We now study the situation where we have repeated eigenvalues, but "not enough" eigenvectors.

Recall, in the case of solving homogeneous equations, if the char. eqn. had repeated roots $r_1 = r_2 = r$, the general solution looked like $x(t) = (c_1 + c_2 t) e^{rt}$, in particular we observed that te^{rt} solved the ODE.

If we were to mimick this procedure, we could try $\bar{x}(t) = \underbrace{\bar{v}_1}_{\text{eigenvector for } \lambda} te^{\lambda t} + \bar{v}_2 e^{\lambda t}$ suitable vector $\bar{v}_2$. Let's see what happens if such a vector solved $\bar{x}' = A\bar{x}$:

$$\bar{x}'(t) = \bar{v}_1 e^{\lambda t} + \bar{v}_1 \lambda t e^{\lambda t} + \bar{v}_2 \lambda e^{\lambda t}$$

$$, \quad A\bar{x}(t) = A\bar{v}_1 te^{\lambda t} + A\bar{v}_2 e^{\lambda t} = \lambda\bar{v}_1 te^{\lambda t} + A\bar{v}_2 e^{\lambda t}$$

since $\bar{v}_1$ is an eigenvector for λ.

equating them we get

$$A\bar{v}_2 e^{\lambda t} = e^{\lambda t}(\bar{v}_1 + \bar{v}_2 \lambda)$$

i.e., $A\bar{v}_2 - \lambda\bar{v}_2 = \bar{v}_1$

or $(A-\lambda I)\bar{v}_2 = \bar{v}_1$ —— ①

One way this is guaranteed is by observing that $(A-\lambda I)^2\bar{v}_2 = (A-\lambda I)\bar{v}_1 = \bar{0}$.

i.e., if $\bar{v}_2$ satisfies ①, then $(A-\lambda I)^2\bar{v}_2 = \bar{0}$.

$\bar{v}_2$ in ① is called a "generalized eigen-vector".

Examples

1) Solve: $\bar{x}' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix}\bar{x}$

Step 1: Find eigenvalues of A:

$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = 0$

$\Rightarrow \underbrace{(1-\lambda)(9-\lambda)+16}_{(\lambda-5)^2} = 0$

$\therefore$ A has repeated eigenvalues :

$$\lambda = 5, 5.$$

Step2 : Finding eigenvectors :

$(A - 5I)\bar{v} = \bar{0}$ leads to one solution (up to scaling)

$$\bar{v}_1 = c \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We find $\bar{v}_2$ by solving $(A - 5I)\bar{v}_2 = \bar{v}_1$:

$$\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

i.e., $4a + 4b = -1$

Pick any solution: $a = 0, \; b = \frac{-1}{4}.$

$\therefore \quad \bar{v}_2 = \begin{bmatrix} 0 \\ \frac{-1}{4} \end{bmatrix} \longrightarrow$ Imp: DO NOT SCALE THIS VECTOR!

Another solution is $\begin{bmatrix} \frac{-1}{8} \\ \frac{-1}{8} \end{bmatrix}.$

We could have solved:

$$\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix},$$

(since $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$ is an eigenvector) to get

$$\bar{v}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

[Observe how $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} -1/8 \\ -1/8 \end{pmatrix}$ are both legitimate vectors, but they are not multiples of each other. There is no unique choice for $\bar{v}_2$!]

Finally we write a general solution:

$$\bar{x}(t) = c_1 \bar{v}_1 e^{\lambda t} + c_2 \left(\bar{v}_1 t e^{\lambda t} + \bar{v}_2 e^{\lambda t} \right)$$

$$= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} t \\ -t \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{5t}$$

or $\bar{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

where $x_1(t) = (c_1 + c_2 t)\, e^{5t}$

$x_2(t) = (-c_1 - c_2 t - c_2)\, e^{5t}$

A 3x3 example:

Solve: $\bar{x}' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix} \bar{x}$

Solution: det $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$.

The eigenvalues are : 1, 2, 2.

(1) eigenvector for $\lambda_1 = 1$: $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(2) eigenvector for $\lambda_2 = 2$: $\bar{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(only one eigenvector up to scaling).

So we take $\bar{v}_3$ to be a generalized eigenvector:

$$(A - 2I)\bar{v}_3 = \bar{v}_2$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$-a + b = 0$$

$$-b = 1$$

$\therefore$ $b = -1$, $a = -1$, c is free, say $c = 0$.

We may choose $\bar{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

Finally, a general solution is

$$\bar{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} +$$

$$c_3 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t e^{2t} + \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} e^{2t} \right)$$

$$\bar{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} -1 \\ -1 \\ t \end{pmatrix} e^{2t}.$$

Complex eigenvalues :

Recall : For 2nd order homogeneous lin. DEs, if $a \pm ib$ were roots of the char. eqn., the general solution was given by

$$x(t) = e^{at} (c_1 \cos bt + c_2 \sin bt),$$

which was obtained by taking real and imaginary parts of the complex valued solutions.

In the case of homogeneous linear systems of 1st order, an analogous solution is obtained: $\bar{x}(t) = e^{at}(\bar{c}_1 \cos bt + \bar{c}_2 \sin bt)$

where $a \pm ib$ are the eigenvalues of the matrix A and $\bar{c}_1, \bar{c}_2$ are obtained using the eigenvectors (but they will not literally be the eigenvectors, since complex eigenvalues have complex valued eigenvectors).

<u>Example</u>:

1. Solve: $$x_1' = 4x_1 - 3x_2$$ $$x_2' = 3x_1 + 4x_2$$

<u>Soln</u>. Let $A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$.

The eigenvalues of this matrix are $4 \pm 3i$.

Let us find eigenvectors:

(i) For $\lambda_1 = 4+3i$:

$$(A - \lambda_1 I)\bar{v} = 0 \rightsquigarrow \begin{bmatrix} 4-(4+3i) & -3 \\ 3 & 4-(4+3i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e., $$\begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Observe that the equations

$$-3ia - 3b = 0,$$

$$3a - 3ib = 0$$

are really not different eqns; the first is a multiple of the second $(\times(-i))$

So we have one eqn in two variables, making one of the variables as free.

$$-3ia = 3b$$

$$\Rightarrow b = -ia$$

Take $a = 1$ then $b = -i$.

So an eigenvector is $v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

ii) $\lambda = 4 - 3i$. Repeat the above process to get $v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$, which is a complex conjugate of v_1! This is not a coincidence.

The eigenvectors for $\lambda = a+ib$ and $\lambda = a-ib$ will always be complex conjugates!

So we need to only calculate one eigenvector, the eigenvector for the conjugate eigenvalue becomes available without any calculation!

Now, we need to extract real valued solutions.

$$\bar{v}_1 e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(4+3i)t}$$

$$= \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{4t} \cdot e^{3it}$$

$$= \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{4t} (\cos 3t + i \sin 3t)$$

$$= \begin{bmatrix} e^{4t}\cos 3t \\ e^{4t}\sin 3t \end{bmatrix} + i \begin{bmatrix} e^{4t}\sin 3t \\ -e^{4t}\cos 3t \end{bmatrix}.$$

The real part is $e^{4t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}$

and the imaginary part is $e^{4t}\begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}$

Therefore, the general solution is

$$\bar{x}(t) = c_1 e^{4t} \begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}.$$

$$\bar{x}(t) = \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix} e^{4t}\cos 3t + \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} e^{4t}\sin 3t.$$

2. Solve: $x_1' = x_1 - 5x_2$; $x_1(0) = 5$

$x_2' = x_1 - x_2$; $x_2(0) = -1$.

Soln. The given system is $\bar{x}' = A\bar{x}$

where $A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$.

Step 1: Eigenvalues: $\lambda_1 = 2i$, $\lambda_2 = -2i$

Step 2: Find ONE eigenvector: Let's do it using $\lambda_1 = 2i$:

$$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-2i)a - 5b = 0 \Rightarrow a = \frac{5}{(1-2i)} b.$$

Pick $b = (1-2i)$, so $a = 5$.

So an eigenvector is $\bar{v}_1 = \begin{bmatrix} 5 \\ 1-2i \end{bmatrix}$

Step 3

Extract the real and imag. parts

of $\bar{v}_1 e^{\lambda_1 t}$: $\begin{bmatrix} 5 \\ 1-2i \end{bmatrix} e^{2it}$

$$= \begin{bmatrix} 5 \\ 1-2i \end{bmatrix} (\cos 2t + i \sin 2t)$$

$$= \begin{bmatrix} 5\cos 2t + i 5 \sin 2t \\ (\cos 2t + 2\sin 2t) + i(-2\cos 2t + \sin 2t) \end{bmatrix}$$

$$= \begin{bmatrix} 5\cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix} + i \begin{bmatrix} 5\sin 2t \\ -2\cos 2t + \sin 2t \end{bmatrix}$$

Step 4

So the general soln. is given by

$$\bar{x}(t) = c_1 \begin{bmatrix} 5\cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} 5\sin 2t \\ \sin 2t - 2\cos 2t \end{bmatrix}.$$

i.e., $x_1(t) = C_1 5\cos 2t + C_2 \cdot 5\sin 2t$

$x_2(t) = (C_1 - 2C_2)\cos 2t + (2C_1 + C_2)\sin 2t$

Finding a particular solution:

$\bar{x}(0) = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \Rightarrow \begin{array}{l} x_1(0) = 5 = 5C_1 \\ x_2(0) = -1 = C_1 - 2C_2 \end{array}$

Using this, we get $C_1 = 1 \Rightarrow C_2 = \frac{C_1 + 1}{2} = 1$

Finally, a particular solution is given by:

$x_1(t) = 5\cos 2t + 5\sin 2t$

$x_2(t) = -\cos 2t + 3\sin 2t$.

Application of first order linear systems:

Consider a 3-stage tank system with three brine tanks T_1, T_2, T_3 containing V_1, V_2, V_3 litres of brine respectively.

Fresh water flows into T_1, while mixed brine flows from T_1 into T_2, then from T_2 into T_3 and then out of T_3.

If the flow rate is constant, and if $x_i(t)$ denotes the amount of salt in Tank i at time $i = 1, 2, 3$ then the process gives us a first order system!

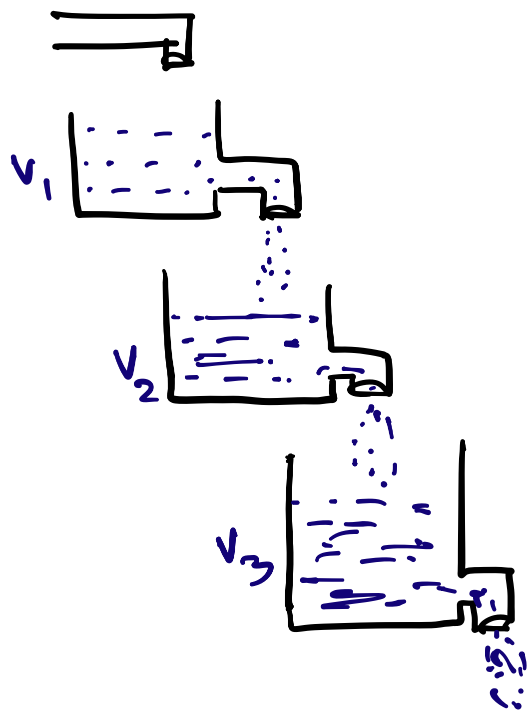
Recall the DE for 1 tank:

$$\frac{dx}{dt} = \begin{pmatrix}\text{rate of}\\\text{inflow}\end{pmatrix} \times \begin{pmatrix}\text{concentration}\\\text{of inflow}\end{pmatrix} - \begin{pmatrix}\text{rate of}\\\text{outflow}\end{pmatrix} \times \begin{pmatrix}\text{conc'n}\\\text{of out-}\\\text{flow}\end{pmatrix}$$

$$\text{Conc'n} = \frac{x(t)}{\text{Volume}}$$

So for a system of cascading tanks:

$$\frac{dx_1}{dt} = 0 - r \cdot \frac{x_1(t)}{V_1}$$



$$\frac{dx_2}{dt} = \frac{r \cdot x_1(t)}{V_1} - r\frac{x_2(t)}{V_2}$$

$$\frac{dx_3}{dt} = \frac{r x_2(t)}{V_2} - \frac{r x_3(t)}{V_3}$$

Since r is constant, V_1, V_2 and V_3 are also constant, and Tank i's outflow is Tank $(i+1)$'s inflow. Let $k_i = \frac{r}{V_i}$ $i = 1, 2, 3$

Then we get

$$x_1' = -k_1 x_1$$
$$x_2' = k_1 x_1 - k_2 x_2$$
$$x_3' = k_2 x_2 - k_3 x_3 .$$

On solving the linear system, we obtain the amount of salt in each tank at any given time t.

<u>Example</u>: Find the amount of salt in each tank at time t for a 3-stage system T_1, T_2, T_3 with $V_1 = 20L$, $V_2 = 40L$ and $V_3 = 50L$ tanks where the flow rate is $5L/s$ and fresh water flows into tank T_1. Initially, there is $9g$ of salt in T_1, $0g$ of salt in T_2 and $0g$ in T_3.

<u>Solution</u>: $V_1 = 20$, $V_2 = 40$, $V_3 = 50$, $r = 5$.

$$\Rightarrow \quad k_1 = \frac{5}{20} = 0.25, \quad k_2 = \frac{5}{40} = 0.125,$$

$$k_3 = \frac{5}{50} = 0.1.$$

Thus the linear system is

$$\bar{x}' = \underbrace{\begin{bmatrix} -0.25 & 0 & 0 \\ 0.25 & -0.125 & 0 \\ 0 & 0.125 & -0.1 \end{bmatrix}}_{A} \bar{x}$$

where $\bar{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$.

First we find the eigenvalues of A:

$$|A - \lambda I| = 0 \rightsquigarrow \begin{vmatrix} -0.25-\lambda & 0 & 0 \\ 0.25 & -0.125-\lambda & 0 \\ 0 & 0.125 & -0.1-\lambda \end{vmatrix} = 0$$

<u>Note:</u>

Since the matrix's upper-triangular, the eigenvalues are the diagonal entries!

$$\lambda_1 = -0.25, \quad \lambda_2 = -0.125, \quad \lambda_3 = -0.1$$

Finding eigenvectors:

1) $\lambda_1 = -0.25$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0.25 & 0.125 & 0 \\ 0 & 0.125 & 0.15 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \mapsto R_2 / 0.025$

$R_{23} \mapsto R_3 / 0.025$

$$\begin{bmatrix} 0 & 0 & 0 \\ 10 & 5 & 0 \\ 0 & 5 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Use the cross-product trick:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10 & 5 & 0 \\ 0 & 5 & 6 \end{vmatrix} = 30\hat{i} - 60\hat{j} + 50\hat{k}$$

$$\therefore \bar{v}_1 = \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}$$

2) $\lambda_2 = -0.125 \rightsquigarrow \bar{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$

3) $\lambda_3 = -0.1 \rightsquigarrow \bar{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Thus, the general solution is

$$\bar{x}(t) = c_1 \bar{v}_1 e^{\lambda_1 t} + c_2 \bar{v}_2 e^{\lambda_2 t} + c_3 \bar{v}_3 e^{\lambda_3 t}$$

$$= c_1 e^{-0.25t} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + c_2 e^{-0.125t} \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$$

$$+ c_3 e^{-0.1t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

i.e., $x_1(t) = 3c_1 e^{-t/4}$

$x_2(t) = -6c_1 e^{-t/4} - c_2 e^{-t/8}$

$x_3(t) = 5c_1 e^{-t/4} - 5c_2 e^{-t/8} + c_3 e^{-t/10}$

Initial conditions: $x_1(0) = 9$

$x_2(0) = 0$

$x_3(0) = 0$.

Using this in the equations above:

$c_1 = 3, \quad c_2 = -18, \quad c_3 = 75.$

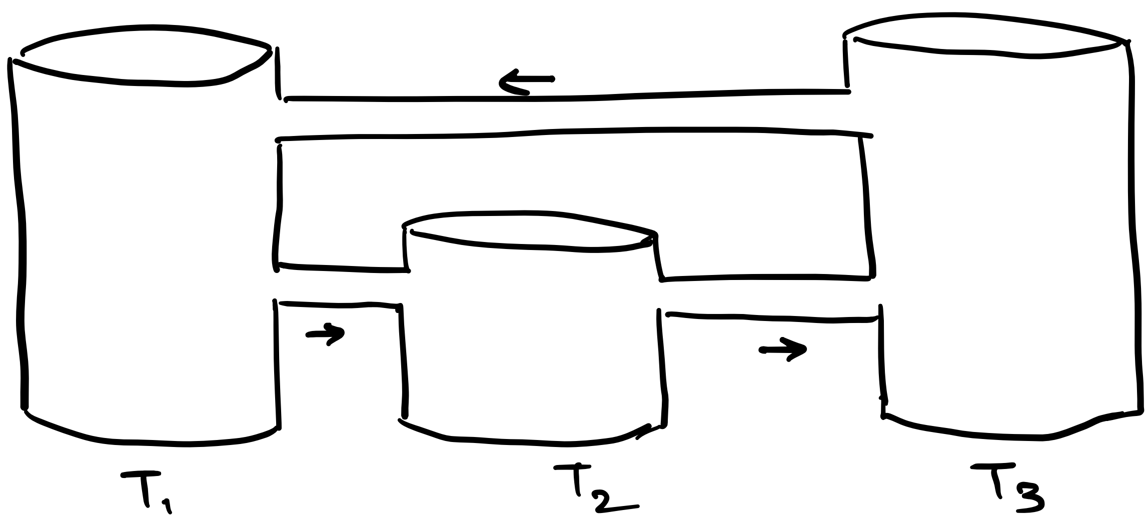
Thus, the amounts of salt in each tank is given by:

$$x_1(t) = 9e^{-t/4}$$

$$x_2(t) = -18e^{-t/4} + 18e^{-t/8}$$

$$x_3(t) = 15e^{-t/4} + 90e^{-t/8} + 75e^{-t/10}.$$

Consider another situation where we have a closed system of three brine tanks with volumes V_1, V_2 and V_3:



The only difference is that now the inflow to tank 1 is from the outflow from tank 3.

With the same notation as before, the process can be described by:

$$\frac{dx_1}{dt} = -k_1 x_1 \qquad + k_3 x_3$$

$$\frac{dx_2}{dt} = k_1 x_1 - k_2 x_2$$

$$\frac{dx_3}{dt} = \qquad k_2 x_2 - k_3 x_3$$

Example Find the amounts $x_1(t)$, $x_2(t)$ and $x_3(t)$ of salt at time t in the three brine tanks in the above setup.

Given: $V_1 = 50$ gal., $V_2 = 25$ gal, $V_3 = 50$ gal, $r = 10$ gal/min.

Solution: The given system is

$$\bar{x}' = \begin{bmatrix} -k_1 & 0 & k_3 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \end{bmatrix} \bar{x}$$

$k_i = \frac{r}{V_i}$ So $k_1 = \frac{10}{50} = 0.2$

$$k_2 = \frac{10}{25} = 0.4$$

$$k_3 = \frac{10}{50} = 0.2$$

$$\Rightarrow A = \begin{bmatrix} -0.2 & 0 & 0.2 \\ 0.2 & -0.4 & 0 \\ 0 & 0.4 & -0.2 \end{bmatrix}$$

Eigenvalues: $0, \ -0.4 \pm 0.2i$

First, we calculate the eigenvector associated to $\lambda = 0$: $\bar{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

Next, pick one of the complex eigenvalues:

$\lambda = -0.4 - 0.2i$

$$A - \lambda I = \begin{bmatrix} 1+i & 0 & 1 \\ 1 & i & 0 \\ 0 & 2 & 1+i \end{bmatrix}.$$

We can find an eigenvector using the cross-product trick here as well!

Pick Row 2 and Row 3:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & i & 0 \\ 0 & 2 & 1+i \end{vmatrix} = \hat{i}\,(i(1+i)) - \hat{j}\,(1+i) + \hat{k}\,(2)$$

$= i - i^2 = i - 1$

$$\therefore \bar{v} = \begin{bmatrix} i-1 \\ -1-i \\ 2 \end{bmatrix}$$

$$\bar{v}e^{\lambda t} = \begin{bmatrix} i-1 \\ -1-i \\ 2 \end{bmatrix} e^{-0.4t} \cdot e^{-0.2it}$$

$$= e^{-0.4t}\left\{\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + i\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}\left(\cos(-0.2t) - i\sin(-0.2t)\right)$$

$$= e^{-0.4t}\left\{\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}\cos(-0.2t) + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\sin(-0.2t)\right\}$$

$$+ i e^{-0.4t}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\cos(-0.2t) - \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}\sin(-0.2t)\right\}$$

Separating the real and imaginary vectors as $\bar{v}_2$ and $\bar{v}_3$,

$-0.4t = -2t/5$

and using $\cos(-0.2t) = \cos(0.2t), = \cos(t/5)$

$\sin(-0.2t) = -\sin(0.2t) = -\sin\left(\frac{t}{5}\right)$

we get:

$$\bar{v}_2 = e^{-2t/5}\left\{\begin{pmatrix}-1\\-1\\2\end{pmatrix}\cos\frac{t}{5} - \begin{pmatrix}1\\-1\\0\end{pmatrix}\sin\frac{t}{5}\right\}$$

$$\bar{v}_3 = e^{-2t/5}\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cos\frac{t}{5} + \begin{pmatrix}-1\\-1\\2\end{pmatrix}\sin\frac{t}{5}\right\}.$$

Thus, the general solution is given by

$$\bar{x}(t) = c_1\bar{v}_1 e^{0t} + c_2\bar{v}_2 + c_3\bar{v}_3 :$$

$$\bar{x}(t) = c_1\begin{pmatrix}2\\1\\2\end{pmatrix} + c_2 e^{-2t/5}\begin{pmatrix}-1\\-1\\2\end{pmatrix}\cos\frac{t}{5} - c_2 e^{\frac{-2t}{5}}\begin{pmatrix}1\\-1\\0\end{pmatrix}\sin\frac{t}{5}$$

$$+ c_3 e^{-2t/5}\begin{pmatrix}1\\-1\\0\end{pmatrix}\cos\frac{t}{5} + c_3 e^{\frac{-2t}{5}}\begin{pmatrix}-1\\-1\\2\end{pmatrix}\sin\frac{t}{5}.$$

$$x_1(t) = 2C_1 + (C_3 - C_2)e^{-\frac{2t}{5}}\cos\frac{t}{5} - (C_3 + C_2)e^{-\frac{2t}{5}}\sin\frac{t}{5}$$

$$x_2(t) = C_1 - (C_2 + C_3)e^{-\frac{2t}{5}}\cos\frac{t}{5} + (C_2 - C_3)e^{-2t/5}\sin\frac{t}{5}$$

$$x_3(t) = 2C_1 + (\ 2C_2\ )e^{-\frac{2t}{5}}\cos\frac{t}{5} + (\ 2C_3\ )e^{-2t/5}\sin\frac{t}{5}$$