HW5 Solutions

1.

(a) The value of this flow is 10. It is not a maximum flow.

(b) The minimum cut is $(\{s,a,b,c\},\{d,t\})$. Its capacity is 11.

2a.

This is is false. Consider a graph with nodes s, v, w, t, edges $(s,v),(v,w),(w,t)$, capacities of 2 on (s,v) and (w,t), and a capacity of 1 on (v,w). Then the maximum flow has value 1, and does not saturate the edge out of s.

2b.

This is false. Consider a graph with nodes s, v_1, v_2, v_3, w, t, edges (s, v_i) and (v_i, w) for each i, and an edge (w,t). There is a capacity of 4 on edge (w,t), and a capacity of 1 on all other edges. Then setting $A = \{s\}$ and $B = V - A$ gives a minimum cut, with capacity 3. But if we add one to every edge then this cut has capacity 6, more than the capacity of 5 on the cut with $B = \{t\}$ and $A = V - B$.

3.

We build the following flow network. There is a node v_i for each client i, a node w_j for each base station j, and an edge (v_i, w_j) of capacity 1 if client i is within range of base station j. We then connect a super-source s to each of the client nodes by an edge of capacity 1, and we connect each of the base station nodes to a super-sink t by an edge of capacity L.

We claim that there is a feasible way to connect all clients to base stations if and only if there is an s-t flow of value n. If there is a feasible connection, then we send one unit of flow from s to t along each of the paths s, v_i, w_j, t, where client i is connected to base station j. This does not violate the capacity conditions, in particular on the edges (w_j, t), due to the load constraints. Conversely, if there is a flow of value n, then there is one with integer values. We connect client i to base station j if the edge (v_i, w_j) carries one unit of flow, and we observe that the capacity condition ensures that no base station is overloaded.

The running is the time required to solve a max-flow problem on a graph with $O(n+k)$ nodes and $O(nk)$ edges.

4.

We build the following flow network. There is a node v_i for each patient i, a node w_j for each hospital j, and an edge (v_i, w_j) of capacity 1 if patient i is within a half hour drive of hospital j. We then connect a super-source s to each of the patient nodes by an edge of capacity 1, and we connect each of the hospital nodes to a super-sink t by an edge of capacity $\lceil n/k \rceil$.

We claim that there is a feasible way to send all patients to hospitals if and only if there is an s-t flow of value n. If there is a feasible way to send patients, then we send one unit of flow from s to t along each of the paths s, v_i, w_j, t, where patient i is sent to hospital j. This does not violate the capacity conditions, in particular on the edges (w_j, t), due to the load constraints. Conversely, if there is a flow of value n, then there is one with integer values. We send patient i to hospital j if the edge (v_i, w_j) carries one unit of flow, and we observe that the capacity condition ensures that no hospital is overloaded.

The running is the time required to solve a max-flow problem on a graph with $O(n+k)$ nodes and $O(nk)$ edges.

5.

(a) Define a flow network as follows. There is a source s, a node x_i representing each balloon i, a node z_i representing each condition c_i, and a sink t. There are edges (s, x_i) of capacity 2, (x_i, z_j) of capacity 1 whenever $c_j \in S_i$, and edges (z_j, t) of capacity k. We then test whether the maximum s-t flow has value nk.

The Ford-Fulkerson algorithm to find a maximum flow has running time $O(|E|C)$, where $|E|$ is the number of edges and C is the total capacity of edges out of s. Here we have $|E| = O(mn)$ and $C = 2m$, so the running time is $O(m^2 n)$.

(b) Break all edges (x_i, z_j). Insert new nodes y_{pj} for each sub-contractor p and condition c_j. Add an edge (x_i, y_{pj}) of capacity 1 when $c_j \in S_i$ and balloon i is produced by sub-contractor p. Add an edge (y_{pj}, c_j) of capacity $k-1$. Again, test whether the maximum s-t flow has value nk.

As in part (a), the running time is $O(|E|C)$. Here $|E|$ is still $O(mn)$, and $C = 2m$, so the running time is $O(m^2 n)$.