MATH1564 K – Linear Algebra with Abstract Vector Spaces
Homework 2

Due Sept. 12, submit to both Canvas-Assignment and Graadescope

1. Solve each of the following liner systems by representing it in matrix form and reduce the augmented matrix to echelon form. If the system has an infinite many solutions, representing them in a parametrization form. Specify if the system has no solution, unique solution, or infinite solutions.

   (i.)
$$\begin{cases} 2x+y+2z &= 2 \\ -x+y-z &= 2 \\ 3x+2y+z &= 2 \\ 5x+4y-z &= 2 \end{cases}$$

   (ii.)
$$\begin{cases} x_1-x_2+2x_3 &= 0 \\ 2x_1-2x_2+4x_3+x_4+2x_5 &= 4 \\ 3x_1+x_2+6x_3+x_5 &= -3 \\ x_1+2x_3+2x_4+x_5 &= 4 \end{cases}$$

   (iii.)
$$\begin{cases} x_1+2x_2+3x_3+4x_4+5x_5 &= 5 \\ x_1+4x_2+6x_3+8x_4+10x_5 &= 10 \\ x_1-x_2+x_3-x_4+x_5 &= 0 \\ -x_1+4x_2+x_3+6x_4+3x_5 &= 5 \end{cases}$$

2. In each of the following, determine the values of a such that the system has: (1) no solution, (2) unique solution, (3) infinitely many solutions.

   (i.)
$$\begin{cases} x_1+x_2+ax_3 &= -1 \\ -x_1+(a-1)x_2+(2-a)x_3 &= a+1 \\ 6x_1+(5a+6)x_2+(7a+7)x_3 &= a^2 \end{cases}$$

   (ii.)
$$\begin{cases} (a+1)x_1+ax_2-ax_3 &= 2+a \\ (a+1)x_1+(a+2)x_2-(a+2)x_3 &= a+4 \\ (a+1)x_1+ax_2+(a^2-6)x_3 &= a^2-2a+4 \\ (2a+2)x_1+2ax_2+(a^2-a-6)x_3 &= a^2-a+6 \end{cases}$$

3. Each of the following gives a description of a matrix A. Determine whether or not such a matrix exists (where m and n can be any positive integers you want; try to use small numbers if you can). If you claim that such a matrix exists then provide a concrete example. If you claim that such a matrix cannot exist then explain carefully why this is the case.
   (i) For **every** $b \in \mathbb{R}^n$ there exist infinitely many solutions to the linear system $AX = b$.
   (ii) There exists $b_1, b_2 \in \mathbb{R}^n$ such that $AX = b_1$ has infinitely many solutions and $AX = b_2$ has no solutions. (In this problem if you are giving a concrete A as an answer, then you also need to give a particular example of b_1 and b_2).
   (iii) There exists $b_1, b_2, b_3 \in \mathbb{R}^n$ such that $AX = b_1$ has infinitely many solutions, $AX = b_2$ has exactly one solution, and $AX = b_3$ has no solution. (In this problem if you are giving a concrete A as an answer, then you also need to give a particular example of b_1, b_2, and b_3).
   (iv) The homogeneous system $AX = 0$ has exactly one solution and the echelon form of A has at least one row of zeroes.
   (v) The triple $(1, 2, 3)$ is a solution to the homogeneous system $AX = 0$ and the echelon form of A has at least three rows of non-zero rows.
   (vi) The triple $(1, 2, 3)$ is a solution to the homogeneous system $AX = 0$ and the echelon form of A has at least two rows which are not zero rows.
   (vii) The set of solutions of the homogeneous system $AX = 0$ is exactly the set $\{(s + t, s, t) : s, t \in \mathbb{R}\}$.
   (viii) The set of solutions of the homogeneous system $AX = 0$ is exactly the set $\{(s - t, s + t) : s, t \in \mathbb{R}\}$.