\sqrt{2}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\} = \frac{1}{\sqrt{2}}\sin(\sqrt{2}t).$$