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CS 4510 – Automata and Complexity Theory

Exam 3

4. (5 pts) Consider the set Q containing only a single 1 if aliens exist or a single 0 if they don't. Is Q decidable or undecidable?

Decidable; this is a finite set, described by:

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$1 \in Q \Rightarrow$ accept

$0 \in Q \Rightarrow 1 \notin Q \Rightarrow$ reject

This algorithm proves the decidability

5. (5 pts) Consider $\{\langle M \rangle \mid M \text{ is a Turing machine}\}$. Is this set countable or uncountable? Prove it.

This set is countable. Every Turing Machine can be described as a string $q_0 w_0 w_1 w_2 \ldots w_k$. Given the typewriter theorem, this set can be considered countable.

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7. (5 pts) Given an upper bound on $K(\langle p_n \rangle)$ where p_n is the nth prime.

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$K(\langle p_n \rangle) \leq K(n) + c$

8. (5 pts) Give an upper bound on $K(f(x))$ for f any computable function.

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$K(f(x)) \leq K(x) + c$

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9. (15 pts) It is not the case that every language is provably decidable or provably undecidable. Use a counting argument to prove that there exists languages which are **unprovably undecidable**.

Assume to the contrary that undecidable languages are provably so. Therefore, there must exist a bijection between the set of undecidable languages and the set of proofs of these languages. We know from the typewriter theorem that the number of proofs is countable as they can be represented by strings. Additionally, the set of all languages is known to be uncountable. Given the decidable languages is countable and the union of two countable languages is countable, and that the union of decidable and undecidable languages is all languages, the set of undecidable languages must be uncountable.

Since the set of proofs is countable and set of undecidable languages is uncountable, there cannot exist a bijection between the two sets.

Therefore, there exists languages which are unprovably undecidable.

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10. (15 pts) Assume Cantor's theorem. Prove that the set of all sets $\mathfrak{M}$ is not a set.

Assume to the contrary that the set of all set $\mathfrak{M}$ is a set. According to cantor's theorem, $|\mathfrak{M}| \lneq |\mathcal{P}(\mathfrak{M})|$. Given this, there exists a set that is in $\mathcal{P}(\mathfrak{M})$ that is not in $\mathfrak{M}$. This is contradictory with the given information, which is that $\mathfrak{M}$ is the set of all sets, not true because there is a set that isn't in $\mathfrak{M}$ that is in $\mathcal{P}(\mathfrak{M})$.

Therefore, $\mathfrak{M}$ is not a set.

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11. (15 pts) Recall for M a Turing machine, the language $L(M)$ is the set of strings M accepts. Consider the language of encodings of Turing machines that do not accept their own encoding.

$$\{\langle M\rangle \mid \langle M\rangle \notin L(M)\}$$

This is obviously a diagonal language. Finish the diagonalization and prove that this language is undecidable. Do not do a reduction. Do not apply Rice's theorem. Do not draw a table.

Consider all M fitting the criteria listed as $M_1, M_2, M_3 \ldots$. Now define $g: \{\langle M\rangle \mid \langle M\rangle \notin (L(M) \cup w)\}$, where w is a string that M does not accept. g does fit the original criteria since $\langle M\rangle$ does not accept it's own encoding, but the list of TM's in g is not neccessarily the same as $M_1, M_2, M_3 \ldots$. Therefore, this leads to a contradiction since g doesn't map exactly to the original list of TM's, and thus the language $\{\langle M\rangle \mid \langle M\rangle \notin L(M)\}$ is undecidable.

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12. (15 pts) Prove $\{\langle M \rangle \mid M \text{ accepts a unary string}\}$ is undecidable by reduction.

Let $U = \{\langle M \rangle / M$ accepts a unary string$\}$ and assume to the contrary that it is decidable with decider D.

U on inputs $\langle M, w \rangle$

simulate D on M:

if $\langle M\ w \rangle \in U$:

accept

else:

reject

This is a reduction of HALT, as any non-unary string would not loop and simply reject with the else statement. Given HALT is undecidable by diagonalization, we can conclude U is undecidable by reduction.

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14. (20 pts) For each of the following, circle only one of T or F. T if the statement is true and F if the statement is false. On the line following the statement, **justify your answer**. You may use more space on the back if necessary.

1. T F A negation of the parallel postulate produces an inconsistency.

It is consistent - consider hyperbolic or euclidean geometry

2. T F C is Turing-complete if $\mathscr{L}(C) \subseteq \mathscr{L}(TM)$

It must also be true that $\mathscr{L}(TM) \subseteq \mathscr{L}(C)$ to show Turing-completeness

3. T F $PM \nvdash Con(PM)$

Set of axioms cannot yield consistency, doing so causes paradoxes

4. T F There is no set with cardinality greater than $\mathcal{P}(\mathbb{N})$

$|\mathcal{P}(\mathcal{P}(\mathbb{N}))| > |\mathcal{P}(\mathbb{N})|$

5. T F $0 \in \mathbb{N}$

0 is the most natural number, according to professor Kadha

(Bonus) Define the largest **number** you can. If your number is above a certain t unspecified threshold, you will receive $\max(15 - l, 0)$ points, where l is the length of your definition.

$t+1!$

+10