MATH1564 K – Linear Algebra with Abstract Vector Spaces
Homework 6

Due 10/24, submit to both Canvas-Assignment and Gradescope

1. (a.) Let $\mathscr{P}_3$ be polynomials over $\mathbb{R}$ of degree less or equal to 3. Let $\mathscr{E} = \langle 1, x, x^2, x^3 \rangle$ and $\mathscr{B} = \langle 1, 1+x, (1+x)^2, (1+x)^3 \rangle$ be two ordered bases for $\mathscr{P}_3$.
Consider the following linear transformations $T, S : \mathscr{P}_3 \to \mathscr{P}_3$ given by

$$Tp(x) = p'(x) \qquad \text{and} \qquad Sp(x) = p(x+1)$$

(i.) Find $[T]_{\mathscr{E}\to\mathscr{E}}$, $[T]_{\mathscr{B}\to\mathscr{E}}$, $[T]_{\mathscr{E}\to\mathscr{B}}$, $[T]_{\mathscr{B}\to\mathscr{B}}$.
(ii.) Find $[S]_{\mathscr{E}\to\mathscr{E}}$, $[S]_{\mathscr{B}\to\mathscr{E}}$, $[S]_{\mathscr{E}\to\mathscr{B}}$, $[S]_{\mathscr{B}\to\mathscr{B}}$.
(iii.) Find $[T\circ S]_{\mathscr{E}\to\mathscr{E}}$, $[T\circ S]_{\mathscr{B}\to\mathscr{E}}$, $[T\circ S]_{\mathscr{E}\to\mathscr{B}}$, $[T\circ S]_{\mathscr{B}\to\mathscr{B}}$.

(b.) Consider $L : \mathscr{P}_3 \to \mathbb{R}^2$ given by $Lp = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}$. Let $\mathscr{B} = \langle 1, x, x^2, x^3 \rangle$ and $E = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$ be two ordered bases for $\mathscr{P}_3$ and $\mathbb{R}^2$ respectively.
(i.) Find $[L]_{\mathscr{B}\to\mathscr{E}}$.
(ii) Find a basis for $null([L]_{\mathscr{B}\to\mathscr{E}})$, and using this information to find a basis for $ker(L)$.
(iii) Find a basis for $ran([L]_{\mathscr{B}\to\mathscr{E}})$, and using this information to find a basis for $ran(L)$.

2. Consider the vector space,

$$\mathscr{W} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}.$$

Let $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and consider the linear transformation $L : \mathscr{W} \to \mathscr{W}$ defined by $L(A) = AH - HA$.
(i.) Show that $L(A)$ is in $\mathscr{W}$ if $A \in \mathscr{W}$.
(ii) Find an ordered basis $\mathscr{B}$ for $\mathscr{W}$. (You will use this basis for the rest of the problem.)
(iii) Find $[L]_{\mathscr{B}\to\mathscr{B}}$.
(iv) Find a basis for $null([L]_{\mathscr{B}\to\mathscr{B}})$, and using this information to find a basis for $ker(L)$.
(v) Find a basis for $ran([L]_{\mathscr{B}\to\mathscr{B}})$, and using this information to find a basis for $ran(L)$.

3. Let $\mathscr{B} = \langle 1, x, x^2 \rangle$ be an order basis for $\mathscr{P}_2$ and $\mathscr{E} = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$. be an ordered basis for $\mathbb{R}^3$. Consider the linear transformation $S : \mathscr{P}_2 \to \mathbb{R}^3$ defined by

$$Sp = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$$

(i) Find $[S]_{\mathscr{B}\to\mathscr{E}}$.
(ii) Find $([S]_{\mathscr{B}\to\mathscr{E}})^{-1}$
(iii) Use (ii) to find $S^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

4. Consider the following ordered bases of $\mathbb{R}^3$:

$$\mathscr{B} = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rangle, \quad \mathscr{C} = \langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \rangle$$

$$\mathscr{E} = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

Find the following matrices of transition from basis to basis:
(i) $[id]_{\mathscr{B}\to\mathscr{E}}$ and $[id]_{\mathscr{C}\to\mathscr{E}}$.
(ii) $[id]_{\mathscr{E}\to\mathscr{B}}$ and $[id]_{\mathscr{E}\to\mathscr{C}}$.
(iii) $[id]_{\mathscr{B}\to\mathscr{C}}$ and $[id]_{\mathscr{C}\to\mathscr{B}}$.

5. Let $\mathscr{W}$ be the subspace in Problem 2 with order basis

$$\mathscr{B} = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle$$

$$\mathscr{C} = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \rangle.$$

Find $[id]_{\mathscr{B}\to\mathscr{C}}$ and $[id]_{\mathscr{C}\to\mathscr{B}}$.

6. Let V be a vector space such that $\dim V = 3$. Assume that $\mathscr{B} = \langle v_1, v_2, v_3 \rangle$ and $\mathscr{C} = \langle w_1, w_2, w_3 \rangle$ are ordered bases of V such that

$$[id]_{\mathscr{B}\to\mathscr{C}} = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 0 & 0 \\ 3 & 5 & 2 \end{pmatrix}$$

(a) Express each of v_1, v_2, v_3 as linear combinations of w_1, w_2, w_3.
(b) Is it true that $w_2 = v_1 + v_3 - v_2$?
(c) Find $[w_1]_{\mathscr{B}}$.