MATH-3012-D HW 07

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TOTAL POINTS

29 / 30

QUESTION 1

1 Q1 **9 / 10**

✓ **+ 5 pts** *(d) correct*

✓ **+ 4 pts** *(a) partial credit*

QUESTION 2

2 Q2 **10 / 10**

✓ **+ 10 pts** *correct*

QUESTION 3

3 C **10 / 10**

✓ **- 0 pts** *Correct*

HW7

9.4 1. (c) $1+\frac{(-a)x}{1!}+\frac{(a^2)x^2}{2!}+\frac{(-a^3)x^3}{3!}+\dots=e^{-ax}$

(d) $1+\frac{a^2x}{1!}+\frac{a^4x}{2!}+\dots=e^{a^2x}$

(e) $a+\frac{a^3x}{1!}+\frac{a^5x^2}{2!}+\dots=a\left(1+\frac{a^2x}{1!}+\frac{a^4x^2}{2!}+\dots\right)=ae^{a^2x}$

(f) $0+\frac{x}{1!}+\frac{2\cdot 2x^2}{2!}+\frac{3\cdot 2^2x^3}{3!}+\dots=x\left(1+\frac{2x}{1!}+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\dots\right)$

$=xe^{2x}$

2. (a) $3, 9, 27, \dots, 3^{n+1}$

(b) $3, 24, 138, \dots, 6(5^n)-3(2^n)$

(c) $x^2 \to 2\frac{x^2}{2!}$

$1, 1, 3, 1, 1, \dots 1$

(d) $-3x^3 \to -18\frac{x^3}{3!}$, $5x^2 \to 10\frac{x^2}{2!}$, $7x \to 7\frac{x^1}{1!}$

$1, 2+7, 4+10, 8-18, 16, 32, 64\dots$

$1, 9, 14, -10, 16, 32, 64, \dots, 2^n$

(e) $\frac{1}{1-x}=1+x+x^2+x^3+\dots$

$1 \to 1\cdot\frac{x^0}{1!}$, $x \to 1\cdot\frac{x^1}{1!}$, $x^2 \to 2\cdot\frac{x^2}{2!}$, $x^3 \to 6\cdot\frac{x^3}{6!}$

$0!, 1!, 2!, 3!, 4!, \dots$

$1, 1, 2, 6, 24, \dots, n!$

(f) $\frac{3}{1-2x}=3+6x+12x^2+24x^3+\dots$

$3\cdot\frac{x^0}{1!}+6\frac{x^1}{1!}+24\cdot\frac{x^2}{2!}+144\frac{x^3}{3!}$

$4, 7, 25, 145, \dots, (1+3\cdot 2^n\cdot n!)$

6. (a)(i) $H \to (1+x)$ $\quad A \to (1+x+\frac{x^2}{2!})$

$W \to (1+x)$ $\quad I \to (1+x+\frac{x^2}{2!})$

$g(x) = (1+x)^2(1+x+\frac{x^2}{2!})^2$

(ii) $M \to (1+x)$ $\quad S \to (1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!})$

$I \to (1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!})$ $\quad P \to (1+x+\frac{x^2}{2!})$

$$g(x) = (1+x)(1+x+\frac{x^2}{2!})(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!})^2$$

(iii) $I \to (1+x+\frac{x^2}{2!})$ $\quad O \to (1+x+\frac{x^2}{2!})$ $\quad R \to (1+x)$ $\quad H \to (1+x)$

$S \to (1+x+\frac{x^2}{2!})$ $\quad M \to (1+x+\frac{x^2}{2!})$ $\quad P \to (1+x)$

$$g(x) = (1+x)^3(1+x+\frac{x^2}{2!})^4$$

(b) $I \to (\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4})$

$$g(x) = (1+x)(1+x+\frac{x^2}{2!})(\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!})(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!})$$

10. (a) $0 \to x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$, $1 \to e^x$, $2 \to x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$, $3 \to e^x$

$$g(x) = (e^x)^2(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots)(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots)$$

$$= (e^{2x})(1/2)(e^x-1)(e^x-1) = \frac{1}{2}(e^{4x}-2e^{3x}+e^{2x})$$

coef$(\frac{x^{20}}{20!})$ in $g(x) \to (\frac{1}{2})(4^{20}-2\cdot 3^{20}+2^{20})$

(b) $g(x) = (e^x - \frac{x^2}{2!})^4 = e^{4x} - 4e^{3x}(\frac{x^2}{2}) + 6e^{2x}(\frac{x^2}{2})^2$

$-4e^x(\frac{x^2}{2})^3 + (\frac{x^2}{2!})^4 = e^{4x} - 2e^{3x}x^2 + \frac{3}{2}e^{2x}x^4$

$-\frac{1}{2}e^x x^6 + \frac{x^8}{16}$

$\text{coef}\left(\frac{x^{20}}{20!}\right)$ in $g(x) = 4^{20} - 2\cdot 3^{18}\cdot\frac{20!}{18!} + \frac{3}{2}\cdot 2^{16}\cdot\frac{20!}{16!} - \frac{1}{2}\cdot 1^{14}\cdot\frac{20!}{14!}$

$= 4^{20} - 2\cdot 3^{18}\cdot 20\cdot 19 + \frac{3}{2}\cdot 2^{16}\cdot 20\cdot 19\cdot 18\cdot 17 - \frac{1}{2}\cdot 20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15$

(c) $g(x) = (e^x - x^3/3!)^4 = e^{4x} - 4e^{3x}\left(\frac{x^3}{6}\right) + 6e^{2x}\left(\frac{x^3}{6}\right)^2 - 4e^x\left(\frac{x^3}{6}\right)^3 + \left(\frac{x^3}{6}\right)^4$

$\text{coef}\left(\frac{x^{20}}{20!}\right)$ in $g(x) = 4^{20} - \frac{2}{3}\cdot 3^{17}\cdot\frac{20!}{17!} + \frac{1}{6}\cdot 2^{14}\cdot\frac{20!}{14!} - \frac{1}{54}\cdot 1^{11}\cdot\frac{20!}{11!}$

$= 4^{20} - \frac{2}{3}\cdot 3^{17}\cdot 20\cdot 19\cdot 18 + \frac{1}{6}\cdot 2^{14}\cdot 20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15 - \frac{1}{54}\cdot 20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15\cdot 14\cdot 13\cdot 12$

(d) $g(x) = (e^x)^3(1 + x^2/2!) = e^{3x} + \frac{1}{2}e^{3x}x^2$

$\text{coef}(x^{20}/20!)$ in $g(x) = 3^{20} + \frac{1}{2}\cdot 3^{18}\cdot\frac{20!}{18!} = 3^{20} + \frac{1}{2}\cdot 3^{18}\cdot 20\cdot 19$

10.2 1. (a) $x^2 - 5x - 6 = (x-6)(x+1)$ → $a_0 = 1 = s + t$

$a_n = s(-1)^n + t(6)^n$ ↗ $a_1 = 3 = -s + 6t$

$a_n = -3/7(-1)^n + 4/7(6)^n,\ n \geq 0$

(b) $2x^2 - 11x + 5 = (2x-1)(x-5)$ → $a_0 = 2 = s + t$

$a_n = s(1/2)^n + t(5)^n$ → $a_1 = -8 = 1/2\,s + 5t$

$a_n = 4(1/2)^n - 2(5)^n,\ n \geq 0$

(d) $x^2 - 6x + 9 = (x-3)^2$ → $a_0 = 5 = s$

$a_n = s(3)^n + tn(3)^n$ ↗ $a_1 = 12 = 3s + 3t$

$a_n = 5(3)^n - n(3)^n = (3)^n(5-n),\ n \geq 0$

1 Q1 **9 / 10**

✓ **+ 5 pts** *(d) correct*

✓ **+ 4 pts** *(a) partial credit*

$\text{coef}\left(\frac{x^{20}}{20!}\right)$ in $g(x) = 4^{20} - 2\cdot3^{18}\cdot\frac{20!}{18!} + \frac{3}{2}\cdot2^{16}\cdot\frac{20!}{16!} - \frac{1}{2}\cdot1^{14}\cdot\frac{20!}{14!}$

$= 4^{20} - 2\cdot3^{18}\cdot20\cdot19 + \frac{3}{2}\cdot2^{16}\cdot20\cdot19\cdot18\cdot17 - \frac{1}{2}\cdot20\cdot19\cdot18\cdot17\cdot16\cdot15$

(c) $g(x) = (e^x - x^3/3!)^4 = e^{4x} - 4e^{3x}\left(\frac{x^3}{6}\right) + 6e^{2x}\left(\frac{x^3}{6}\right)^2 - 4e^x\left(\frac{x^3}{6}\right)^3 + \left(\frac{x^3}{6}\right)^4$

$\text{coef}\left(\frac{x^{20}}{20!}\right)$ in $g(x) = 4^{20} - \frac{2}{3}\cdot3^{17}\cdot\frac{20!}{17!} + \frac{1}{6}\cdot2^{14}\cdot\frac{20!}{14!} - \frac{1}{54}\cdot1^{11}\cdot\frac{20!}{11!}$

$= 4^{20} - \frac{2}{3}\cdot3^{17}\cdot20\cdot19\cdot18 + \frac{1}{6}\cdot2^{14}\cdot20\cdot19\cdot18\cdot17\cdot16\cdot15 - \frac{1}{54}\cdot20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14\cdot13\cdot12$

(d) $g(x) = (e^x)^3(1 + x^2/2!) = e^{3x} + \frac{1}{2}e^{3x}x^2$

$\text{coef}(x^{20}/20!)$ in $g(x) = 3^{20} + \frac{1}{2}\cdot3^{18}\cdot\frac{20!}{18!} = 3^{20} + \frac{1}{2}\cdot3^{18}\cdot20\cdot19$

10.2 1. (a) $x^2 - 5x - 6 = (x-6)(x+1)$

$a_n = s(-1)^n + t(6)^n$ $\rightarrow$ $a_0 = 1 = s + t$, $a_1 = 3 = -s + 6t$

$a_n = -3/7(-1)^n + 4/7(6)^n,\ n \geq 0$

(b) $2x^2 - 11x + 5 = (2x-1)(x-5)$

$a_n = s(1/2)^n + t(5)^n$ $\rightarrow$ $a_0 = 2 = s + t$, $a_1 = -8 = 1/2 s + 5t$

$a_n = 4(1/2)^n - 2(5)^n,\ n \geq 0$

(d) $x^2 - 6x + 9 = (x-3)^2$

$a_n = s(3)^n + tn(3)^n$ $\rightarrow$ $a_0 = 5 = s$, $a_1 = 12 = 3s + 3t$

$a_n = 5(3)^n - n(3)^n = (3)^n(5-n),\ n \geq 0$

(e) $x^2+2x+2 = (x-(-1+i))(x-(-1-i))$

$-1-i = \sqrt{2}(\cos(5\pi/4) + i\sin(5\pi/4)) = \sqrt{2}(\cos(3\pi/4) - i\sin(3\pi/4))$

$-1+i = \sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))$

$a_n = (\sqrt{2})^n(s\cos(3\pi n/4) + t\sin(3\pi n/4))$

$a_0 = s = 1$

$a_1 = \sqrt{2}(-\sqrt{2}/2 + t\frac{\sqrt{2}}{2}) = -1 + t = 3 \to t = 4$

$a_n = (\sqrt{2})^n(\cos(3\pi n/4) + 4\sin(3\pi n/4))$

9. $a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 3$

$\hookrightarrow a_n = a_{n-1} + a_{n-2} \to x^2 - x - 1$

$a_n = s\left(\frac{1+\sqrt{5}}{2}\right)^n + t\left(\frac{1-\sqrt{5}}{2}\right)^n \to a_0 = 1,\ a_1 = 1$

$s + t = 1$

$s + s\sqrt{5} + t - t\sqrt{5} = 2$

$s - t = 1/\sqrt{5}$

$s = \frac{\sqrt{5}+1}{2\sqrt{5}} \quad t = \frac{\sqrt{5}-1}{2\sqrt{5}}$

$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1},$

$n \geq 0$

12. • top chip is blue → one below cannot be blue, so $n-2$ chips give a_{n-2} ways to sort

• top chip not blue → $n-1$ chips give a_{n-1} ways to sort

$a_0 = 1,\ a_1 = 4,\ a_2 = 4^2 - 1 = 15$

$a_n = 3a_{n-1} + 3a_{n-2} \quad \to a_n = s\left(\frac{3+\sqrt{21}}{2}\right)^n + t\left(\frac{3-\sqrt{21}}{2}\right)^n$

$x^2 - 3x - 3 = 0 \qquad s + t = 1$

$x = \frac{3 \pm \sqrt{9+12}}{2} = \frac{3 \pm \sqrt{21}}{2} \qquad \frac{(s+t)(3) + (s-t)(\sqrt{21})}{2} = 4$

$(s-t)(\sqrt{21}) = 5$

$s - t = 5/\sqrt{21} \to s = \frac{5+\sqrt{21}}{2\sqrt{21}},\ t = \frac{5-\sqrt{21}}{2\sqrt{21}}$

$$a_n = \left(\frac{5+\sqrt{21}}{2\sqrt{21}}\right)\left(\frac{3+\sqrt{21}}{2}\right)^n + \left(\frac{5-\sqrt{21}}{2\sqrt{21}}\right)\left(\frac{3-\sqrt{21}}{2}\right)^n, \quad n \geq 0$$

2 Q2 10 / 10

✓ **+ 10 pts** *correct*

3 C **10 / 10**

✓ **- 0 pts** *Correct*