MATH 3215 Assignment 11

For Problems 1–4, state the null hypothesis, the alternative hypothesis, the test statistic, the test (in the form "the test rejects H_0 if ..."), and the p-value. Determine whether we accept or reject the null hypothesis based on both approaches: via the test itself and via the p-value. Compute each numerical value to at least two digits after the decimal point (or more digits if necessary). Note that the p-value can sometimes be very large (e.g., > 0.5) or very small (e.g., extremely close to 0), which just means that it is easy to make a decision.

1. The average body temperature of healthy individuals is at most 98.6 degrees Fahrenheit, but a scientist believes that the average body temperature has increased over time and is now greater. To prove this, 100 healthy individuals are selected and their average temperature is 98.74 degrees with a sample standard deviation of 1.1 degrees. Does this support the scientist's claim at the 5 percent significance level? Assume that body temperatures follow a normal distribution.

We have $H_0 : \mu \leq 98.6$ and $H_1 : \mu > 98.6$. The test statistic is

$$\frac{\sqrt{n}}{S}(\bar{X} - \mu_0) = \frac{\sqrt{100}}{1.1}(98.74 - 98.6) \approx 1.27.$$

The test rejects H_0 if $\frac{\sqrt{n}}{S}(\bar{X} - \mu_0) > t_{0.05,99} \approx 1.66$, which is not the case. Hence we accept H_0, i.e., we cannot prove the scientist's claim. The p-value is

$$1 - F_{99}\Big(\frac{\sqrt{100}}{1.1}(98.74 - 98.6)\Big) \approx 0.10 > 0.05,$$

where F_{99} denotes the CDF of the t-distribution with 99 degrees of freedom. This gives the same conclusion.

2. A sample of 10 fish were caught at lake A and their PCB concentrations were measured. The resulting data in parts per million were

$$11.5, \quad 10.8, \quad 11.6, \quad 9.4, \quad 12.4, \quad 11.4, \quad 12.2, \quad 11, \quad 10.6, \quad 10.8.$$

In addition, a sample of 8 fish were caught at lake B and their levels of PCB were

$$11.8, \quad 12.1, \quad 11.9, \quad 12.5, \quad 11.7, \quad 12.1, \quad 10.9, \quad 12.6.$$

It is known that PCB concentrations are normally distributed and have variances 0.09 and 0.16 at lake A and lake B respectively. Can we reject at the 1 percent level of significance the claim that the average PCB concentration level at lake A is higher?

We have $H_0 : \mu_1 \geq \mu_2$ and $H_1 : \mu_1 < \mu_2$. The test statistic is

$$\frac{\bar{X} - \bar{Y}}{\sqrt{\sigma_1^2/n + \sigma_2^2/m}} = \frac{11.17 - 11.95}{\sqrt{0.09/10 + 0.16/8}} \approx -4.58.$$

The test rejects H_0 if the test statistic is smaller than $-z_{0.01,99} \approx -2.33$. Hence H_0 is rejected. The p-value is

$$\Phi\Big(\frac{11.17 - 11.95}{\sqrt{0.09/10 + 0.16/8}}\Big) \approx 0 < 0.01,$$

which gives the same conclusion.

3. The viscosity of two different brands of car oil is measured, resulting in data for Brand 1:

$$10.62, \quad 10.58, \quad 10.33, \quad 10.72, \quad 10.44, \quad 10.74;$$

and data for Brand 2:

$$10.50, \quad 10.52, \quad 10.58, \quad 10.62, \quad 10.55, \quad 10.51, \quad 10.53.$$

Test at significance level 0.01 the hypothesis that the mean viscosity of Brand 1 is equal to that of Brand 2, assuming that the populations have normal distributions with equal variances.

We have $H_0 : \mu_1 = \mu_2$ and $H_1 : \mu_1 \neq \mu_2$. Moreover,

$$\bar{X} \approx 10.5717, \quad S_1^2 \approx 0.0257, \quad \bar{Y} \approx 10.5443, \quad S_2^2 \approx 0.0018,$$

so

$$S_p^2 = \frac{5S_1^2 + 6S_2^2}{5+6} \approx 0.0127,$$

and the test statistic is

$$T = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{1/6 + 1/7}} \approx 0.4371.$$

The test rejects H_0 if $|T| > t_{0.01/2,11} \approx 3.106$, which is not the case. Hence H_0 is accepted. The p-value is

$$2\,F_{11}(-|T|) \approx 0.65 > 0.1,$$

where F_{11} denotes the CDF of the t-distribution with 11 degrees of freedom. The same conclusion follows.

4. A gun-like apparatus has recently been designed to replace needles in administering vaccines. The apparatus can be set to inject different amounts of the serum, but because of random fluctuations, the actual amount injected is normally distributed with a mean equal to the setting and with an unknown variance σ^2. It has been decided that the apparatus would be unsafe to use if $\sigma \geq 0.1$ (null hypothesis). If a random sample of 50 injections resulted in a sample standard deviation of 0.08, does this support the safety of the apparatus at the level of significance 0.01?

We have $H_0 : \sigma \geq 0.1$ and $H_1 : \sigma < 0.1$. The test statistic is

$$\frac{n-1}{\sigma_0^2} S^2 = \frac{49}{0.1^2} 0.08^2 = 31.36.$$

The χ^2-test rejects H_0 if

$$\frac{n-1}{\sigma_0^2} S^2 < x_{1-0.01} \approx 28.94,$$

where $x_{1-0.01}$ denotes the quantile of order 0.01 for χ^2_{49}. This does not hold, so H_0 is accepted, i.e., the evidence is not sufficient to support the safety of the apparatus. The p-value is

$$F\Big(\frac{49}{0.1^2} 0.08^2\Big) \approx 0.0235 > 0.01,$$

where F denotes the CDF of χ^2_{49}. The conclusion is the same.

5. A standard drug is known to be effective in 72 percent of the cases in which it is used to treat a certain infection. A new drug has been developed and testing has found it to be effective in 42 cases out of 50. This is potential evidence that the new drug is more effective than the old one. What is the null hypothesis and what is the alternative hypothesis? Find the p-value in two ways: (a) using the binomial CDF and (b) using normal approximation with correction for continuity.

Suppose that the new drug works with probability p. We test $H_0 : p \le 0.72$ against $H_1 : p > 0.72$. The binomial p-value is

$$1 - F_{50,0.72}(42) \approx 0.0158,$$

where $F_{50,0.72}$ denotes the CDF of $\mathsf{Bin}(50, 0.72)$. The normal approximated p-value is

$$1 - \Phi\Big(\frac{42.5 - 50 \cdot 0.72}{\sqrt{50 \cdot 0.72 \cdot 0.28}}\Big) \approx 0.0203.$$

6. Suppose that we observe i.i.d. $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ where μ and σ^2 are unknown. Consider testing $H_0 : \sigma^2 \ge \sigma_0^2$ against $H_1 : \sigma^2 < \sigma_0^2$. Let F denote the CDF of χ^2_{n-1}, and for $\beta \in (0, 1)$, let $x_\beta = F^{-1}(1 - \beta)$ denote the quantile of order $1 - \beta$ for χ^2_{n-1}. Derive the one-sided χ^2-test at significance level α and determine the p-value (in terms of a quantile and F).

The test rejects H_0 if $\frac{n-1}{\sigma_0^2} S^2 < x_{1-\alpha}$. The p-value is $F\big(\frac{n-1}{\sigma_0^2} S^2\big)$.

7. Given i.i.d. $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ where σ is unknown, consider testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \ne \mu_0$. Let $t_\alpha = F^{-1}(1 - \alpha)$ denote the quantile of order $1 - \alpha$ for the t-distribution with $n - 1$ degrees of freedom, where F denotes the CDF of the distribution. Test 1 rejects H_0 if

$$\frac{\sqrt{n}}{S} |\bar{X} - \mu_0| > t_{\alpha/2},$$

where $\bar{X}$ is the sample mean and S^2 is the sample variance. Test 2 rejects H_0 if

$$2\, F\Big(- \frac{\sqrt{n}}{S} |\bar{X} - \mu_0| \Big) < \alpha,$$

that is, the p-value is smaller than α. Are Tests 1 and 2 equivalent (i.e., one rejects H_0 if and only if the other rejects H_0)? Why?

Yes. We have

$$\begin{aligned} F\Big(- \frac{\sqrt{n}}{S} |\bar{X} - \mu_0| \Big) < \alpha/2 \quad &\iff \quad -\frac{\sqrt{n}}{S} |\bar{X} - \mu_0| < F^{-1}(\alpha/2) \\ &\iff \quad \frac{\sqrt{n}}{S} |\bar{X} - \mu_0| > -F^{-1}(\alpha/2) = F^{-1}(1 - \alpha/2) = t_{\alpha/2}. \end{aligned}$$

8. Suppose that we observe $X \sim \mathsf{Bin}(n, p)$ where p is unknown. Consider testing $H_0 : p \ge p_0$ against $H_1 : p < p_0$. Let F denote the CDF of $\mathsf{Bin}(n, p_0)$. Test 1 rejects H_0 if

$$X < \min\big\{k : F(k) \ge \alpha\big\}.$$

Test 2 rejects H_0 if

$$F(X) < \alpha,$$

that is, the p-value is smaller than α. Are Tests 1 and 2 equivalent (i.e., one rejects H_0 if and only if the other rejects H_0)? Why?

Yes. If $X < \min\big\{k : F(k) \geq \alpha\big\}$, then $F(X) < \alpha$.

On the other hand, since F is nondecreasing, if $X \geq \min\big\{k : F(k) \geq \alpha\big\}$, then $F(X) \geq \alpha$. As a result, if $F(X) < \alpha$, then $X < \min\big\{k : F(k) \geq \alpha\big\}$.