MATH1564 K – Linear Algebra with Abstract Vector Spaces
Homework 3

Due Sept. 19, submit to both Canvas-Assignment and Graadescope

1. In each of the following you are given a set and two operations: A 'sum', acting between two elements in the set, and a 'multiplication by scalar', acting between one element in the set and a scalar from $\mathbb{R}$. In each case determine whether the set with these two operations gives a vector space over $\mathbb{R}$. If it is a vector space then prove this fact. If it is not a vector space then show this by giving a counterexample.

   i. The set $S = \{f \in P_3(\mathbb{R}) : f(2) = f(3)\}$ with the usual operations of summation and multiplication by scalar defined for polynomials.

   ii. The set
$$\{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x - y + 2z = 0\}$$
   with the usual operations of summation and multiplication by scalar defined for n-tuples.

   iii. The set $\mathbb{R}^2$ with the operations
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ 0 \end{pmatrix}$$
   and
$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 0 \end{pmatrix}.$$

   iv. The set $\mathbb{R}^2$ with the operations (note the locations of y_2 in the definition)
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_2 \\ x_2 + y_2 \end{pmatrix}$$
   and
$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}.$$

   v. The set $\mathbb{R}^2$ with the operations
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - 3 \\ x_2 + y_2 - 2 \end{pmatrix}$$
   and
$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 - 3\alpha + 3 \\ \alpha x_2 - 2\alpha + 2 \end{pmatrix}.$$

vi. The set $\mathbb{R}^2$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2\alpha x_1 \\ 2\alpha x_2 \end{pmatrix}.$$

vii. The set $\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R} : x_1 > 0, x_2 > 0\}$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^\alpha \\ x_2^\alpha \end{pmatrix}.$$

2. Let V be a vector space over $\mathbb{R}$ and let $W \subset V$ and $U \subset V$ be two subspaces of V. The following claims are either true or false. Determine whether they are true or false and prove or disprove using a counterexample accordingly.
   a. $U \cap W$ is also a subspace of V.
   b. $U \cup W$ is also a subspace of V.
   c. We define the following subset of V:
   $$U + W := \{u + w : u \in U, w \in W\}.$$
   In this part of the question the claim is: $U + W$ is a subspace of V.

3. In each of the following you are given a statement, which may be true or false. Determine whether the statement is correct and show how you reached this conclusion.
   i. $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \in \text{span}\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$
   ii. $2 + 3x + 2x^2 - x^3 \in \text{span}\{1 - x^3, 2 + x + x^2, 3 - x\}$
   iii. $\text{span}\{\begin{pmatrix} 5 & -2 \\ -5 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}\} \subseteq \text{span}\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$
   iv. $\text{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\} = \text{span}\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}$

v. $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\}$ is a spanning set for $\mathbb{R}^2$.

vi. $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ is a spanning set for $\mathbb{R}^2$.

vii. $\{1 - x + x^2, x - x^2 + x^3, 1 + x^2 - x^3, x^3\}$ is a spanning set for $P_3(x)$, the set of polynomials over $\mathbb{R}$ of degree at most 3.

vii. $\{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix}\}$ is a spanning set for $M_{2\times 2}(\mathbb{R})$.