MATH-3012-D Exam 2

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TOTAL POINTS

94 / 100

QUESTION 1

1 Q1 **25 / 25**

(a)

✓ **+ 8 pts** *Correct*

(b)

✓ **+ 8 pts** *Correct.*

(c)

✓ **+ 1 pts** *Point out $$11000$$ has prime factors $$2, 5, 11$$.*

✓ **+ 2 pts** *Calculate $$\phi(11000) = 4000$$.*

(d)

✓ **+ 4 pts** *Correct.*

+ 2 *Point adjustment*

QUESTION 2

2 Q2 **3 / 10**

✓ **+ 1 pts** *$$a_1 = 3$$.*

+ 2 *Point adjustment*

QUESTION 3

3 Q3 **25 / 25**

✓ **+ 15 pts** *(a) correct*

✓ **+ 5 pts** *(b) correct*

✓ **+ 1 pts** *(c) a_0=1, a_1=0*

+ 4 *Point adjustment*

QUESTION 4

4 Q4 **12 / 12**

✓ **+ 12 pts** *Correct*

QUESTION 5

5 Q5 **15 / 16**

(d)

✓ **- 1 pts** *partial*

QUESTION 6

6 Q6 **12 / 12**

✓ **- 0 pts** *Correct*

QUESTION 7

7 Curve **2 / 0**

✓ **+ 2 pts** *Correct*

Q1. (8+8+5+4=25 points)

(a) Find the number of permutations of the 26 alphabet letters in which none of the patterns MATH, JOYS, or FIELD appear.

(b) How many are there functions from a set A of 9 elements to the set $\{1,2,3,4,5,6\}$, where each of 1,2,3,4 is in the image.

(c) Among $\{1,2,\ldots,11000\}$ how many are NOT co-prime with 11000. Answer should be a number, like 350.

(d) How many are there ways to give 11 distinct books to 6 children so that each gets at least one book?

(a) C_1: MATH appears
C_2: JOYS appears
C_3: FIELD appears

$N = 26!$
$N(C_1) = 23! = N(C_2)$
$N(C_3) = 22!$
$N(C_1C_2) = 20!$
$N(C_1C_3) = N(C_2C_3) = 19!$
$N(C_1C_2C_3) = 16!$

$$\boxed{26! - (2\cdot 23! + 22!) + (20! + 2\cdot 19!) - 16!}$$

(b) C_1: 1 not in image
C_2: 2 not in image
C_3: 3 not in image
C_4: 4 not in image

$N = 6^9$
$N(C_i) = 5^9$
$N(C_iC_j) = 4^9$
$N(C_iC_jC_k) = 3^9$
$N(C_iC_jC_kC_l) = 2^9$

$$\boxed{6^9 - \binom{4}{1}5^9 + \binom{4}{2}4^9 - \binom{4}{3}3^9 + 2^9}$$

(c) $\frac{11000}{5} = \frac{2200}{5} = \frac{440}{5} = \frac{88}{2} = \frac{44}{2} = \frac{22}{2} = 11$

$11000 = 2^3 \cdot 5^3 \cdot 11$

$\phi(11000) = 11000\left(\frac{1}{2}\right)\left(\frac{4}{5}\right)\left(\frac{10}{11}\right)$
$= \frac{10000\cdot 4}{10} = 4000$

$11000 - 4000 - 1 = \boxed{6999}$

(d) $\text{surj}(11,6) = \sum_{j=0}^{6} (-1)^j \binom{6}{j} (6-j)^{11}$

$$= \boxed{6^{11} - \binom{6}{1}5^{11} + \binom{6}{2}4^{11} - \binom{6}{3}3^{11} + \binom{6}{4}2^{11} - 6}$$

1 Q1 **25 / 25**

(a)

✓ **+ 8 pts** *Correct*

(b)

✓ **+ 8 pts** *Correct.*

(c)

✓ **+ 1 pts** *Point out $$11000$$ has prime factors $$2, 5, 11$$.*

✓ **+ 2 pts** *Calculate $$\phi(11000) = 4000$$.*

(d)

✓ **+ 4 pts** *Correct.*

+ 2 *Point adjustment*

Q2. (10 points. Show work! Guessing has no credit.)

A ternary sequence is a sequence of 0, 1, 2. Let a_n be the number of ternary sequences of length n not having any of 00 and 01 as a substring. Find a recurrence formula for the sequence (a_n) and calculate a_1, a_2, a_3. (You don't have to solve the recurrence relation.)

start with 0 $\rightarrow$ 01, 02 $\rightarrow$ $2a_{n-2}$ $\qquad$ $a_1 = 0, 1, 2 \rightarrow 3$

start with 1,2 $\rightarrow$ 1_, 2_ $\rightarrow$ $2a_{n-1}$ $\qquad$ $a_2 = 01, 02, 11, 12, 10, 21, 22, 20 \rightarrow 8$

$$a_n = 2a_{n-1} + 2a_{n-2}, \quad a_1 = 3, \ a_2 = 8$$

$$x^2 - 2x - 2 = 0$$

$$x = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$$

$$\boxed{a_n = t(1+\sqrt{3})^n + s(1-\sqrt{3})^n}$$

$$a_1 = t + t\sqrt{3} + s - s\sqrt{3} = (t+s) + \sqrt{3}(t-s) = 3$$

$$a_2 = t(4+2\sqrt{3}) + s(4-2\sqrt{3}) = 4(t+s) + 2\sqrt{3}(t-s) = 8$$

$$4(t+s) + 2\sqrt{3}(t-s) = 8$$
$$2(t+s) + 2\sqrt{3}(t-s) = 6$$
$$2(t+s) = 2$$
$$t+s = 1$$

$$\sqrt{3}(t-s) = 2$$
$$t - s = 2/\sqrt{3}$$
$$t + s = 1$$
$$2t = \frac{\sqrt{3}+2}{\sqrt{3}} \rightarrow t = \frac{\sqrt{3}+2}{2\sqrt{3}}$$

$$s = 1 - \frac{2+\sqrt{3}}{2\sqrt{3}} = \frac{\sqrt{3}-2}{2\sqrt{3}}$$

$$a_n = \frac{2+\sqrt{3}}{2\sqrt{3}}(1+\sqrt{3})^n - \frac{2-\sqrt{3}}{2\sqrt{3}}(1-\sqrt{3})^n$$

$$a_1 = 3, \ a_2 = 8, \ a_3 = 2(3) + 2(8) = 6 + 16 = 22$$

2 Q2 **3 / 10**

✓ **+ 1 pts** *$$a_1 = 3$$.*

+ 2 *Point adjustment*

Q3. (15+5+5=25 points)

(a) Solve the recursion equation

$$x_n = x_{n-1} + 2x_{n-2}; \qquad x_0 = 8, x_1 = 1.$$

(b) Find the general solution for the recurrence relation $b_n = 5b_{n-1} - 6b_{n-2} + 2(5)^n$.

(c) Assume that $a_0 = 1, b_0 = 2$ and for $n \geq 1$ one has

$$a_n = 2a_{n-1} - b_{n-1}$$
$$b_n = a_{n-1} + 3b_{n-1}.$$

Find a recurrence relation for the sequence (a_n) and find enough initial values needed for the recurrence relation. (You don't have to solve the recurrence relation).

(a) $x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$X_n = t(2)^n + s(-1)^n$

$X_0 = t + s = 8$

$X_1 = 2t - s = 1$

$3t = 9$

$t = 3$

$s = 5$

$\boxed{X_n = 3(2)^n + 5(-1)^n}$

(b) $x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

$b_n = t(3)^n + s(2)^n + b_n^p$

$b_n^p = A\cdot 5^n \rightarrow A\cdot 5^n - 5A\cdot 5^{n-1} + 6A\cdot 5^{n-2} = 2\cdot 5^n$

$25A - 25A + 6A = 50$

$A = 50/6 = 25/3$

$\boxed{b_n = t(3)^n + s(2)^n + \frac{25}{3}(5)^n}$

(c) $a_1 = 2(1) - 2 = 0$

$b_1 = 1 + 3(2) = 7$

$a_n = 2a_{n-1} - \frac{1}{3}b_n + \frac{1}{3}a_{n-1}$

$a_n = \frac{7}{3}a_{n-1} - \frac{1}{3}b_n$

$a_n = t(7/3)^n + a_n^p$

$a_n^p = A\cdot b_n \rightarrow A\, b_n - \frac{7}{3}Ab_{n-1} = -\frac{1}{3}b_n$

$b_n =$

$3A\cdot b_n - 7A\, b_{n-1} = -b_n$

$3A\cdot 7 - 7A\cdot 2 = -7$

$21A - 14A = -7$

$7A = -7$

$A = -1$

$a_n = t(7/3)^n - b_n$

$a_0 = t - 2 = 1 \Rightarrow t = 3$

$\boxed{a_n = 3(7/3)^n - b_n, \quad a_0 = 1 \quad a_1 = 0 \quad b_0 = 2 \quad b_1 = 7}$

3 Q3 **25 / 25**

✓ **+ 15 pts** *(a) correct*

✓ **+ 5 pts** *(b) correct*

✓ **+ 1 pts** *(c) a_0=1, a_1=0*

+ 4 *Point adjustment*

Q4. (6+6=12 points.) An n-signal is a sequence of length n of flags from an unlimited supply of flags of colors A, B, C, D, E.

(i) Find the number of 20-signals containing at most 12 flags of each colors. You can give your answer in the form: The answer is equal to the coefficient of x^k, or $k!$ times the coefficient of x^k, in $f(x)$, with explicit value of k and formula of $f(x)$.

(ii) Find the number of 20-signals containing a even number of A flags and an odd number of B flags (no restriction on the numbers of other flags). The answer should be of the form similar to $3(2^{15}) + 7(5^{18})$.

(i) $A, B, C, D, E = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{12}}{12!}$

$f(x) = \left(\sum_{n=0}^{12} \frac{x^n}{n!}\right)^5$ Answer $= 20! \cdot$ coefficient(x^{20}) in $f(x)$

(ii) $A \to 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = (1, 0, 1, 0, \ldots) = \frac{e^x + e^{-x}}{2}$

$B \to \frac{x^1}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = (0, 1, 0, 1, \ldots) = \frac{e^x - e^{-x}}{2}$

$C, D, E \to 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots = e^x$

$g(x) = e^{3x}\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \left(\frac{e^{2x} - e^{-2x}}{4}\right)e^{3x} = \frac{e^{5x} - e^{x}}{4}$

$\frac{1}{4}(5^{20} - 1^{20}) = \boxed{\frac{1}{4}(5^{20}) - \frac{1}{4}}$

4 Q4 **12 / 12**

✓ **+ 12 pts** *Correct*

Q5. (6+4+3+3=16 points)

a. Draw the Young diagram of the partition the $7 = 1+1+1+2+2$, then find the transpose of this partition.

b. Let c_n be the number of partitions of n where each of summands 1, 2, and 3 occurs at most 5 times. Find the generating function of (c_n).

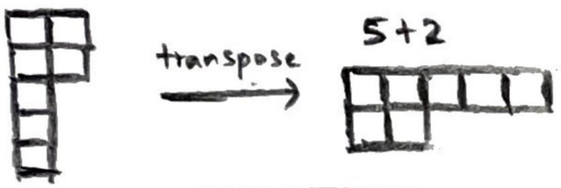
c. Find the number of partitions of 11 where summand 2 occurs at most 4 times. (Answer must be an integer like 100. You might want to use the expansion given on page 1.)

d. Find the generating function for the number of non-negative integer solutions of

$$\begin{cases} 2x+3y+4z=n \\ y \geq 5 \text{ and } y \text{ is odd, } 2 \leq z \leq 11 \end{cases}$$

2+2+1+1+1

(a)



5+2

(b) $(1-x^6)(1-x^{12})(1-x^{18})\cdot\prod_{k=1}^{\infty}\frac{1}{1-x^k}=g(x)$

(c) $(1-x^{10})\cdot\prod_{n=0}^{\infty}\frac{1}{1-x^n} \Rightarrow \text{coef}(x^{11}) \Rightarrow 56-1=\boxed{55}$

(d) $2x \to x^0+x^2+x^4+x^6+\dots=\frac{1}{1-x^2}$

$3y \to y^{15}+y^{21}+y^{27}+\dots=\frac{x^{15}}{1-x^6}$

$4z \to x^8+x^{12}+\dots+x^{40}+x^{44}=\left(\frac{1-x^{40}}{1-x^4}\right)x^8$

$$\boxed{g(x)=\frac{x^{23}}{(1-x^2)(1-x^6)(1-x^4)}}$$

5 Q5 **15 / 16**

(d)

✓ **- 1 pts** *partial*

Q6. (6+6=12 points)

(i) There are 19 balls of colors A and unlimited supplies of balls of colors B, C, and D. Let a_n be the number of different ways to choose n balls from the above collection, where the number of B balls is odd. Write down the generating function for the sequence (a_n). (answer should be in the form of the quotient of 2 polynomials.)

(ii) If $\frac{(1+x^5)^6}{(1-x)^3}$ is the generating function of the sequence (a_n), what is exact the value of a_{10}?

(i) $A \to x^0+x^1+x^2+\dots+x^{18}+x^{19} = \frac{1-x^{20}}{1-x}$ $\qquad C, D \to x^0+x^1+x^2+\dots = \frac{1}{1-x}$

$B \to x^1+x^3+x^5+\dots = \frac{x}{1-x^2}$

$$\boxed{g(x) = \frac{x(1-x^{20})}{(1-x)^2(1-x^2)}}$$

(ii) $(1+x^5)^6 = 1+\binom{6}{1}x^5+\binom{6}{2}x^{10}+\dots$ HOT $\qquad \frac{1}{(1-x)^3} = \sum_{n=0}^{\infty}\binom{n+3-1}{n}x^n$

$\frac{(1+x^5)^6}{(1-x)^3} = \binom{n+3-1}{n}x^n + \binom{6}{1}\binom{n+3-1}{n}x^{n+5} + \binom{6}{2}\binom{n+3-1}{n}x^{n+10}$

$\text{coef}(x^{10}) = \binom{12}{10}x^{10} + 6\binom{7}{5}x^{10} + 15\binom{2}{0}$

$= \boxed{\binom{12}{2}+6\binom{7}{2}+15} = \frac{12\times 11}{2}+6\cdot\frac{7\times 6}{2}+15$

$= 66+6(21)+15$

$= 66+126+15$

$= 192+15 = \boxed{207}$

6 Q6 **12 / 12**

✓ **- 0 pts** *Correct*

7 Curve 2 / 0

✓ **+ 2 pts** *Correct*

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3012 Exam 2

Instructions.

- Work on papers, scan and submit to Gradescope.
- Please refrain from disturbing your fellow students. **You can begin to scan only when the proctor gives the instruction (5 minutes before the end).** Gradescope site will be open only at that time.
- **Show work**, except where otherwise explicitly stated. You might loose part or even all credit of a question if you don't show work.
- Leave answer in the form of products, quotients, sums of factorials or combination numbers $\binom{n}{r}$, or permutation number $P(n, r)$; unless otherwise instructed. Box you answers except for proof questions.
- Notes, books, tablets, calculators are not allowed. You can have pens (pencils) and your ID on your desk. Print your name on each sheet (top left corner). Scratch papers, if needed, will be provided.
- Strict: No redundant solutions. Cross out anything not relevant to the solutions. If you give two answers/solutions to the same questions, the worse one will be graded. A wrong statement in a correct solution might get penalized.
- The last 5 minutes are for scanning and uploading your work. Have your cellphone fully charged and practice scanning and uploading beforehand. Anybody who writes on exam papers during the scanning period will get a 0 for the exam/quiz.
- There are 6 problems. Please sure to have all of them.

Some formulas

$$\mathrm{Surj}(n, k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + 42x^{10} + 56x^{11} + 77x^{12} + 101x^{13} + \ldots.$$