MATH 3215 Assignment 3

1. Let random variable X be the number of days that a patient will need to be in the hospital. Suppose that X has PMF

$$f(x) = \frac{5-x}{10}, \quad x = 1, 2, 3, 4.$$

The patient is to receive \$200 from an insurance company for each of the first two days in the hospital and \$100 for each day after the first two days. Find the expectation and the standard deviation of the payment for the (entire) hospitalization.

The expectation is

$$\frac{4}{10} \cdot 200 + \frac{3}{10} \cdot 400 + \frac{2}{10} \cdot 500 + \frac{1}{10} \cdot 600 = 360.$$

The variance is

$$\frac{4}{10} \cdot (200-360)^2 + \frac{3}{10} \cdot (400-360)^2 + \frac{2}{10} \cdot (500-360)^2 + \frac{1}{10} \cdot (600-360)^2 = 20400,$$

so the standard deviation is $\sqrt{20400}$.

2. A school has 20 classes: 16 with 25 students in each, 3 with 100 students in each, and 1 with 300 students in each.

(a) What is the average class size (i.e., number of students divided by number of classes)?

(b) Select a student uniformly at random from the set of all students. Let X be the size of the class to which this student belongs. What is the PMF of X?

(c) What is the mean (i.e., expectation) of X?

(a) $1000/20 = 50$

(b) $f(25) = 2/5$, $f(100) = 3/10$, $f(300) = 3/10$

(c) $\mathbb{E}[X] = 25 \cdot 2/5 + 100 \cdot 3/10 + 300 \cdot 3/10 = 10 + 30 + 90 = 130$

3. (a) Suppose that X has PDF

$$f(x) = \begin{cases} x-8 & \text{for } 8 \le x < 9, \\ 10-x & \text{for } 9 \le x < 10, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the mean and the variance of X.

(b) Suppose that X oz is the weight of an article sold by a manufacturer for a fixed price of \$2. The cost of production is known to be related to the weight by $\$(X/16 + 0.35)$. Moreover, suppose that the manufacturer needs to refund the purchase money if the weight of the article is less than 8.25 oz. Express the profit per article in terms of X using the indicator function. (See Section 1.4 of the lecture notes if you are not sure what an indicator is.)

(c) What is the expected profit per article? (That is, compute the expectation.)

(a) $\mathbb{E}[X] = 9$ and $\mathsf{Var}(X) = \int_8^{10}(x-9)^2 f(x)\,dx = 2\int_0^1 y^2(1-y)\,dy = 1/6$

(b) The profit is $Y = 2 - (X/16 + 0.35) - 2 \cdot \mathbb{1}\{X < 8.25\}$.

(c) Its expectation is

$$\mathbb{E}[Y] = 1.65 - \mathbb{E}[X/16] - 2 \cdot \mathbb{P}\{X < 8.25\} = 1.65 - 9/16 - 2\int_8^{8.25}(x-8)\,dx = 1.025.$$

4. The weekly demand for propane gas is X thousands of gallons, where X is a random variable with PDF $f(x) = 4x^3e^{-x^4}$ for $x \geq 0$ and $f(x) = 0$ otherwise. If the stockpile consists of two thousands of gallons at the beginning of a week, what is the probability of not being able to meet the demand during the week?

$$\mathbb{P}\{X > 2\} = \int_2^\infty 4x^3e^{-x^4}\,dx = -e^{-x^4}\big|_2^\infty = e^{-16} \approx 0$$

5. A random variable X is said to have an exponential distribution with parameter $\lambda > 0$ if its PDF is $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and $f(x) = 0$ otherwise. Derive the CDF, the mean, and the variance of X.

The CDF is

$$F(x) = \int_0^x \lambda e^{-\lambda y}\,dy = -e^{-\lambda y}\big|_0^x = 1 - e^{-\lambda x}$$

for $x \geq 0$ and $F(x) = 0$ otherwise. The mean is (using integration by parts)

$$\mathbb{E}[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx = -\frac{1}{\lambda}e^{-\lambda x}(\lambda x + 1)\Big|_0^\infty = \frac{1}{\lambda}.$$

Moreover, since

$$\mathbb{E}[X^2] = \int_0^\infty x^2 \cdot \lambda e^{-\lambda x}\,dx = -\frac{1}{\lambda^2}e^{-\lambda x}(\lambda^2x^2 + 2\lambda x + 2)\Big|_0^\infty = \frac{2}{\lambda^2},$$

we obtain

$$\mathsf{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

6. Let X be uniformly distributed in $[0,1]$. Let $Y = aX + b$ for constants $a > 0$ and $b \in \mathbb{R}$. Find the CDF and then the PDF of Y.

The CDF of Y is

$$F(t) = \mathbb{P}\{Y \leq t\} = \mathbb{P}\{aX + b \leq t\} = \mathbb{P}\left\{X \leq \frac{t-b}{a}\right\} = \frac{t-b}{a}$$

if $t \in [b, b+a]$, $F(t) = 0$ if $t < b$, and $F(t) = 1$ if $t \geq b + a$. The PDF of Y is therefore

$$f(t) = F'(t) = 1/a$$

for $t \in [b, b+a]$ and $f(t) = 0$ otherwise.

7. Let X be uniformly distributed in $[0,1]$.

(a) Compute the MGF $M(t) := \mathbb{E}[e^{tX}]$ of X for $t \in \mathbb{R}$.

(b) By differentiating the MGF, compute $\mathbb{E}[X^n]$ for $n = 1, 2, 3$. Then verify your answers by direct computation.

(c) (Optional, not graded) Compute $\mathbb{E}[X^n]$ for every positive integer n. (Hint: Use the Taylor expansion of the MGF around $t = 0$.)

The MGF is

$$M(t) = \int_0^1 e^{tx}\,dx = \frac{e^{tx}}{t}\bigg|_0^1 = \frac{e^t - 1}{t} = \frac{\sum_{k=0}^{\infty}(t^k/k!) - 1}{t} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}.$$

On the other hand, the Taylor series for the MGF around zero is

$$M(t) = \sum_{n=0}^{\infty} M^{(n)}(0) \cdot \frac{t^n}{n!}.$$

Therefore, by matching coefficients, we see that $\mathbb{E}[X^n] = M^{(n)}(0) = \frac{1}{n+1}$.

8. A satellite system consists of 4 components and can function adequately if at least 2 of the 4 components are in working condition. If each component is, independently, in working condition with probability 0.6, what is the probability that the system functions adequately?

$$\binom{4}{2} \cdot 0.6^2 \cdot 0.4^2 + \binom{4}{3} \cdot 0.6^3 \cdot 0.4^1 + \binom{4}{4} \cdot 0.6^4 \cdot 0.4^0 = \frac{513}{625} = 0.8208$$

9. Five cards are selected at random without replacement from a standard 52-card deck. Let X be the number of face cards (kings, queens, or jacks) in the selected five cards.

(a) How many possible combinations are there for the five cards selected?

(b) How many possible combinations are there if exactly x of the five cards are face cards?

(c) What is the PMF of X?

(It suffices to give formulas for the above quantities without simplification.)

$$\binom{52}{5}, \qquad \binom{12}{x}\binom{40}{5-x}, \qquad f(x) = \frac{\binom{12}{x}\binom{40}{5-x}}{\binom{52}{5}}, \quad x = 0, 1, 2, \ldots, 5$$