MATH 3215 Assignment 6

1. Compute the conditional PDF $f_{X|Y}(x|y)$ of X given $Y = y$ for the examples from previous assignments (you may use intermediate results computed there):

(a) The joint PDF of (X, Y) is $f(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)$ for $x \in (0, 1)$, $y \in (0, 2)$, and $f(x, y) = 0$ otherwise.

(b) The joint PDF of (X, Y) is $f(x, y) = xe^{-x-y}$ for $x > 0$, $y > 0$, and $f(x, y) = 0$ otherwise.

(c) The joint PDF (X, Y) is $f(x, y) = 2$ for $x \in (0, y)$, $y \in (0, 1)$, and $f(x, y) = 0$ otherwise.

(a) We have

$$f_Y(y) = \int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dx = \frac{3y + 4}{14},$$

so

$$f_{X|Y}(x|y) = \frac{(6/7)(x^2 + xy/2)}{(3y + 4)/14} = \frac{12x^2 + 6xy}{3y + 4}$$

for $x \in (0, 1)$.

(b) Same as the marginal PDF.

(c) We have $f_{X|Y}(x|y) = \frac{2}{2y} = \frac{1}{y}$ for $x \in (0, y)$.

2. Compute the covariance $\mathsf{Cov}(X, Y)$ and the correlation (coefficient) $\mathsf{Corr}(X, Y)$ in Problem 1(b) and (c) above (again, you may use intermediate results from previous assignments).

(b) Both are equal to zero because X and Y are independent.

(c) Recall that the marginal PDFs are $f_X(x) = 2 - 2x$ for $x \in (0, 1)$ and $f_Y(y) = 2y$ for $y \in (0, 1)$. Then the marginal means are $\mathbb{E}[X] = 1/3$ and $\mathbb{E}[Y] = 2/3$. The marginal variances are $\mathsf{Var}(X) = 1/18$ and $\mathsf{Var}(Y) = 1/18$. Then we have

$$\mathsf{Cov}(X, Y) = \int_0^1 \int_0^y 2\,(x - 1/3)(y - 2/3)\,dxdy = 1/36, \qquad \mathsf{Corr}(X, Y) = \frac{1/36}{1/18} = 1/2.$$

3. Let (X, Y) be a joint pair of real-valued random variables. Let f_X and f_Y denote their marginal PDFs/PMFs. Show that $f_{X|Y}(x|y) = f_X(x)$ for all $x, y \in \mathbb{R}$ if and only if $f_{Y|X}(y|x) = f_Y(y)$ for all $x, y \in \mathbb{R}$.

In view of the definition of a conditional PDF/PMF, both statements are equivalent to saying that X and Y are independent.

4. A product is classified according to the number of defects X it contains and the label of the factory Y that produces it. We know that X takes values in $\{0, 1, 2\}$ and Y takes values in $\{1, 2\}$. Moreover, suppose that (X, Y) has joint PMF $f(x, y)$ satisfying $f(0, 1) = 1/8$, $f(0, 2) = 1/8$, $f(1, 1) = 1/4$, $f(1, 2) = 1/8$, and $f(2, 1) = 1/8$.

(a) Compute $f(2, 2)$ and the marginal PMFs f_X and f_Y.

(b) Compute the means and variances of X and Y respectively.

(c) Compute $\mathsf{Cov}(X, Y)$ and $\mathsf{Corr}(X, Y)$.

(a) We have $f(2,2) = 1 - 1/8 - 1/8 - 1/4 - 1/8 - 1/8 = 1/4$. The marginals are $f_X(0) = 1/4$, $f_X(1) = 3/8$, $f_X(2) = 3/8$, $f_Y(1) = 1/2$, and $f_Y(2) = 1/2$.

(b) We have $\mathbb{E}[X] = 3/8 + 2 \cdot 3/8 = 9/8$ and $\mathsf{Var}(X) = 3/8 + 2^2 \cdot 3/8 - (9/8)^2 = 39/64$. Moreover, $\mathbb{E}[Y] = 3/2$ and $\mathsf{Var}(Y) = 1/4$.

(c) We have $\mathsf{Cov}(X,Y) = (-9/8)(-1/2)/8 + (-9/8)(1/2)/8 + (-1/8)(-1/2)/4 + (-1/8)(1/2)/8 + (7/8)(-1/2)/8 + (7/8)(1/2)/4 = 1/16$ and $\mathsf{Corr}(X,Y) = 1/16/\sqrt{(39/64)(1/4)} = 1/\sqrt{39}$.

5. Consider n independent trials, each of which results in an outcome in $\{1,2,3\}$ with respective probabilities p_1, p_2, and $1 - p_1 - p_2$. Let N_i denote the number of trials that result in outcome i, for $i = 1,2,3$. Compute $\mathsf{Cov}(N_1, N_2)$ (express the answer in terms of n, p_1, p_2).

(Hint: Write $N_1 = \sum_{j=1}^n X_j$ and $N_2 = \sum_{j=1}^n Y_j$ for $X_j := \mathbb{1}\{\text{trial } j \text{ results in outcome } 1\}$ and $Y_j := \mathbb{1}\{\text{trial } j \text{ results in outcome } 2\}$.)

Since the pairs (X_j, Y_j) are independent across $j = 1, \dots, n$, we have

$$\begin{aligned}\mathsf{Cov}(N_1, N_2) &= \sum_{j=1}^n \mathsf{Cov}(X_j, Y_j) = n\,\mathbb{E}[(X_1 - p_1)(Y_1 - p_2)] \\ &= n\big[p_1(1-p_1)(-p_2) + p_2(-p_1)(1-p_2) + (1-p_1-p_2)(-p_1)(-p_2)\big] = -np_1p_2.\end{aligned}$$

6. For random variables X and Y, show that

$$\mathsf{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

We have

$$\begin{aligned}\mathsf{Cov}(X,Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] - \mathbb{E}[X]\,\mathbb{E}[Y] + \mathbb{E}[X]\,\mathbb{E}[Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].\end{aligned}$$

7. Recall that if X and Y are independent random variables, then $\mathsf{Cov}(X,Y) = 0$. It follows that $\mathsf{Corr}(X,Y) = 0$, that is, they are uncorrelated. This problem shows that the converse is not true in general.

Let X and Y be independent random variables defined by

$$X = \begin{cases} 0 & \text{with probability } 1/2, \\ 1 & \text{with probability } 1/2, \end{cases} \qquad Y = \begin{cases} -1 & \text{with probability } 1/2, \\ 1 & \text{with probability } 1/2. \end{cases}$$

Let $Z := XY$. Show that X and Z are uncorrelated, but not independent.

(Hint: Consider the events $\{Z = 1\}$ and $\{X = 0\}$.)

We have $\mathbb{E}[X] = 1/2$ and $\mathbb{E}[Y] = 0$, so $\mathbb{E}[Z] = \mathbb{E}[X]\,\mathbb{E}[Y] = 0$. Therefore,

$$\mathsf{Cov}(X,Z) = (-1/2)(0)/4 + (-1/2)(0)/4 + (1/2)(-1)/4 + (1/2)(1)/4 = 0.$$

However, X and Z are not independent because

$$\mathbb{P}\{Z = 1,\, X = 0\} = 0 \neq (1/4)(1/2) = \mathbb{P}\{Z = 1\}\,\mathbb{P}\{X = 0\}.$$

8. A real-valued random variable X has PDF/PMF $f_X(x)$ and expectation $\mathbb{E}[X] = \mu$. Given that $X = x$, a real-valued random variable Y is assumed to have conditional expectation

$$\mathbb{E}[Y \mid X = x] := \int_{\mathbb{R}} y \cdot f_{Y|X}(y|x)\, dy = x + c$$

for a constant $c \in \mathbb{R}$. Compute the marginal expectation of Y (in terms of μ and c).

We have

$$\begin{aligned}
\mathbb{E}[Y] &= \int y\, f_Y(y)\, dy \\
&= \int y \Big(\int f(x,y)\, dx \Big)\, dy \\
&= \int y \Big(\int f_{Y|X}(y|x)\, f_X(x)\, dx \Big)\, dy \\
&= \int \Big(\int y\, f_{Y|X}(y|x)\, dy \Big)\, f_X(x)\, dx \\
&= \int (x + c)\, f_X(x)\, dx = \mathbb{E}[X] + c = \mu + c.
\end{aligned}$$

9. An automobile repair shop makes an initial estimate, X hundreds of dollars, of the amount of money needed to fix a car after an accident. Suppose that X has the uniform distribution over $[5, 9]$. Given that $X = x$, the final payment Y has the uniform distribution over $[x, x+2]$. What is the (marginal) expectation of Y?

(There are at least three ways to solve this problem: 1. Give an argument based on intuition; 2. Do direct computation; 3. Use the previous problem (recommended).)

The expectation of X is 7, and the expectation of Y should be 8. We can nevertheless do a naive but rigorous computation to obtain this result.

We have $f_X(x) = 1/4$ for $x \in [5, 9]$, and $f_X(x) = 0$ otherwise. Moreover, $f_{Y|X}(y|x) = 1/2$ for $y \in [x, x+2]$, and $f_{Y|X}(y|x) = 0$ otherwise. Hence

$$f(x,y) = f_{Y|X}(y|x) \cdot f_X(x) = 1/8$$

for $x \in [5, 9]$, $y \in [x, x+2]$, and $f(x,y) = 0$ otherwise. Then the marginal PDF of Y is

$$f_Y(y) = \begin{cases} \int_5^y \frac{1}{8}\, dx = \frac{y-5}{8} & \text{for } y \in [5, 7), \\ \int_{y-2}^y \frac{1}{8}\, dx = \frac{1}{4} & \text{for } y \in [7, 9), \\ \int_{y-2}^9 \frac{1}{8}\, dx = \frac{11-y}{8} & \text{for } y \in [9, 11]. \end{cases}$$

Therefore, the marginal expectation of Y is

$$\mathbb{E}[Y] = \int_5^7 y\, \frac{y-5}{8}\, dy + \int_7^9 \frac{y}{4}\, dy + \int_9^{11} y\, \frac{11-y}{8}\, dy = 8.$$