MATH1564 K – Linear Algebra with Abstract Vector Spaces
Homework 4

Due Sept. 28, submit to both Canvas-Assignment and Gradescope

1. In each of the following you are given a set, determine whether it is linearly independent or linearly dependent, show how you reach your conclusion.
   i. $\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$
   ii. $\{1 - x^3, 2 + x + x^2, 3 - x, 1 + x + x^2 + x^3\}$
   iii. $\{f(x) = \sin^2 x, g(x) = \cos^2(x), h(x) = 1\}$ (Note that $h(x)$ is the constant function which is equal to 1 for every x).

2. Let V be a vector space and w_1, w_2, w_3 in V be such that $\{w_1, w_2, w_3\}$ is linearly independent. Prove or disprove the following claims.
   i. The set $\{w_1 + w_2 + w_3, w_2 + w_3, w_3\}$ is linearly independent.
   ii. The set $\{w_1 + 2w_2 + w_3, w_2 + w_3, w_1 + w_2\}$ is linearly independent.

3. Let V be a vector space and let $S \subset V$ and $T \subset V$ be two finite subsets of V. Prove or disprove the following claims.
   i. If $S \subset T$ and S is linearly independent then T is linearly independent.
   ii. If $S \subset T$ and T is linearly independent then S is linearly independent.
   iii. If S and T are linearly independent then $S \cap T$ is either empty or linearly independent (Remark: sometimes people consider an empty set to be linearly independent).
   iv. If S and T are linearly independent then $S \cup T$ is linearly independent.
   v. If $W = \text{span}S$ and $U = \text{span}T$ then $W + U = \text{span}(S \cup T)$.

      (The sum of two subspaces X and Y of some vector space V is defined as $X + Y = \{x + y : x \in X, y \in Y\}$.)

4. Find a basis to the following spaces and determine the dimension of each of these spaces.
   i. $\text{span}\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}\}$.
   ii. $\text{span}\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ 3 \\ 7 \end{pmatrix}\}$.

5. i. Let $v_1, ..., v_n \in \mathbb{R}^m$ and denote by A the matrix who's columns are $v_1, ..., v_n$ that is

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}$$

Denote: $L(A) := \{b \in \mathbb{R}^m : (A|b) \text{ has a solution } \}$. Prove that $L(A) = \text{span}\{v_1, ..., v_n\}$.

ii. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 10 \\ 1 & 4 & -1 \\ 2 & 2 & 4 \end{pmatrix}.$$

Find a basis for $L(A)$ and the dimension of $L(A)$.

6. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.

i. Let V be a vector space which satisfies $\dim V = 3$ and let $v_1, v_2, v_3 \in V$ be such that $\{v_1, v_2\}$ are linearly independent, $\{v_2, v_3\}$ are linearly independent, and $\{v_3, v_1\}$ are linearly independent. Then $\{v_1, v_2, v_3\}$ is a basis for V.

ii. Let V be a vector space and $v_1, ..., v_n \in V$ then: $\{v_1, ..., v_n\}$ is linearly independent iff $\dim\big(\text{span}\{v_1, ..., v_n\}\big) = n$.

iii. Let V be a vector space and let $V_1, V_2, V_3 \subset V$ be such that $V_1 + V_2 = V_1 + V_3$ and $\dim V_2 = \dim V_3$ then $V_2 = V_3$. (The sum of two subspaces was defined in Problem 3.v).