

## Part 1. Background

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## NONLOCAL GAMES

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We introduce definitions associated with nonlocal games and strategies that will be used throughout.

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### 1. GAMES AND STRATEGIES

0008 **Definition 1.1** (Two-player one-round games). A *two-player one-round game*  $\mathfrak{G}$  is specified by a tuple  $(\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$  where

- (1)  $\mathcal{X}$  and  $\mathcal{Y}$  are finite sets (called the *question alphabets*),
- (2)  $\mathcal{A}$  and  $\mathcal{B}$  are finite sets (called the *answer alphabets*),
- (3)  $\mu$  is a probability distribution over  $\mathcal{X} \times \mathcal{Y}$  (called the *question distribution*), and
- (4)  $D : \mathcal{X} \times \mathcal{Y} \times \mathcal{A} \times \mathcal{B} \rightarrow \{0, 1\}$  is a function (called the *decision predicate*).

0009 **Definition 1.2** (Tensor product strategies). A *tensor product strategy*  $S$  for a game  $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$  is a tuple  $(|\psi\rangle, A, B)$  where

- $|\psi\rangle$  is a pure quantum state in  $\mathcal{H}_A \otimes \mathcal{H}_B$  for finite dimensional complex Hilbert spaces  $\mathcal{H}_A, \mathcal{H}_B$ ,
- $A$  is a set  $\{A^x\}$  such that for every  $x \in \mathcal{X}$ ,  $A^x = \{A_a^x\}_{a \in \mathcal{A}}$  is a POVM over  $\mathcal{H}_A$ , and
- $B$  is a set  $\{B^y\}$  such that for every  $y \in \mathcal{Y}$ ,  $B^y = \{B_b^y\}_{b \in \mathcal{B}}$  is a POVM over  $\mathcal{H}_B$ .

000A **Definition 1.3** (Tensor product value). The *tensor product value* of a tensor product strategy  $S = (|\psi\rangle, A, B)$  with respect to a game  $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$  is defined as

$$\text{val}^*(\mathfrak{G}, S) = \sum_{x, y, a, b} \mu(x, y) D(x, y, a, b) \langle \psi | A_a^x \otimes B_b^y | \psi \rangle .$$

For  $v \in [0, 1]$  we say that the strategy  $S$  *passes (or wins)*  $\mathfrak{G}$  *with probability*  $v$  if  $\text{val}^*(\mathfrak{G}, S) \geq v$ . The *tensor product value* of  $\mathfrak{G}$  is defined as

$$\text{val}^*(\mathfrak{G}) = \sup_S \text{val}^*(\mathfrak{G}, S) ,$$

where the supremum is taken over all tensor product strategies  $S$  for  $\mathfrak{G}$ .

*Remark 1.4.* Unless specified otherwise, all strategies considered in this paper are tensor product strategies, and we simply call them *strategies*. Similarly, we refer to  $\text{val}^*(\mathfrak{G})$  as the *value* of the game  $\mathfrak{G}$ .

**Definition 1.5** (Projective strategies). We say that a strategy  $S = (|\psi\rangle, A, B)$  is *projective* if all the measurements  $\{A_a^x\}_a$  and  $\{B_b^y\}_b$  are projective. 000B

*Remark 1.6.* A game  $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$  is *symmetric* if the question and answer alphabets are the same for both players (i.e.  $\mathcal{X} = \mathcal{Y}$  and  $\mathcal{A} = \mathcal{B}$ ), the distribution  $\mu$  is symmetric (i.e.  $\mu(x, y) = \mu(y, x)$ ), and the decision predicate  $D$  treats both players symmetrically (i.e. for all  $x, y, a, b$ ,  $D(x, y, a, b) = D(y, x, b, a)$ ). Furthermore, we call a strategy  $S = (|\psi\rangle, A, B)$  *symmetric* if  $|\psi\rangle$  is a state in  $\mathcal{H} \otimes \mathcal{H}$ , for some Hilbert space  $\mathcal{H}$ , that is invariant under permutation of the two factors, and the measurement operators of both players are identical. We specify symmetric games  $\mathfrak{G}$  and symmetric strategies  $S$  using a more compact notation: we write  $\mathfrak{G} = (\mathcal{X}, \mathcal{A}, \mu, D)$  and  $S = (|\psi\rangle, M)$  where  $M$  denotes the set of measurement operators for both players. 000C

**Lemma 1.7.** Let  $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$  be a symmetric game such that  $\text{val}^*(\mathfrak{G}) = 1 - \varepsilon$  for some  $\varepsilon \geq 0$ . Then there exists a symmetric and projective strategy  $S = (|\psi\rangle, M)$  such that  $\text{val}^*(\mathfrak{G}, S) \geq 1 - 2\varepsilon$ . 000D

*Proof.* By definition there exists a strategy  $S' = (|\psi'\rangle, A, B)$  such that  $\text{val}^*(\mathfrak{G}, S') \geq 1 - 2\varepsilon$ . Using Naimark's theorem (for example as formulated in [?, Theorem 4.2]) we can assume without loss of generality that  $|\psi'\rangle \in \mathbb{C}_{A'}^d \otimes \mathbb{C}_{B'}^d$  for some integer  $d$  and that for every  $x$  and  $y$ ,  $A^x$  and  $B^y$  is a projective measurement. Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B |\psi'\rangle_{A'B'} + |1\rangle_A |0\rangle_B |\psi'_\tau\rangle_{A'B'}) \in (\mathbb{C}_A^2 \otimes \mathbb{C}_{A'}^d) \otimes (\mathbb{C}_B^2 \otimes \mathbb{C}_{B'}^d),$$

where  $|\psi'_\tau\rangle$  is obtained from  $|\psi'\rangle$  by permuting the two players' registers. Observe that  $|\psi\rangle$  is invariant under permutation of  $AA'$  and  $BB'$ .

For any question  $x \in \mathcal{X} = \mathcal{Y}$ , define the measurement  $M^x = \{M_a^x\}_{a \in \mathcal{A}}$  acting on the Hilbert space  $\mathbb{C}^2 \otimes \mathbb{C}^d$  as follows:

$$M_a^x = |0\rangle\langle 0| \otimes A_a^x + |1\rangle\langle 1| \otimes B_a^x.$$

When Alice receives question  $x$ , she measures  $M^x$  on registers  $AA'$ , and when Bob receives question  $y$ , he measures  $M^y$  on registers  $BB'$ . Using that by assumption the decision predicate  $D$  for  $\mathfrak{G}$  is symmetric, it is not hard to verify that  $\text{val}^*(\mathfrak{G}, S) = \text{val}^*(\mathfrak{G}, S')$ .  $\square$

**Definition 1.8.** Let  $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$  be a game, and let  $S = (|\psi\rangle, A, B)$  be a strategy for  $\mathfrak{G}$  such that the spaces  $\mathcal{H}_A \simeq \mathcal{H}_B$  canonically. Let  $S \subseteq \mathcal{X} \times \mathcal{Y}$  denote the support of the question distribution  $\mu$ , i.e. the set of  $(x, y)$  such that  $\mu(x, y) > 0$ . We say that  $S$  is a *commuting strategy* for  $\mathfrak{G}$  if for all question pairs  $(x, y) \in S$ , we have  $[A_a^x, B_b^y] = 0$  for all  $a \in \mathcal{A}, b \in \mathcal{B}$ , where  $[A, B] = AB - BA$  denotes the commutator. 000E

**Definition 1.9** (Consistent measurements). Let  $\mathcal{A}$  be a finite set, let  $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$  be a state, and  $\{M_a\}_{a \in \mathcal{A}}$  a projective measurement on  $\mathcal{H}$ . We say that  $\{M_a\}_{a \in \mathcal{A}}$  is *consistent on  $|\psi\rangle$*  if and only if 000F

$$\forall a \in \mathcal{A}, \quad M_a \otimes \text{Id}_B |\psi\rangle = \text{Id}_A \otimes M_a |\psi\rangle.$$

000G **Definition 1.10** (Consistent strategies). Let  $S = (|\psi\rangle, A, B)$  be a projective strategy with state  $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ , for some Hilbert space  $\mathcal{H}$ , which is defined on question alphabets  $\mathcal{X}$  and  $\mathcal{Y}$  and answer alphabets  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. We say that the strategy  $S$  is *consistent* if for all  $x \in \mathcal{X}$ , the measurement  $\{A_a^x\}_{a \in \mathcal{A}}$  is consistent on  $|\psi\rangle$  and if for all  $y \in \mathcal{Y}$ , the measurement  $\{B_b^y\}_{b \in \mathcal{B}}$  is consistent on  $|\psi\rangle$ .

000H **Definition 1.11.** We say that a strategy  $S$  for a game  $\mathfrak{G}$  is *PCC* if it is projective, consistent, and commuting for  $\mathfrak{G}$ . Additionally, we say that a PCC strategy  $S$  is *SPCC* if it is furthermore symmetric.

*Remark 1.12.* A strategy  $S = (|\psi\rangle, A, B)$  for a symmetric game  $\mathfrak{G} = (\mathcal{X}, \mathcal{X}, \mathcal{A}, \mathcal{A}, \mu, D)$  is called *synchronous* if it holds that for every  $x \in \mathcal{X}$  and  $a \neq b \in \mathcal{A}$ ,  $\langle \psi | A_a^x \otimes B_b^x | \psi \rangle = 0$ ; in other words, the players never return different answers when simultaneously asked the same question. As shown in [?] the condition for a finite-dimensional strategy of being synchronous is equivalent to the condition that it is projective, consistent, and moreover  $|\psi\rangle$  is a maximally entangled state. (The equivalence is extended to infinite-dimensional strategies, as well as correlations induced by limits of finite-dimensional strategies, in [?].)

000J **Definition 1.13** (Entanglement requirements of a game). For all games  $\mathfrak{G}$  and  $\nu \in [0, 1]$ , let  $\mathcal{E}(\mathfrak{G}, \nu)$  denote the minimum integer  $d$  such that there exists a finite dimensional tensor product strategy  $S$  that achieves success probability at least  $\nu$  in the game  $\mathfrak{G}$  with a state  $|\psi\rangle$  whose Schmidt rank is at most  $d$ . If there is no finite dimensional strategy that achieves success probability  $\nu$ , then define  $\mathcal{E}(\mathfrak{G}, \nu)$  to be  $\infty$ .

## 2. DISTANCE MEASURES

We introduce several distance measures that are used throughout.

000K **Definition 2.1** (Distance between states).

Let  $\{|\psi_n\rangle\}_{n \in \mathbb{N}}$  and  $\{|\psi'_n\rangle\}_{n \in \mathbb{N}}$  be two families of states in the same space  $\mathcal{H}$ . For some function  $\delta : \mathbb{N} \rightarrow [0, 1]$  we say that  $\{|\psi_n\rangle\}$  and  $\{|\psi'_n\rangle\}$  are  $\delta$ -close, denoted as  $|\psi\rangle \approx_\delta |\psi'\rangle$ , if  $\| |\psi_n\rangle - |\psi'_n\rangle \|^2 = O(\delta(n))$ . (For convenience we generally leave the dependence of the states and  $\delta$  on the indexing parameter  $n$  implicit.)

000L **Definition 2.2** (Consistency between POVMs). Let  $\mathcal{X}$  be a finite set and  $\mu$  a distribution on  $\mathcal{X}$ . Let  $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$  be a quantum state, and for all  $x \in \mathcal{X}$ ,  $\{A_a^x\}$  and  $\{B_b^x\}$  POVMs. We write

$$A_a^x \otimes I_B \simeq_\delta I_A \otimes B_b^x$$

on state  $|\psi\rangle$  and distribution  $\mu$  if

$$\mathbb{E}_{x \sim \mu} \sum_{a \neq b} \langle \psi | A_a^x \otimes B_b^x | \psi \rangle \leq O(\delta).$$

In this case, we say that  $\{A_a^x\}$  and  $\{B_b^x\}$  are  $\delta$ -consistent on  $|\psi\rangle$ .

Note that a consistent measurement according to Definition 1.9 is 0-consistent with itself, under the singleton distribution, according to Definition 2.2 (and vice-versa).

**Definition 2.3** (Distance between POVMs). Let  $\mathcal{X}$  be a finite set and  $\mu$  a distribution on  $\mathcal{X}$ . Let  $|\psi\rangle \in \mathcal{H}$  be a quantum state, and for all  $x \in \mathcal{X}$ ,  $\{M_a^x\}$  and  $\{N_a^x\}$  two POVMs on  $\mathcal{H}$ . We say that  $\{M_a^x\}$  and  $\{N_a^x\}$  are  $\delta$ -close on state  $|\psi\rangle$  and under distribution  $\mu$  if

$$\mathbb{E}_{x \sim \mu} \sum_a \|(M_a^x - N_a^x)|\psi\rangle\|^2 \leq O(\delta),$$

and we write  $M_a^x \approx_\delta N_a^x$  to denote this when the state  $|\psi\rangle$  and distribution  $\mu$  are clear from context. This distance is referred to as the *state-dependent* distance.

**Definition 2.4** (Distance between strategies). Let  $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$  be a nonlocal game and let  $S = (\psi, A, B)$ ,  $S' = (\psi', A', B')$  be strategies for  $\mathfrak{G}$ . For  $\delta \in [0, 1]$  we say that  $S$  is  $\delta$ -close to  $S'$  if the following conditions hold.

- (1) The states  $|\psi\rangle, |\psi'\rangle$  are states in the same Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$  and are  $\delta$ -close.
- (2) For all  $x \in \mathcal{X}, y \in \mathcal{Y}$ , we have  $A_a^x \approx_\delta (A')_a^x$  and  $B_b^y \approx_\delta (B')_b^y$ , with the approximations holding under the distribution  $\mu$ , and on either  $|\psi\rangle$  or  $|\psi'\rangle$ .

We record several useful facts about the consistency measure and the state-dependent distance without proof. Readers are referred to Sections 4.4 and 4.5 in [?] for additional discussion and proofs.

**Fact 2.5** (Fact 4.13 and Fact 4.14 in [?]). For POVMs  $\{A_a^x\}$  and  $\{B_a^x\}$ , the following hold.

- (1) If  $A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$  then  $A_a^x \otimes I_B \approx_\delta I_A \otimes B_a^x$ .
- (2) If  $A_a^x \otimes I_B \approx_\delta I_A \otimes B_a^x$  and  $\{A_a^x\}$  and  $\{B_a^x\}$  are projective measurements, then  $A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$ .
- (3) If  $A_a^x \otimes I_B \approx_\delta I_A \otimes B_a^x$  and either  $\{A_a^x\}$  or  $\{B_a^x\}$  is a projective measurement, then  $A_a^x \otimes I_B \simeq_{\delta^{1/2}} I_A \otimes B_a^x$ .

**Fact 2.6** (Fact 4.20 in [?]). Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  be finite sets, and let  $D$  be a distribution over question pairs  $(x, y)$ . Let  $\{A_{a,b}^x\}$  and  $\{B_{a,b}^x\}$  be operators whose outcomes range over the product set  $\mathcal{A} \times \mathcal{B}$ . Suppose a set of operators  $\{C_{a,c}^y\}$ , whose outcomes range over the product set  $\mathcal{A} \times \mathcal{C}$ , satisfies the condition  $\sum_{a,c} (C_{a,c}^y)^\dagger C_{a,c}^y \leq \text{Id}$  for all  $y$ . If  $A_{a,b}^x \approx_\delta B_{a,b}^x$  on average over  $x$  sampled from the corresponding marginal of distribution  $D$ , then  $C_{a,c}^y A_{a,b}^x \approx_\delta C_{a,c}^y B_{a,b}^x$  on average over  $(x, y)$  sampled from  $D$ .

*Proof.* Fix questions  $x, y$  and answers  $a \in \mathcal{A}, b \in \mathcal{B}$ . We have then that

(2.6.1)

$$\sum_c \|(C_{a,c}^y A_{a,b}^x - C_{a,c}^y B_{a,b}^x)|\psi\rangle\|^2 = \sum_c \langle\psi| (A_{a,b}^x - B_{a,b}^x)^\dagger (C_{a,c}^y)^\dagger (C_{a,c}^y) (A_{a,b}^x - B_{a,b}^x) |\psi\rangle$$

(2.6.2)

$$\leq \langle\psi| (A_{a,b}^x - B_{a,b}^x)^\dagger (A_{a,b}^x - B_{a,b}^x) |\psi\rangle$$

(2.6.3)

$$= \|(A_{a,b}^x - B_{a,b}^x)|\psi\rangle\|^2$$

where the inequality follows from the fact that  $\sum_c (C_{a,c}^y)^\dagger C_{a,c}^y \leq \sum_{a,c} (C_{a,c}^y)^\dagger C_{a,c}^y \leq \text{Id}$ . Thus we obtain the desired conclusion

$$\mathbb{E}_{(x,y) \sim D} \sum_{a,b,c} \|(C_{a,c}^y A_{a,b}^x - C_{a,c}^y B_{a,b}^x)|\psi\rangle\|^2 \leq \mathbb{E}_{(x,y) \sim D} \sum_{a,b} \|(A_{a,b}^x - B_{a,b}^x)|\psi\rangle\|^2 \leq \delta.$$

□

000U **Fact 2.7.** *Let  $\mathcal{X}, \mathcal{A}$  denote finite sets, and let  $\mathcal{G}$  denote a set of functions  $g : \mathcal{X} \rightarrow \mathcal{A}$ . Let  $\{A_a^x\}, \{B_a^x\}$  be POVMs indexed by  $\mathcal{X}$  and outcomes in  $\mathcal{A}$ . Let  $\{S_g^x\}$  denote a set of operators such that for all  $x \in \mathcal{X}$ ,  $\sum_g (S_g^x)^\dagger S_g^x \leq \text{Id}$ . If  $A_a^x \approx_\delta B_a^x$  on average over  $x$ , then  $S_g^x A_{g(x)}^x \approx_\delta S_g^x B_{g(x)}^x$ .*

*Proof.* We expand:

$$\begin{aligned} \mathbb{E}_x \sum_g \left\| S_g^x (A_{g(x)}^x - B_{g(x)}^x) |\psi\rangle \right\|^2 &= \mathbb{E}_x \sum_g \langle \psi | (A_{g(x)}^x - B_{g(x)}^x)^\dagger (S_g^x)^\dagger S_g^x (A_{g(x)}^x - B_{g(x)}^x) | \psi \rangle \\ &= \mathbb{E}_x \sum_a \langle \psi | (A_a^x - B_a^x)^\dagger \left( \sum_{g: g(x)=a} (S_g^x)^\dagger S_g^x \right) (A_a^x - B_a^x) | \psi \rangle \\ &\leq \mathbb{E}_x \sum_a \langle \psi | (A_a^x - B_a^x)^\dagger (A_a^x - B_a^x) | \psi \rangle \\ &= \mathbb{E}_x \sum_a \| (A_a^x - B_a^x) |\psi\rangle \|^2. \end{aligned}$$

The inequality follows from the fact that  $\sum_{g: g(x)=a} (S_g^x)^\dagger S_g^x \leq \sum_g (S_g^x)^\dagger S_g^x \leq \text{Id}$ . The last line is at most  $\delta$  by assumption, and we obtain the desired conclusion.  $\square$

000V **Lemma 2.8.** *Let  $\mathcal{A}$  be a finite set. Let  $\{A_a^x\}_{a \in \mathcal{A}}$  be a projective measurement and let  $\{B_a^x\}_{a \in \mathcal{A}}$  be a set of matrices. If  $A_a^x \approx_\delta B_a^x$ , then for all subsets  $S \subseteq \mathcal{A}$ , we have*

$$\sum_{a \in S} A_a^x \approx_\delta \sum_{a \in S} A_a^x \cdot B_a^x.$$

*Proof.* We expand:

$$\begin{aligned} \mathbb{E}_x \left\| \sum_{a \in S} (A_a^x - A_a^x \cdot B_a^x) |\psi\rangle \right\|^2 &= \mathbb{E}_x \left\| \sum_{a \in S} A_a^x \cdot (A_a^x - B_a^x) |\psi\rangle \right\|^2 \\ &= \mathbb{E}_x \sum_{a \in S} \langle \psi | (A_a^x - B_a^x)^\dagger A_a^x (A_a^x - B_a^x) | \psi \rangle \\ &\leq \mathbb{E}_x \sum_a \| (A_a^x - B_a^x) |\psi\rangle \|^2. \end{aligned}$$

In the second line we used the projectivity of  $\{A_a^x\}$ , and in the third line we used that  $A_a^x \leq \text{Id}$ .  $\square$

000W **Fact 2.9** (Triangle inequality, Fact 4.28 in [?]). *If  $A_a^x \approx_\delta B_a^x$  and  $B_a^x \approx_\epsilon C_a^x$ , then  $A_a^x \approx_{\delta+\epsilon} C_a^x$ .*

000X **Fact 2.10** (Triangle inequality for “ $\simeq$ ”, Proposition 4.29 in [?]). *If  $A_a^x \otimes I_B \simeq_\epsilon I_A \otimes B_a^x$ ,  $C_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$ , and  $C_a^x \otimes I_B \simeq_\gamma I_A \otimes D_a^x$ , then  $A_a^x \otimes I_B \simeq_{\epsilon+2\sqrt{\delta+\gamma}} I_A \otimes D_a^x$ .*

000Y **Fact 2.11** (Data processing, Fact 4.26 in [?]). *Suppose  $A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$ . Then  $A_{[f(\cdot)=b]}^x \otimes I_B \simeq_\delta I_A \otimes B_{[f(\cdot)=b]}^x$ .*

The state-dependent distance is the right tool for reasoning about the closeness of measurement operators in a strategy. The following lemma ensures that, when two families of measurements are close on a state, changing from one family of measurement to the other only introduces a small error to the value of the strategy.

**Lemma 2.12.** *Let  $\{A_{a,b}^x\}$ ,  $\{B_{a,b,c}^x\}$ ,  $\{C_{a,c}^x\}$  be POVMs. Suppose  $\{B_{a,b,c}^x\}$  is projective, and*

$$\begin{aligned} A_{a,b}^x \otimes \text{Id}_B &\approx_\delta \text{Id}_A \otimes B_{a,b}^x, \\ C_{a,c}^x \otimes \text{Id}_B &\approx_\delta \text{Id}_A \otimes B_{a,c}^x. \end{aligned}$$

*Then the following approximate commutation relation holds:*

$$[A_{a,b}^x, C_{a,c}^x] \otimes I_B \approx_\delta 0.$$

*Proof.* Applying Fact 2.6 to  $C_{a,c}^x \otimes \text{Id}_B \approx_\delta \text{Id}_A \otimes B_{a,c}^x$  and  $\{A_{a,b}^x \otimes \text{Id}_B\}$ , we have

$$(2.12.1) \quad A_{a,b}^x C_{a,c}^x \otimes \text{Id}_B \approx_\delta A_{a,b}^x \otimes B_{a,c}^x. \quad 0010$$

Similarly, applying Fact 2.6 to  $A_{a,b}^x \otimes \text{Id}_B \approx_\delta \text{Id}_A \otimes B_{a,b}^x$  and  $\{\text{Id}_A \otimes B_{a,c}^x\}$ , and using the fact that  $\{B_{a,b,c}^x\}$  is projective, we have

$$(2.12.2) \quad \begin{aligned} A_{a,b}^x \otimes B_{a,c}^x &\approx_\delta \text{Id}_A \otimes B_{a,c}^x B_{a,b}^x \\ &= \text{Id}_A \otimes B_{a,b,c}^x. \end{aligned} \quad 0011$$

Combining Equations (2.12.1) and (2.12.2), we have

$$(2.12.3) \quad A_{a,b}^x C_{a,c}^x \otimes \text{Id}_B \approx_\delta \text{Id}_A \otimes B_{a,b,c}^x. \quad 0012$$

A similar argument gives

$$(2.12.4) \quad C_{a,c}^x A_{a,b}^x \otimes \text{Id}_B \approx_\delta \text{Id}_A \otimes B_{a,b,c}^x. \quad 0013$$

The claim follows from Equations (2.12.3) and (2.12.4).  $\square$

The following lemma is a slightly modified version of [?, Fact 4.34].

**Lemma 2.13.** *Let  $k \geq 0$  be an integer and let  $\varepsilon > 0$ . Let  $\mathcal{X}$  be a finite set and  $\mu$  a distribution over  $\mathcal{X}$ . For each  $1 \leq i \leq k$  let  $\mathcal{G}_i$  be a set of functions  $g_i : \mathcal{Y} \rightarrow \mathcal{R}_i$  and for each  $x \in \mathcal{X}$  let  $\{G_g^{i,x}\}_{g \in \mathcal{G}_i}$  be a projective measurement. Suppose that for all  $i \in \{1, \dots, k\}$ ,  $\mathcal{G}_i$  satisfies the following property: for any two  $g_i \neq g'_i \in \mathcal{G}_i$ , the probability that  $g_i(y) = g'_i(y)$  over a uniformly random  $y \in \mathcal{Y}$  is at most  $\varepsilon$ .*

*Let  $\{A_{g_1, g_2, \dots, g_k}^x\}$  be a projective measurement with outcomes  $(g_1, \dots, g_k) \in \mathcal{G}_1 \times \dots \times \mathcal{G}_k$ . For each  $1 \leq i \leq k$ , suppose that on average over  $x \sim \mu$  and  $y \in \mathcal{Y}$  sampled uniformly at random,*

$$(2.13.1) \quad A_{[\text{eval}_y(\cdot)=a_i]}^x \otimes I_B \simeq_\delta I_A \otimes G_{[\text{eval}_y(\cdot)=a_i]}^{i,x}. \quad 0015$$

*Define the POVM family  $\{C_{g_1, g_2, \dots, g_k}^x\}$ , for  $x \in \mathcal{X}$ , by*

$$C_{g_1, g_2, \dots, g_k}^x = G_{g_k}^{k,x} \dots G_{g_2}^{2,x} G_{g_1}^{1,x} G_{g_2}^{2,x} \dots G_{g_k}^{k,x}.$$

*Then on average over  $x \sim \mu$  and  $y \in \mathcal{Y}$  sampled uniformly at random,*

$$(2.13.2) \quad A_{[\text{eval}_y(\cdot)=(a_1, a_2, \dots, a_k)]}^x \otimes I_B \simeq_{k(\delta+\varepsilon)^{1/2}} I_A \otimes C_{[\text{eval}_y(\cdot)=(a_1, a_2, \dots, a_k)]}^x. \quad 0016$$

*Proof.* The proof is identical to the one given in [?, Fact 4.34], with the only modification needed to insert the dependence on  $x$  for all measurements considered.  $\square$

**Lemma 2.14** (Fact 4.35 in [?]). *Let  $\mathcal{D}$  be a distribution on  $(x, y_1, y_2) \in \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2$ . For  $i \in \{1, 2\}$  let  $\mathcal{G}_i$  be a collection of functions  $g_i : \mathcal{Y}_i \rightarrow \mathcal{R}_i$  and let  $\{(G_i)_g^x\}_{g \in \mathcal{G}_i}$  be families of measurements such that  $\{(G_2)_g^x\}_g$  is projective for every  $x$ . Suppose further that for every  $(x, y_1)$  it holds that for  $g_2 \neq g'_2 \in \mathcal{G}_2$  the probability, on average over  $y_2$  chosen from  $\mathcal{D}$  conditioned on  $(x, y_1)$ , that  $g_2(y_2) = g'_2(y_2)$  is at most  $\eta$ . Let*

$\{A_{a_1, a_2}^{x, y_1, y_2}\}$  be a family of projective measurements with outcomes  $(a_1, a_2) \in \mathcal{R}_1 \times \mathcal{R}_2$  such that for  $i \in \{1, 2\}$ ,

$$0018 \quad (2.14.1) \quad A_{a_i}^{x, y_1, y_2} \otimes \text{Id} \simeq_\delta \text{Id} \otimes (G_i)_{[\text{eval}_{y_i}(\cdot)=a_i]}^x$$

and

$$0019 \quad (2.14.2) \quad A_{a_1, a_2}^{x, y_1, y_2} \otimes \text{Id} \simeq_\delta \text{Id} \otimes A_{a_1, a_2}^{x, y_1, y_2}.$$

Define a family of measurements  $\{J_{g_1, g_2}^x\}$  as

$$001A \quad (2.14.3) \quad J_{g_1, g_2}^x = (G_2)_{g_2}^x (G_1)_{g_1}^x (G_2)_{g_2}^x.$$

Then there is a

$$001B \quad (2.14.4) \quad \delta_{\text{pasting}} = \delta_{\text{pasting}}(\eta, \delta) = \text{poly}(\eta, \delta)$$

such that

$$001C \quad (2.14.5) \quad A_{a_1, a_2}^{x, y_1, y_2} \otimes \text{Id} \simeq_{\delta_p} \text{Id} \otimes J_{[\text{eval}_{y_1}(\cdot)=a_1, \text{eval}_{y_2}(\cdot)=a_2]}^x.$$

### 3. CHAPTERS

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