MATH 3215 Assignment 2

1. (Discrete/continuous mixed up) Suppose that a random variable X has CDF

$$F(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \le x < 1 \\ 2/3 & 1 \le x < 2 \\ 11/12 & 2 \le x < 3 \\ 1 & x \ge 3. \end{cases}$$

Determine the following probabilities: (a) $\mathbb{P}\{X > 1/2\}$; (b) $\mathbb{P}\{2 < X \le 4\}$; (c) $\mathbb{P}\{X < 3\}$; (d) $\mathbb{P}\{X = 1\}$. (Think carefully about (c) and (d).)

(a) $\mathbb{P}\{X > 1/2\} = 1 - \mathbb{P}\{X \le 1/2\} = 1 - 1/4 = 3/4$

(b) $\mathbb{P}\{X \le 4\} - \mathbb{P}\{X \le 2\} = 1 - 11/12 = 1/12$

(c) $\mathbb{P}\{X < 3\} = 11/12$

(d) $\mathbb{P}\{X = 1\} = 2/3 - 1/2 = 1/6$

2. Suppose that the mount of time (in hours) that a computer functions before breaking down is a continuous random variable with PDF $f(x) = \lambda e^{-x/100}$ for $x \ge 0$ and $f(x) = 0$ for $x < 0$.

(a) What is λ?

(b) What is the probability that the computer will function between 50 to 150 hours before breaking down?

Since $1 = \int_0^\infty f(x)\,dx = \int_0^\infty \lambda e^{-x/100}\,dx = -100\lambda e^{-x/100}\big|_0^\infty = 100\lambda$, we have $\lambda = 0.01$.

The probability is $\mathbb{P}\{50 \le X \le 150\} = \int_{50}^{150} f(x)\,dx = \int_{50}^{150} 0.01 e^{-x/100}\,dx = -e^{-x/100}\big|_{50}^{150} \approx 0.3834$.

3. An insurance company writes a policy to the effect that an amount of money A must be paid if some event E occurs within a year. If the company estimates that E will occur within a year with probability p, what should it charge the customer (in terms of A and p) so that its expected profit will be 10 percent of A?

Let S be the amount charged. Then $S - pA = 0.1A$, so $S = (0.1 + p)A$.

4. Suppose that a discrete random variable X has PMF $f(x) = x/10$ for $x \in \{1, 2, 3, 4\}$. What is $\mathbb{E}[X(5 - X)]$?

$\mathbb{E}[X(5 - X)] = 4 \cdot 1/10 + 6 \cdot 2/10 + 6 \cdot 3/10 + 4 \cdot 4/10 = 5$

5. Suppose that the PDF of X is given by $f(x) = a + bx^2$ for $x \in [0, 1]$ and $f(x) = 0$ otherwise, where a and b are fixed constants. If $\mathbb{E}[X] = 3/5$, find a and b.

We have $1 = \int_0^1 f(x)\,dx = a + b/3$ and $3/5 = \int_0^1 x f(x)\,dx = a/2 + b/4$. It follows that $b = 6/5$ and $a = 3/5$.

6. For a continuous random variable X with CDF F, the median of X is defined as the value m such that $F(m) = \mathbb{P}\{X \le m\} = 1/2$. Find the median of the random variable with PDF:

(a) $f(x) = 1$ for $x \in [0,1]$ and $f(x) = 0$ otherwise;

(b) $f(x) = e^{-x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$.

(a) 1/2

(b) $1/2 = F(m) = \int_0^m f(x)\,dx = \int_0^m e^{-x}\,dx = -e^{-x}\big|_0^m = 1 - e^{-m}$, so $m = \log 2$

7. For a real-valued random variable X, what is the constant c that minimizes $\mathbb{E}[(X-c)^2]$?

We have $\mathbb{E}[(X-c)^2] = \mathbb{E}[X^2 - 2cX + c^2] = \mathbb{E}[X^2] - 2c\,\mathbb{E}[X] + c^2$, so the minimizer is $c = \mathbb{E}[X]$.

8. Let $S := \{x_1, \ldots, x_n\}$ be a set of n real numbers. Take a uniformly random number from the set S and call it X. Next, set $y_i = a + bx_i$ for $i = 1, \ldots, n$, where a and b are fixed real numbers. Define $T := \{y_1, \ldots, y_n\}$, and let Y be a uniformly random number from the set T.

(a) Express $\mathbb{E}[X]$, $\mathbb{E}[X^2]$, and $\mathsf{Var}(X)$ in terms of $x_1, \ldots, x_n$.
$\mathbb{E}[X] = \bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$
$\mathbb{E}[X^2] = \frac{1}{n}\sum_{i=1}^n x_i^2$
$\mathsf{Var}(X) = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$

(b) Verify that $\mathsf{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ using the expressions from part (a).
We have

$$\begin{aligned}
\sum_{i=1}^n (x_i - \bar{x})^2 &= \sum_{i=1}^n (x_i^2 - 2x_i\bar{x} + \bar{x}^2) \\
&= \sum_{i=1}^n x_i^2 - 2\Big(\sum_{i=1}^n x_i\Big)\bar{x} + n\bar{x}^2 \\
&= \sum_{i=1}^n x_i^2 - 2n\bar{x}\cdot\bar{x} + n\bar{x}^2 \\
&= \sum_{i=1}^n x_i^2 - n\bar{x}^2.
\end{aligned}$$

Dividing both sides by n yields $\mathsf{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

(c) Express $\mathbb{E}[Y]$ and $\mathsf{Var}(Y)$ in terms of $\mathbb{E}[X]$, $\mathsf{Var}(X)$, a, and b (show intermediate steps).

$$\mathbb{E}[Y] = \frac{1}{n}\sum_{i=1}^n y_i = \frac{1}{n}\sum_{i=1}^n (a + bx_i) = a + \frac{b}{n}\sum_{i=1}^n x_i = a + b\,\mathbb{E}[X]$$

$$\mathsf{Var}(Y) = \frac{1}{n}\sum_{i=1}^n (y_i - \bar{y})^2 = \frac{1}{n}\sum_{i=1}^n (a + bx_i - a - b\bar{x})^2 = \frac{1}{n}\sum_{i=1}^n b^2(x_i - \bar{x})^2 = b^2\,\mathsf{Var}(X)$$