MATH 3215 Assignment 5

1. Let (X, Y) have joint PDF $f(x, y) = c$ for $x \in (0, y)$, $y \in (0, 1)$ and $f(x, y) = 0$ otherwise, where c is a constant.

(a) What is the constant c?

(b) Compute the marginal PDFs of X and Y.

(c) Determine whether X and Y are independent.

Since

$$1 = \int_0^1 \int_0^y c \, dx \, dy = \frac{c}{2},$$

we obtain $c = 2$. Moreover,

$$f_X(x) = \int_x^1 2 \, dy = 2 - 2x, \qquad f_Y(y) = \int_0^y 2 \, dx = 2y$$

for $x \in (0, 1)$ and $y \in (0, 1)$ respectively. Hence $f(x, y) \neq f_X(x) \cdot f_Y(y)$, so X and Y are not independent.

2. Let X and Y denote the numbers we see when rolling two dice.

(a) Is the event $\{X + Y = 7\}$ independent of the event $\{X = 4\}$?

(b) Is the event $\{X + Y = 6\}$ independent of the event $\{Y = 2\}$?

We have

$$\mathbb{P}\{X = 4, Y = 3\} = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = \mathbb{P}\{X + Y = 7\} \cdot \mathbb{P}\{X = 4\},$$

so yes for part (a). Moreover,

$$\mathbb{P}\{X = 4, Y = 2\} = \frac{1}{36} \neq \frac{5}{36} \cdot \frac{1}{6} = \mathbb{P}\{X + Y = 6\} \cdot \mathbb{P}\{Y = 2\},$$

so no for part (b).

3. (Not to be graded for correctness. To be more precise, the problem should state that a uniformly random card is chosen and then a uniformly random face of it is revealed.)

Of three cards, one is painted red on both sides; one is painted black on both sides; and one is painted red on one side and black on the other. A card is randomly chosen and placed on a table. If the side facing up is red, what is the probability that the other side is also red?

Out of three red faces, two cases have the other side being red too, so the probability is $2/3$.

4. As an indicator of whether a male has prostate cancer, doctors often perform a test that measures the level of the prostate specific antigen (PSA). Although higher PSA levels are indicative of cancer, the test is unreliable: The probability that a noncancerous man will have an elevated PSA level is 0.1, with this probability increasing to 0.25 if the man does have cancer. If, based on other factors, a physician is 70 percent certain that a male has prostate cancer, what is the conditional probability that he has the cancer given that the test indicates an elevated PSA level?

Let E denote the event of cancer and F denote the event of elevated PSA level. Then

$$\mathcal{P}(E \mid F) = \frac{\mathcal{P}(F \mid E) \cdot \mathcal{P}(E)}{\mathcal{P}(F)} = \frac{\mathcal{P}(F \mid E) \cdot \mathcal{P}(E)}{\mathcal{P}(F \mid E) \cdot \mathcal{P}(E) + \mathcal{P}(F \mid E^c) \cdot \mathcal{P}(E^c)}$$
$$= \frac{0.25 \cdot 0.7}{0.25 \cdot 0.7 + 0.1 \cdot 0.3} = \frac{35}{41}.$$

5. An insurance company classifies 20 percent of the population into the "good risk" category, 50 percent into "average risk", and 30 percent into "bad risk". The records indicate that the probabilities that good, average, and bad risk individuals will be involved in an accident over a 1-year span are, respectively, 0.05, 0.15, and 0.30. What overall proportion of people have accidents in a fixed year? If an individual had no accidents in the year, what is the probability that the person is in the good risk category?

Let E denote the event of having accidents. Let G, A, and B denote the event of being in the three categories respectively. The overall proportion of having accidents is

$$\mathcal{P}(E) = 0.2 \cdot 0.05 + 0.5 \cdot 0.15 + 0.3 \cdot 0.3 = 0.175.$$

The conditional probability is

$$\mathcal{P}(G \mid E^c) = \frac{\mathcal{P}(E^c \mid G) \cdot \mathcal{P}(G)}{\mathcal{P}(E^c)} = \frac{(1 - 0.05) \cdot 0.2}{1 - 0.175} = \frac{38}{165}.$$

6. The number of times that an individual contracts a cold in a given year is a Poisson random variable with parameter equal to 3. Suppose that a new drug reduces the Poisson parameter to 2 for 75 percent of the population. For the other 25 percent of the population, the drug has no effect. Given that a random individual from the population tries the drug for the year and has 0 colds in that time, how likely is it that the drug was actually effective for this individual?

Let X denote the number of colds, and let Y be the indicator of whether the drug is effective. Then we can compute

$$\mathbb{P}\{Y = 1 \mid X = 0\} = \frac{\mathbb{P}\{X = 0 \mid Y = 1\} \cdot \mathbb{P}\{Y = 1\}}{\mathbb{P}\{X = 0\}}$$
$$= \frac{\mathbb{P}\{X = 0 \mid Y = 1\} \cdot \mathbb{P}\{Y = 1\}}{\mathbb{P}\{X = 0 \mid Y = 1\} \cdot \mathbb{P}\{Y = 1\} + \mathbb{P}\{X = 0 \mid Y = 0\} \cdot \mathbb{P}\{Y = 0\}}$$
$$= \frac{e^{-2} \cdot 3/4}{e^{-2} \cdot 3/4 + e^{-3} \cdot 1/4} = \frac{3\, e^{-2}}{3\, e^{-2} + e^{-3}}.$$

7. Consider independent uniform random variables $X_1, \ldots, X_n$ over $(0, 1)$. Find the CDF and then the PDF of the random variable $M := \max\{X_1, \ldots, X_n\}$.

(Hint: If events $E_1, \ldots, E_n$ are independent, then $\mathbb{P}\{E_1 \cap \cdots \cap E_n\} = \mathbb{P}\{E_1\} \times \cdots \times \mathbb{P}\{E_n\}$.)

The CDF of M is $F(t) = 0$ for $t \le 0$, $F(t) = 1$ for $t \ge 1$, and

$$F(t) = \mathbb{P}\{M \le t\} = \mathbb{P}\{X_1 \le t, \ldots, X_n \le t\} = \mathbb{P}\{X_1 \le t\} \times \cdots \times \mathbb{P}\{X_n \le t\} = t^n$$

for $t \in (0, 1)$. The PDF of M is $f(t) = 0$ for $t \notin (0, 1)$ and $f(t) = n\, t^{n-1}$ for $t \in (0, 1)$.

8. Recall that an exponential random variable X with parameter $\lambda > 0$ has PDF $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and $f(x) = 0$ otherwise.

(a) Show that $\mathbb{P}\{X > s + t \mid X > t\} = \mathbb{P}\{X > s\}$, where $s, t \geq 0$. (This is called the memoryless property.)

(b) Let $X_1, \ldots, X_n$ be independent exponential random variables with respective parameters $\lambda_1, \ldots, \lambda_n$. Show that $M := \min\{X_1, \ldots, X_n\}$ is an exponential random variable with parameter $\sum_{i=1}^{n} \lambda_i$.
(Hint: Consider its CDF.)

We have $\mathbb{P}\{X > s\} = \int_s^\infty \lambda e^{-\lambda x}\, dx = e^{-\lambda s}$, and, on the other hand,

$$\mathbb{P}\{X > s + t \mid X > t\} = \frac{\mathbb{P}\{X > s + t\}}{\mathbb{P}\{X > t\}} = e^{-\lambda(s+t)}/e^{-\lambda t} = e^{-\lambda s}.$$

Let F be the CDF of M. Then

$$\begin{aligned} 1 - F(t) = \mathbb{P}\{M > t\} &= \mathbb{P}\{X_1 > t, \ldots, X_n > t\} \\ &= \mathbb{P}\{X_1 > t\} \times \cdots \times \mathbb{P}\{X_n > t\} \\ &= e^{-\lambda_1 t} \times \cdots \times e^{-\lambda_n t} = e^{-(\sum_{i=1}^{n} \lambda_i)\, t}. \end{aligned}$$

Hence F is the CDF of the exponential distribution with parameter $\sum_{i=1}^{n} \lambda_i$.