MATH 3215 Assignment 10

For Problems 2–5, state/compute the null hypothesis, the alternative hypothesis, the test statistic, the test (in the form "the test rejects H_0 if ..."), and the p-value. Determine whether we accept or reject the hypothesis based on both approaches: via the test itself and via the p-value. Compute each numerical value to two digits after the decimal point.

1. The breaking strength of a certain type of cloth is to be measured for a sample of 20 specimens, and the average breaking strength of the sample is 182 psi. The underlying distribution is normal with an unknown mean θ psi and a standard deviation equal to 3 psi. Suppose also that based on previous experience we feel that the unknown mean has a prior distribution that is normally distributed with mean 200 psi and standard deviation 2 psi. Determine a two-sided (Bayesian/posterior) confidence interval that contains θ with probability 0.95.

The posterior mean is

$$\frac{n\sigma^2}{n\sigma^2+\tau^2}\bar{X}+\frac{\tau^2}{n\sigma^2+\tau^2}\mu=\frac{20\cdot 2^2}{20\cdot 2^2+3^2}182+\frac{3^2}{20\cdot 2^2+3^2}200\approx 183.82,$$

and the posterior variance is

$$\frac{\sigma^2\tau^2}{n\sigma^2+\tau^2}=\frac{2^2\cdot 3^2}{20\cdot 2^2+3^2}\approx 0.404.$$

Hence the confidence interval is

$$(183.82-z_{0.025}\sqrt{0.404},183.82-z_{0.025}\sqrt{0.404})\approx(182.57,185.07).$$

2. A large population of mice in a laboratory has an average weight 32 grams with a standard deviation of 4 grams. A scientist weighs a random sample of 25 mice for an experiment. If the sample mean of the 25 mice is 30.4 grams, would this be significant evidence, at the 5 percent level of significance, against the hypothesis that the average weight of all mice is indeed 32 grams?

We have $H_0:\mu=32$ and $H_1:\mu\neq 32$. The test statistic is

$$\frac{\sqrt{n}}{\sigma}(\bar{X}-\mu_0)=\frac{\sqrt{25}}{4}(30.4-32)=-2.$$

The test rejects H_0 if $\frac{\sqrt{n}}{\sigma}|\bar{X}-\mu_0|>z_{0.05/2}\approx 1.96$, so H_0 is rejected. In terms of the p-value, we have

$$\mathbb{P}\left\{|Z|>\frac{\sqrt{25}}{4}|30.4-32|\right\}=\Phi(-2)+1-\Phi(2)\approx 0.0455<0.05,$$

so we reach the same conclusion of rejecting H_0.

3. In a certain chemical process, a particular solution that is to be used as a reactant is supposed to have a pH of exactly 8.20. A method for determining pH for solutions of this type is known to give measurements that are normally distributed with a mean equal to the actual pH and with a standard deviation of 0.02. Suppose 10 independent measurements yielded the following pH values:

8.18, 8.23, 8.16, 8.15, 8.17, 8.21, 8.22, 8.16, 8.19, 8.18.

What conclusion can be drawn at the significance level 0.01?

We have $H_0 : \mu = 8.2$ and $H_1 : \mu \neq 8.2$. The test statistic is

$$\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu_0) = \frac{\sqrt{10}}{0.02}(8.185 - 8.2) \approx -2.37.$$

The test rejects H_0 if $\frac{\sqrt{n}}{\sigma}|\bar{X} - \mu_0| > z_{0.01/2} \approx 2.58$, which is not the case, so H_0 is accepted. In terms of the p-value, we have

$$\mathbb{P}\left\{|Z| > \frac{\sqrt{10}}{0.02}|8.185 - 8.2|\right\} \approx 2\,\Phi(-2.37) \approx 0.018 > 0.01,$$

so we reach the same conclusion of accepting H_0.

4. The weights of salmon grown at a commercial hatchery are normally distributed with a standard deviation of 1.2 pounds. The hatchery claims (treated as H_0) that the mean weight of this year's crop is at least 7.6 pounds. Suppose a random sample of 16 fish yields an average weight of 7.2 pounds. Is this strong enough evidence to reject the hatchery's claim at the 5 percent level of significance?

We have $H_0 : \mu \geq 7.6$ and $H_1 : \mu < 7.6$. The test statistic is

$$\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu_0) = \frac{\sqrt{16}}{1.2}(7.2 - 7.6) \approx -1.33.$$

The test rejects H_0 if $\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu_0) < -z_{0.05} \approx -1.65$, which is not the case, so H_0 is accepted. In terms of the p-value, we have

$$\mathbb{P}\left\{Z < \frac{\sqrt{16}}{1.2}(7.2 - 7.6)\right\} \approx \Phi(-1.33) \approx 0.09 > 0.05,$$

so we reach the same conclusion of accepting H_0.

5. A new toothpaste is supposed to keep cavities of children in a certain age group no more than 3 times per year on average (treated as H_0). Cavities per year for this age group are normal with standard deviation 1. A study of 2500 children who used this toothpaste found an average of 3.05 cavities per child. Do these data, at the 1 percent level of significance, contradict the claimed function of the toothpaste?

We have $H_0 : \mu \leq 3$ and $H_1 : \mu > 3$. The test statistic is

$$\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu_0) = \frac{\sqrt{2500}}{1}(3.05 - 3) = 2.5.$$

The test rejects H_0 if $\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu_0) > z_{0.01} \approx 2.33$, so H_0 is rejected. In terms of the p-value, we have

$$\mathbb{P}\left\{Z > \frac{\sqrt{2500}}{1}(3.05 - 3)\right\} \approx 1 - \Phi(2.5) \approx 0.0062 < 0.01,$$

so we reach the same conclusion of rejecting H_0.

6. (a) Consider a trial in which a jury must decide between the hypothesis that the defendant is guilty and the hypothesis that the defendant is innocent. In the framework of hypothesis testing and the U.S. legal system, which of the hypotheses should be the null hypothesis?

(b) A pharmaceutical company has recently developed a new drug for migraine headaches. The company claims that the mean time it takes for a drug to enter the bloodstream is less than 10 minutes. To prove its claim, what should the company take as the null hypothesis?

(a) Null hypothesis: innocent. Alternative hypothesis: guilty.

(b) Null hypothesis: takes more than 10 minutes. Alternative hypothesis: takes less than 10 minutes.

As said in class, the alternative hypothesis if what the test is trying to prove. We would like to reject the null and accept the alternative with high confidence (the probability of error is bounded by the significance level). Therefore we have the above choices.

7. Given i.i.d. random variables $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ where σ^2 is known. Let ϕ denote the two-sided Z-test at significance level $\alpha \in (0, 1)$ for testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. Let I denote the $(1 - \alpha)$ two-sided confidence interval for μ. Prove that ϕ rejects H_0 if and only if $\mu_0 \notin I$.

The two-sided Z-test rejects $H_0 : \mu = \mu_0$ if

$$\frac{\sqrt{n}}{\sigma}|\bar{X} - \mu_0| > z_{\alpha/2}.$$

The $(1 - \alpha)$ two-sided confidence interval for μ is

$$I = \Big[\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\Big].$$

Therefore, $\mu_0 \notin I$ if and only if

$$|\bar{X} - \mu_0| > z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \iff \quad \frac{\sqrt{n}}{\sigma}|\bar{X} - \mu_0| > z_{\alpha/2}.$$

8. (optional problem, no grade)

Suppose that we are given i.i.d. $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ where σ is known.

(a) Consider testing $H_0 : \mu \geq \mu_0$ against $H_1 : \mu < \mu_0$. Recall that the p-value for the Z-test is

$$\Phi\Big(\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu_0)\Big),$$

where Φ is the CDF of $\mathcal{N}(0, 1)$. Recall that $\bar{X}$ is typically viewed as the value of the sample mean, but we now view $\bar{X}$ as a random variable. Show that the p-value is a uniform random variable over $(0, 1)$ under H_0. (Hint: Use Problem 9 of Assignment 7.)

(b) Consider testing $H_0 : \mu \leq \mu_0$ against $H_1 : \mu > \mu_0$. What is the p-value in this case? Is it a uniform random variable over $(0, 1)$ under H_0?

(c) For $Z \sim \mathcal{N}(0, 1)$, what is the CDF of $|Z|$ in terms of Φ?

(d) Consider testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. What is the p-value in this case? Is it a uniform random variable over $(0, 1)$ under H_0?

(a) Since Φ is the CDF of $\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu_0)$, this follows immediately from Problem 9 of Assignment 7.

(b) The p-value in this case is

$$1 - \Phi\Big(\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu_0)\Big) = \Phi\Big(\frac{\sqrt{n}}{\sigma}(\mu_0 - \bar{X})\Big).$$

Since $\frac{\sqrt{n}}{\sigma}(\mu_0 - \bar{X}) \sim \mathcal{N}(0, 1)$, the same result follows.

(c) We have, for $t \geq 0$,

$$\mathbb{P}\{|Z| \leq t\} = \Phi(t) - \Phi(-t).$$

(d) The p-value in this case is

$$1 - \Phi\Big(\frac{\sqrt{n}}{\sigma}|\bar{X} - \mu_0|\Big) + \Phi\Big(-\frac{\sqrt{n}}{\sigma}|\bar{X} - \mu_0|\Big).$$

Since

$$\Phi\Big(\frac{\sqrt{n}}{\sigma}|\bar{X} - \mu_0|\Big) - \Phi\Big(-\frac{\sqrt{n}}{\sigma}|\bar{X} - \mu_0|\Big)$$

is uniform over $(0, 1)$, so is the p-value.