Fall 2022, MATH 3215-J, Exam 3 (30 pts)

Name:

GT ID:

- Regular exam time is 75 min.
- Open-book/notes. Calculators are allowed. No communication in any form.
- Please clearly indicate what each formula/number is referring to. All the solutions are supposed to be short and can fit in the given space.
- Consider skipping a part if you get stuck somewhere.
- If there is a mistake in a problem statement making it unsolvable, skip it and you will be awarded full points for that problem.

1. Team A will play against other teams, once per team with independent win/loss outcome. Suppose that Team A beats each of 400 "weak" teams with probability 0.8 and beats each of 150 "strong" teams with probability 0.4. Use the CDF Φ of $\mathcal{N}(0,1)$ to approximate the probabilities of the following events (no need to worry about correction for continuity or strict vs. non-strict inequalities):

(a) (2 pts) Team A beats fewer than 330 out of the 400 "weak" teams.

(b) (2 pts) Team A beats more than 370 out of the 550 teams in total.

(c) (2 pts) Team A beats more "weak" teams than "strong" teams.

(a) Let X be the number of wins against "weak" teams. Then $X \sim \mathrm{Bin}(400, 0.8)$, so $\mathbb{E}[X] = 320$ and $\mathrm{Var}(X) = 64$. It follows that

$$\mathbb{P}\{X < 330\} = \mathbb{P}\left\{\frac{X-320}{8} < \frac{330-320}{8}\right\} \approx \Phi(5/4).$$

(b) Let Y be the number of wins against "strong" teams. Then $Y \sim \mathrm{Bin}(150, 0.4)$, so $\mathbb{E}[Y] = 60$ and $\mathrm{Var}(Y) = 36$. Hence, $X + Y$ is approximately $\mathcal{N}(380, 100)$. It follows that

$$\mathbb{P}\{X+Y > 370\} = \mathbb{P}\left\{\frac{X+Y-380}{10} > \frac{370-380}{10}\right\} \approx 1 - \Phi(-1) = \Phi(1).$$

(c) We have that $X - Y$ is approximately $\mathcal{N}(260, 100)$, so

$$\mathbb{P}\{X > Y\} = \mathbb{P}\left\{\frac{X-Y-260}{10} > \frac{-260}{10}\right\} \approx \Phi(26) \approx 1.$$

2. Suppose that we are given i.i.d. $X_1, \ldots, X_n$ from the Pareto distribution with PDF $f(x) = \lambda\theta^\lambda x^{-(\lambda+1)}$ for $x \geq \theta$ and $f(x) = 0$ otherwise, where $\lambda > 0$ is known and θ is the parameter.

(a) (2 pts) What is the likelihood at θ?

(b) (1 pt) What is the MLE of θ?

The likelihood is $\lambda^n \theta^{n\lambda}(x_1 \cdots x_n)^{-(\lambda+1)}$ for $\theta \leq x_i$ for all $i = 1, \ldots, n$ and is zero otherwise. To maximize this function over θ, we should take $\theta = \min_{1 \leq i \leq n} x_i$. Therefore, the MLE of θ is

$$\hat{\theta} = \min_{1 \leq i \leq n} X_i.$$

3. Suppose that we are given i.i.d. $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \theta)$ where the mean μ is known and the variance θ is to be estimated.

(Remark: We usually denote θ by σ^2. Here we use θ so that there is no confusion whether σ or σ^2 is the parameter to be estimated.)

(a) (1 pt) What is the joint PDF of $X_1, \ldots, X_n$?

(b) (1 pt) What is the log-likelihood at θ?

(c) (2 pts) What is the maximum likelihood estimator (MLE) of θ?

(a) The joint PDF of $(X_1, \ldots, X_n)$ is

$$f(x_1, \ldots, x_n \mid \theta) = \frac{1}{(2\pi\theta)^{n/2}} \exp\left(\frac{-\sum_{i=1}^n (x_i - \mu)^2}{2\theta}\right).$$

(b) The log-likelihood is

$$\frac{-\sum_{i=1}^n (x_i - \mu)^2}{2\theta} - \frac{n}{2}\log(\theta) - \frac{n}{2}\log(2\pi).$$

(c) Differentiating the log-likelihood with respect to θ and setting the result to zero, we obtain

$$\frac{\sum_{i=1}^n (x_i - \mu)^2}{2\theta^2} - \frac{n}{2\theta} = 0.$$

Therefore, the MLE of θ is $\frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2$.

4. Suppose that we are given i.i.d. $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \theta)$ where the mean μ is known and the variance θ is to be estimated. Consider the estimator $\hat{\theta} := \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2$.

(a) (1 pt) What is the bias of $\hat{\theta}$?

(b) (1 pt) What is the distribution of $(X_i - \mu)/\sqrt{\theta}$?

(c) (1 pt) What is the distribution of $n\hat{\theta}/\theta$? (Hint: Write it as a sum and use part (b).)

(d) (1 pt) Using the fact that $\mathbb{E}[Z^4] = 3$ for $Z \sim \mathcal{N}(0,1)$, compute $\mathrm{Var}(Z^2)$.

(e) (1 pt) What is the variance of $n\hat{\theta}/\theta$?

(f) (1 pt) What is the variance of $\hat{\theta}$?

(a) $\mathbb{E}[\hat{\theta}] - \theta = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[(X_i - \mu)^2] - \theta = \frac{1}{n}\sum_{i=1}^n \theta - \theta = 0$

(b) $\mathcal{N}(0,1)$ (need to indicate the parameters)

(c) $n\hat{\theta}/\theta = \sum_{i=1}^n \left(\frac{X_i-\mu}{\sqrt{\theta}}\right)^2 \sim \chi_n^2$

(d) $\mathrm{Var}(Z^2) = \mathbb{E}[Z^4] - (\mathbb{E}[Z^2])^2 = 3 - 1 = 2$

(e) $\mathrm{Var}(n\hat{\theta}/\theta) = \mathrm{Var}\left(\sum_{i=1}^n \left(\frac{X_i-\mu}{\sqrt{\theta}}\right)^2\right) = \sum_{i=1}^n \mathrm{Var}\left(\left(\frac{X_i-\mu}{\sqrt{\theta}}\right)^2\right) = \sum_{i=1}^n 2 = 2n$

(f) $\mathrm{Var}(\hat{\theta}) = (\theta^2/n^2)\,\mathrm{Var}(\hat{\theta}) = 2\theta^2/n$

5. Consider two independent samples of sizes n_1 and n_2 from $\mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathcal{N}(\mu_2, \sigma_2^2)$ respectively. Let S_1^2 and S_2^2 be the respective sample variances. Let $\mathcal{F}$ denote the F-distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom. Recall that

$$R := \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim \mathcal{F}.$$

Note that R takes values in $(0, \infty)$. Let G denote the CDF of $\mathcal{F}$. For $\alpha \in (0,1)$, let $g_\alpha := G^{-1}(1-\alpha)$ be the quantile of order $1-\alpha$.

(a) (1 pt) What is the upper bound c_1 such that $\mathbb{P}\{R < c_1\} = \alpha$? (Express c_1 as a quantile.)

(b) (1 pt) What is the lower bound c_2 such that $\mathbb{P}\{R > c_2\} = \alpha$? (Express c_2 as a quantile.)

(c) (2 pts) Derive a one-sided confidence interval that contains σ_2^2/σ_1^2 with probability $1-\alpha$, based on the upper bound c_1.

(d) (1 pt) State the other one-sided confidence interval that contains σ_2^2/σ_1^2 with probability $1-\alpha$, based on the lower bound c_2.

(e) (1 pt) State a two-sided confidence interval that contains σ_2^2/σ_1^2 with probability $1-\alpha$.

(a) $c_1 = G^{-1}(\alpha) = g_{1-\alpha}$

(b) $c_2 = G^{-1}(1-\alpha) = g_\alpha$

(c) We have $\alpha = \mathbb{P}\{R < g_{1-\alpha}\} = \mathbb{P}\left\{\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < g_{1-\alpha}\right\} = \mathbb{P}\{\sigma_2^2/\sigma_1^2 < g_{1-\alpha} S_2^2/S_1^2\}$. It follows that $\mathbb{P}\{\sigma_2^2/\sigma_1^2 > g_{1-\alpha} S_2^2/S_1^2\} = 1-\alpha$, so the confidence interval is $(g_{1-\alpha} S_2^2/S_1^2, \infty)$.

(d) $(0, g_\alpha S_2^2/S_1^2)$

(e) $(g_{1-\alpha/2} S_2^2/S_1^2, g_{\alpha/2} S_2^2/S_1^2)$

6. In the setting of Bayesian estimation, suppose that we observe $X_1 \sim \mathcal{N}(\theta, 2)$ and have the prior distribution $\theta \sim \mathcal{N}(3, 2)$.

(a) (1 pt) What is the posterior distribution of θ? (You can use the formulas introduced in class.)

(b) (2 pts) Suppose that we observe an additional $X_2 \sim \mathcal{N}(\theta, 2)$ independent of X_1, and use the posterior from part (a) as the new prior. What is the new posterior distribution of θ?

(c) (2 pts) Suppose that we observe independent $X_1, X_2 \sim \mathcal{N}(\theta, 2)$ at the beginning, and use the original prior $\theta \sim \mathcal{N}(3, 2)$. What is the posterior distribution of θ? Is it the same as the posterior in part (b)?

The general formula for the posterior distribution is

$$\mathcal{N}\left(\frac{n\sigma^2}{n\sigma^2+\tau^2}\bar{X} + \frac{\tau^2}{n\sigma^2+\tau^2}\mu, \frac{\sigma^2\tau^2}{n\sigma^2+\tau^2}\right).$$

(a) With $n = 1$, $\bar{X} = X_1$, $\tau^2 = 2$, $\mu = 3$, and $\sigma^2 = 2$, the posterior distribution is

$$\mathcal{N}\left(\frac{X_1+3}{2}, 1\right).$$

(b) With $n = 1$, $\bar{X} = X_2$, $\tau^2 = 2$, $\mu = \frac{X_1+3}{2}$, and $\sigma^2 = 1$, the new posterior is

$$\mathcal{N}\left(\frac{X_2}{3} + \frac{X_1+3}{3}, \frac{2}{3}\right).$$

(c) With $n = 2$, $\bar{X} = \frac{X_1+X_2}{2}$, $\tau^2 = 2$, $\mu = 3$, and $\sigma^2 = 2$, the new posterior is

$$\mathcal{N}\left(\frac{X_1+X_2}{3} + 1, \frac{2}{3}\right),$$

same as that in part (b).