Part 1. **Background**

0006   **NONLOCAL GAMES**

Contents

We introduce definitions associated with nonlocal games and strategies that will be used throughout.

0007

1. GAMES AND STRATEGIES

0008   **Definition 1.1** (Two-player one-round games)**.** A *two-player one-round game* $\mathfrak{G}$ is specified by a tuple $(\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ where

    (1) $\mathcal{X}$ and $\mathcal{Y}$ are finite sets (called the *question alphabets*),
    (2) $\mathcal{A}$ and $\mathcal{B}$ are finite sets (called the *answer alphabets*),
    (3) μ is a probability distribution over $\mathcal{X} \times \mathcal{Y}$ (called the *question distribution*), and
    (4) $D : \mathcal{X} \times \mathcal{Y} \times \mathcal{A} \times \mathcal{B} \to \{0,1\}$ is a function (called the *decision predicate*).

0009   **Definition 1.2** (Tensor product strategies)**.** A *tensor product strategy* S for a game $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ is a tuple $(|\psi\rangle, A, B)$ where

    • $|\psi\rangle$ is a pure quantum state in $\mathcal{H}_A \otimes \mathcal{H}_B$ for finite dimensional complex Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$,
    • A is a set $\{A^x\}$ such that for every $x \in \mathcal{X}$, $A^x = \{A^x_a\}_{a \in \mathcal{A}}$ is a POVM over $\mathcal{H}_A$, and
    • B is a set $\{B^y\}$ such that for every $y \in \mathcal{Y}$, $B^y = \{B^y_b\}_{b \in \mathcal{B}}$ is a POVM over $\mathcal{H}_B$.

000A   **Definition 1.3** (Tensor product value)**.** The *tensor product value* of a tensor product strategy $S = (|\psi\rangle, A, B)$ with respect to a game $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ is defined as

$$\mathrm{val}^*(\mathfrak{G}, S) = \sum_{x, y, a, b} \mu(x, y) \, D(x, y, a, b) \, \langle\psi|A^x_a \otimes B^y_b |\psi\rangle .$$

For $v \in [0,1]$ we say that the strategy S *passes (or wins)* $\mathfrak{G}$ *with probability* v if $\mathrm{val}^*(\mathfrak{G}, S) \geq v$. The *tensor product value* of $\mathfrak{G}$ is defined as

$$\mathrm{val}^*(\mathfrak{G}) = \sup_S \mathrm{val}^*(\mathfrak{G}, S) ,$$

where the supremum is taken over all tensor product strategies S for $\mathfrak{G}$.

Remark 1.4. Unless specified otherwise, all strategies considered in this paper are tensor product strategies, and we simply call them *strategies*. Similarly, we refer to $\mathrm{val}^*(\mathfrak{G})$ as the *value* of the game $\mathfrak{G}$.

Definition 1.5 (Projective strategies)**.** We say that a strategy $S = (|\psi\rangle, A, B)$ is   000B
projective if all the measurements $\{A_a^x\}_a$ and $\{B_b^y\}_b$ are projective.

Remark 1.6. A game $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ is *symmetric* if the question and   000C
answer alphabets are the same for both players (i.e. $\mathcal{X} = \mathcal{Y}$ and $\mathcal{A} = \mathcal{B}$), the distribution μ is symmetric (i.e. $\mu(x, y) = \mu(y, x)$), and the decision predicate D treats both players symmetrically (i.e. for all x, y, a, b, $D(x, y, a, b) = D(y, x, b, a)$). Furthermore, we call a strategy $S = (|\psi\rangle, A, B)$ *symmetric* if $|\psi\rangle$ is a state in $\mathcal{H} \otimes \mathcal{H}$, for some Hilbert space $\mathcal{H}$, that is invariant under permutation of the two factors, and the measurement operators of both players are identical. We specify symmetric games $\mathfrak{G}$ and symmetric strategies S using a more compact notation: we write $\mathfrak{G} = (\mathcal{X}, \mathcal{A}, \mu, D)$ and $S = (|\psi\rangle, M)$ where M denotes the set of measurement operators for both players.

Lemma 1.7. *Let* $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ *be a symmetric game such that* $\mathrm{val}^*(\mathfrak{G}) =$ 000D
$1 - \varepsilon$ *for some* $\varepsilon \geq 0$. *Then there exists a symmetric and projective strategy* $S = (|\psi\rangle, M)$ *such that* $\mathrm{val}^*(\mathfrak{G}, S) \geq 1 - 2\varepsilon$.

Proof. By definition there exists a strategy $S' = (|\psi'\rangle, A, B)$ such that $\mathrm{val}^*(\mathfrak{G}, S') \geq 1 - 2\varepsilon$. Using Naimark's theorem (for example as formulated in [**?**, Theorem 4.2]) we can assume without loss of generality that $|\psi'\rangle \in \mathbb{C}_{A'}^d \otimes \mathbb{C}_{B'}^d$ for some integer d and that for every x and y, A^x and B^y is a projective measurement. Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle_A |1\rangle_B |\psi'\rangle_{A'B'} + |1\rangle_A |0\rangle_B |\psi'_\tau\rangle_{A'B'}\big) \in (\mathbb{C}_A^2 \otimes \mathbb{C}_{A'}^d) \otimes (\mathbb{C}_B^2 \otimes \mathbb{C}_{B'}^d)\,,$$

where $|\psi'_\tau\rangle$ is obtained from $|\psi\rangle$ by permuting the two players' registers. Observe that $|\psi\rangle$ is invariant under permutation of AA' and BB'.

For any question $x \in \mathcal{X} = \mathcal{Y}$, define the measurement $M^x = \{M_a^x\}_{a \in \mathcal{A}}$ acting on the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^d$ as follows:

$$M_a^x = |0\rangle\langle 0| \otimes A_a^x + |1\rangle\langle 1| \otimes B_a^x\,.$$

When Alice receives question x, she measures M^x on registers AA', and when Bob receives question y, he measures M^y on registers BB'. Using that by assumption the decision predicate D for $\mathfrak{G}$ is symmetric, it is not hard to verify that $\mathrm{val}^*(\mathfrak{G}, S) = \mathrm{val}^*(\mathfrak{G}, S')$. $\square$

Definition 1.8. Let $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ be a game, and let $S = (|\psi\rangle, A, B)$ 000E
be a strategy for $\mathfrak{G}$ such that the spaces $\mathcal{H}_A \simeq \mathcal{H}_B$ canonically. Let $S \subseteq \mathcal{X} \times \mathcal{Y}$ denote the support of the question distribution μ, i.e. the set of (x, y) such that $\mu(x, y) > 0$. We say that S is a *commuting strategy for* $\mathfrak{G}$ if for all question pairs $(x, y) \in S$, we have $[A_a^x, B_b^y] = 0$ for all $a \in \mathcal{A}, b \in \mathcal{B}$, where $[A, B] = AB - BA$ denotes the commutator.

Definition 1.9 (Consistent measurements)**.** Let $\mathcal{A}$ be a finite set, let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ 000F
a state, and $\{M_a\}_{a \in \mathcal{A}}$ a projective measurement on $\mathcal{H}$. We say that $\{M_a\}_{a \in \mathcal{A}}$ *is consistent on* $|\psi\rangle$ if and only if

$$\forall a \in \mathcal{A}\,, \quad M_a \otimes \mathrm{Id}_B\, |\psi\rangle = \mathrm{Id}_A \otimes M_a\, |\psi\rangle\,.$$

000G **Definition 1.10** (Consistent strategies)**.** Let $S = (|\psi\rangle, A, B)$ be a projective strategy with state $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$, for some Hilbert space $\mathcal{H}$, which is defined on question alphabets $\mathcal{X}$ and $\mathcal{Y}$ and answer alphabets $\mathcal{A}$ and $\mathcal{B}$, respectively. We say that the strategy S is *consistent* if for all $x \in \mathcal{X}$, the measurement $\{A_a^x\}_{a \in \mathcal{A}}$ is consistent on $|\psi\rangle$ and if for all $y \in \mathcal{Y}$, the measurement $\{B_b^y\}_{b \in \mathcal{B}}$ is consistent on $|\psi\rangle$.

000H **Definition 1.11.** We say that a strategy S for a game $\mathfrak{G}$ is *PCC* if it is projective, consistent, and commuting for $\mathfrak{G}$. Additionally, we say that a PCC strategy S is *SPCC* if it is furthermore symmetric.

Remark 1.12. A strategy $S = (|\psi\rangle, A, B)$ for a symmetric game $\mathfrak{G} = (\mathcal{X}, \mathcal{X}, \mathcal{A}, \mathcal{A}, \mu, D)$ is called *synchronous* if it holds that for every $x \in \mathcal{X}$ and $a \neq b \in \mathcal{A}$, $\langle \psi | A_a^x \otimes B_b^x | \psi \rangle = 0$; in other words, the players never return different answers when simultaneously asked the same question. As shown in [**?**] the condition for a finite-dimensional strategy of being synchronous is equivalent to the condition that it is projective, consistent, and moreover $|\psi\rangle$ is a maximally entangled state. (The equivalence is extended to infinite-dimensional strategies, as well as correlations induced by limits of finite-dimensional strategies, in [**?**].)

000J **Definition 1.13** (Entanglement requirements of a game)**.** For all games $\mathfrak{G}$ and $\nu \in [0, 1]$, let $\mathcal{E}(\mathfrak{G}, \nu)$ denote the minimum integer d such that there exists a finite dimensional tensor product strategy S that achieves success probability at least ν in the game $\mathfrak{G}$ with a state $|\psi\rangle$ whose Schmidt rank is at most d. If there is no finite dimensional strategy that achieves success probability ν, then define $\mathcal{E}(\mathfrak{G}, \nu)$ to be ∞.

2. Distance measures

We introduce several distance measures that are used throughout.

000K **Definition 2.1** (Distance between states)**.**
Let $\{|\psi_n\rangle\}_{n \in \mathbb{N}}$ and $\{|\psi'_n\rangle\}_{n \in \mathbb{N}}$ be two families of states in the same space $\mathcal{H}$. For some function $\delta : \mathbb{N} \to [0, 1]$ we say that $\{|\psi_n\rangle\}$ and $\{|\psi'_n\rangle\}$ are δ-*close*, denoted as $|\psi\rangle \approx_\delta |\psi'\rangle$, if $\||\psi_n\rangle - |\psi'_n\rangle\|^2 = O(\delta(n))$. (For convenience we generally leave the dependence of the states and δ on the indexing parameter n implicit.)

000L **Definition 2.2** (Consistency between POVMs)**.** Let $\mathcal{X}$ be a finite set and μ a distribution on $\mathcal{X}$. Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a quantum state, and for all $x \in \mathcal{X}$, $\{A_a^x\}$ and $\{B_a^x\}$ POVMs. We write

$$A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$$

on state $|\psi\rangle$ and distribution μ if

$$\mathbb{E}_{x \sim \mu} \sum_{a \neq b} \langle \psi | A_a^x \otimes B_b^x | \psi \rangle \leq O(\delta) \ .$$

In this case, we say that $\{A_a^x\}$ and $\{B_a^x\}$ are δ-*consistent* on $|\psi\rangle$.

Note that a consistent measurement according to Definition 1.9 is 0-consistent with itself, under the singleton distribution, according to Definition 2.2 (and vice-versa).

Definition 2.3 (Distance between POVMs). Let $\mathcal{X}$ be a finite set and μ a 000M
distribution on $\mathcal{X}$. Let $|\psi\rangle \in \mathcal{H}$ be a quantum state, and for all $x \in \mathcal{X}$, $\{M_a^x\}$
and $\{N_a^x\}$ two POVMs on $\mathcal{H}$. We say that $\{M_a^x\}$ *and* $\{N_a^x\}$ *are δ-close on state*
$|\psi\rangle$ *and under distribution* μ if

$$\mathbb{E}_{x \sim \mu} \sum_a \left\| (M_a^x - N_a^x)|\psi\rangle \right\|^2 \leq O(\delta) \ ,$$

and we write $M_a^x \approx_\delta N_a^x$ to denote this when the state $|\psi\rangle$ and distribution μ are
clear from context. This distance is referred to as the *state-dependent* distance.

Definition 2.4 (Distance between strategies). Let $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ be a 000N
nonlocal game and let $S = (\psi, A, B)$, $S' = (\psi', A', B')$ be strategies for $\mathfrak{G}$. For
$\delta \in [0, 1]$ we say that S *is δ-close to* S' if the following conditions hold.
   (1) The states $|\psi\rangle, |\psi'\rangle$ are states in the same Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and are
       δ-close.
   (2) For all $x \in \mathcal{X}, y \in \mathcal{Y}$, we have $A_a^x \approx_\delta (A')_a^x$ and $B_b^y \approx_\delta (B')_b^y$, with the
       approximations holding under the distribution μ, and on either $|\psi\rangle$ or $|\psi'\rangle$.

We record several useful facts about the consistency measure and the state-
dependent distance without proof. Readers are referred to Sections 4.4 and 4.5
in [**?**] for additional discussion and proofs.

Fact 2.5 (Fact 4.13 and Fact 4.14 in [**?**]). *For POVMs $\{A_a^x\}$ and $\{B_a^x\}$, the* 000P
following hold.
   (1) *If $A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$ then $A_a^x \otimes I_B \approx_\delta I_A \otimes B_a^x$.* 000Q
   (2) *If $A_a^x \otimes I_B \approx_\delta I_A \otimes B_a^x$ and $\{A_a^x\}$ and $\{B_a^x\}$ are projective measurements,*
       then $A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$. 000R
   (3) *If $A_a^x \otimes I_B \approx_\delta I_A \otimes B_a^x$ and either $\{A_a^x\}$ or $\{B_a^x\}$ is a projective measurement,*
       then $A_a^x \otimes I_B \simeq_{\delta^{1/2}} I_A \otimes B_a^x$. 000S

Fact 2.6 (Fact 4.20 in [**?**]). *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be finite sets, and let D be a distribution* 000T
over question pairs (x, y). Let $\{A_{a,b}^x\}$ and $\{B_{a,b}^x\}$ be operators whose outcomes range
over the product set $\mathcal{A} \times \mathcal{B}$. Suppose a set of operators $\{C_{a,c}^y\}$, whose outcomes range
over the product set $\mathcal{A} \times \mathcal{C}$, satisfies the condition $\sum_{a,c} (C_{a,c}^y)^\dagger C_{a,c}^y \leq \mathrm{Id}$ for all y.
If $A_{a,b}^x \approx_\delta B_{a,b}^x$ on average over x sampled from the corresponding marginal of
distribution D, then $C_{a,c}^y A_{a,b}^x \approx_\delta C_{a,c}^y B_{a,b}^x$ on average over (x, y) sampled from D.

Proof. Fix questions x, y and answers $a \in \mathcal{A}, b \in \mathcal{B}$. We have then that

(2.6.1)
$$\sum_c \left\| (C_{a,c}^y A_{a,b}^x - C_{a,c}^y B_{a,b}^x)|\psi\rangle \right\|^2 = \sum_c \langle\psi| (A_{a,b}^x - B_{a,b}^x)^\dagger (C_{a,c}^y)^\dagger (C_{a,c}^y)(A_{a,b}^x - B_{a,b}^x)|\psi\rangle$$

$$(2.6.2) \qquad\qquad\qquad \leq \langle\psi| (A_{a,b}^x - B_{a,b}^x)^\dagger (A_{a,b}^x - B_{a,b}^x)|\psi\rangle$$

$$(2.6.3) \qquad\qquad\qquad = \left\| (A_{a,b}^x - B_{a,b}^x)|\psi\rangle \right\|^2$$

where the inequality follows from the fact that $\sum_c (C_{a,c}^y)^\dagger C_{a,c}^y \leq \sum_{a,c} (C_{a,c}^y)^\dagger C_{a,c}^y \leq$
Id. Thus we obtain the desired conclusion

$$\mathbb{E}_{(x,y) \sim D} \sum_{a,b,c} \left\| (C_{a,c}^y A_{a,b}^x - C_{a,c}^y B_{a,b}^x)|\psi\rangle \right\|^2 \leq \mathbb{E}_{(x,y) \sim D} \sum_{a,b} \left\| (A_{a,b}^x - B_{a,b}^x)|\psi\rangle \right\|^2 \leq \delta.$$

$\square$

000U **Fact 2.7.** *Let $\mathcal{X}, \mathcal{A}$ denote finite sets, and let $\mathcal{G}$ denote a set of functions $g : \mathcal{X} \to \mathcal{A}$. Let $\{A_a^x\}$, $\{B_a^x\}$ be POVMs indexed by $\mathcal{X}$ and outcomes in $\mathcal{A}$. Let $\{S_g^x\}$ denote a set of operators such that for all $x \in \mathcal{X}$, $\sum_g (S_g^x)^\dagger S_g^x \leq \mathrm{Id}$. If $A_a^x \approx_\delta B_a^x$ on average over x, then $S_g^x A_{g(x)}^x \approx_\delta S_g^x B_{g(x)}^x$.*

Proof. We expand:

$$\mathbb{E}_x \sum_g \left\| S_g^x (A_{g(x)}^x - B_{g(x)}^x) |\psi\rangle \right\|^2 = \mathbb{E}_x \sum_g \langle\psi|(A_{g(x)}^x - B_{g(x)}^x)^\dagger (S_g^x)^\dagger S_g^x (A_{g(x)}^x - B_{g(x)}^x)|\psi\rangle$$

$$= \mathbb{E}_x \sum_a \langle\psi|(A_a^x - B_a^x)^\dagger \Big(\sum_{g:g(x)=a} (S_g^x)^\dagger S_g^x \Big)(A_a^x - B_a^x)|\psi\rangle$$

$$\leq \mathbb{E}_x \sum_a \langle\psi|(A_a^x - B_a^x)^\dagger (A_a^x - B_a^x)|\psi\rangle$$

$$= \mathbb{E}_x \sum_a \|(A_a^x - B_a^x)|\psi\rangle\|^2.$$

The inequality follows from the fact that $\sum_{g:g(x)=a}(S_g^x)^\dagger S_g^x \leq \sum_g (S_g^x)^\dagger (S_g^x) \leq \mathrm{Id}$. The last line is at most δ by assumption, and we obtain the desired conclusion. $\square$

000V **Lemma 2.8.** *Let $\mathcal{A}$ be a finite set. Let $\{A_a^x\}_{a \in \mathcal{A}}$ be a projective measurement and let $\{B_a^x\}_{a \in \mathcal{A}}$ be a set of matrices. If $A_a^x \approx_\delta B_a^x$, then for all subsets $S \subseteq \mathcal{A}$, we have*

$$\sum_{a \in S} A_a^x \approx_\delta \sum_{a \in S} A_a^x \cdot B_a^x.$$

Proof. We expand:

$$\mathbb{E}_x \left\| \sum_{a \in S} \Big(A_a^x - A_a^x \cdot B_a^x \Big)|\psi\rangle \right\|^2 = \mathbb{E}_x \left\| \sum_{a \in S} A_a^x \cdot \Big(A_a^x - B_a^x \Big)|\psi\rangle \right\|^2$$

$$= \mathbb{E}_x \sum_{a \in S} \langle\psi|(A_a^x - B_a^x)^\dagger A_a^x (A_a^x - B_a^x)|\psi\rangle$$

$$\leq \mathbb{E}_x \sum_a \|(A_a^x - B_a^x)|\psi\rangle\|^2.$$

In the second line we used the projectivity of $\{A_a^x\}$, and in the third line we used that $A_a^x \leq \mathrm{Id}$. $\square$

000W **Fact 2.9** (Triangle inequality, Fact 4.28 in [**?**])**.** *If $A_a^x \approx_\delta B_a^x$ and $B_a^x \approx_\epsilon C_a^x$, then $A_a^x \approx_{\delta+\epsilon} C_a^x$.*

000X **Fact 2.10** (Triangle inequality for "$\simeq$", Proposition 4.29 in [**?**])**.** *If $A_a^x \otimes I_B \simeq_\varepsilon I_A \otimes B_a^x$, $C_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$, and $C_a^x \otimes I_B \simeq_\gamma I_A \otimes D_a^x$, then $A_a^x \otimes I_B \simeq_{\varepsilon + 2\sqrt{\delta+\gamma}} I_A \otimes D_a^x$.*

000Y **Fact 2.11** (Data processing, Fact 4.26 in [**?**])**.** *Suppose $A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$. Then $A_{[f(\cdot)=b]}^x \otimes I_B \simeq_\delta I_A \otimes B_{[f(\cdot)=b]}^x$.*

The state-dependent distance is the right tool for reasoning about the closeness of measurement operators in a strategy. The following lemma ensures that, when two families of measurements are close on a state, changing from one family of measurement to the other only introduces a small error to the value of the strategy.

Lemma 2.12. *Let* $\{A_{a,b}^x\}$, $\{B_{a,b,c}^x\}$, $\{C_{a,c}^x\}$ *be POVMs. Suppose* $\{B_{a,b,c}^x\}$ *is projective, and*

$$A_{a,b}^x \otimes \mathrm{Id}_B \approx_\delta \mathrm{Id}_A \otimes B_{a,b}^x \,,$$
$$C_{a,c}^x \otimes \mathrm{Id}_B \approx_\delta \mathrm{Id}_A \otimes B_{a,c}^x \,.$$

Then the following approximate commutation relation holds:

$$[A_{a,b}^x, C_{a,c}^x] \otimes I_B \approx_\delta 0 \,.$$

Proof. Applying Fact 2.6 to $C_{a,c}^x \otimes \mathrm{Id}_B \approx_\delta \mathrm{Id}_A \otimes B_{a,c}^x$ and $\{A_{a,b}^x \otimes \mathrm{Id}_B\}$, we have

$$(2.12.1) \qquad\qquad A_{a,b}^x C_{a,c}^x \otimes \mathrm{Id}_B \approx_\delta A_{a,b}^x \otimes B_{a,c}^x \,.$$

Similarly, applying Fact 2.6 to $A_{a,b}^x \otimes \mathrm{Id}_B \approx_\delta \mathrm{Id}_A \otimes B_{a,b}^x$ and $\{\mathrm{Id}_A \otimes B_{a,c}^x\}$, and using the fact that $\{B_{a,b,c}^x\}$ is projective, we have

$$(2.12.2) \qquad\qquad \begin{aligned} A_{a,b}^x \otimes B_{a,c}^x &\approx_\delta \mathrm{Id}_A \otimes B_{a,c}^x B_{a,b}^x \\ &= \mathrm{Id}_A \otimes B_{a,b,c}^x \,. \end{aligned}$$

Combining Equations (2.12.1) and (2.12.2), we have

$$(2.12.3) \qquad\qquad A_{a,b}^x C_{a,c}^x \otimes \mathrm{Id}_B \approx_\delta \mathrm{Id}_A \otimes B_{a,b,c}^x \,.$$

A similar argument gives

$$(2.12.4) \qquad\qquad C_{a,c}^x A_{a,b}^x \otimes \mathrm{Id}_B \approx_\delta \mathrm{Id}_A \otimes B_{a,b,c}^x \,.$$

The claim follows from Equations (2.12.3) and (2.12.4). $\qquad\square$

The following lemma is a slightly modified version of [**?**, Fact 4.34].

Lemma 2.13. *Let* $k \geq 0$ *be an integer and let* $\varepsilon > 0$. *Let* $\mathcal{X}$ *be a finite set and* μ *a distribution over* $\mathcal{X}$. *For each* $1 \leq i \leq k$ *let* $\mathcal{G}_i$ *be a set of functions* $g_i : \mathcal{Y} \to \mathcal{R}_i$ *and for each* $x \in \mathcal{X}$ *let* $\{G_g^{i,x}\}_{g \in \mathcal{G}_i}$ *be a projective measurement. Suppose that for all* $i \in \{1, \dots, k\}$, $\mathcal{G}_i$ *satisfies the following property: for any two* $g_i \neq g_i' \in \mathcal{G}_i$, *the probability that* $g_i(y) = g_i'(y)$ *over a uniformly random* $y \in \mathcal{Y}$ *is at most* ε.

Let $\{A_{g_1, g_2, \dots, g_k}^x\}$ *be a projective measurement with outcomes* $(g_1, \dots, g_k) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_k$. *For each* $1 \leq i \leq k$, *suppose that on average over* $x \sim \mu$ *and* $y \in \mathcal{Y}$ *sampled uniformly at random,*

$$(2.13.1) \qquad\qquad A_{[\mathrm{eval}_y(\cdot)_i = a_i]}^x \otimes I_B \simeq_\delta I_A \otimes G_{[\mathrm{eval}_y(\cdot) = a_i]}^{i,x} \,.$$

Define the POVM family $\{C_{g_1, g_2, \dots, g_k}^x\}$, *for* $x \in \mathcal{X}$, *by*

$$C_{g_1, g_2, \dots, g_k}^x = G_{g_k}^{k,x} \cdots G_{g_2}^{2,x} G_{g_1}^{1,x} G_{g_2}^{2,x} \cdots G_{g_k}^{k,x} \,.$$

Then on average over $x \sim \mu$ *and* $y \in \mathcal{Y}$ *sampled uniformly at random,*

$$(2.13.2) \quad A_{[\mathrm{eval}_y(\cdot) = (a_1, a_2, \dots, a_k)]}^x \otimes I_B \simeq_{k(\delta + \varepsilon)^{1/2}} I_A \otimes C_{[\mathrm{eval}_y(\cdot) = (a_1, a_2, \dots, a_k)]}^x \,.$$

Proof. The proof is identical to the one given in [**?**, Fact 4.34], with the only modification needed to insert the dependence on x for all measurements considered. $\qquad\square$

Lemma 2.14 (Fact 4.35 in [**?**])**.** *Let* $\mathcal{D}$ *be a distribution on* $(x, y_1, y_2) \in \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2$. *For* $i \in \{1, 2\}$ *let* $\mathcal{G}_i$ *be a collection of functions* $g_i : \mathcal{Y}_i \to \mathcal{R}_i$ *and let* $\{(G_i)_g^x\}_{g \in \mathcal{G}_i}$ *be families of measurements such that* $\{(G_2)_g^x\}_g$ *is projective for every* x. *Suppose further that for every* (x, y_1) *it holds that for* $g_2 \neq g_2' \in \mathcal{G}_2$ *the probability, on average over* y_2 *chosen from* $\mathcal{D}$ *conditioned on* (x, y_1), *that* $g_2(y_2) = g_2'(y_2)$ *is at most* η. *Let*

$\{A^{x,y_1,y_2}_{a_1,a_2}\}$ *be a family of projective measurements with outcomes* $(a_1, a_2) \in \mathcal{R}_1 \times \mathcal{R}_2$ *such that for* $i \in \{1, 2\}$,

0018 (2.14.1)
$$A^{x,y_1,y_2}_{a_i} \otimes \mathrm{Id} \simeq_\delta \mathrm{Id} \otimes (G_i)^x_{[\mathrm{eval}_{y_i}(\cdot) = a_i]}$$

and

0019 (2.14.2)
$$A^{x,y_1,y_2}_{a_1,a_2} \otimes \mathrm{Id} \simeq_\delta \mathrm{Id} \otimes A^{x,y_1,y_2}_{a_1,a_2} \ .$$

Define a family of measurements $\{J^x_{g_1,g_2}\}$ *as*

001A (2.14.3)
$$J^x_{g_1,g_2} = (G_2)^x_{g_2} (G_1)^x_{g_1} (G_2)^x_{g_2} \ .$$

Then there is a

001B (2.14.4)
$$\delta_{pasting} = \delta_{pasting}(\eta, \delta) = \mathrm{poly}(\eta, \delta)$$

such that

001C (2.14.5)
$$A^{x,y_1,y_2}_{a_1,a_2} \otimes \mathrm{Id} \simeq_{\delta_p} \mathrm{Id} \otimes J^x_{[\mathrm{eval}_{y_1}(\cdot) = a_1, \mathrm{eval}_{y_2}(\cdot) = a_2]} \ .$$

3. Chapters

Preliminaries
    (1) Introduction
Background
    (2) Nonlocal games
    (3) Complexity theory
    (4) Operator algebras
Warm-up
    (5) Pauli braiding
Overview
    (6) The argument

(7) Question reduction
(8) Answer reduction
(9) Recursive compression
Building blocks
    (10) Classical low-degree test
    (11) Quantum low-degree test
Related tools
    (12) Parallel repetition
Extensions
    (13) TBD