ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 1

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Please remember to write your name and GTID in the homework submitted.

Problem 1 Neftci's book (Second Edition) Chapter 5, exercise 1

You are given two discrete random variables X, Y that assume the possible values 0, 1 according to the following joint distribution:

	$\mathbb{P}(Y=1)$	$\mathbb{P}(Y=0)$
$\mathbb{P}(X=1)$	0.2	0.4
$\mathbb{P}(X=0)$	0.15	0.25

(a) What are the marginal distributions of X and Y?

(b) Are X and Y independent?

(c) Calculate $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(d) Calculate the conditional distribution $\mathbb{P}[X|Y=1]$.

(e) Obtain the conditional expectation $\mathbb{E}[X|Y=1]$ and the conditional variance $\mathbb{V}[X|Y=1]$. (Hints: We can calculate conditional variance in similar way as we do for unconditional variance. In this question, $\mathbb{V}[X|Y=1] = \mathbb{E}[(X-\mathbb{E}[X])^2|Y=1]$)

Answers:

(a)

	$\mathbb{P}(Y=1)$	$\mathbb{P}(Y=0)$	Marginal X
$\mathbb{P}(X=1)$	0.2	0.4	0.6
$\mathbb{P}(X=0)$	0.15	0.25	0.4
Marginal Y	0.35	0.65	

(b)

$$F(X=1, Y=1) = 0.6 * 0.35 = 0.21$$
$$F(X=1, Y=0) = 0.6 * 0.65 = 0.39$$
$$F(X=0, Y=1) = 0.4 * 0.35 = 0.14$$
$$F(X=0, Y=0) = 0.4 * 0.65 = 0.26$$

$$F(X,Y) \neq F(X)F(Y)$$

Therefore, X and Y are not independent.

(c)

$$\mathbb{E}[X] = 1 * 0.6 + 0 * 0.4 = 0.6$$
$$\mathbb{E}[Y] = 1 * 0.35 + 0 * 0.65 = 0.35$$

(d)

$$\mathbb{P}(X = 1|Y = 1) = \frac{\mathbb{P}(X = 1, Y = 1)}{\mathbb{P}(Y = 1)} = \frac{0.2}{0.35} = \frac{4}{7}$$
$$\mathbb{P}(X = 0|Y = 1) = \frac{\mathbb{P}(X = 0, Y = 1)}{\mathbb{P}(Y = 1)} = \frac{0.15}{0.35} = \frac{3}{7}$$

(e)

$$\begin{aligned}
\mathbb{E}[X|Y = 1] &= 1 * \mathbb{P}(X = 1|Y = 1) + 0 * \mathbb{P}(X = 0|Y = 1) \\
&= 1 * \frac{4}{7} + 0 * \frac{3}{7} \\
&= \frac{4}{7} \\
\mathbb{V}[X|Y = 1] &= \mathbb{E}[(X - \mathbb{E}[X])^2|Y = 1] \\
&= \left(1 - \frac{4}{7}\right)^2 \mathbb{P}(X = 1|Y = 1) + \left(0 - \frac{4}{7}\right)^2 \mathbb{P}(X = 0|Y = 1) \\
&= \left(\frac{3}{7}\right)^2 \frac{4}{7} + \left(\frac{4}{7}\right)^2 \frac{3}{7} \\
&= \frac{12}{49} \approx 0.24490
\end{aligned}$$

Problem 2 Neftci's book (Second Edition) Chapter 5, exercise 2

We let the random variable X_n be defined as,

$$X_n = \sum_{i=1}^{n} B_i$$

where each B_i $(i = 1, 2, \cdots, n)$ is distributed according to

$$B_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p. \end{cases}$$

B_i's are independent and identically distributed.

(a) Calculate the probabilities $\mathbb{P}(X_4 > k)$ for $k = 0, 1, 2, 4$ and plot the distribution function. (Hints: let $p = 1/2$ when you plot the distribution function. The matlab function for binomial distribution are binopdf( ) and binocdf( ).)

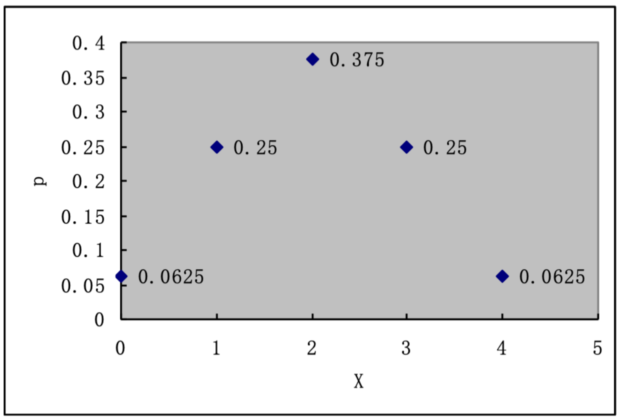
(b) Calculate the expected value and the variance of X_n for $n = 3$.

Answers:

(a)

$$\begin{aligned}
\mathbb{P}(X_4 > 0) &= 1 - \mathbb{P}(X_4 = 0) \\
&= 1 - \binom{4}{0} p^0 (1-p)^4 \\
&= 1 - (1-p)^4 \\
\mathbb{P}(X_4 > 1) &= 1 - \mathbb{P}(X_4 = 0) - \mathbb{P}(X_4 = 1) \\
&= 1 - \binom{4}{0} p^0 (1-p)^4 - \binom{4}{1} p^1 (1-p)^3 \\
&= 1 - (1-p)^4 - 4p(1-p)^3 \\
\mathbb{P}(X_4 > 2) &= 1 - \mathbb{P}(X_4 = 0) - \mathbb{P}(X_4 = 1) - \mathbb{P}(X_4 = 2) \\
&= 1 - \binom{4}{0} p^0 (1-p)^4 - \binom{4}{1} p^1 (1-p)^3 - \binom{4}{2} p^2 (1-p)^2 \\
&= 1 - (1-p)^4 - 4p(1-p)^3 - 6p^2(1-p)^2 \\
\mathbb{P}(X_4 > 4) &= 0
\end{aligned}$$

The PDF graph is shown as follow:



(b)

$$\begin{aligned}
\mathbb{E}[X_n] &= np \\
\mathbb{V}[X_n] &= np(1-p)
\end{aligned}$$

Therefore, $\mathbb{E}[X_3] = 3p$, $\mathbb{V}[X_3] = 3p(1-p)$

Problem 3 Neftci's book (Second Edition) Chapter 5, exercise 3

We say that Z is exponentially distributed with parameter $\lambda > 0$ if the distribution function of Z is given by:

$$\mathbb{P}(Z \leq z) = \begin{cases} 1 - e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

(a) Determine and plot the density function (probability density function) of Z.

(b) Calculate $\mathbb{E}[Z]$.

(c) Obtain the variance of Z.

(d) Suppose Z_1 and Z_2 are both distributed as exponential with parameter λ and are independent. Calculate the distribution of their sum:

$$S = Z_1 + Z_2$$

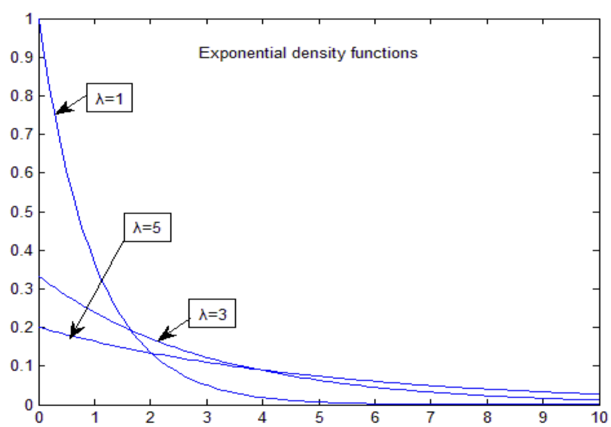
(e) Calculate the mean and the variance of S.

Answers:

(a)

$$f(z) = \frac{\partial \mathbb{P}(Z \leq z)}{\partial z} = \begin{cases} \lambda e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

The PDF graph:



(b)

$$\begin{aligned} \mathbb{E}[Z] &= \int_0^\infty z f(z) dz \\ &= [ze^{-\lambda z}]_0^\infty + \int_0^\infty e^{-\lambda z} dz \\ &= \frac{1}{\lambda} \end{aligned}$$

(c)

$$\mathbb{E}[Z^2] = \int_0^\infty z^2 f(z) dz = \frac{2}{\lambda^2}$$

$$\mathbb{V}[Z] = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 = \frac{1}{\lambda^2}$$

(d)

$$\begin{aligned} f(s) &= \int_0^s \lambda e^{-\lambda z} \lambda e^{-\lambda(s-z)} dz \\ &= \lambda^2 e^{-\lambda s} \int_0^s dz \\ &= \lambda^2 s e^{-\lambda s} \end{aligned}$$

(e)

$$\begin{aligned} \mathbb{E}[S] &= \int_0^\infty s\lambda^2 s e^{-\lambda s} ds \\ &= \lambda^2 \int_0^\infty s^2 e^{-\lambda s} ds \\ &= \frac{2}{\lambda} \\ \mathbb{E}[S^2] &= \int_0^\infty s^2 \lambda^2 s e^{-\lambda s} ds \\ &= \lambda^2 \int_0^\infty s^3 e^{-\lambda s} ds \\ &= \frac{6}{\lambda^2} \\ \mathbb{V}[S] &= \mathbb{E}[S^2] - (\mathbb{E}[S])^2 \\ &= \frac{2}{\lambda^2} \end{aligned}$$

Problem 4 Neftci's book (Second Edition) Chapter 5, exercise 4

A random variable Z has Poisson distribution if

$$\begin{aligned} p(k) &= \mathbb{P}(Z = k) \\ &= \frac{\lambda^k e^{-\lambda}}{k!} \end{aligned}$$

for $k = 0, 1, 2, \cdots$

(a) Use the expansion

$$e^\lambda = 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{3!} + \cdots$$

to show that

$$\sum_{k=0}^{\infty} p(k) = 1$$

(b) Calculate the mean $\mathbb{E}[Z]$ and the variance $\mathbb{V}[Z]$.

Answers:

(a)

Since

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{3!} + \cdots$$

$$\begin{aligned}
\sum_{k=0}^{\infty} p(k) &= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} e^{\lambda} = 1
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbb{E}[Z] &= \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} \\
&= \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-1)!} \\
&= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[Z^2] &= \sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!} \\
&= \sum_{k=1}^{\infty} k \frac{\lambda^k e^{-\lambda}}{(k-1)!} \\
&= e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^k}{(k-1)!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} (x+1) \frac{\lambda^{x+1}}{x!} \\
&= \lambda e^{-\lambda} \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} + \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \\
&= \lambda^2 + \lambda \\
\mathbb{V}[Z] &= \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 \\
&= \lambda^2 + \lambda - \lambda^2 = \lambda
\end{aligned}$$

Problem 5

Let us now try something more interesting.
A person has 5 coins in his pocket. Two have both sides being heads, one has both sides being tails, and two are normal. The coins cannot be distinguished unless one looks at them.

(a) The person closes his eyes, picks a coin from pocket at random, and tosses it. What is the probability that the down-side of the coin is heads?

(b) He opens his eyes and sees that the up-side of the coin is heads. What is the probability that the downside is also heads (namely, this is a two-heads coin).

(c) Without looking at the other side of the coin, he tosses it again. What is the probability that the downside is heads?

(d) Now he looks at the upside of the coin and it is heads. What is the probability that the downside of the coin is heads?

Hints:

1) To get yourself started, let D denote the event that a two-heads coin is picked, N denote the event that a normal coin is picked, and Z be the event that a two-tails coin is picked.

   Let H_{L_i} (and H_{U_i}) denote the event that the down-side (and the up-side) of the coin on the i^{th} toss is heads.

2) You may find

$$\mathbb{P}(B_1|A) = \frac{\mathbb{P}(B_1 \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B_1)\mathbb{P}(B_1)}{\mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2)}$$

   useful.

Answers:

(a)

$$\begin{aligned}\mathbb{P}(H_{L_1}) &= \mathbb{P}(H_{L_1}|D)\mathbb{P}(D) + \mathbb{P}(H_{L_1}|N)\mathbb{P}(N) + \mathbb{P}(H_{L_1}|Z)\mathbb{P}(Z) \\ &= 1 * \frac{2}{5} + \frac{1}{2} * \frac{2}{5} + 0 * \frac{1}{5} = \frac{3}{5}\end{aligned}$$

(b)

$$\mathbb{P}(H_{L_1}|H_{U_1}) = \frac{\mathbb{P}(H_{L_1} \cap H_{U_1})}{\mathbb{P}(H_{U_1})} = \frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}$$

(c)

$$\begin{aligned}\mathbb{P}(H_{L_2}|H_{U_1}) &= \frac{\mathbb{P}(H_{L_2}\cap H_{U_1})}{\mathbb{P}(H_{U_1})}\\ &= \frac{\mathbb{P}(H_{L_2}\cap H_{U_1}|D)\mathbb{P}(D)+\mathbb{P}(H_{L_2}\cap H_{U_1}|N)\mathbb{P}(N)+\mathbb{P}(H_{L_2}\cap H_{U_1}|Z)\mathbb{P}(Z)}{\mathbb{P}(H_{U_1})}\\ &= \frac{1*\frac{2}{5}+\frac{1}{4}*\frac{2}{5}+0*\frac{1}{5}}{\frac{3}{5}} = \frac{5}{6}\end{aligned}$$

(d)

Similarly,

$$\begin{aligned}\mathbb{P}(H_{L_2}|H_{U_1}\cap H_{U_2}) &= \frac{\mathbb{P}(H_{L_2}\cap H_{U_1}\cap H_{U_2})}{\mathbb{P}(H_{U_1}\cap H_{U_2})}\\ &= \frac{1*\frac{2}{5}+0*\frac{2}{5}+0*\frac{1}{5}}{1*\frac{2}{5}+\frac{1}{4}*\frac{2}{5}+0*\frac{1}{5}} = \frac{4}{5}\end{aligned}$$

Problem 6

Below is a classical interview question from investment banks. Only a few people have got it completely correct. Now that you have learnt enough to work it out, think it through and... enjoy!

I will roll a single die no more than three times. You can stop me immediately after the first roll, or immediately after the second, or you can wait for the third. I will pay you the same amount of dollars as the number turned up on my last roll (roll number three unless you stop me sooner).

(a) What is your playing strategy?

(b) If you were running this game, how much would you charge players for repeated plays of the game?

(Hint: take a note at how 'the law of large numbers' come into play here)

(c) Suppose instead an amended game is played: I roll a single die three times without pause, and the payoff to player is the maximum of the three rolls. What is the expected payoff to the player? Can you tell up front whether the original or amended game has the higher expected payoff?

Answers:

(a)

Expected value of asking for third roll

$$\mathbb{E}[R_3] = \frac{1+2+3+4+5+6}{6} = 3.5.$$

Expected value of asking for second roll

$$\mathbb{E}[R_2] = \frac{1}{2}*3.5+\frac{1}{2}*\frac{4+5+6}{3} = 4.25.$$

Therefore, the player should stop if he/she gets number 5 or 6 from the first roll, otherwise continue until he/she gets 4, 5 or 6.

(b)

Expected payoff of the game is

$$\mathbb{E}[R_1] = \frac{1}{6} * 5 + \frac{1}{6} * 6 + \frac{1}{6} * 4 * \mathbb{E}[R_2] = 4.67.$$

So in a repeated game, the player is actually getting expected payoff (Law of large number). Thus, you can charge the expected value ($4.67) plus a commission.

(c)

Now, for the amended game,

$$\begin{aligned}
\mathbb{P}(\$1 \text{ payoff}) &= \left(\frac{1}{6}\right)^3 = \frac{1}{216} \\
\mathbb{P}(\$2 \text{ payoff}) &= \left(\frac{2}{6}\right)^3 - \frac{1}{216} = \frac{7}{216} \\
\mathbb{P}(\$3 \text{ payoff}) &= \left(\frac{3}{6}\right)^3 - \frac{1}{216} - \frac{7}{216} = \frac{19}{216} \\
\mathbb{P}(\$4 \text{ payoff}) &= \left(\frac{4}{6}\right)^3 - \frac{1}{216} - \frac{7}{216} - \frac{19}{216} = \frac{37}{216} \\
\mathbb{P}(\$5 \text{ payoff}) &= \left(\frac{5}{6}\right)^3 - \frac{1}{216} - \frac{7}{216} - \frac{19}{216} - \frac{37}{216} = \frac{61}{216} \\
\mathbb{P}(\$6 \text{ payoff}) &= 1 - \frac{1}{216} - \frac{7}{216} - \frac{19}{216} - \frac{37}{216} - \frac{61}{216} = \frac{91}{216}
\end{aligned}$$

Therefore, the expected payoff for the amended game is:

$$\mathbb{E}[R] = \sum_{k=1}^{6} k * \mathbb{P}(\$ \text{ k payoff}) = \$\frac{119}{24} \approx \$4.96.$$

Since $4.96 > $4.667, the amended game has higher expected payoff.

Problem 7

Suppose that you are on a game show and there are 10 doors in front of you. You know that there is a prize behind one of them and nothing behind the other 9. You have to choose a door that contains the prize in order to win the game. Once the game starts, you choose a door. After that, the game show host promises that instead of immediately opening your chosen door to reveal its content, he will open one of the other 9 doors that has nothing behind it. He will then give you the option to either keep your choice or choose another door. You may assume that the host is completely impartial – not malicious in any way. For instance, if you choose door 3, he will open one door, say door 5, to reveal that it is empty. Should you continue with door 3 or choose another door? Calculate the probability that the prize is behind your chosen door before the game show host shows you an empty door and the probability of you winning the prize by changing to a different door after seeing the revealed empty door.

Answers:

The probability of finding the prize before host reveals a door that is empty is $\frac{1}{10}$.

The probability of finding the prize by changing to another door after the host has revealed an empty door is:

$$\begin{aligned}
P(\text{find the prize}) = {} & P(\text{chose the wrong door at first}) * P(\text{switch to the right one}) \\
& + P(\text{chose the right door at first}) * P(\text{switch to the right one}) \\
& = \frac{9}{10} * \frac{1}{8} + \frac{1}{10} * 0 = \frac{9}{80}
\end{aligned}$$

Problem 8

Suppose that Mr. Warren Buffet and Mr. Zhao Danyang agree to meet at a specified place between 12 pm and 1 pm. Suppose each person arrives between 12 pm and 1 pm at random with uniform probability. What is the distribution function for the length of the time that the first to arrive has to wait for the other?

Answers:

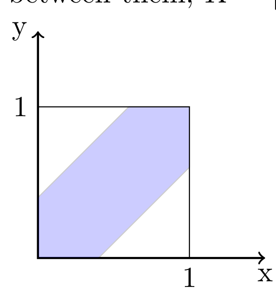
- Let X be the time Buffet arrives.

$$f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{o.w} \end{cases}$$

- Let Y be the time Zhao Danyang arrives.

$$f_Y(y) = \begin{cases} 1 & y \in [0,1] \\ 0 & \text{o.w} \end{cases}$$

- Let K be the time length between them, $K = |X - Y|$. Since each person arrive at random



with uniform probability,

$$\begin{aligned}
P(K \leq k) &= 1 - P(|X - Y| > k) \\
&= 1 - \int_0^{1-k} \int_{x+k-1}^{1} dydx - \int_k^1 \int_0^{x-k+1} dydx \\
&= 1 - (1-k)^2 = -k^2 + 2k
\end{aligned}$$

$$f(k) = \frac{\partial P(K \leq k)}{\partial k} = 2 - 2k$$

Problem 9

Take a stick of unit length and break it into three pieces, choosing the two break points at random. (The break points are assumed to be chosen simultaneously). What is the probability that the three pieces can be used to form a triangle? Hint: the necessary and sufficient condition for the three pieces to form a triangle is that the total length of any two pieces must be greater than the length of the remaining piece.

Answers:

Let X, Y, $1-X-Y$ be the length of three pieces, and X & Y and uniform distributed between 0 and 1.

Conditioned on $1-X-Y \geq 0$, X, Y have joint density

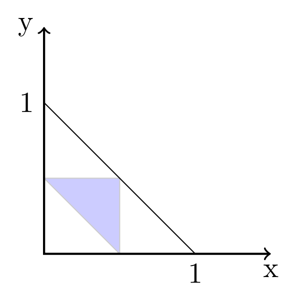
$$f_{X,Y}(x,y) = \begin{cases} 2 & x \in [0,1], y \in [0, 1-x] \\ 0 & \text{otherwise} \end{cases}$$

In order to form a triangle, the sum of lengths of any two pieces must exceed the length of the third. Thus:

$$\begin{aligned} X + (1-X-Y) &> Y \\ X + Y &> (1-X-Y) \\ Y + (1-X-Y) &> X, \end{aligned}$$

This leads to:

$$\begin{aligned} Y &< 0.5 \\ X &< 0.5 \\ X+Y &> 0.5, \end{aligned}$$



The shaded region is the feasible region and it is 1/4 of the total possible region.

$$P = \int_0^{0.5} \int_{0.5-x}^{0.5} f_{X,Y}(x,y)dydx = \int_0^{0.5} \int_{0.5-x}^{0.5} 2dydx = \frac{1}{4}$$

Problem 10

Let X, Y be continuous random variables with following joint probability density function:

$$f(x,y) = Cxy^2 \qquad \forall 0 \leq x \leq y \leq 1$$

1 What is the acceptable value of C?

2 Are X and Y independent?

3 Compute the following:

(a) $f(x)$ and $\mathbb{E}[X]$

(b) $f(y)$ and $\mathbb{E}[Y]$

(c) $\mathbb{E}[X|Y=y]$

(d) $\mathbb{E}[Y|X=x]$

Answers:

1

$$\begin{aligned}
&\int_0^1 \int_0^y Cxy^2 dxdy = 1 \\
\implies &\int_0^1 Cy^2 \left(\frac{x^2}{2}\right)\Big|_0^y dy = 1 \\
\implies &\int_0^1 C\frac{y^4}{2} dy = 1 \\
\implies &C = 10
\end{aligned}$$

2

$$\begin{aligned}
f_X(x) &= \int_x^1 10xy^2 dy = \frac{10}{3}(x - x^4) \\
f_Y(y) &= \int_0^y 10xy^2 dy = 5y^4
\end{aligned}$$

$f(x,y) \neq f_X(x)f_Y(y)$. Therefore, X and Y are not independent.

3

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} xf_X(x)dx = \int_0^1 x\frac{10}{3}(x - x^4)dx = \frac{5}{9} \\
\mathbb{E}[Y] &= \int_{-\infty}^{\infty} yf_Y(y)dy = \int_0^1 y5y^4 dy = \frac{5}{6} \\
f(x|Y=y) &= \frac{f(x,y)}{f_Y(y)} = \frac{2x}{y^2}1_{\{0\le x\le y\le 1\}} \\
f(y|X=x) &= \frac{f(x,y)}{f_X(x)} = \frac{3y^2}{1-x^3}1_{\{0\le x\le y\le 1\}} \\
\mathbb{E}[X|Y=y] &= \int_0^y xf(x|Y=y)dx = \int_0^y x\frac{2x}{y^2}dx = \frac{2y}{3} \\
\mathbb{E}[Y|X=x] &= \int_x^1 yf(y|X=x)dy = \int_x^1 y\frac{3y^2}{1-x^3}dx = \frac{3(1-x^4)}{4(1-x^3)}
\end{aligned}$$

Problem 11

Given a continuous random variable X, find the value of c for which $\mathbb{E}[(X–c)^2]$ is minimized.

Answers:

$$\begin{aligned}
\mathbb{E}[(X-c)^2] &= \mathbb{E}[X^2–2Xc+c^2] \\
&= \mathbb{E}[X^2]–2c\mathbb{E}[X]+c^2 \\
f(c) &= c^2 - 2\mathbb{E}[X]*c+\mathbb{E}[X^2]
\end{aligned}$$

To minimize $f(c)$, c has to be $-\left(-\frac{2\mathbb{E}[X]}{2}\right) = \mathbb{E}[X]$, to check it is the minimum value, $\frac{\partial^2 f}{\partial c^2} = 2 > 0$.

Problem 12

A random variable X has probability density function:

$$f(x) = \begin{cases} cxe^{-k^2x^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) Find the value of c which makes $f(x)$ a valid probability density function;

(b) Calculate the $\mathbb{E}[X]$ and $\mathbb{V}[X]$.

Hint: The standard normal distribution has probability density function:

$$f(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}$$

Answers:

(a)

$$\int_{-\infty}^{\infty} f(x)dx = 1 \implies \int_0^{\infty} cxe^{-k^2x^2}dx = \frac{c}{2k^2} = 1 \implies c = 2k^2$$

(b)

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^{\infty} xf(x)dx = \int_0^{\infty} 2k^2x^2e^{-k^2x^2}dx = -xe^{-k^2x^2}\Big|_0^{\infty} + \int_0^{\infty} e^{-k^2x^2}dx \\
&= \int_0^{\infty} e^{-k^2x^2}dx = \frac{\sqrt{\pi}}{|k|}\int_0^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}2k^2x^2}d(\sqrt{2}|k|x) = \frac{\sqrt{\pi}}{2|k|} \\
\mathbb{E}[X^2] &= \int_0^{\infty} x^2f(x)dx = \int_0^{\infty} 2k^2x^3e^{-k^2x^2}dx \\
&= \frac{1}{k^2}\int_0^{\infty} te^{-t}dt = \frac{1}{k^2}(-te^{-t}\Big|_0^{\infty} + \int_0^{\infty} e^{-t}dt) \\
&= \frac{1}{k^2}(-te^{-t} - e^{-t})\Big|_0^{\infty} = \frac{1}{k^2} \\
\mathbb{V}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{k^2} - \frac{\pi}{4k^2}
\end{aligned}$$

Problem 13

Two players A and B play a marble game. Each player has both a red and blue marble. They *randomly* present one marble to each other. If both present red, A wins \$3. If both present blue, A wins \$1. If the colors of the two marbles do not match, B wins \$2. Is it better to be A, or B, or does it matter?

Answers:

$$
\begin{aligned}
\mathbb{E}[R_A] &= \frac{1}{4} * 3 + \frac{1}{4} * 1 = 1 \\
\mathbb{E}[R_B] &= \frac{1}{2} * 2 = 1 \\
\mathbb{V}[R_A] &= \frac{1}{4} * 2^2 + \frac{1}{4} * 0^2 + \frac{1}{2} * 1^2 = \frac{3}{2} \\
\mathbb{V}[R_B] &= \frac{1}{2} * 1^2 + \frac{1}{2} * 1^2 = 1
\end{aligned}
$$

The expected payoff of A and B are the same. Because B's variance is smaller than A's variance, B is less risky than A. Thus, it's better to be B. The host should ask for at least \$1, which is the expected payoff of A and B.

Problem 14

There is a bag of 100 coins, 99 of them are fair coins and the rest one is a coin with both sides being heads. You randomly drew a coin from the bag and tossed it 10 times. The result was 10 heads. What's the probability that you drew the coin with both sides to be heads?

Answers:

By Bayes' Theorem:

$$
\begin{aligned}
\mathbb{P}(\text{2 Heads Coin}|\text{10 Heads Toss}) &= \frac{\mathbb{P}(\text{10 Heads Toss}|\text{2 Heads Coin})\mathbb{P}(\text{2 Heads Coin})}{\mathbb{P}(\text{10 Heads Toss})} \\
&= 1 * 0.01/(1 * 0.01 + 0.99 * 0.5^{10}) = 0.91184
\end{aligned}
$$

Problem 15

There is a deck of 50 cards. Each card has a number from 1 to 50 and nothing else. The dealer will shuffle the deck and reveal the top 4 cards one by one (donate the numbers on the top four cards to be A, B, C, D, A being the first card and D being the forth). If the numbers on the 4 cards revealed are in ascending order (not necessarily consecutive, as long as $A < B < C < D$) you will win \$10, otherwise you lose \$1. Would you play? How about a deck of 100 cards?

Answers:

In fact, it does not matter how many cards there are in the deck, because you will always have equivalent chances of getting each of the combination of four cards from the deck, with $\binom{n}{4}$ combinations. ($n \geq 4$, n is the total number of cards). Of all the sequences of each combination, there is only one sequence that satisfies a win, that is $A < B < C < D$. Also notice that the chance of getting this particular sequence in each of the $\binom{n}{4}$ combinations is always $\frac{1}{4!} = \frac{1}{24}$. So the expected payoff is always $\frac{1}{24} * 10 - \frac{23}{24} * 1 < 0$.

Problem 16

Three people, A, B and C are in a three-man duel. A is a bad shooter that shoots with 1/3 accuracy. B is moderate that shoots with 2/3 accuracy. C is a perfect shooter that shoots on target all the time. In this duel A will shoot first, then B and then C. The cycle goes on until only one person is left. All of the three people are perfectly smart and rational. They'll do their best to optimize their chance of survival. What is A's best course of action? And what's the chance of survival?

(Hint: A can choose to shoot at either a person or deliberately miss the target.)

Answers:

To maximize their chances each of the three people would prefer to be left with a weaker opponent. So B would not shoot at A in preference to C, and C will not shoot at A in preference to B. Therefore A will not be shot at until B or C is dead and he or she will either be left standing with B or C, and C will either go first or second. Now consider the following situations:

- Probability of A, shooting first, surviving against B is given by (donate by p):

$$
\begin{aligned}
p &= \mathbb{P}(\text{A hits B}) + \mathbb{P}(\text{A misses B}) * \mathbb{P}(\text{B misses A}) * p \\
p &= \frac{1}{3} + \frac{2}{3} * \frac{1}{3} * p \\
\implies p &= \frac{3}{7}
\end{aligned}
$$

- Probability of A, shooting second, surviving against B is given by:

$$
\begin{aligned}
p &= \mathbb{P}(\text{B misses A}) * (\mathbb{P}(\text{A hits B}) + \mathbb{P}(\text{A misses B}) * p) \\
p &= \frac{1}{3} * (\frac{1}{3} + \frac{2}{3} * p) \\
\implies p &= \frac{1}{7}
\end{aligned}
$$

- Probability of A, shooting first, surviving against C is given by:

$$
\begin{aligned}
p &= \mathbb{P}(\text{A hits C}) + \mathbb{P}(\text{A misses C}) * \mathbb{P}(\text{C misses A}) * p \\
p &= \frac{1}{3} + \frac{2}{3} * 0 * p \\
\implies p &= \frac{1}{3}
\end{aligned}
$$

- Probability of A, shooting second, surviving against C is given by:

$$\begin{aligned} p &= \mathbb{P}(\text{C misses A}) * (\mathbb{P}(\text{A hits C}) + \mathbb{P}(\text{A misses C}) * p) \\ p &= 0 * (\frac{1}{3} + \frac{2}{3} * p) \\ \implies p &= 0 \end{aligned}$$

So, A's probability of surviving from each position is:

- B, with A shooting first: 3/7
- C, with A shooting first: 1/3
- B, with A shooting second: 1/7
- C, with A shooting second: 0

If A chooses to deliberately miss, then A is guaranteed to shoot first when facing B or C alone. A can also choose to shoot C and try to face B in the end for a higher survival probability compared to facing C, but it is not guaranteed. By comparing survival probabilities, it is clear that A is better off to shoot first than to face B alone, not to mention the latter case is not guaranteed. So A is best off not to shoot anyone since the advantage A gains by having the first shot exceeds any possible benefit of facing B rather than C. A should deliberately miss.

Given that A is neither going to shoot at B or C, or be shot at by either B or C until one is dead, B and C are essentially in a two-person duel, the winner to face A. They cannot improve their chances by forgoing a shot, so they shoot and B goes first. B wins that 2/3 of the time, C 1/3.

The survival probability

- for A is $2/3 * 3/7 + 1/3 * 1/3 = 25/63$.
- for B is $2/3 * 4/7 = 24/63$.
- for C is $1/3 * 2/3 = 14/63$.

Problem 17

Suppose in a society where there are equal numbers of men and women. There is a 50% chance for each child that a couple gives birth to is a girl and the genders of their children are mutually independent. Suppose in this strange and primitive society every couple prefers a girl and they will continue to have more children until they get a girl and once they have a girl they will stop having more children, what will eventually happen to the gender ratio of population in this society?

Answers:

The preference of each couple will not affect gender distribution of the society, as long as there is no interference of the "natural probability" (e.g. gender biased abortion). Because each additional child born into the society will still have 50-50 chance to be a boy or a girl. As long as this is maintained, the gender ratio will not change.

Problem 18

A person shoots basketball 100 times and scores 1 point if he makes one shot, 0 point otherwise. He has already made the first shot and missed the second one. For the following shots, the probability of making each shot is his score before the shot divided by total shots before the shot, i.e. if he has scored 13 points out of first 20 shots, then his next shot has a probability of 13/20 to score. What's the probability of scoring exactly 66 points after making 100 shots (including the first two)?

Answers:

Let $(n, k), (k \in [1, n-1], n \geq 3)$ be the event that the player scores k baskets after n throws. From the problem know the scoring probability for $k+1$ throw $P_{k+1} = k/n$.

Obviously, we have $P_3 = 1/2$, so

$\mathbb{P}(3,1) = 1–P_3 = 1/2$
$\mathbb{P}(3,2) = P_3 = 1/2$
$\mathbb{P}(4,1) = \mathbb{P}((4,1)|(3,1)) * \mathbb{P}(3,1) = (1–1/3) * 1/2 = 1/3$
$\mathbb{P}(4,2) = \mathbb{P}((4,2)|(3,1)) * \mathbb{P}(3,1) + \mathbb{P}((4,2)|(3,2)) * \mathbb{P}(3,2) = 1/3 * 1/2 + (1–2/3) * 1/3 = 1/3$
$\mathbb{P}(4,3) = \mathbb{P}((4,3)|(3,2)) * \mathbb{P}(3,2) = 2/3 * 1/2 = 1/3$

Both $(3, n)$ and $(4, n)$ situation indicates that $\mathbb{P}(n, k) = 1/(n-1)$ for $k \in [1, n-1], n \geq 3)$. Let's assume this is true and try to induct $\mathbb{P}(n+1, k)$. If our assumption stands, we should have $\mathbb{P}(n+1, k) = 1/n$:

$$\begin{aligned}\mathbb{P}(n+1,k) &= \mathbb{P}((n+1,k)|(n,k)) * \mathbb{P}(n,k) + \mathbb{P}((n+1,k)|(n,k-1)) * \mathbb{P}(n,k-1) \\ &= (1-k/n) * 1/(n-1) + (k-1)/n * 1/(n-1) = 1/n, \quad k \in [1,n].\end{aligned}$$

So we can conclude that $\mathbb{P}(n, k) = 1/(n-1)$ for $k \in [1, n-1], n \geq 3)$.

As a result, $\mathbb{P}(100, 66) = 1/(100-1) = 1/99$.