Fall 2022, MATH 3215-J, Exam 1 (30 pts)

Name:

GT ID:

- Start at 2:00pm and stop at 3:15pm (75 min).
- Open-book/notes. Calculators are allowed, but no communications, smartphones, laptops, internet, etc.
- The answers are supposed to be short and should fit in the provided space. Please clearly indicate what each formula/number is referring to.
- Consider skipping a part if you get stuck somewhere.

1. Choose a number X uniformly at random from the union of intervals $(-2, 1) \cup (5, 7)$. What is the probability density function (PDF) $f(x)$ of the random variable X? (That is, determine the values $f(x)$ with the domain of x clearly specified.) (2 pts)

The PDF of X is $f(x) = 1/5$ for $x \in (-2, 1) \cup (5, 7)$ and $f(x) = 0$ otherwise.

2. Suppose that a random variable X takes values (only) in $(0, 2)$.

(i) If $\mathbb{P}\{X \le x\} = \frac{1}{8}x^3$ for $x \in (0, 2)$, what is the PDF $f(x)$ of X for $x \in (0, 2)$? (2 pts)

(ii) Is it possible that $\mathbb{P}\{X \le x\} = \frac{1}{4}x^3$ for $x \in (0, 2)$? (1 pt)

Differentiating the CDF, we obtain $f(x) = \frac{3}{8}x^2$ for $x \in (0, 2)$.

No to the other question, because $\mathbb{P}\{X \le 2\}$ should be equal to 1.

3. There are 2 red cards and 3 black cards in a box. We draw 2 cards uniformly at random (without replacement) from the box and X of them are red.

(i) What is the sample space S of the random variable X? (1 pt)

(ii) What is the probability mass function (PMF) $f(x)$ of X? (That is, determine the values of $f(x)$ at all possible x.) (2 pts)

The sample space is $S = \{0, 1, 2\}$.

There are $\binom{5}{2} = 10$ possible ways of choosing 2 cards. One way has 2 red cards, $2 \times 3 = 6$ ways have 1 red card, and 3 ways have 0 red cards. The PMF is given by $f(0) = 3/10$, $f(1) = 6/10 = 3/5$, and $f(2) = 1/10$.

4. **(There is a typo — $\mathbb{E}[X] = 3$ should be $\mathbb{E}[X] = 3/2$. You get 2 pts for part (ii) for free.)**

A random variable X takes values in $\{0, 2, 4\}$ with PMF denoted by $f(x)$. Suppose that $f(0) = 1/2$ and $\mathbb{E}[X] = 3$.

(i) Evaluate $f(0) + f(2) + f(4)$. (1 pt)

(ii) Evaluate $f(2)$ and $f(4)$. (2 pts)

We have $3/2 = \mathbb{E}[X] = 0f(0) + 2f(2) + 4f(4) = 2f(2) + 4f(4)$ and $1 = f(0) + f(2) + f(4) = 1/2 + f(2) + f(4)$. Hence $f(2) = 1/4$ and $f(4) = 1/4$.

5. A random variable X takes values in $\{-2, 0, 2\}$ with PMF denoted by $f(x)$. Suppose that its mean is $\mathbb{E}[X] = 0$ and its variance is $\mathrm{Var}(X) = 4$. Evaluate $f(0)$ and $f(2)$. (2 pts)

We have $0 = \mathbb{E}[X] = -2f(-2) + 0f(0) + 2f(2)$, so $f(2) = f(-2)$. Moreover, $4 = \mathrm{Var}(X) = 4f(-2) + 0f(0) + 4f(2)$, so $f(-2) + f(2) = 1$. We conclude that $f(-2) = 1/2$, $f(0) = 0$, and $f(2) = 1/2$.

6. (i) Does $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$ always hold for real-valued random variable X? Why? (2 pts)

(ii) Give an example of a real-valued random variable X such that $\mathbb{E}[X^2] = (\mathbb{E}[X])^2$. (1 pt)

(i) Yes, because $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \geq 0$.

(ii) $X = 0$ with probability 1.

7. Suppose that the time for a student to complete an assignment is $(0.5 + 2X)$ hours, where the random variable X has PDF $f(x) = c/x^3$ for $x \geq 1$ and $f(x) = 0$ for $x < 1$.

(i) Determine the constant c. (2 pts)

(ii) Compute the expectation of the number of hours taken by the student to complete the assignment. (2 pts)

(iii) What is the probability that the student takes longer than 4.5 hours to complete the assignment? (2 pts)

(i) We have $1 = \int_1^\infty c/x^3\,dx = \frac{-c}{2}x^{-2}\big|_1^\infty = c/2$, so $c = 2$.

(ii) We have $\mathbb{E}[X] = \int_1^\infty x \cdot 2/x^3\,dx = -2x^{-1}\big|_1^\infty = 2$, so $\mathbb{E}[0.5 + 2X] = 0.5 + 2\,\mathbb{E}[X] = 4.5$.

(iii) We have $\mathbb{P}\{0.5 + 2X > 4.5\} = \mathbb{P}\{X > 2\} = \int_2^\infty 2/x^3\,dx = -x^{-2}\big|_2^\infty = 1/4$.

8. The number of defective items produced daily at a factory follows the Poisson distribution Poi(3). What is the probability that the number of defective items produced in a day is no larger than its expected value? (Simplify and express the final answer using the constant e without plugging in its numerical value.) (2 pts)

Recall that $X \sim \text{Poi}(3)$ has PMF $3^x e^{-3}/x!$ for $x = 0, 1, 2, \ldots,$ and the expected value is 3. Hence the probability is $\mathbb{P}\{X \leq 3\} = e^{-3} + 3e^{-3} + 9e^{-3}/2 + 27e^{-3}/6 = 13e^{-3}$.

9. Suppose that X has cumulative distribution function (CDF) $F(x) = 1 - e^{-5x}$ for $x \geq 0$ and $F(x) = 0$ for $x < 0$.

(i) What is the moment generating function (MGF) $M(t)$ of X where $t < 5$? (2 pts)

(ii) What is the mean of X? (2 pts)

(i) The PDF is $f(x) = F'(x) = 5\,e^{-5x}$ for $x > 0$ and $f(x) = 0$ otherwise. Hence we have $M(t) = \int_0^\infty e^{tx} \cdot 5\,e^{-5x}\,dx = \frac{5}{t-5} e^{(t-5)x}\Big|_0^\infty = \frac{5}{5-t}$.

(ii) We have $M'(t) = \frac{5}{(5-t)^2}$, so $\mathbb{E}[X] = M'(0) = 1/5$.

10. Five people drive to work (in their own cars) but there are only three parking spots available. Suppose that in a day each person goes to work with probability p independently from other people. What is the probability that there will not be enough parking spots for all the people that go to work? (Simplify and express the final answer in terms of p.) (2 pts)

The number of people X that go to work in a day is a $\mathrm{Bin}(5, 0.8)$ random variable. The probability is thus $\mathbb{P}\{X \geq 4\} = \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5 = 5p^4 - 5p^5 + p^5 = 5p^4 - 4p^5$.