Part 1. **Background**

0006

NONLOCAL GAMES

Contents

We introduce definitions associated with nonlocal games and strategies that will be used throughout.

1. Games and strategies

Definition 1.1 (Two-player one-round games)**.** A *two-player one-round game* $\mathfrak{G}$ is specified by a tuple $(\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ where

(1) $\mathcal{X}$ and $\mathcal{Y}$ are finite sets (called the *question alphabets*),
(2) $\mathcal{A}$ and $\mathcal{B}$ are finite sets (called the *answer alphabets*),
(3) μ is a probability distribution over $\mathcal{X} \times \mathcal{Y}$ (called the *question distribution*), and
(4) $D : \mathcal{X} \times \mathcal{Y} \times \mathcal{A} \times \mathcal{B} \to \{0, 1\}$ is a function (called the *decision predicate*).

Definition 1.2 (Tensor product strategies)**.** A *tensor product strategy* S for a game $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ is a tuple $(|\psi\rangle, A, B)$ where

- $|\psi\rangle$ is a pure quantum state in $\mathcal{H}_A \otimes \mathcal{H}_B$ for finite dimensional complex Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$,
- A is a set $\{A^x\}$ such that for every $x \in \mathcal{X}$, $A^x = \{A^x_a\}_{a \in \mathcal{A}}$ is a POVM over $\mathcal{H}_A$, and
- B is a set $\{B^y\}$ such that for every $y \in \mathcal{Y}$, $B^y = \{B^y_b\}_{b \in \mathcal{B}}$ is a POVM over $\mathcal{H}_B$.

Definition 1.3 (Tensor product value)**.** The *tensor product value* of a tensor product strategy $S = (|\psi\rangle, A, B)$ with respect to a game $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ is defined as

$$\mathrm{val}^*(\mathfrak{G}, S) = \sum_{x, y, a, b} \mu(x, y)\, D(x, y, a, b)\, \langle \psi | A^x_a \otimes B^y_b \, |\psi\rangle \; .$$

For $v \in [0, 1]$ we say that the strategy S *passes (or wins)* $\mathfrak{G}$ *with probability* v *if* $\mathrm{val}^*(\mathfrak{G}, S) \geq v$. The *tensor product value* of $\mathfrak{G}$ is defined as

$$\mathrm{val}^*(\mathfrak{G}) = \sup_S \mathrm{val}^*(\mathfrak{G}, S) \; ,$$

where the supremum is taken over all tensor product strategies S for $\mathfrak{G}$.

Remark 1.4. Unless specified otherwise, all strategies considered in this paper are tensor product strategies, and we simply call them *strategies*. Similarly, we refer to $\mathrm{val}^*(\mathfrak{G})$ as the *value* of the game $\mathfrak{G}$.

Definition 1.5 (Projective strategies)**.** We say that a strategy $S = (|\psi\rangle, A, B)$ is *projective* if all the measurements $\{A^x_a\}_a$ and $\{B^y_b\}_b$ are projective.

Remark 1.6. A game $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ is *symmetric* if the question and answer alphabets are the same for both players (i.e. $\mathcal{X} = \mathcal{Y}$ and $\mathcal{A} = \mathcal{B}$), the distribution μ is symmetric (i.e. $\mu(x, y) = \mu(y, x)$), and the decision predicate D treats both players symmetrically (i.e. for all $x, y, a, b,\ D(x, y, a, b) = D(y, x, b, a)$). Furthermore, we call a strategy $S = (|\psi\rangle, A, B)$ *symmetric* if $|\psi\rangle$ is a state in $\mathcal{H} \otimes \mathcal{H}$, for some Hilbert space $\mathcal{H}$, that is invariant under permutation of the two factors, and the measurement operators of both players are identical. We specify symmetric games $\mathfrak{G}$ and symmetric strategies S using a more compact notation: we write $\mathfrak{G} = (\mathcal{X}, \mathcal{A}, \mu, D)$ and $S = (|\psi\rangle, M)$ where M denotes the set of measurement operators for both players.

Lemma 1.7. *Let $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ be a symmetric game such that $\mathrm{val}^*(\mathfrak{G}) = 1 - \varepsilon$ for some $\varepsilon \geq 0$. Then there exists a symmetric and projective strategy $S = (|\psi\rangle, M)$ such that $\mathrm{val}^*(\mathfrak{G}, S) \geq 1 - 2\varepsilon$.*

Proof. By definition there exists a strategy $S' = (|\psi'\rangle, A, B)$ such that $\mathrm{val}^*(\mathfrak{G}, S') \geq 1 - 2\varepsilon$. Using Naimark's theorem (for example as formulated in [**?**, Theorem 4.2]) we can assume without loss of generality that $|\psi'\rangle \in \mathbb{C}^d_{A'} \otimes \mathbb{C}^d_{B'}$ for some integer d and that for every x and y, A^x and B^y is a projective measurement. Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle_A|1\rangle_B|\psi'\rangle_{A'B'} + |1\rangle_A|0\rangle_B|\psi'_\tau\rangle_{A'B'}\big) \in (\mathbb{C}^2_A \otimes \mathbb{C}^d_{A'}) \otimes (\mathbb{C}^2_B \otimes \mathbb{C}^d_{B'})\ ,$$

where $|\psi'_\tau\rangle$ is obtained from $|\psi\rangle$ by permuting the two players' registers. Observe that $|\psi\rangle$ is invariant under permutation of AA' and BB'.

For any question $x \in \mathcal{X} = \mathcal{Y}$, define the measurement $M^x = \{M^x_a\}_{a\in\mathcal{A}}$ acting on the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^d$ as follows:

$$M^x_a = |0\rangle\langle 0| \otimes A^x_a + |1\rangle\langle 1| \otimes B^x_a\ .$$

When Alice receives question x, she measures M^x on registers AA', and when Bob receives question y, he measures M^y on registers BB'. Using that by assumption the decision predicate D for $\mathfrak{G}$ is symmetric, it is not hard to verify that $\mathrm{val}^*(\mathfrak{G}, S) = \mathrm{val}^*(\mathfrak{G}, S')$. □

Definition 1.8. Let $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ be a game, and let $S = (|\psi\rangle, A, B)$ be a strategy for $\mathfrak{G}$ such that the spaces $\mathcal{H}_A \simeq \mathcal{H}_B$ canonically. Let $S \subseteq \mathcal{X} \times \mathcal{Y}$ denote the support of the question distribution μ, i.e. the set of (x, y) such that $\mu(x, y) > 0$. We say that S is a *commuting strategy for* $\mathfrak{G}$ if for all question pairs $(x, y) \in S$, we have $[A^x_a, B^y_b] = 0$ for all $a \in \mathcal{A}, b \in \mathcal{B}$, where $[A, B] = AB - BA$ denotes the commutator.

Definition 1.9 (Consistent measurements)**.** Let $\mathcal{A}$ be a finite set, let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ a state, and $\{M_a\}_{a\in\mathcal{A}}$ a projective measurement on $\mathcal{H}$. We say that $\{M_a\}_{a\in\mathcal{A}}$ *is consistent on* $|\psi\rangle$ if and only if

$$\forall a \in \mathcal{A}\ , \quad M_a \otimes \mathrm{Id}_B\ |\psi\rangle = \mathrm{Id}_A \otimes M_a\ |\psi\rangle\ .$$

Definition 1.10 (Consistent strategies)**.** Let $S = (|\psi\rangle, A, B)$ be a projective strategy with state $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$, for some Hilbert space $\mathcal{H}$, which is defined on question alphabets $\mathcal{X}$ and $\mathcal{Y}$ and answer alphabets $\mathcal{A}$ and $\mathcal{B}$, respectively. We say that the strategy S is *consistent* if for all $x \in \mathcal{X}$, the measurement $\{A^x_a\}_{a\in\mathcal{A}}$ is consistent on $|\psi\rangle$ and if for all $y \in \mathcal{Y}$, the measurement $\{B^y_b\}_{b\in\mathcal{B}}$ is consistent on $|\psi\rangle$.

Definition 1.11. We say that a strategy S for a game $\mathfrak{G}$ is *PCC* if it is projective, consistent, and commuting for $\mathfrak{G}$. Additionally, we say that a PCC strategy S is *SPCC* if it is furthermore symmetric.

Remark 1.12. A strategy $S = (|\psi\rangle, A, B)$ for a symmetric game $\mathfrak{G} = (\mathcal{X}, \mathcal{X}, \mathcal{A}, \mathcal{A}, \mu, D)$ is called *synchronous* if it holds that for every $x \in \mathcal{X}$ and $a \neq b \in \mathcal{A}$, $\langle\psi|A_a^x \otimes B_b^x|\psi\rangle = 0$; in other words, the players never return different answers when simultaneously asked the same question. As shown in [**?**] the condition for a finite-dimensional strategy of being synchronous is equivalent to the condition that it is projective, consistent, and moreover $|\psi\rangle$ is a maximally entangled state. (The equivalence is extended to infinite-dimensional strategies, as well as correlations induced by limits of finite-dimensional strategies, in [**?**].)

Definition 1.13 (Entanglement requirements of a game)**.** For all games $\mathfrak{G}$ and $\nu \in [0,1]$, let $\mathcal{E}(\mathfrak{G}, \nu)$ denote the minimum integer d such that there exists a finite dimensional tensor product strategy S that achieves success probability at least ν in the game $\mathfrak{G}$ with a state $|\psi\rangle$ whose Schmidt rank is at most d. If there is no finite dimensional strategy that achieves success probability ν, then define $\mathcal{E}(\mathfrak{G}, \nu)$ to be ∞.

2. Distance measures

We introduce several distance measures that are used throughout.

Definition 2.1 (Distance between states)**.** Let $\{|\psi_n\rangle\}_{n\in\mathbb{N}}$ and $\{|\psi'_n\rangle\}_{n\in\mathbb{N}}$ be two families of states in the same space $\mathcal{H}$. For some function $\delta : \mathbb{N} \to [0,1]$ we say that $\{|\psi_n\rangle\}$ and $\{|\psi'_n\rangle\}$ are *δ-close*, denoted as $|\psi\rangle \approx_\delta |\psi'\rangle$, if $\||\psi_n\rangle - |\psi'_n\rangle\|^2 = O(\delta(n))$. (For convenience we generally leave the dependence of the states and δ on the indexing parameter n implicit.)

Definition 2.2 (Consistency between POVMs)**.** Let $\mathcal{X}$ be a finite set and μ a distribution on $\mathcal{X}$. Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a quantum state, and for all $x \in \mathcal{X}$, $\{A_a^x\}$ and $\{B_a^x\}$ POVMs. We write

$$A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$$

on state $|\psi\rangle$ and distribution μ if

$$\mathop{\mathbb{E}}_{x\sim\mu} \sum_{a\neq b} \langle\psi|A_a^x \otimes B_b^x|\psi\rangle \leq O(\delta) \ .$$

In this case, we say that $\{A_a^x\}$ and $\{B_a^x\}$ are *δ-consistent* on $|\psi\rangle$.

Note that a consistent measurement according to Definition 1.9 is 0-consistent with itself, under the singleton distribution, according to Definition 2.2 (and vice-versa).

Definition 2.3 (Distance between POVMs)**.** Let $\mathcal{X}$ be a finite set and μ a distribution on $\mathcal{X}$. Let $|\psi\rangle \in \mathcal{H}$ be a quantum state, and for all $x \in \mathcal{X}$, $\{M_a^x\}$ and $\{N_a^x\}$ two POVMs on $\mathcal{H}$. We say that $\{M_a^x\}$ *and* $\{N_a^x\}$ *are δ-close on state* $|\psi\rangle$ *and under distribution* μ if

$$\mathop{\mathbb{E}}_{x\sim\mu} \sum_a \|(M_a^x - N_a^x)|\psi\rangle\|^2 \leq O(\delta) \ ,$$

and we write $M_a^x \approx_\delta N_a^x$ to denote this when the state $|\psi\rangle$ and distribution μ are clear from context. This distance is referred to as the *state-dependent* distance.

Definition 2.4 (Distance between strategies)**.** Let $\mathfrak{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mu, D)$ be a nonlocal game and let $S = (\psi, A, B)$, $S' = (\psi', A', B')$ be strategies for $\mathfrak{G}$. For $\delta \in [0,1]$ we say that S *is δ-close to* S' if the following conditions hold.

(1) The states $|\psi\rangle, |\psi'\rangle$ are states in the same Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and are δ-close.
(2) For all $x \in \mathcal{X}, y \in \mathcal{Y}$, we have $A^x_a \approx_\delta (A')^x_a$ and $B^y_b \approx_\delta (B')^y_b$, with the approximations holding under the distribution μ, and on either $|\psi\rangle$ or $|\psi'\rangle$.

We record several useful facts about the consistency measure and the state-dependent distance without proof. Readers are referred to Sections 4.4 and 4.5 in [**?**] for additional discussion and proofs.

Fact 2.5 (Fact 4.13 and Fact 4.14 in [**?**])**.** *For POVMs $\{A^x_a\}$ and $\{B^x_a\}$, the following hold.*

(1) *If $A^x_a \otimes I_B \simeq_\delta I_A \otimes B^x_a$ then $A^x_a \otimes I_B \approx_\delta I_A \otimes B^x_a$.*
(2) *If $A^x_a \otimes I_B \approx_\delta I_A \otimes B^x_a$* and *$\{A^x_a\}$ and $\{B^x_a\}$ are projective measurements, then $A^x_a \otimes I_B \simeq_\delta I_A \otimes B^x_a$.*
(3) *If $A^x_a \otimes I_B \approx_\delta I_A \otimes B^x_a$ and either $\{A^x_a\}$ or $\{B^x_a\}$ is a projective measurement, then $A^x_a \otimes I_B \simeq_{\delta^{1/2}} I_A \otimes B^x_a$.*

Fact 2.6 (Fact 4.20 in [**?**])**.** *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be finite sets, and let D be a distribution over question pairs (x, y). Let $\{A^x_{a,b}\}$ and $\{B^x_{a,b}\}$ be operators whose outcomes range over the product set $\mathcal{A} \times \mathcal{B}$. Suppose a set of operators $\{C^y_{a,c}\}$, whose outcomes range over the product set $\mathcal{A} \times \mathcal{C}$, satisfies the condition $\sum_{a,c} (C^y_{a,c})^\dagger C^y_{a,c} \leq \mathrm{Id}$ for all y. If $A^x_{a,b} \approx_\delta B^x_{a,b}$ on average over x sampled from the corresponding marginal of distribution D, then $C^y_{a,c} A^x_{a,b} \approx_\delta C^y_{a,c} B^x_{a,b}$ on average over (x, y) sampled from D.*

Proof. Fix questions x, y and answers $a \in \mathcal{A}, b \in \mathcal{B}$. We have then that

$$\sum_c \left\| (C^y_{a,c} A^x_{a,b} - C^y_{a,c} B^x_{a,b}) |\psi\rangle \right\|^2 = \sum_c \langle \psi | (A^x_{a,b} - B^x_{a,b})^\dagger (C^y_{a,c})^\dagger (C^y_{a,c}) (A^x_{a,b} - B^x_{a,b}) |\psi\rangle \tag{2.6.1}$$

$$\leq \langle \psi | (A^x_{a,b} - B^x_{a,b})^\dagger (A^x_{a,b} - B^x_{a,b}) |\psi\rangle \tag{2.6.2}$$

$$= \left\| (A^x_{a,b} - B^x_{a,b}) |\psi\rangle \right\|^2 \tag{2.6.3}$$

where the inequality follows from the fact that $\sum_c (C^y_{a,c})^\dagger C^y_{a,c} \leq \sum_{a,c} (C^y_{a,c})^\dagger C^y_{a,c} \leq \mathrm{Id}$. Thus we obtain the desired conclusion

$$\mathop{\mathbb{E}}_{(x,y)\sim D} \sum_{a,b,c} \left\| (C^y_{a,c} A^x_{a,b} - C^y_{a,c} B^x_{a,b}) |\psi\rangle \right\|^2 \leq \mathop{\mathbb{E}}_{(x,y)\sim D} \sum_{a,b} \left\| (A^x_{a,b} - B^x_{a,b}) |\psi\rangle \right\|^2 \leq \delta.$$

□

Fact 2.7. *Let $\mathcal{X}, \mathcal{A}$ denote finite sets, and let $\mathcal{G}$ denote a set of functions $g : \mathcal{X} \to \mathcal{A}$. Let $\{A^x_a\}$, $\{B^x_a\}$ be POVMs indexed by $\mathcal{X}$ and outcomes in $\mathcal{A}$. Let $\{S^x_g\}$ denote a set of operators such that for all $x \in \mathcal{X}$, $\sum_g (S^x_g)^\dagger S^x_g \leq \mathrm{Id}$. If $A^x_a \approx_\delta B^x_a$ on average over x, then $S^x_g A^x_{g(x)} \approx_\delta S^x_g B^x_{g(x)}$.*

Proof. We expand:

$$\begin{aligned}
\mathop{\mathbb{E}}_x \sum_g \left\| S_g^x (A_{g(x)}^x - B_{g(x)}^x)|\psi\rangle \right\|^2 &= \mathop{\mathbb{E}}_x \sum_g \langle\psi|(A_{g(x)}^x - B_{g(x)}^x)^\dagger (S_g^x)^\dagger S_g^x (A_{g(x)}^x - B_{g(x)}^x)|\psi\rangle \\
&= \mathop{\mathbb{E}}_x \sum_a \langle\psi|(A_a^x - B_a^x)^\dagger \Big(\sum_{g:g(x)=a} (S_g^x)^\dagger S_g^x \Big)(A_a^x - B_a^x)|\psi\rangle \\
&\leq \mathop{\mathbb{E}}_x \sum_a \langle\psi|(A_a^x - B_a^x)^\dagger (A_a^x - B_a^x)|\psi\rangle \\
&= \mathop{\mathbb{E}}_x \sum_a \left\|(A_a^x - B_a^x)|\psi\rangle\right\|^2 .
\end{aligned}$$

The inequality follows from the fact that $\sum_{g:g(x)=a} (S_g^x)^\dagger S_g^x \leq \sum_g (S_g^x)^\dagger (S_g^x) \leq \mathrm{Id}$. The last line is at most δ by assumption, and we obtain the desired conclusion. □

Lemma 2.8. *Let $\mathcal{A}$ be a finite set. Let $\{A_a^x\}_{a\in\mathcal{A}}$ be a projective measurement and let $\{B_a^x\}_{a\in\mathcal{A}}$ be a set of matrices. If $A_a^x \approx_\delta B_a^x$, then for all subsets $S \subseteq \mathcal{A}$, we have*

$$\sum_{a\in S} A_a^x \approx_\delta \sum_{a\in S} A_a^x \cdot B_a^x .$$

Proof. We expand:

$$\begin{aligned}
\mathop{\mathbb{E}}_x \left\| \sum_{a\in S} \Big(A_a^x - A_a^x \cdot B_a^x \Big)|\psi\rangle \right\|^2 &= \mathop{\mathbb{E}}_x \left\| \sum_{a\in S} A_a^x \cdot \Big(A_a^x - B_a^x \Big)|\psi\rangle \right\|^2 \\
&= \mathop{\mathbb{E}}_x \sum_{a\in S} \langle\psi|(A_a^x - B_a^x)^\dagger A_a^x (A_a^x - B_a^x)|\psi\rangle \\
&\leq \mathop{\mathbb{E}}_x \sum_a \left\|(A_a^x - B_a^x)|\psi\rangle\right\|^2 .
\end{aligned}$$

In the second line we used the projectivity of $\{A_a^x\}$, and in the third line we used that $A_a^x \leq \mathrm{Id}$. □

Fact 2.9 (Triangle inequality, Fact 4.28 in [?])**.** *If $A_a^x \approx_\delta B_a^x$ and $B_a^x \approx_\epsilon C_a^x$, then $A_a^x \approx_{\delta+\epsilon} C_a^x$.*

Fact 2.10 (Triangle inequality for "$\simeq$", Proposition 4.29 in [?])**.** *If $A_a^x \otimes I_B \simeq_\varepsilon I_A \otimes B_a^x$, $C_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$, and $C_a^x \otimes I_B \simeq_\gamma I_A \otimes D_a^x$, then $A_a^x \otimes I_B \simeq_{\varepsilon+2\sqrt{\delta+\gamma}} I_A \otimes D_a^x$.*

Fact 2.11 (Data processing, Fact 4.26 in [?])**.** *Suppose $A_a^x \otimes I_B \simeq_\delta I_A \otimes B_a^x$. Then $A_{[f(\cdot)=b]}^x \otimes I_B \simeq_\delta I_A \otimes B_{[f(\cdot)=b]}^x$.*

The state-dependent distance is the right tool for reasoning about the closeness of measurement operators in a strategy. The following lemma ensures that, when two families of measurements are close on a state, changing from one family of measurement to the other only introduces a small error to the value of the strategy.

Lemma 2.12. *Let $\{A_{a,b}^x\}$, $\{B_{a,b,c}^x\}$, $\{C_{a,c}^x\}$ be POVMs. Suppose $\{B_{a,b,c}^x\}$ is projective, and*

$$\begin{aligned}
A_{a,b}^x \otimes \mathrm{Id}_B &\approx_\delta \mathrm{Id}_A \otimes B_{a,b}^x , \\
C_{a,c}^x \otimes \mathrm{Id}_B &\approx_\delta \mathrm{Id}_A \otimes B_{a,c}^x .
\end{aligned}$$

Then the following approximate commutation relation holds:

$$[A_{a,b}^x, C_{a,c}^x] \otimes I_B \approx_\delta 0 .$$

Proof. Applying Fact 2.6 to $C^x_{a,\,c} \otimes \mathrm{Id}_B \approx_\delta \mathrm{Id}_A \otimes B^x_{a,\,c}$ and $\{A^x_{a,b} \otimes \mathrm{Id}_B\}$, we have

$$A^x_{a,\,b} C^x_{a,\,c} \otimes \mathrm{Id}_B \approx_\delta A^x_{a,\,b} \otimes B^x_{a,\,c}\,. \tag{2.12.1}$$

Similarly, applying Fact 2.6 to $A^x_{a,\,b} \otimes \mathrm{Id}_B \approx_\delta \mathrm{Id}_A \otimes B^x_{a,\,b}$ and $\{\mathrm{Id}_A \otimes B^x_{a,c}\}$, and using the fact that $\{B^x_{a,b,c}\}$ is projective, we have

$$\begin{aligned} A^x_{a,\,b} \otimes B^x_{a,\,c} &\approx_\delta \mathrm{Id}_A \otimes B^x_{a,\,c} B^x_{a,\,b} \\ &= \mathrm{Id}_A \otimes B^x_{a,\,b,\,c}\,. \end{aligned} \tag{2.12.2}$$

Combining Equations (2.12.1) and (2.12.2), we have

$$A^x_{a,\,b} C^x_{a,\,c} \otimes \mathrm{Id}_B \approx_\delta \mathrm{Id}_A \otimes B^x_{a,\,b,\,c}\,. \tag{2.12.3}$$

A similar argument gives

$$C^x_{a,\,c} A^x_{a,\,b} \otimes \mathrm{Id}_B \approx_\delta \mathrm{Id}_A \otimes B^x_{a,\,b,\,c}\,. \tag{2.12.4}$$

The claim follows from Equations (2.12.3) and (2.12.4). □

The following lemma is a slightly modified version of [**?**, Fact 4.34].

Lemma 2.13. *Let $k \ge 0$ be an integer and let $\varepsilon > 0$. Let $\mathcal{X}$ be a finite set and μ a distribution over $\mathcal{X}$. For each $1 \le i \le k$ let $\mathcal{G}_i$ be a set of functions $g_i : \mathcal{Y} \to \mathcal{R}_i$ and for each $x \in \mathcal{X}$ let $\{G^{i,\,x}_g\}_{g \in \mathcal{G}_i}$ be a projective measurement. Suppose that for all $i \in \{1, \dots, k\}$, $\mathcal{G}_i$ satisfies the following property: for any two $g_i \ne g'_i \in \mathcal{G}_i$, the probability that $g_i(y) = g'_i(y)$ over a uniformly random $y \in \mathcal{Y}$ is at most ε.*

Let $\left\{A^x_{g_1,\,g_2,\,\dots\,,\,g_k}\right\}$ be a projective measurement with outcomes $(g_1, \dots, g_k) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_k$. For each $1 \le i \le k$, suppose that on average over $x \sim \mu$ and $y \in \mathcal{Y}$ sampled uniformly at random,

$$A^x_{[\mathrm{eval}_y(\cdot)_i = a_i]} \otimes I_B \simeq_\delta I_A \otimes G^{i,\,x}_{[\mathrm{eval}_y(\cdot) = a_i]}\,. \tag{2.13.1}$$

Define the POVM family $\{C^x_{g_1,\,g_2,\,\dots\,,\,g_k}\}$, for $x \in \mathcal{X}$, by

$$C^x_{g_1,\,g_2,\,\dots\,,\,g_k} = G^{k,\,x}_{g_k} \cdots G^{2,\,x}_{g_2}\, G^{1,\,x}_{g_1}\, G^{2,\,x}_{g_2} \cdots G^{k,\,x}_{g_k}\,.$$

Then on average over $x \sim \mu$ and $y \in \mathcal{Y}$ sampled uniformly at random,

$$A^x_{[\mathrm{eval}_y(\cdot) = (a_1,\,a_2,\,\dots,\,a_k)]} \otimes I_B \simeq_{k(\delta+\varepsilon)^{1/2}} I_A \otimes C^x_{[\mathrm{eval}_y(\cdot) = (a_1,\,a_2,\,\dots,\,a_k)]}\,. \tag{2.13.2}$$

Proof. The proof is identical to the one given in [**?**, Fact 4.34], with the only modification needed to insert the dependence on x for all measurements considered. □

Lemma 2.14 (Fact 4.35 in [**?**])**.** *Let $\mathcal{D}$ be a distribution on $(x, y_1, y_2) \in \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2$. For $i \in \{1, 2\}$ let $\mathcal{G}_i$ be a collection of functions $g_i : \mathcal{Y}_i \to \mathcal{R}_i$ and let $\{(G_i)^x_g\}_{g \in \mathcal{G}_i}$ be families of measurements such that $\{(G_2)^x_g\}_g$ is projective for every x. Suppose further that for every (x, y_1) it holds that for $g_2 \ne g'_2 \in \mathcal{G}_2$ the probability, on average over y_2 chosen from $\mathcal{D}$ conditioned on (x, y_1), that $g_2(y_2) = g'_2(y_2)$ is at most η. Let $\{A^{x,y_1,y_2}_{a_1,a_2}\}$ be a family of projective measurements with outcomes $(a_1, a_2) \in \mathcal{R}_1 \times \mathcal{R}_2$ such that for $i \in \{1, 2\}$,*

$$A^{x,y_1,y_2}_{a_i} \otimes \mathrm{Id} \simeq_\delta \mathrm{Id} \otimes (G_i)^x_{[\mathrm{eval}_{y_i}(\cdot) = a_i]} \tag{2.14.1}$$

and

$$A^{x,y_1,y_2}_{a_1,a_2} \otimes \mathrm{Id} \simeq_\delta \mathrm{Id} \otimes A^{x,y_1,y_2}_{a_1,a_2}\,. \tag{2.14.2}$$

Define a family of measurements $\{J^x_{g_1,g_2}\}$ *as*

$$J^x_{g_1,g_2} \;=\; (G_2)^x_{g_2}(G_1)^x_{g_1}(G_2)^x_{g_2}\ . \tag{2.14.3}$$

Then there is a

$$\delta_{pasting} \;=\; \delta_{pasting}(\eta,\delta) \;=\; \mathrm{poly}(\eta,\delta) \tag{2.14.4}$$

such that

$$A^{x,y_1,y_2}_{a_1,a_2} \otimes \mathrm{Id} \simeq_{\delta_p} \mathrm{Id} \otimes J^x_{[\mathrm{eval}_{y_1}(\cdot)=a_1,\mathrm{eval}_{y_2}(\cdot)=a_2]}\ . \tag{2.14.5}$$

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