E-402-STFO PROBLEMS FOR MODULE A

CREATED BY HENNING ULFARSSON

These problems deal with algorithmic solutions to common problems in multivariable calculus. You get a perfect score for this module by getting 75 points or more.

1. The Frenet-Serret frame (20 points)

Problem 1 (10 points). Let $a(t), b(t), c(t)$ be three functions and define the curve $\mathcal{C}$ by

$$\vec{\mathbf{r}}(t) = a(t)\hat{\mathbf{i}} + b(t)\hat{\mathbf{j}} + c(t)\hat{\mathbf{k}}.$$

The *Frenet-Serret frame* is defined as the following three vectors

$$\hat{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{||\vec{\mathbf{r}}'(t)||}$$
$$\hat{\mathbf{N}}(t) = \frac{\hat{\mathbf{T}}'(t)}{||\hat{\mathbf{T}}'(t)||}$$
$$\hat{\mathbf{B}}(t) = \hat{\mathbf{T}}(t) \times \hat{\mathbf{N}}(t)$$

Write a function `mAp1(r,t0)` that inputs a vector `r` with three coordinates, each containing a function of the variable `t`, as well as a real number `t0`. The function should output the Frenet-Serret frame, at the point on the curve corresponding to $t = t0$, as a 3-tuple of vectors. For example:

- Input: `mAp1(vector([t,t**2,t**3]),0`

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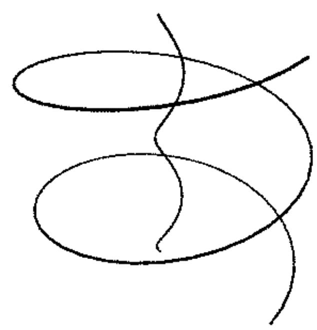

Figure 1. Figure for Problem (1)

- When we calculate the Frenet-Serret with the formulas above we get a complicated looking triple of vectors that simplify to $T = (1, 0, 0)$, $N = (0, 1, 0)$, $B = (0, 0, 1)$ when we plug in $t = 0$.
- Output: `((1, 0, 0), (0, 1, 0), (0, 0, 1))`

Problem 2 (10 points)**.** This problem is a continuation of Problem 1. The *turning radius* of the curve is defined as

$$\rho(t) = \frac{||\vec{\mathbf{r}}'(t)||^3}{||\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)||}$$

We then define the *evolute* of the curve $\vec{\mathbf{r}}(t)$ as

$$\vec{\mathbf{r}}_c(t) = \vec{\mathbf{r}}(t) + \rho(t)\hat{\mathbf{T}}(t)$$

Write a function `mAp2(r,t0)` that inputs a vector `r` with three coordinates, each containing a function of the variable `t`, as well as a real number `t0`. The function should output the the point on the evolute corresponding to $t = t0$. For example:

- Input: `mAp2(vector([t,t**2,t**3]),0`
- Output: `(1/2, 0, 0))`

2. Analysing critical points (40 points)

Problem 3 (10 points)**.** Let $f(x, y)$ be a function with first partial derivatives. Write a function `mAp3(f)` that inputs a function `f` of two variables `x` and `y`. The function should output the critical points of the function in a list ordered first by the first coordinate, and then by the second. For example:

- Input: `mAp3(x**3 + y**3 - 3*x*y)`
- The gradient of this function is $(3x^2 - 3y, 3y^2 - 3x)$ which is $(0, 0)$ at the points $(0, 0)$, $(1, 1)$
- Output: `(0, 0), (1, 1)`

Problem 4 (10 points)**.** This problem is a continuation of Problem 3. Write a function `mAp4(f,P,u)` that inputs a function `f`, as before, a point `P` in the plane, and a unit vector `u`. The function should output the directional derivative of the function in the direction of the vector at the given point. For example:

- Input: `mAp4(y**4+2*x*y**3+x**2*y**2,(0,1),vector([1,2])/sqrt(5))`
- Output: `2*sqrt(5)`

Problem 5 (20 points)**.** Use the functions from the previous two problems to write a function `mAp5(f)` that is able to do "elementary analysis of local minima, maxima and saddle points" of the function `f`. The function should input a function $f(x, y)$ as before and operate as follows: First the function (or you) chooses a small $\epsilon > 0$ and an integer $n > 0$. For each critical point P the function should try to guess whether that point is a local maximum, a local minimum or a saddle point, by choosing points $P_1 = (x_1, y_1), P_2 = (x_2, y_2), \ldots, P_n = (x_n, y_n)$ on a circle with radius ϵ and center P, calculate the directional derivatives $D_{\vec{u}_n} f(P_n)$ where $\vec{u}_n$ is a unit vector with the same direction as $\vec{PP_n}$ and infer that "the function f has a local minum if [students fill in here]", "has a local maximum if [students fill in here]" and "has a saddle point if [students fill in here]".

The function should output a list of of tuples, where each tuple has two coordinates, the first containing the critical point, and the second containing 1 for a local maximum, -1 for a local minimum and 0 for a saddle point. For example:

- Input: `mAp5(x**3 + y**3 - 3*x*y)`
- Output: `[((0, 0), 0), ((1, 1), -1)]`

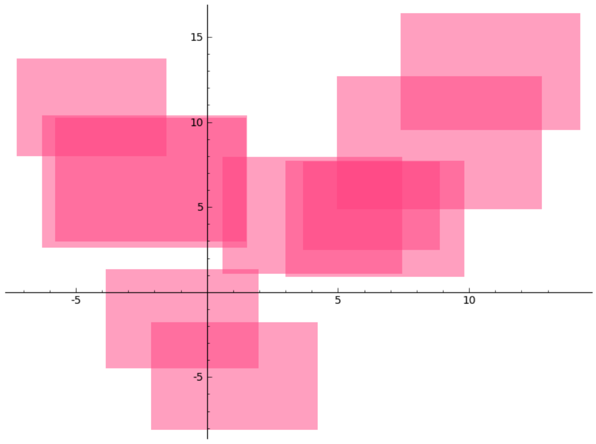

FIGURE 2. Figure for Problem (6)

3. Simply connected (40 points)

Problem 6 (10 points)**.** Write a function `mAp6(Cs,Ls)` that inputs two equally long lists. The first list contains coordinates in the form `(x,y)`, where `x` and `y` are real numbers. The second list contains positive real numbers. The coordinates `Cs[i]` in the list `Cs` specify a the lower left corner of a square in the plane with side lengths `Ls[i]`. The function should output whether the union of the squares is connected or not. For example:

- Input: `mAp6([(-2.1, -8.1), (0.59, 1.1), (-3.9, -4.5), (3.7, 2.5), (5.0, 4.9), (-6.3, 2.6), (3.0, 0.91), (-7.3, 8.0), (7.4, 9.5), (-5.8, 3.0)], [6.3, 6.9, 5.8, 5.2, 7.8, 7.8, 6.8, 5.7, 6.9, 7.3])`
- The squares corresponding to the input are shown in Figure 2.
- Output: `True`

Problem 7 (30 points)**.** This problem is a continuation of Problem 6. Write a function `mAp7(Cs,Ls)` that inputs two equally long lists of the same type as before. The function should output whether the union of the squares is simply-connected or not. For example:

- Input: `mAp6([(-2.1, -8.1), (0.59, 1.1), (-3.9, -4.5), (3.7, 2.5), (5.0, 4.9), (-6.3, 2.6), (3.0, 0.91), (-7.3, 8.0), (7.4, 9.5), (-5.8, 3.0)], [6.3, 6.9, 5.8, 5.2, 7.8, 7.8, 6.8, 5.7, 6.9, 7.3])`
- The squares corresponding to the input are shown in Figure 2.
- Output: `True`

School of Computer Science, Reykjavik University, Menntavegi 1, 101 ReykjavÃdk, Iceland
E-mail address: `henningu@ru.is`