RELATION PROBLEM SET

Problem Set 7.1

1. If a set a has n elements, how many relations are there from A to A ?
2. If A set has m elements and B has n elements. How many relations are there from A to B and vice-versa ?
3. List the ordered pairs in the relation R from A = {0, 1, 2, 3, 4} to B = {0, 1, 2, 3} where $(a, b) \in R$ if and only if (*i*) $a = b$, (*ii*) $a + b = 3$, (*iii*) $a \times b$, (*iv*) $a \,!\, b$, (*v*) $gcd\,(a, b) = 1$, (*vi*) lcm $(a, b) = 2$.
4. Let $A = Z^+$, the positive integers, and R be the relation defined by $a\,Rb$ if and only if there exists a k in Z^+ so that $a = b^k$. Which of the following belong to R?

   (*a*) (4, 16) (*b*) (1, 3) (*c*) (4, 2) (*d*) (2, 8) (*e*) (4, 4) (*f*) (2, 16)
5. Consider the relation R from X to Y

   X = {1, 2, 3}; Y = {7, 8} and R = {(1, 7), (2, 7), (1, 8), (3, 8)}. Find

   (*i*) R^{-1} (the inverse of R) (*ii*) R′ (complement of R).
6. The relation R on the set {1, 2, 3, 4, 5} is defined by the rule $(x, y) \in R$ if 3 divides $x - y$. Find

   (*i*) the elements of R (*ii*) the elements of R^{-1} (*iii*) the domain of R

   (*iv*) the range of R (*v*) the domain of R^{-1} (*vi*) the range of R^{-1}
7. The relation R on the set {1, 2, 3, 4, 5} defined by the rule $(x, y) \in R$ if $x + y \leq 6$. Find

   (*i*) the elements of R (*ii*) the elements of R^{-1} (*iii*) the domain of R

   (*iv*) the range of R (*v*) the domain of R^{-1} (*vi*) the range of R^{-1}
8. Let R be the relation R = {(a, b) : a divides b} on the set of positive integer. Find (*i*) R^{-1} (*ii*) R′.
9. Let P = {(1, 2), (2, 4), (3, 3)} and Q = {(1, 3), (2, 4), (4, 2)}. Find $P \cup Q$, $P \cap Q$, dom. (P), dom. (Q), dom. $(P \cup Q)$, ran (P), ran (Q) and ran $(P \cap Q)$. Show that

$$\text{dom. } (P \cup Q) = \text{dom. } (P) \cup \text{dom. } (Q)$$
$$\text{ran } (P \cap Q) \subseteq \text{ran } (P) \cap \text{ran } (Q)$$

10. Let A = {2, 4} and B = {6, 8, 10} and define binary relations R and S from A to B as follows:

$$\text{for all } (x, y) \in A \times B, \qquad xRy \Leftrightarrow x \,/\, y$$
$$\text{for all } (x, y) \in A \times B, \qquad xSy \Leftrightarrow y - 4 = x$$

State explicitly which ordered pairs are in $A \times B$, R, S, $R \cup S$ and $R \cap S$.

11. Give example of a relation on the set of positive integers which is
   (i) symmetric and reflexive but not transitive;
   (ii) reflexive and transitive but not symmetric;
   (iii) symmetric, transitive but not reflexive;
   (iv) reflexive but neither symmetric nor transitive;
   (v) neither symmetric nor antisymmetric.
12. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and / or transitive, where $(x, y) \in R$ if and only if
   (a) $x \neq y$ (b) $xy \geq 1$ (c) $x = y \pmod 7$ (d) x is a multiple of y
   (e) $x = y^2$ (f) $x \geq y^2$ (g) $x \leq y + 1$ (h) $|x + y| = 2$
   (i) $y - x + 2$ is a prime number (j) $|y - x| + 2$ is a prime
13. Below is the list of relations among people. For each of the following relations, state whether the relation is reflexive, symmetric, antisymmetric or transitive
   (a) xRy stands for x is a child of y
   (b) xRy stands for x is a spouse of y
   (c) xRy stands for x is a wife of y
   (d) xRy stands for x is a superior of y
   (e) xRy stands for x and y have the same parents.
14. Give an example of a relation that is both symmetric and antisymmetric.
15. Prove that the relation R defined on the set of positive integers $(x, y) \in R$ if $x - y$ is divisible by 5 is an equivalence relation.
16. Which of the relations defined in the set of real numbers are equivalence relation?
   (i) aRb if and only if $|a| = |b|$
   (ii) aRb if and only if $a \geq b$
   (iii) aRb if and only if $|a| > |b|$
17. Show that the relation 'is congruent modulo 4 to' on the set of integers {0, 1, 2, ..., 10} is an equivalence relation.
18. Determine which of the following are equivalence relations and / or partial ordering relations for the given sets.
   (a) A = {lines in the plane}; xRy if and only if x is parallel to y.
   (b) A = {the set of rural numbers}; xRy if and only if $|x - y| \leq 7$.
19. (a) Let $x = \{1, 2, 3, 4\}$. If for $x, y \in X$.
   $R = \{(x, y) : x - y$ is an integral non-zero multiple of $2\}$
   $S = \{(x, y) : x - y$ is an integral non-zero multiple of $3\}$.
   (b) If $x = \{1, 2, 3, ...\}$ what is $R \cap S$ for R and S as defined in (a).
20. Let $A = \{1, 2, 3, 6\}$. If for $x, y \in A$.

$$R = \{(x, y) : x \leq y\}$$
$$S = \{(x, y) : x \text{ divides } y\}$$

   Write R and S as sets and find $R \cap S$.
21. For a set X with n elements, find how many relations on X there are which are
   (a) symmetric
   (b) antisymmetric
   (c) reflexive
   (d) irreflexive
   (e) reflexive and symmetric
   (f) reflexive but not symmetric
   (g) symmetric but not reflexive
   (h) neither reflexive nor irreflexive
22. Show that the relation R on a set A is reflexive if and only if the inverse relation R^{-1} is reflexive.
23. Show that the relation R on the set of all triangles in the plane defined by
   $R = \{(a, b)$: triangle a is similar to triangle $b\}$ is an equivalence relation.

24. Show that the relation R over the set of all straight lines in the plane defined by 'is perpendicular to' is symmetric but neither reflexive nor transitive.

25. Show that the relation 'less than or equal to' on the set of integers is a partial order.

26. Let A = {all words in the English language}. If $x, y \in A$, define $x \sim y$ if and only if x and y have the same number of letters, show that the relation is an equivalence relation and find the equivalence class of the word student.

27. Let $A = \{a, b, c, d\}$, and consider the relation

$$R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}$$

Show that R is a partial ordering.

28. If R be a relation in the set of integers Z defined by

$$R = \{(x, y) : x \in Z, y \in Z, (x - y) \text{ is multiple of } 3\}$$

Show that it is an equivalence relation. What is the equivalence class of 0? How many equivalence classes are there?

29. Let A be the set of non-zero integers and let R be the relation on A × A defined by

$$(a, b)\ R\ (c, d) \Leftrightarrow ad = bc$$

(*a*) Show that R is an equivalence relation

(*b*) Find [(1, 2)], *i.e.*, equivalence class of (1, 2).

30. A relation R is defined on $A = Z^+ \times Z^+$ by $(a, b)\ R\ (c, d) \Leftrightarrow a + b = b + c$.

(*a*) Show that R is an equivalence relation.

(*b*) Find [(2, 5)], *i.e.*, the equivalence class of (2, 5).

(*c*) Interpreting A as the co-ordinate plane, give a geometrical description of the partition of A into equivalence classes.

31. Define ρ on the set R × R of ordered pairs of real numbers as follows. For all $(a, b), (c, d) \in R \times R$.

$$(a, b)\ \rho\ (c, d) \Leftrightarrow a = c$$

(*a*) Show that ρ is an equivalence relation.

(*b*) Describe the distinct equivalence class of ρ. Interpreting R × R as the co-ordinate place, give a geometrical description of the equivalence classes.

32. Let A = {0, 1, 2, 3, 4}. Show that the relation R = {(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)} is an equivalence relation. Find the distinct equivalence classes of R.

33. Let R be the relation of congruence modulo 3. Which of the following equivalence classes are equal?

$$[7], [-4], [-6], [17], [4], [27], [19]$$

34. Let R and S be relations on a set A

(*a*) if R is reflexive, so is R^{-1}.

(*b*) if R and S are reflexive, then so are $R \cap S$ and $R \cup S$.

(*c*) if R is symmetric, so are R^{-1} and R'.

(*d*) if R and S are symmetric, so are $R \cap S$ and $R \cup S$.

35. If R and S be two relations from A to B. Show that

(*i*) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ (*ii*) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

36. Let A = {1, 2, 3} and B = {1, 2, 3, 4}. Let R and S be two relations from A to B defined by R = {(1, 1), (2, 2), (3, 3)} and S = {(1, 1), (1, 2), (1, 3), (1, 4)}. Find (*i*) $R \cup S$ (*ii*) $R \cap S$ (*iii*) R^{-1} (*iv*) S^{-1} and show that $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.

37. Let $A = \{a, b, c, d\}$ and $R = \{(a, b), (a, a), (b, a), (b, b), (c, c), (d, d), (d, e), (d, e), (e, d), (e, e)\}$ and $S = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, c), (c, d), (d, e), (e, d)\}$ be the equivalence relations on A. Determine the partitions corresponding to followings (*i*) R^{-1} (*ii*) $R \cap S$.

ANSWERS 7.1

1. 2^n

2. $2^{m \times n}$

3. (*i*) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

   (*ii*) $\{(0, 3), (1, 2), (2, 1), (3, 0)\}$

   (*iii*) $\{(1, 0), (2, 0), (3, 0), (4, 0), (2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$

   (*iv*) $\{(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0), (3, 3), (4, 0)\}$

   (*v*) $\{(0, 1), (1, 0), (1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3)\}$

   (*vi*) $\{(1, 2), (2, 1), (2, 2)\}$

4. (*a*) No, No, Yes, No, Yes, No

5. (*i*) $R^{-1} = \{(7, 1), (7, 2), (8, 1), (8, 3)\}$

   (*ii*) $R' = \{(3, 7), (2, 8)\}$

6. (*i*) $R = \{(1, 1), (1, 4), (2, 2), (2, 5), (3, 3), (4, 1), (4, 4), (5, 2), (5, 5)\}$

   (*ii*) $R^{-1} = \{(1, 1), (4, 1), (2, 2), (5, 2), (3, 3), (1, 4), (4, 4), (2, 5), (5, 5)\}$

(iii) dom (R) = {1, 2, 3, 4, 5}

(iv) ran (R) = {1, 2, 3, 4, 5}

(v) dom (R^{-1}) = dom (R)

(vi) ran (R^{-1}) = ran (R)

7. $R = R^{-1}$ = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1)}

dom R = ran R = dom R^{-1} = ran R^{-1} = {1, 2, 3, 4, 5}

8. (i) R = {(a, b) : b divides a}

(ii) R = {(a, b) : a does not divide b}

9. P ∪ Q = {(1, 2), (1, 3), (2, 4), (3, 3), (4, 2)}

P ∩ Q = {(2, 4)}, dom P = {1, 2, 3}, dom Q = {1, 2, 4}, dom (P ∪ Q) = {1, 2, 3, 4} ran (P) = (2, 3, ran (Q) = {2, 3, 4} ran (P ∩ Q) = {4}.

10. A × B = {(2, 6), (2, 8), (2, 10), (4, 6), (4, 8), (4, 10)}

R = {(2, 6), (2, 8), (2, 10), (4, 8)}, S = {(2, 6), (4, 8)} R ∪ S = R, R ∩ S = S

11. (i) R = {(1, 1), (2, 2), (3, 3), (2, 3), (3, 2), (1, 2), (2, 1)} in A = {1, 2, 3}

(ii) Let A = {1, 2, 3, ...}, R is defined by the 'x is a divisor of y' in A

(iii) R = {(1, 1), (2, 2), (1, 2), (2, 1)} in A = {1, 2, 3}

(iv) R = {(1, 1), (1, 2), (2, 2), (3, 3), (2, 3)} in A = {1, 2, 3}

(v) R = {(a, b), (a, c), (c, a)} in A = {a, b, c}

12. (a) symmetric (b) symmetric, transitive (c) reflexive, symmetric, transitive

(d) reflexive, transitive (e) antitransitive (f) antisymmetric, transitive

(g) reflexive (h) symmetric (i) reflexive

(j) reflexive, symmetric

13. (a) antisymmetric (b) symmetric (c) antisymmetric

(d) antisymmetric, transitive (e) reflexive, symmetric, transitive

14. The identity relation is both symmetric and antisymmetric. More generally, any relation in which xRy true only if $x = y$ is both symmetric and antisymmetric. A = {a, b, c}, S = {(a, a), (b, b)}.

16. (i) Equivalence (ii) Not (iii) Not

18. (a) It is an equivalence relation but not a partial ordering relation since R is not antisymmetric.

(b) Not transitive, therefore, it is neither.

19. (a) R = {(1, 3), (3, 1), (2, 4), (4, 2)}, S = {(1, 4), (4, 1)}, R ∪ S = {(1, 3), (3, 1), (2, 4), (4, 2), (1, 4), (4, 1)}, R ∩ S = φ

(b) R ∩ S = {(x, y) : x – y is a non-zero multiple of 6}

20. R = {(1, 1), (1, 2), (1, 3), (1, 6), (2, 2), (2, 3), (2, 6), (3, 3), (3, 6), (6, 6)}, S = {(1, 2), (1, 3), (1, 6), (2, 6), (3, 6)}, R ∩ S = {(1, 2), (1, 3), (1, 6), (2, 6), (3, 6)} = S

21. (a) $2^{(n^2+n)/2}$ (b) $2^n \cdot 3^{n(n-1)/2}$ (c) 2^{n^2-n} (d) $2^{(n^2-n)}$

(e) $2^{(n^2-n)/2}$ (f) $2^{n^2-n} - 2^{(n^2-n)/2}$ (g) $2^{(n^2+n)/2} - 2^{(n^2-n)/2}$ (h) $2^{n^2} - 2 \cdot 2^{n^2-n}$

28. [0] = {0, ± 3, ± 6, ± 9, ± 12, ...}. There are three equivalence classes [0], [1] and [2].

29. [(1, 2)] = {(1, 2), (– 1, – 2), (2, 4), (– 2, – 4), (3, 6), (– 3, – 6), ...}.

30. (b) [(2, 5)] = {(1, 4), (2, 5), (3, 6), (4, 7), ...}. (c) The straight line of gradient 1

31. The distinct equivalence classes are all sets of ordered pairs {(x, y) ∈ R × R / x = a} for each real number a. The equivalence classes consist of all vertical lines in the cartesian plane.

33. [7] = [4] = [19], [– 4] = [17], [– 6] = [27].

36. (i) {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)} (ii) {(1, 1)}

(iii) {(1, 1), (2, 2), (3, 3)} (iv) {(1, 1), (2, 1), (3, 1), (4, 1)}

37. (i) {(a, b), (c), (d, e)} (ii) {(a), (b), (c), (d, e)}