MATH 3215 Assignment 9

Show intermediate steps if there is any. Give the formula for each answer and compute the numerical value with two digits after the decimal point.

1. Suppose that we are given i.i.d. $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ where μ and σ are unknown. Let t_α be the quantile of order $1-\alpha$ for the t-distribution with $n-1$ degrees of freedom. Let S^2 be the sample variance. For $\alpha \in (0,1)$, derive the $(1-\alpha)$ one-sided confidence interval that provides a lower bound on μ.

The random variable

$$T = \frac{\sqrt{n}}{S}(\bar{X} - \mu)$$

follows the t-distribution with $n-1$ degrees of freedom. Therefore,

$$1 - \alpha = \mathbb{P}\left\{\frac{\sqrt{n}}{S}(\bar{X} - \mu) \le t_{\alpha,n-1}\right\} = \mathbb{P}\left\{\bar{X} - \mu \le \frac{S}{\sqrt{n}} t_{\alpha,n-1}\right\} = \mathbb{P}\left\{\bar{X} - \frac{S}{\sqrt{n}} t_{\alpha,n-1} \le \mu\right\}.$$

It follows that the confidence interval is

$$\left(\bar{X} - t_{\alpha,n-1}\frac{S}{\sqrt{n}}, \infty\right).$$

2. Fix $\lambda > 0$ and $\alpha \in (0,1)$. Given i.i.d. Poisson random variables $X_1, \ldots, X_n \sim \mathsf{Poi}(\lambda)$, use normal approximation to find a $(1-\alpha)$ two-sided confidence interval for λ. (Follow the strategy for binomial estimation discussed in class. No need to do the half-unit correction for continuity.)

By the central limit theorem, $\frac{\bar{X}-\lambda}{\sqrt{\lambda/n}}$ is approximately $\mathcal{N}(0,1)$. We estimate λ by $\bar{X}$. Then

$$1 - \alpha \approx \mathbb{P}\left\{-z_{\alpha/2} < \frac{\bar{X} - \lambda}{\sqrt{\bar{X}/n}} < z_{\alpha/2}\right\} = \mathbb{P}\left\{-z_{\alpha/2}\sqrt{\bar{X}/n} < \bar{X} - \lambda < z_{\alpha/2}\sqrt{\bar{X}/n}\right\}$$

so the confidence interval is

$$\left(\bar{X} - z_{\alpha/2}\sqrt{\bar{X}/n}, \bar{X} + z_{\alpha/2}\sqrt{\bar{X}/n}\right).$$

3. Let $X_1, \ldots, X_n, X_{n+1}$ be i.i.d. $\mathcal{N}(\mu, 1)$ random variables where μ is unknown.

(a) Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. What is the distribution of $X_{n+1} - \bar{X}$?

(b) Fix $\alpha \in (0,1)$. Suppose that $\bar{X}$ is known but X_{n+1} is unknown. Find a two-sided interval that contains X_{n+1} with probability $1-\alpha$.

(a) The distribution is $\mathcal{N}(0, 1 + 1/n)$.

(b) We have

$$\begin{aligned}1 - \alpha &= \mathbb{P}\left\{-z_{\alpha/2} < \frac{X_{n+1} - \bar{X}}{\sqrt{1 + 1/n}} < z_{\alpha/2}\right\} \\ &= \mathbb{P}\left\{-z_{\alpha/2}\sqrt{1+1/n} < X_{n+1} - \bar{X} < z_{\alpha/2}\sqrt{1+1/n}\right\},\end{aligned}$$

so the desired interval is

$$\left(\bar{X} - z_{\alpha/2}\sqrt{1+1/n}, \bar{X} + z_{\alpha/2}\sqrt{1+1/n}\right).$$

4. A manufacturer produces computer chips, each of which is independently acceptable with probability p. To obtain a 99% two-sided confidence interval for p, a sample of 300 chips has been taken, and 260 of them are acceptable.

(a) What approximate confidence interval do we obtain from this sample?

(b) If the goal is to make the length of the confidence interval smaller than 0.06, how many chips do we need to sample in total approximately?

(a) We have $\hat{p} = 260/300$ and $n = 300$ so that

$$\Delta := z_{0.01/2}\sqrt{\hat{p}(1-\hat{p})/n} \approx 0.05055.$$

The confidence interval is therefore

$$(\hat{p} - \Delta, \hat{p} + \Delta) \approx (0.82, 0.92).$$

(b) Solving

$$2z_{0.01/2}\sqrt{\hat{p}(1-\hat{p})/n} \leq 0.06,$$

we obtain $n \geq 852$ (something close to this is sufficient).

5. The PCB concentration of a fish caught in Lake Michigan was measured by a technique that is known to result in an error of measurement that is normally distributed with a standard deviation of 0.08 ppm (parts per million). Suppose the results of 10 independent measurements of this fish are

$$11.2, \quad 12.4, \quad 10.8, \quad 11.6, \quad 12.5, \quad 10.1, \quad 11.0, \quad 12.2, \quad 12.4, \quad 10.6.$$

Find the 95% two-sided and one-sided confidence intervals for the average PCB level of this fish.

We have $\bar{X} = 11.48$, so the confidence intervals are

$$\begin{aligned}
&(11.48 - z_{0.025} \cdot 0.08/\sqrt{10}, 11.48 + z_{0.025} \cdot 0.08/\sqrt{10}) \approx (11.43, 11.53),\\
&(11.48 - z_{0.05} \cdot 0.08/\sqrt{10}, \infty) \approx (11.44, \infty),\\
&(-\infty, 11.48 + z_{0.05} \cdot 0.08/\sqrt{10}) \approx (-\infty, 11.52).
\end{aligned}$$

6. A sample of 20 cigarettes is tested to determine nicotine content and the average value observed was 1.2 mg. Find the 99% two-sided and one-sided confidence intervals for the mean nicotine content of a cigarette, if the population standard deviation is unknown but the sample standard deviation is 0.2 mg.

The confidence intervals are

$$\begin{aligned}
&(1.2 - t_{0.005,19} \cdot 0.2/\sqrt{20}, 1.2 + t_{0.005,19} \cdot 0.2/\sqrt{20}) \approx (1.07, 1.33),\\
&(1.2 - t_{0.01,19} \cdot 0.2/\sqrt{20}, \infty) \approx (1.09, \infty),\\
&(-\infty, 1.2 + t_{0.01,19} \cdot 0.2/\sqrt{20}) \approx (-\infty, 1.31).
\end{aligned}$$

7. The capacities (in ampere-hours) of 10 batteries were recorded as follows:

$$140, \quad 136, \quad 150, \quad 144, \quad 148, \quad 152, \quad 138, \quad 141, \quad 143, \quad 151.$$

Assume that the capacities are normally distributed. Compute the 90% two-sided and one-sided confidence intervals for the variance σ^2.

The sample variance is $S^2 \approx 32.233$. The confidence intervals are

$$\Big(\frac{9 \cdot 32.233}{x_{0.05,9}}, \frac{9 \cdot 32.233}{x_{0.95,9}}\Big) \approx (17.15, 87.25),$$
$$\Big(\frac{9 \cdot 32.233}{x_{0.1,9}}, \infty\Big) \approx (19.76, \infty),$$
$$\Big(0, \frac{9 \cdot 32.233}{x_{0.9,9}}\Big) \approx (0, 69.60),$$

where $x_{\alpha,n}$ denotes the quantile of order $1-\alpha$ of the distribution χ^2_n.

8. Independent random samples are taken from the output of two machines on a production line. The weight of each item is of interest. From the first machine, a sample of size 36 is taken, with sample mean weight of 120 grams and a sample variance of 4. From the second machine, a sample of size 64 is taken, with a sample mean weight of 130 grams and a sample variance of 5. It is assumed that the weights of items from the first machine are normally distributed with mean μ_1 and variance σ_1^2 and that the weights of items from the second machine are normally distributed with mean μ_2 and variance σ_2^2.

(a) Find the 99% two-sided and one-sided confidence intervals for $\mu_1 - \mu_2$, if $\sigma_1^2 = 4$ and $\sigma_2^2 = 5$.

(b) Find the 99% two-sided and one-sided confidence intervals for $\mu_1 - \mu_2$, if $\sigma_1 = \sigma_2 = \sigma$ for some unknown $\sigma > 0$.

(a) The confidence intervals are

$$(120 - 130 - z_{0.005}\sqrt{4/36 + 5/64}, 120 - 130 + z_{0.005}\sqrt{4/36 + 5/64}) \approx (-11.12, -8.88),$$
$$(120 - 130 - z_{0.01}\sqrt{4/36 + 5/64}, \infty) \approx (-11.01, \infty),$$
$$(-\infty, 120 - 130 + z_{0.01}\sqrt{4/36 + 5/64}) \approx (-\infty, -8.99).$$

(b) We have

$$S_p^2 = \frac{35 \cdot 4 + 63 \cdot 5}{35 + 63} \approx 4.64.$$

Hence the confidence intervals are

$$(120 - 130 - t_{0.005,98}S_p\sqrt{1/36 + 1/64}, 120 - 130 + t_{0.005,98}S_p\sqrt{1/36 + 1/64}) \approx (-11.18, -8.82),$$
$$(120 - 130 - t_{0.01,98}S_p\sqrt{1/36 + 1/64}, \infty) \approx (-11.06, \infty),$$
$$(-\infty, 120 - 130 + t_{0.01,98}S_p\sqrt{1/36 + 1/64}) \approx (-\infty, -8.94).$$