MATH1564 K – Linear Algebra with Abstract Vector Spaces
Homework 7

Due 10/30, submit to both Canvas-Assignment and Gradescope

1. Consider the subspace of $\mathbb{R}^4$

$$\mathscr{V} = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a - b + c - d = 0 \text{ and } a + d = 0 \right\}$$

of $\mathbb{R}^4$.
(a) Find a basis for $\mathscr{V}$.
(b) Find an orthonormal basis $\mathscr{C}$ for $\mathscr{V}$.
(c) Let $x = (1,2,3,4)^T \in \mathbb{R}^4$. Find the orthogonal projection of x onto the space $\mathscr{V}$: $proj_{\mathscr{V}} x$.
(d) Find an orthonormal basis of $\mathbb{R}^4$ that contains the vectors from $\mathscr{C}$ from (b).
(e) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space $\mathscr{V}$ with respect to the basis that you obtained from (d).
(f) Find the matrix representation of the orthogonal projection of $\mathbb{R}^4$ onto the space $\mathscr{V}$ with respect to the standard basis of $\mathbb{R}^4$.
(g) Use the answer from (f) to calculate $proj_{\mathscr{V}} x$

2. Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = [v_1|v_2|v_3]$ with v_1, v_2, v_3 are the columns of A.
(i) Use Gram-Schmidt process to construct an orthogonal set $\{q_1, q_2, q_3\}$ sucht that for $j = 1, 2, 3$,

$$span\{q_1, \dots, q_j\} = span\{v_1, \dots, v_j\}.$$

(ii) Use the answer from (i), find r_{ij}, for $1 \le i \le j \le 3$ such that

$$v_1 = r_{11} q_1, \quad v_2 = r_{12} q_1 + r_{22} q_2, \quad v_3 = r_{13} q_1 + r_{23} q_2 + r_{33} q_3.$$

(iii) Denote $Q = [q_1|q_2|q_3]$ and $R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$. Show that indeed $A = QR$ and $ran(A) = ran(Q)$.
(iv) Show that $Q^T Q = I_{3\times 3}$ and $QQ^T = q_1 q_1^T + q_2 q_2^T + q_3 q_3^T$. Therefore QQ^T is the orthogonal projection onto $ran(Q) = ran(A)$.

(v) Let $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}$. Show that b is not in $ran(A)$.

(vi) Find $proj_{ran(A)}b$.

(vii) Use results from (v) show that $Ax = b$ has no solution.

(viii) Use (iii) and (iv) to find x_* such that $Ax_* = QQ^T b$.

(ix) Show that $x_* = (A^T A)^{-1} A^T b$.

(x) Show that $\|b - Ax_*\| \leq \|b - Ax\|$ for any $x \in \mathbb{R}^3$. (Thus, x_* is called the *least square* solution of $Ax = b$. In elementary linear algebra text book, x_* is usually given by (viii) and not solved using (vii). However (vii) is more numerically stable than (viii) and conceptually better explained – it is easier to prove (ix) using (vii); and can be readily generalized to more general setting, for example when the columns of A are not linearly independent.)