MATH-3012-D HW 03

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TOTAL POINTS

29 / 30

QUESTION 1

1 Q1 **10 / 10**

✓ **+ 10 pts** *Correct*

QUESTION 2

2 Q2 **9 / 10**

✓ **+ 7 pts** *Some mistake*

+ 2 *Point adjustment*

No. 17. No solution when c = 11,13,14,15,16, 17, 19. When c = 12: x = 118 - 165k, y = -10 +14k. When c = 18: x = 177-165k, y = -15 +14k.

QUESTION 3

3 C **10 / 10**

✓ **- 0 pts** *Correct*

Homework 3

Q1 4.1

1. (a) $\sum_{i=0}^{n} (2n-1) = \frac{n(2n-1)(2n+1)}{3}$

Base case: $n=1$, LHS $=1$, RHS $=1$, true for $n=1$

Assume true for $n=k \rightarrow 1^2+3^2+\ldots+(2k-1)^2 = \frac{1}{3}(k(2k-1)(2k+1))$

Check $\rightarrow 1^2+3^2+\ldots+(2k-1)^2+(2k+1)^2 = \frac{1}{3}((k+1)(2k+1)(2k+3))$

$$1^2+3^2+\ldots+(2k-1)^2+(2k+1)^2 = \frac{1}{3}(k(2k-1)(2k+1))+(2k+1)^2$$
$$= \frac{4}{3}k^3 - \frac{1}{3}k + 4k^2 + 4k + 1$$
$$= \frac{1}{3}(4k^3+12k^2+11k+3)$$
$$= \frac{1}{3}(k+1)(4k^2+8k+3)$$
$$= \frac{1}{3}(k+1)(2k+1)(2k+3)$$

* If true for $n=k$, then true for $n=k+1$ is proven. Therefore, the statement is true for all $n \geq 1$ by MIP.

2. (b) $\sum_{i=1}^{n} i(2^i) = 2+(n-1)2^{n+1}$

Base case: $n=1$, LHS $=2$, RHS $=2$, true for $n=1$

Assume true for $n=k \rightarrow 2+8+24+\ldots+k\cdot 2^k = 2+(k-1)\cdot 2^{k+1}$

Check $\rightarrow 2+8+24+\ldots+k\cdot 2^k+(k+1)\cdot 2^{k+1} = 2+(k)\cdot 2^{k+2}$

$$2+8+24+\ldots+k\cdot 2^k+(k+1)\cdot 2^{k+1} = 2+(k-1)\cdot 2^{k+1}+(k+1)\cdot 2^{k+1}$$
$$= 2+k\cdot 2^{k+1} - 2^{k+1} + k\cdot 2^{k+1} + 2^{k+1}$$
$$= 2+2k\cdot 2^{k+1}$$
$$= 2+k\cdot 2^{k+2}$$

* If true for $n=k$, then true for $n=k+1$ is proven. Therefore, the statement is true for all $n \geq 1$ by MIP

4. $n^3 - n = 3M,\ M \in \mathbb{Z}$

Base case: $n=1$, LHS: 0, RHS: $M=0$, true for $n=1$

Assume true for $n=k \rightarrow k^3 - k = 3A,\ A \in \mathbb{Z}$

Check $\rightarrow (k+1)^3 - (k+1) = 3B,\ B \in \mathbb{Z}$

$$k^3+3k^2+3k+1-k-1 = k^3-k+3k^2+3k$$
$$= 3A + 3k^2 + 3k$$
$$= 3(A+k^2+k)$$
$$= 3B,\ B = A+k^2+k \in \mathbb{Z}$$

* If true for $n=k$, then true for $n=k+1$ is proven. Therefore, the statement is true for all $n \in \mathbb{N} (n \geq 1)$ by MIP

24. (a) $a_3 = 2+1 = 3$, $a_4 = 3+2 = 5$, $a_5 = 5+3 = 8$, $a_6 = 8+5 = 13$, $a_7 = 13+8 = 21$

(b) Base case: $n=1$, LHS: 1, RHS: $7/4$, true for $n=1$

Assume true for F_{n-1} and F_{n-2}, show true for F_n

$$F_n = F_{n-1} + F_{n-2} < (7/4)^{n-2} + (7/4)^{n-1}$$
$$< (7/4)^{n-2}(1 + 7/4)$$
$$< (7/4)^{n-2}(11/4) = (7/4)^{n-2}(44/16) < (7/4)^{n-2}(49/16)$$
$$= (7/4)^n$$

* Therefore $F_n < (7/4)^n$ is proven by MIP

1 Q1 **10 / 10**

✓ **+ 10 pts** *Correct*

Q2
4.3

6. · Base case: $n=2$, $a_2 \mid b_2$ is true

· Assume true for $n=k$, $k \geq 2$

↳ By induction, when $a = a_1 a_2 \cdots a_k$ and $b = b_1 b_2 \cdots b_k$, $a \mid b$ holds true

· $a_{k+1} \mid b_{k+1}$ is also true

· Therefore $(a \cdot a_{k+1}) \mid (b \cdot b_{k+1}) \rightarrow (a_1 a_2 \cdots a_{k+1}) \mid (b_1 b_2 \cdots b_{k+1})$ holds true

· Thus a_i divides b_i for all $n \geq 2$ by MIP

10. · Base case: $n=1$, $8 \mid (0)$ is true as $8 \cdot 0 = 0$

· Assume true for $n = 2k-1$, $k \geq 1$

· Test $n = 2k+1$

$$\frac{8}{(2k+1)^2-1} = \frac{8}{4k^2+4k} = \frac{2}{k^2+k} = \frac{2}{k(k+1)} \rightarrow$$ $k(k+1)$ must be divisible by 2 as either k or $k+1$ is even.

· There, by MIP, $8 \mid (n^2-1)$ for odd positive integer n

12. $$\begin{array}{r} -10 \\ 12 \overline{)\,-115} \\ \underline{-120} \\ 5 \end{array}$$

$-115 = (-10) \cdot 12 + 5$

$q = -10$, $r = 5$

19. We want $\frac{5n+18}{n} = k$, $k \in \mathbb{Z}$

$\frac{5n+18}{n} = 5 + \frac{18}{n}$. For this to be an integer, $\frac{18}{n}$ must be an integer. Therefore, the set of n making this true are 18's divisors (not inverses as $n \in \mathbb{Z}^+$):

$$\{1, 2, 3, 6, 9, 18\}$$

Q2
4.4

1\. (c) $4001 = (1)\cdot 2689 + 1312$ $\quad \gcd(2689, 4001) = 1$

$2689 = (2)\cdot 1312 + 65$ $\quad 1 = 1\cdot 5 - 2\cdot 2$

$1312 = (20)\cdot 65 + 12$ $\quad = 5 - 2(12 - 2\cdot 5)$

$65 = (5)\cdot 12 + 5$ $\quad = -2\cdot 12 + 5\cdot 5$

$12 = (2)\cdot 5 + 2$ $\quad = -2\cdot 12 + 5(65 - 5\cdot 12)$

$5 = (2)\cdot 2 + 1$ $\quad = 5\cdot 65 - 27(1312 - 20\cdot 65)$

$2 = (2)\cdot 1$ $\quad = -27\cdot 1312 + 545(2689 - 2\cdot 1312)$

$2689(1662) + 4001(-1117) = 1$ $\quad = 545\cdot 2689 - 1117(4001 - 1\cdot 2689)$

$= -1117\cdot 4001 + 1662\cdot 2689$

4\. Let $d = \gcd(a,b)$, therefore $d|a$ and $d|b$.
Therefore, for $s,t \in \mathbb{Z}$, $d = as + bt$.
$\frac{a}{d}\cdot\frac{n}{n} = \frac{na}{nd}$ and $\frac{b}{d}\cdot\frac{n}{n} = \frac{nb}{nd}$ are integers
such that $nd|na$ and $nd|nb$, where
$nd = nas + nbt$. therefore, $\gcd(na, nb) = nd = n\cdot\gcd(a,b)$ → proven by using the Euclidean algorithm

8\. Given $\gcd(a,b) = 1$, for some $x, y \in \mathbb{Z}$, $xa + yb = 1$.
$\frac{c}{ab} = \frac{c(xa+yb)}{ab} = \frac{xc}{b} + y\frac{c}{a}$. ~~This is only an integer~~
if x, y, c/b, and c/a are integers, which is true by given statements for x and y and that $a|c$ and $b|c$.
This would not necessarily work for $\gcd(a,b) \neq 1$, unless the value of $\gcd(a,b)$ is a divisor of x, y, c/a, and c/b.

13. $7n+4 = 1(5n+3) + (2n+1)$

$5n+3 = 2(2n+1) + (n+1)$

$2n+1 = 1(n+1) + n$

$(n+1) = n+1$

$$
\begin{aligned}
1 &= (n+1) - n \\
&= (n+1) - ((2n+1) - (n+1)) \\
&= 2(n+1) - (2n+1) \\
&= 2(5n+3 - 2(2n+1)) - (2n+1) \\
&= 2(5n+3) - 5(2n+1) \\
&= 2(5n+3) - 5((7n+4) - (5n+3)) \\
&= 7(5n+3) - 5(7n+4)
\end{aligned}
$$

Therefore, using Euclidean Algorithm, we can conclude that $\gcd(5n+3, 7n+4) = 1$

17. $d = \gcd(a,b)$

if $d \nmid c$, then $84x + 990y = c$ has no solution

$990 = 11(84) + 66$

$84 = 1(66) + 18$

$66 = 3(18) + 12$

$18 = 1(12) + 6$

$12 = 2(6)$

$\hookrightarrow c = \{11, 13, 14, 15, 16, 17, 19, 20\}$

$c = 12 \rightarrow 12 = 66 - 3 \cdot 18 = 66 - 3(84 - 66) = 4 \cdot 66 - 3 \cdot 84$
$= 4(990 - 11 \cdot 84) - 3 \cdot 84 = 4 \cdot 990 - 47 \cdot 84$

$c = 18 \rightarrow 18 = 84 - 66 = 84 - (990 - 11 \cdot 84) = 12 \cdot 84 - 990$

2 Q2 **9 / 10**

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