# MODAL ANALYSIS OF DAMPED OSCILLATOR

## INTRODUCTION

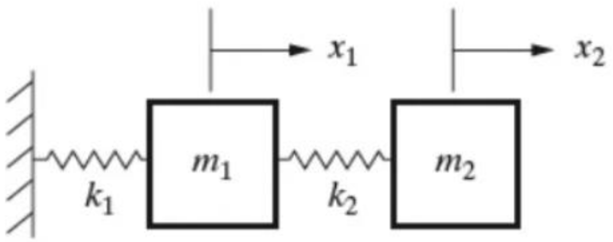
Damped harmonic oscillators are vibrating systems for which the amplitude of vibration decreases over time. Since nearly all physical systems involve considerations such as air resistance, friction, and intermolecular forces where energy in the system is lost to heat or sound, accounting for damping is important in realistic oscillatory systems.

Real Life Examples includes Yo-yo, clock pendulum, or guitar string: after starting the yo-yo, clock, or guitar string vibrating, the vibration slows down and stops over time, corresponding to the decay of sound volume or amplitude in general.

When it is a not necessity for amplitude of vibrations to die down instantaneously a damped oscillator with damping coefficient $(\xi)$ $0<\xi<1$ is used.

This is a study of damped oscillator on varying initial conditions and the damping ratios.

**Free Vibration of Undamped System**




Equation of motion

$$m_1\ddot{x_1} + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x_2} + k_2(x_2 - x_1) = 0$$

Matrix form $$\begin{bmatrix} m1 & 0 \\ 0 & m2 \end{bmatrix}\ddot{x} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}x = 0$$

Response of the system

$$x(t) = \sum_{i=1}^{2} \{\hat{u}\}_i \left(\{\hat{u}\}_i^T[M]\{x(0)\}\cos(w_i t) + \frac{\{\hat{u}\}_i^T[M]\{\dot{x(0)}\}\sin(w_i t)}{w_i}\right)$$

## Graphs

Effect of various parameters on mode shapes:

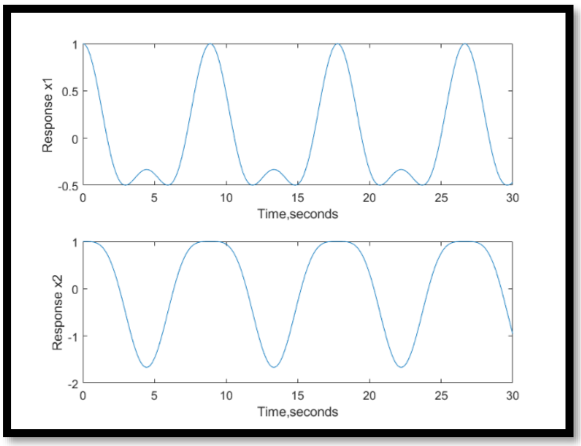
$$x(0) = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\dot{x(0)} = \begin{Bmatrix} \dot{x_1} \\ \dot{x_2} \end{Bmatrix}$$

**Case 1** Response to initial displacement

$$x(0) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

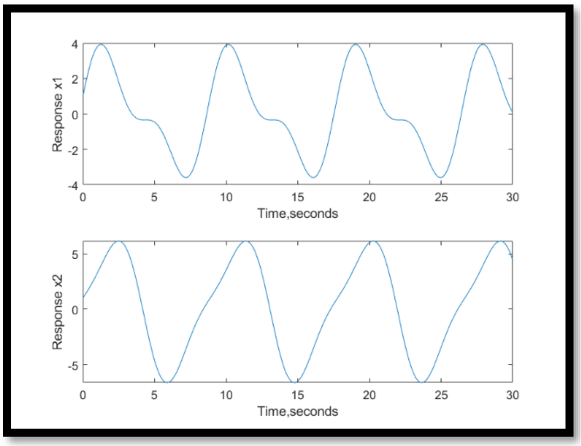
$$\dot{x}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$




**Case 2** Response to initial displacement and initial velocity

$$x(0) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

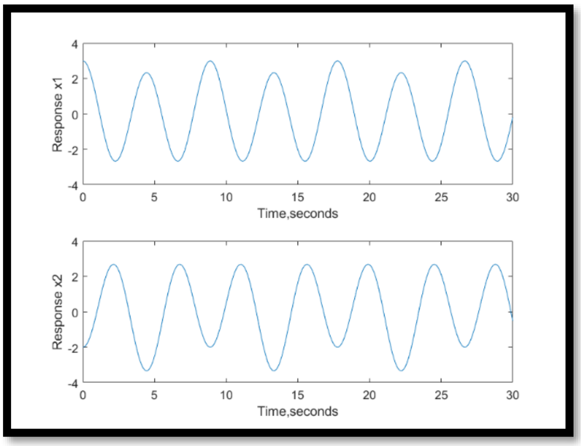
$$\dot{x}(0) = \begin{Bmatrix} 4 \\ 2 \end{Bmatrix}$$



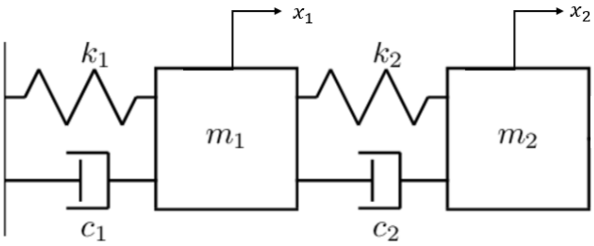
**Case 3** Response to initial displacement (Opposite Direction)

$x(0) = \begin{Bmatrix} 3 \\ -2 \end{Bmatrix}$

$\dot{x(0)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$




**Free Vibration of Damped System**




Equation of motion

$$m_1\ddot{x_1} + (c_1 + c_2)\dot{x_1} - c_2\dot{x_2} - k_2x_2 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x_2} + c_2(\dot{x_2} - \dot{x_1}) + k_2(x_2 - x_1) = 0$$

Matrix form $$\begin{bmatrix} m1 & 0 \\ 0 & m2 \end{bmatrix}\ddot{x} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ c_2 & c_2 \end{bmatrix}\dot{x} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}x = 0$$

$$[m]\ddot{x} + [c]\dot{x} + [k]x = 0$$

Here, damping matrix can be expressed as a linear combination of mass and stiffness matrices

$$c \ = \ \alpha[m] + \ \beta[k]$$

Where α and β are constants and c are Rayleigh's proportional damping

$$[m]\ddot{x} + [\alpha[m] + \ \beta[k]]\dot{x} + [k]x = 0$$

Solution vector x is the linear combination of natural modes of undamped system

$$x(t) = [X]q(t)$$

Where q is the Modal participation coefficients

$$[X]^T[m][X]\ddot{q(t)} + [\alpha[X]^T[m][X] + \beta[X]^T[k][X]]\dot{q}(t) + [X]^T[k][X]q(t) = 0$$

The above expression results to

$$\ddot{q(t)} + (\alpha + {w_i}^2\beta)\dot{q} + {w_i}^2 q = 0$$

Where ${w_i}^2$ is the $i^{th}$ natural frequency of the undamped system

$$\alpha + {w_i}^2\beta = 2\xi_i w_i$$

Where $\xi$ is the modal damping ratio of the $i^{th}$ normal mode.

The final expression is

$$\ddot{q(t)} + 2\xi_i w_i \dot{q(t)} + {w_i}^2 q(t) = 0$$

The solution of the equation is

$$q(t) = A_i * e^{-\xi_i w_i t} \sin{(w_{di}t + \Phi)}$$

On Substituting,

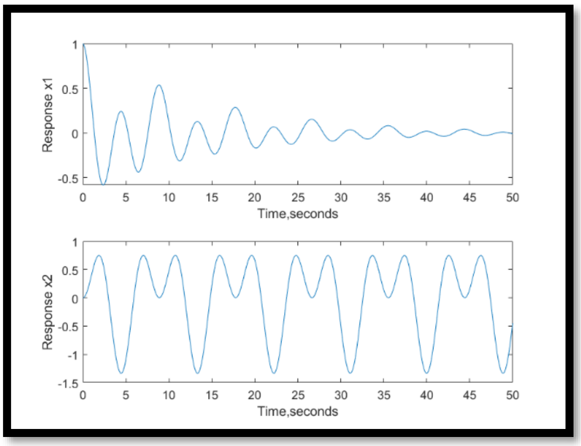
Amplitude of the system:

$$A_i(t) = \sum_{i=1}^{2} \{\hat{u}\}_i \left(\frac{\{\hat{u}\}_i^T[M]\{q(0)\}\cos(w_{di}t)}{\sqrt{1-\xi^2}} + \frac{\{\hat{u}\}_i^T[M]\{\dot{q(0)}\}\sin(w_{di}t)}{w_{di}}\right)$$

**Effect of various parameters on mode shapes:**

**Case1:** Response to Initial Displacement and no velocity

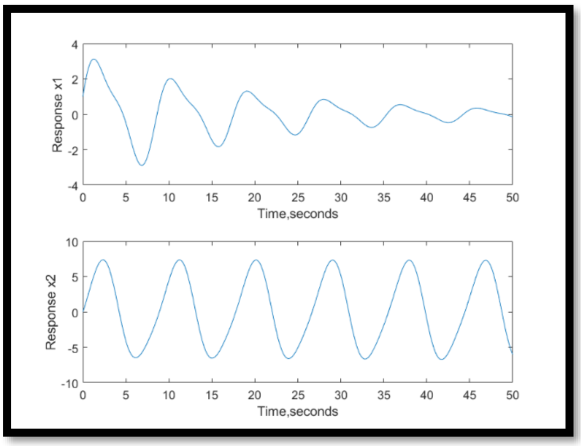
$$x(0) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$
$$\dot{x(0)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$




**Case2:** Response to Initial Displacement and initial velocity

$$x(0) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$
$$\dot{x(0)} = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$$

$$\xi = \begin{Bmatrix} 0.05 \\ 0.1 \end{Bmatrix}$$

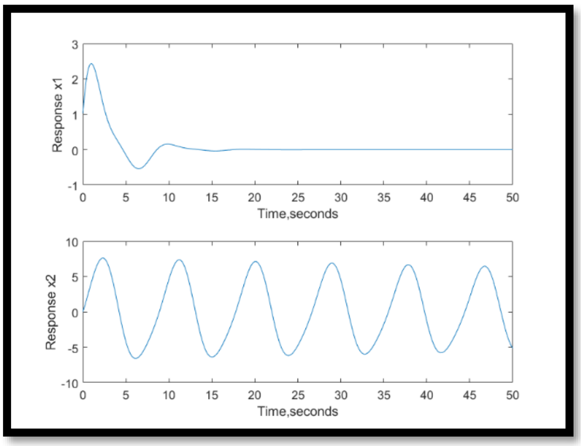


**Case3:** Changing of damping ratio

$x(0) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

$\dot{x(0)} = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$

$\xi = \begin{Bmatrix} 0.3 \\ 0.1 \end{Bmatrix}$




**MATLAB CODE FOR DETERMINING EQUATION OF MOTION**

```
clear
clc

%% create symbolic variables

syms x1 real;
syms x2 real;
syms m1 real;
syms m2 real;
syms x1dot real;
syms x2dot real;
syms x1ddot real;
syms x2ddot real;
syms k1 real;
syms k2 real;
syms c1 real;
syms c2 real;
T = 0.5*(m1*x1dot^2 + m2*x2dot^2 );
U = 0.5*(k1*x1^2 + k2*(x2 - x1)^2 );
D = 0.5*(c1*x1dot^2 + c2*(x2dot- x1dot)^2);
L = T - U;
```

```
dL_dx1dot = diff(L, x1dot);
d_dtdl_dx1dot = diff(dL_dx1dot,x1dot)*x1ddot;
dD_dx1dot = diff(D, x1dot);
dL_dx2dot = diff(L, x2dot);
d_dtdl_dx2dot = diff(dL_dx2dot,x2dot)*x2ddot;
dD_dx2dot = diff(D, x2dot);
dL_dx1 = diff(L, x1);
dL_dx2 = diff(L, x2);
Eqn1 = d_dtdl_dx1dot - dL_dx1+dD_dx1dot
Eqn2 = d_dtdl_dx2dot - dL_dx2+dD_dx2dot
```

**MATLAB CODE FOR MODAL ANALYSIS**

**CASE 1**: UNDAMPED

```
clc;
M=[10 0;0 5]
K=[15 -5;-5 5]
A=inv(M)*K
I=eye(2);
E=eig(A);
X1=adjoint(A-(E(1)*I))
X2=adjoint(A-(E(2)*I))
P(:,1)=X1(:,1)
P(:,2)=X2(:,1)
t1=X1(:,1).'*M*X1(:,1)
t2=X2(:,1).'*M*X2(:,1)
a1=1/(sqrt(t1))
a2=1/(sqrt(t2))
xn1=a1*X1(:,1)
xn2=a2*X2(:,1)
Pn(:,1)=xn1
Pn(:,2)=xn2
Mn=Pn.'*M*Pn
Kn=Pn.'*K*Pn
w1=sqrt(Kn(1)/Mn(1))
w2=sqrt(Kn(4)/Mn(4))
tf=input("Enter Final Time")
t=0:0.1:tf

x0=[3;-2];
v0=[0;0];
```

```
q=tf/0.1;
w=[w1;w2];
y=zeros(size(2,q));

for j=1:2


xt=Pn(:,j)*(Pn(:,j)'*M*x0*cos(w(j).*t)+(Pn(:,j)'*M*v0
*sin(w(j).*t)/w(j)));
    y=y+xt;
end
for i =1:2
    subplot(2,1,i)
    plot(t,y(i,:))
    xlabel('Time,seconds');
    ylabel(['Response x',num2str(i)]);
end
```

**CASE 2:** DAMPED

```
clear;
M=[10 0;0 5];
K=[15 -5;-5 5];
zeta=[0.07;0.03];
[u,l]=eig(K,M)
for s=1:2
    alpha=sqrt(u(:,s)'*M*u(:,s));
    u(:,s)=u(:,s)/alpha;
end
x0=input('Enter the initial displacement column
vector:');
v0=input('Enter the initial velocity column
vector:');
tf=input('Enter the final time:');
t=0:0.1:tf
q=tf/0.1;
x=zeros(size(2,q));
for j=1:2
    w(j)=sqrt(l(j,j));
    wd(j)=w(j)*sqrt(1-zeta(j)^2);
    xt=u(:,j)*(u(:,j)'*M*x0*cos(w(j).*t)/sqrt(1-
zeta(j)^2)+u(:,j)'*M*v0/w(j)*sin(w(j).*t)/wd(j));
    x=x+xt;
```

```
end
for i=1:2
    x(i,:)=x(i,:).*exp(-zeta(i)*x(i).*t);
end
for r=1:2
    subplot(2,1,r)
    plot(t,x(r,:))
    xlabel('Time,seconds');
    ylabel(['Response x',num2str(r)]);
end
```

**CONCLUSION**

The effect of damping on the vibration response of a system containing two degree of freedom was analysed and found that if damping ratio is more the amplitude keeps on decreasing and dies down for one mass system and behaves like undamped vibration in other mass system. The bifurcation diagram illustrated the systems response dependant on different parameters. The responses shows the effects of multiple sinusoids even though the first mass is much greater than the second, this is because the first stiffness in very large this keeps the mass from gaining too much velocity, so the smaller mass can effectively transfer its momentum to the larger.