

Part 1. Background

0006

NONLOCAL GAMES

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We introduce definitions associated with nonlocal games and strategies that will be used throughout.

1. GAMES AND STRATEGIES

0008 **Definition 1.1** (Two-player one-round games). A *two-player one-round game* \mathfrak{G} is specified by a tuple $(\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ where

- (1) \mathcal{X} and \mathcal{Y} are finite sets (called the *question alphabets*),
- (2) \mathcal{A} and \mathcal{B} are finite sets (called the *answer alphabets*),
- (3) μ is a probability distribution over $\mathcal{X} \times \mathcal{Y}$ (called the *question distribution*), and
- (4) $D : \mathcal{X} \times \mathcal{Y} \times \mathcal{A} \times \mathcal{B} \rightarrow \{0, 1\}$ is a function (called the *decision predicate*).

0009 **Definition 1.2** (Tensor product strategies). A *tensor product strategy* S for a game $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ is a tuple $(|\psi\rangle, A, B)$ where

- $|\psi\rangle$ is a pure quantum state in $\mathcal{H}_A \otimes \mathcal{H}_B$ for finite dimensional complex Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$,
- A is a set $\{A^x\}$ such that for every $x \in \mathcal{X}$, $A^x = \{A_a^x\}_{a \in \mathcal{A}}$ is a POVM over \mathcal{H}_A , and
- B is a set $\{B^y\}$ such that for every $y \in \mathcal{Y}$, $B^y = \{B_b^y\}_{b \in \mathcal{B}}$ is a POVM over \mathcal{H}_B .

000A **Definition 1.3** (Tensor product value). The *tensor product value* of a tensor product strategy $S = (|\psi\rangle, A, B)$ with respect to a game $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ is defined as

$$\text{val}^*(\mathfrak{G}, S) = \sum_{x, y, a, b} \mu(x, y) D(x, y, a, b) \langle \psi | A_a^x \otimes B_b^y | \psi \rangle .$$

For $v \in [0, 1]$ we say that the strategy S *passes (or wins)* \mathfrak{G} *with probability* v if $\text{val}^*(\mathfrak{G}, S) \geq v$. The *tensor product value* of \mathfrak{G} is defined as

$$\text{val}^*(\mathfrak{G}) = \sup_S \text{val}^*(\mathfrak{G}, S) ,$$

where the supremum is taken over all tensor product strategies S for \mathfrak{G} .

Remark 1.4. Unless specified otherwise, all strategies considered in this paper are tensor product strategies, and we simply call them *strategies*. Similarly, we refer to $\text{val}^*(\mathfrak{G})$ as the *value* of the game \mathfrak{G} .

Definition 1.5 (Projective strategies). We say that a strategy $S = (|\psi\rangle, A, B)$ is *projective* if all the measurements $\{A_a^x\}_a$ and $\{B_b^y\}_b$ are projective. 000B

Remark 1.6. A game $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ is *symmetric* if the question and answer alphabets are the same for both players (i.e. $\mathcal{X} = \mathcal{Y}$ and $\mathcal{A} = \mathcal{B}$), the distribution μ is symmetric (i.e. $\mu(x, y) = \mu(y, x)$), and the decision predicate D treats both players symmetrically (i.e. for all x, y, a, b , $D(x, y, a, b) = D(y, x, b, a)$). Furthermore, we call a strategy $S = (|\psi\rangle, A, B)$ *symmetric* if $|\psi\rangle$ is a state in $\mathcal{H} \otimes \mathcal{H}$, for some Hilbert space \mathcal{H} , that is invariant under permutation of the two factors, and the measurement operators of both players are identical. We specify symmetric games \mathfrak{G} and symmetric strategies S using a more compact notation: we write $\mathfrak{G} = (\mathcal{X}, \mathcal{A}, \mu, D)$ and $S = (|\psi\rangle, M)$ where M denotes the set of measurement operators for both players. 000C

Lemma 1.7. Let $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ be a symmetric game such that $\text{val}^*(\mathfrak{G}) = 1 - \varepsilon$ for some $\varepsilon \geq 0$. Then there exists a symmetric and projective strategy $S = (|\psi\rangle, M)$ such that $\text{val}^*(\mathfrak{G}, S) \geq 1 - 2\varepsilon$. 000D

Proof. By definition there exists a strategy $S' = (|\psi'\rangle, A, B)$ such that $\text{val}^*(\mathfrak{G}, S') \geq 1 - 2\varepsilon$. Using Naimark's theorem (for example as formulated in [?, Theorem 4.2]) we can assume without loss of generality that $|\psi'\rangle \in \mathbb{C}_{A'}^d \otimes \mathbb{C}_{B'}^d$, for some integer d and that for every x and y , A^x and B^y is a projective measurement. Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B |\psi'\rangle_{A'B'} + |1\rangle_A |0\rangle_B |\psi'_\tau\rangle_{A'B'}) \in (\mathbb{C}_A^2 \otimes \mathbb{C}_{A'}^d) \otimes (\mathbb{C}_B^2 \otimes \mathbb{C}_{B'}^d),$$

where $|\psi'_\tau\rangle$ is obtained from $|\psi'\rangle$ by permuting the two players' registers. Observe that $|\psi\rangle$ is invariant under permutation of AA' and BB' .

For any question $x \in \mathcal{X} = \mathcal{Y}$, define the measurement $M^x = \{M_a^x\}_{a \in \mathcal{A}}$ acting on the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^d$ as follows:

$$M_a^x = |0\rangle\langle 0| \otimes A_a^x + |1\rangle\langle 1| \otimes B_a^x.$$

When Alice receives question x , she measures M^x on registers AA' , and when Bob receives question y , he measures M^y on registers BB' . Using that by assumption the decision predicate D for \mathfrak{G} is symmetric, it is not hard to verify that $\text{val}^*(\mathfrak{G}, S) = \text{val}^*(\mathfrak{G}, S')$. \square

Definition 1.8. Let $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ be a game, and let $S = (|\psi\rangle, A, B)$ be a strategy for \mathfrak{G} such that the spaces $\mathcal{H}_A \simeq \mathcal{H}_B$ canonically. Let $S \subseteq \mathcal{X} \times \mathcal{Y}$ denote the support of the question distribution μ , i.e. the set of (x, y) such that $\mu(x, y) > 0$. We say that S is a *commuting strategy* for \mathfrak{G} if for all question pairs $(x, y) \in S$, we have $[A_a^x, B_b^y] = 0$ for all $a \in \mathcal{A}, b \in \mathcal{B}$, where $[A, B] = AB - BA$ denotes the commutator. 000E

Definition 1.9 (Consistent measurements). Let \mathcal{A} be a finite set, let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ be a state, and $\{M_a\}_{a \in \mathcal{A}}$ a projective measurement on \mathcal{H} . We say that $\{M_a\}_{a \in \mathcal{A}}$ is *consistent on $|\psi\rangle$* if and only if 000F

$$\forall a \in \mathcal{A}, \quad M_a \otimes \text{Id}_B |\psi\rangle = \text{Id}_A \otimes M_a |\psi\rangle.$$

Definition 1.10 (Consistent strategies). Let $S = (|\psi\rangle, A, B)$ be a projective strategy with state $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$, for some Hilbert space \mathcal{H} , which is defined on question alphabets \mathcal{X} and \mathcal{Y} and answer alphabets \mathcal{A} and \mathcal{B} , respectively. We say that the strategy S is *consistent* if for all $x \in \mathcal{X}$, the measurement $\{A_a^x\}_{a \in \mathcal{A}}$ is 000G

consistent on $|\psi\rangle$ and if for all $y \in \mathcal{Y}$, the measurement $\{B_b^y\}_{b \in \mathcal{B}}$ is consistent on $|\psi\rangle$.

000H **Definition 1.11.** We say that a strategy S for a game \mathfrak{G} is *PCC* if it is projective, consistent, and commuting for \mathfrak{G} . Additionally, we say that a PCC strategy S is *SPCC* if it is furthermore symmetric.

Remark 1.12. A strategy $S = (|\psi\rangle, A, B)$ for a symmetric game $\mathfrak{G} = (\mathcal{X}, \mathcal{X}, \mathcal{A}, \mathcal{A}, \mu, D)$ is called *synchronous* if it holds that for every $x \in \mathcal{X}$ and $a \neq b \in \mathcal{A}$, $\langle \psi | A_a^x \otimes B_b^x | \psi \rangle = 0$; in other words, the players never return different answers when simultaneously asked the same question. As shown in [?] the condition for a finite-dimensional strategy of being synchronous is equivalent to the condition that it is projective, consistent, and moreover $|\psi\rangle$ is a maximally entangled state. (The equivalence is extended to infinite-dimensional strategies, as well as correlations induced by limits of finite-dimensional strategies, in [?].)

000J **Definition 1.13** (Entanglement requirements of a game). For all games \mathfrak{G} and $\nu \in [0, 1]$, let $\mathcal{E}(\mathfrak{G}, \nu)$ denote the minimum integer d such that there exists a finite dimensional tensor product strategy S that achieves success probability at least ν in the game \mathfrak{G} with a state $|\psi\rangle$ whose Schmidt rank is at most d . If there is no finite dimensional strategy that achieves success probability ν , then define $\mathcal{E}(\mathfrak{G}, \nu)$ to be ∞ .

2. DISTANCE MEASURES

We introduce several distance measures that are used throughout.

000K **Definition 2.1** (Distance between states).

Let $\{|\psi_n\rangle\}_{n \in \mathbb{N}}$ and $\{|\psi'_n\rangle\}_{n \in \mathbb{N}}$ be two families of states in the same space \mathcal{H} . For some function $\delta : \mathbb{N} \rightarrow [0, 1]$ we say that $\{|\psi_n\rangle\}$ and $\{|\psi'_n\rangle\}$ are δ -close, denoted as $|\psi\rangle \approx_\delta |\psi'\rangle$, if $\| |\psi_n\rangle - |\psi'_n\rangle \|^2 = O(\delta(n))$. (For convenience we generally leave the dependence of the states and δ on the indexing parameter n implicit.)

000L **Definition 2.2** (Consistency between POVMs). Let \mathcal{X} be a finite set and μ a distribution on \mathcal{X} . Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a quantum state, and for all $x \in \mathcal{X}$, $\{A_a^x\}$ and $\{B_a^x\}$ POVMs. We write

$$A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$$

on state $|\psi\rangle$ and distribution μ if

$$\mathbb{E}_{x \sim \mu} \sum_{a \neq b} \langle \psi | A_a^x \otimes B_b^x | \psi \rangle \leq O(\delta).$$

In this case, we say that $\{A_a^x\}$ and $\{B_a^x\}$ are δ -consistent on $|\psi\rangle$.

Note that a consistent measurement according to Definition 1.9 is 0-consistent with itself, under the singleton distribution, according to Definition 2.2 (and vice-versa).

000M **Definition 2.3** (Distance between POVMs). Let \mathcal{X} be a finite set and μ a distribution on \mathcal{X} . Let $|\psi\rangle \in \mathcal{H}$ be a quantum state, and for all $x \in \mathcal{X}$, $\{M_a^x\}$ and $\{N_a^x\}$ two POVMs on \mathcal{H} . We say that $\{M_a^x\}$ and $\{N_a^x\}$ are δ -close on state $|\psi\rangle$ and under distribution μ if

$$\mathbb{E}_{x \sim \mu} \sum_a \|(M_a^x - N_a^x)|\psi\rangle\|^2 \leq O(\delta),$$

and we write $M_a^x \approx_\delta N_a^x$ to denote this when the state $|\psi\rangle$ and distribution μ are clear from context. This distance is referred to as the *state-dependent* distance.

Definition 2.4 (Distance between strategies). Let $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ be a nonlocal game and let $S = (\psi, A, B)$, $S' = (\psi', A', B')$ be strategies for \mathfrak{G} . For $\delta \in [0, 1]$ we say that S is δ -close to S' if the following conditions hold. 000N

- (1) The states $|\psi\rangle, |\psi'\rangle$ are states in the same Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and are δ -close.
- (2) For all $x \in \mathcal{X}, y \in \mathcal{Y}$, we have $A_a^x \approx_\delta (A')_a^x$ and $B_b^y \approx_\delta (B')_b^y$, with the approximations holding under the distribution μ , and on either $|\psi\rangle$ or $|\psi'\rangle$.

We record several useful facts about the consistency measure and the state-dependent distance without proof. Readers are referred to Sections 4.4 and 4.5 in [?] for additional discussion and proofs.

Fact 2.5 (Fact 4.13 and Fact 4.14 in [?]). For POVMs $\{A_a^x\}$ and $\{B_a^x\}$, the following hold. 000P

- (1) If $A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$ then $A_a^x \otimes I_B \approx_\delta I_A \otimes B_a^x$. 000Q
- (2) If $A_a^x \otimes I_B \approx_\delta I_A \otimes B_a^x$ and $\{A_a^x\}$ and $\{B_a^x\}$ are projective measurements, then $A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$. 000R
- (3) If $A_a^x \otimes I_B \approx_\delta I_A \otimes B_a^x$ and either $\{A_a^x\}$ or $\{B_a^x\}$ is a projective measurement, then $A_a^x \otimes I_B \simeq_{\delta^{1/2}} I_A \otimes B_a^x$. 000S

Fact 2.6 (Fact 4.20 in [?]). Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be finite sets, and let D be a distribution over question pairs (x, y) . Let $\{A_{a,b}^x\}$ and $\{B_{a,b}^x\}$ be operators whose outcomes range over the product set $\mathcal{A} \times \mathcal{B}$. Suppose a set of operators $\{C_{a,c}^y\}$, whose outcomes range over the product set $\mathcal{A} \times \mathcal{C}$, satisfies the condition $\sum_{a,c} (C_{a,c}^y)^\dagger C_{a,c}^y \leq \text{Id}$ for all y . If $A_{a,b}^x \approx_\delta B_{a,b}^x$ on average over x sampled from the corresponding marginal of distribution D , then $C_{a,c}^y A_{a,b}^x \approx_\delta C_{a,c}^y B_{a,b}^x$ on average over (x, y) sampled from D . 000T

Proof. Fix questions x, y and answers $a \in \mathcal{A}, b \in \mathcal{B}$. We have then that

$$(2.6.1) \quad \sum_c \|(C_{a,c}^y A_{a,b}^x - C_{a,c}^y B_{a,b}^x)|\psi\rangle\|^2 = \sum_c \langle\psi|(A_{a,b}^x - B_{a,b}^x)^\dagger (C_{a,c}^y)^\dagger (C_{a,c}^y) (A_{a,b}^x - B_{a,b}^x)|\psi\rangle$$

$$(2.6.2) \quad \leq \langle\psi|(A_{a,b}^x - B_{a,b}^x)^\dagger (A_{a,b}^x - B_{a,b}^x)|\psi\rangle$$

$$(2.6.3) \quad = \|(A_{a,b}^x - B_{a,b}^x)|\psi\rangle\|^2$$

where the inequality follows from the fact that $\sum_c (C_{a,c}^y)^\dagger C_{a,c}^y \leq \sum_{a,c} (C_{a,c}^y)^\dagger C_{a,c}^y \leq \text{Id}$. Thus we obtain the desired conclusion

$$\mathbb{E}_{(x,y) \sim D} \sum_{a,b,c} \|(C_{a,c}^y A_{a,b}^x - C_{a,c}^y B_{a,b}^x)|\psi\rangle\|^2 \leq \mathbb{E}_{(x,y) \sim D} \sum_{a,b} \|(A_{a,b}^x - B_{a,b}^x)|\psi\rangle\|^2 \leq \delta.$$

□

Fact 2.7. Let \mathcal{X}, \mathcal{A} denote finite sets, and let \mathcal{G} denote a set of functions $g : \mathcal{X} \rightarrow \mathcal{A}$. Let $\{A_a^x\}, \{B_a^x\}$ be POVMs indexed by \mathcal{X} and outcomes in \mathcal{A} . Let $\{S_g^x\}$ denote a set of operators such that for all $x \in \mathcal{X}$, $\sum_g (S_g^x)^\dagger S_g^x \leq \text{Id}$. If $A_a^x \approx_\delta B_a^x$ on average over x , then $S_g^x A_{g(x)}^x \approx_\delta S_g^x B_{g(x)}^x$. 000U

Proof. We expand:

$$\begin{aligned}
\mathbb{E}_x \sum_g \left\| S_g^x (A_{g(x)}^x - B_{g(x)}^x) |\psi\rangle \right\|^2 &= \mathbb{E}_x \sum_g \langle \psi | (A_{g(x)}^x - B_{g(x)}^x)^\dagger (S_g^x)^\dagger S_g^x (A_{g(x)}^x - B_{g(x)}^x) | \psi \rangle \\
&= \mathbb{E}_x \sum_a \langle \psi | (A_a^x - B_a^x)^\dagger \left(\sum_{g:g(x)=a} (S_g^x)^\dagger S_g^x \right) (A_a^x - B_a^x) | \psi \rangle \\
&\leq \mathbb{E}_x \sum_a \langle \psi | (A_a^x - B_a^x)^\dagger (A_a^x - B_a^x) | \psi \rangle \\
&= \mathbb{E}_x \sum_a \|(A_a^x - B_a^x) |\psi\rangle\|^2.
\end{aligned}$$

The inequality follows from the fact that $\sum_{g:g(x)=a} (S_g^x)^\dagger S_g^x \leq \sum_g (S_g^x)^\dagger S_g^x \leq \text{Id}$. The last line is at most δ by assumption, and we obtain the desired conclusion. \square

000V **Lemma 2.8.** *Let \mathcal{A} be a finite set. Let $\{A_a^x\}_{a \in \mathcal{A}}$ be a projective measurement and let $\{B_a^x\}_{a \in \mathcal{A}}$ be a set of matrices. If $A_a^x \approx_\delta B_a^x$, then for all subsets $S \subseteq \mathcal{A}$, we have*

$$\sum_{a \in S} A_a^x \approx_\delta \sum_{a \in S} A_a^x \cdot B_a^x.$$

Proof. We expand:

$$\begin{aligned}
\mathbb{E}_x \left\| \sum_{a \in S} (A_a^x - A_a^x \cdot B_a^x) |\psi\rangle \right\|^2 &= \mathbb{E}_x \left\| \sum_{a \in S} A_a^x \cdot (A_a^x - B_a^x) |\psi\rangle \right\|^2 \\
&= \mathbb{E}_x \sum_{a \in S} \langle \psi | (A_a^x - B_a^x)^\dagger A_a^x (A_a^x - B_a^x) | \psi \rangle \\
&\leq \mathbb{E}_x \sum_a \|(A_a^x - B_a^x) |\psi\rangle\|^2.
\end{aligned}$$

In the second line we used the projectivity of $\{A_a^x\}$, and in the third line we used that $A_a^x \leq \text{Id}$. \square

000W **Fact 2.9** (Triangle inequality, Fact 4.28 in [?]). *If $A_a^x \approx_\delta B_a^x$ and $B_a^x \approx_\epsilon C_a^x$, then $A_a^x \approx_{\delta+\epsilon} C_a^x$.*

000X **Fact 2.10** (Triangle inequality for “ \simeq ”, Proposition 4.29 in [?]). *If $A_a^x \otimes I_B \simeq_\epsilon I_A \otimes B_a^x$, $C_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$, and $C_a^x \otimes I_B \simeq_\gamma I_A \otimes D_a^x$, then $A_a^x \otimes I_B \simeq_{\epsilon+2\sqrt{\delta+\gamma}} I_A \otimes D_a^x$.*

000Y **Fact 2.11** (Data processing, Fact 4.26 in [?]). *Suppose $A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$. Then $A_{[f(\cdot)=b]}^x \otimes I_B \simeq_\delta I_A \otimes B_{[f(\cdot)=b]}^x$.*

The state-dependent distance is the right tool for reasoning about the closeness of measurement operators in a strategy. The following lemma ensures that, when two families of measurements are close on a state, changing from one family of measurement to the other only introduces a small error to the value of the strategy.

000Z **Lemma 2.12.** *Let $\{A_{a,b}^x\}$, $\{B_{a,b,c}^x\}$, $\{C_{a,c}^x\}$ be POVMs. Suppose $\{B_{a,b,c}^x\}$ is projective, and*

$$\begin{aligned}
A_{a,b}^x \otimes \text{Id}_B &\approx_\delta \text{Id}_A \otimes B_{a,b}^x, \\
C_{a,c}^x \otimes \text{Id}_B &\approx_\delta \text{Id}_A \otimes B_{a,c}^x.
\end{aligned}$$

Then the following approximate commutation relation holds:

$$[A_{a,b}^x, C_{a,c}^x] \otimes I_B \approx_\delta 0.$$

Proof. Applying Fact 2.6 to $C_{a,c}^x \otimes \text{Id}_B \approx_\delta \text{Id}_A \otimes B_{a,c}^x$ and $\{A_{a,b}^x \otimes \text{Id}_B\}$, we have

$$(2.12.1) \quad A_{a,b}^x C_{a,c}^x \otimes \text{Id}_B \approx_\delta A_{a,b}^x \otimes B_{a,c}^x. \quad 0010$$

Similarly, applying Fact 2.6 to $A_{a,b}^x \otimes \text{Id}_B \approx_\delta \text{Id}_A \otimes B_{a,b}^x$ and $\{\text{Id}_A \otimes B_{a,c}^x\}$, and using the fact that $\{B_{a,b,c}^x\}$ is projective, we have

$$(2.12.2) \quad \begin{aligned} A_{a,b}^x \otimes B_{a,c}^x &\approx_\delta \text{Id}_A \otimes B_{a,c}^x B_{a,b}^x \\ &= \text{Id}_A \otimes B_{a,b,c}^x. \end{aligned} \quad 0011$$

Combining Equations (2.12.1) and (2.12.2), we have

$$(2.12.3) \quad A_{a,b}^x C_{a,c}^x \otimes \text{Id}_B \approx_\delta \text{Id}_A \otimes B_{a,b,c}^x. \quad 0012$$

A similar argument gives

$$(2.12.4) \quad C_{a,c}^x A_{a,b}^x \otimes \text{Id}_B \approx_\delta \text{Id}_A \otimes B_{a,b,c}^x. \quad 0013$$

The claim follows from Equations (2.12.3) and (2.12.4). \square

The following lemma is a slightly modified version of [?, Fact 4.34].

Lemma 2.13. *Let $k \geq 0$ be an integer and let $\varepsilon > 0$. Let \mathcal{X} be a finite set and μ a distribution over \mathcal{X} . For each $1 \leq i \leq k$ let \mathcal{G}_i be a set of functions $g_i : \mathcal{Y} \rightarrow \mathcal{R}_i$ and for each $x \in \mathcal{X}$ let $\{G_g^{i,x}\}_{g \in \mathcal{G}_i}$ be a projective measurement. Suppose that for all $i \in \{1, \dots, k\}$, \mathcal{G}_i satisfies the following property: for any two $g_i \neq g'_i \in \mathcal{G}_i$, the probability that $g_i(y) = g'_i(y)$ over a uniformly random $y \in \mathcal{Y}$ is at most ε .* 0014

Let $\{A_{g_1, g_2, \dots, g_k}^x\}$ be a projective measurement with outcomes $(g_1, \dots, g_k) \in \mathcal{G}_1 \times \dots \times \mathcal{G}_k$. For each $1 \leq i \leq k$, suppose that on average over $x \sim \mu$ and $y \in \mathcal{Y}$ sampled uniformly at random,

$$(2.13.1) \quad A_{[\text{eval}_y(\cdot)=a_i]}^x \otimes I_B \simeq_\delta I_A \otimes G_{[\text{eval}_y(\cdot)=a_i]}^{i,x}. \quad 0015$$

Define the POVM family $\{C_{g_1, g_2, \dots, g_k}^x\}$, for $x \in \mathcal{X}$, by

$$C_{g_1, g_2, \dots, g_k}^x = G_{g_k}^{k,x} \dots G_{g_2}^{2,x} G_{g_1}^{1,x} G_{g_2}^{2,x} \dots G_{g_k}^{k,x}.$$

Then on average over $x \sim \mu$ and $y \in \mathcal{Y}$ sampled uniformly at random,

$$(2.13.2) \quad A_{[\text{eval}_y(\cdot)=(a_1, a_2, \dots, a_k)]}^x \otimes I_B \simeq_{k(\delta+\varepsilon)^{1/2}} I_A \otimes C_{[\text{eval}_y(\cdot)=(a_1, a_2, \dots, a_k)]}^x. \quad 0016$$

Proof. The proof is identical to the one given in [?, Fact 4.34], with the only modification needed to insert the dependence on x for all measurements considered. \square

Lemma 2.14 (Fact 4.35 in [?]). *Let \mathcal{D} be a distribution on $(x, y_1, y_2) \in \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2$. For $i \in \{1, 2\}$ let \mathcal{G}_i be a collection of functions $g_i : \mathcal{Y}_i \rightarrow \mathcal{R}_i$ and let $\{(G_i)_g^x\}_{g \in \mathcal{G}_i}$ be families of measurements such that $\{(G_2)_g^x\}_g$ is projective for every x . Suppose further that for every (x, y_1) it holds that for $g_2 \neq g'_2 \in \mathcal{G}_2$ the probability, on average over y_2 chosen from \mathcal{D} conditioned on (x, y_1) , that $g_2(y_2) = g'_2(y_2)$ is at most η . Let $\{A_{a_1, a_2}^{x, y_1, y_2}\}$ be a family of projective measurements with outcomes $(a_1, a_2) \in \mathcal{R}_1 \times \mathcal{R}_2$ such that for $i \in \{1, 2\}$,* 0017

$$(2.14.1) \quad A_{a_i}^{x, y_1, y_2} \otimes \text{Id} \simeq_\delta \text{Id} \otimes (G_i)_{[\text{eval}_{y_i}(\cdot)=a_i]}^x \quad 0018$$

and

$$0019 \quad (2.14.2) \quad A_{a_1, a_2}^{x, y_1, y_2} \otimes \text{Id} \simeq_{\delta} \text{Id} \otimes A_{a_1, a_2}^{x, y_1, y_2} .$$

Define a family of measurements $\{J_{g_1, g_2}^x\}$ as

$$001A \quad (2.14.3) \quad J_{g_1, g_2}^x = (G_2)_{g_2}^x (G_1)_{g_1}^x (G_2)_{g_2}^x .$$

Then there is a

$$001B \quad (2.14.4) \quad \delta_{\text{pasting}} = \delta_{\text{pasting}}(\eta, \delta) = \text{poly}(\eta, \delta)$$

such that

$$001C \quad (2.14.5) \quad A_{a_1, a_2}^{x, y_1, y_2} \otimes \text{Id} \simeq_{\delta_p} \text{Id} \otimes J_{[\text{eval}_{y_1}(\cdot)=a_1, \text{eval}_{y_2}(\cdot)=a_2]}^x .$$

3. CHAPTERS

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