ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 5

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Problem 1

(a) $R_n^{(L)} = \sum_{j=1}^n W_{t_{j-1}}^2 \Delta W_j$

$R_n^{(R)} = \sum_{j=1}^n W_{t_j}^2 \Delta W_j$

$R_n^{(M)} = \sum_{j=1}^n \frac{W_{t_{j-1}}^2 + W_{t_j}^2}{2} \Delta W_j$

(b) Ito uses the left endpoints:

$$\int_0^t W_s^2 dW_s = \lim_{|\Delta| \to 0} \sum_{j=1}^n W_{t_{j-1}}^2 \Delta W_j$$

(c) Using independence of increments and $E[\Delta W_j] = 0$:

$E\left[W_{t_{j-1}}^2 \Delta W_j\right] = 0$

$E\left[W_{t_j}^2 \Delta W_j\right] = 0$

Midpoint term also has zero mean by symmetry.

Thus: $E[R_n^{(L)}] = E[R_n^{(R)}] = E[R_n^{(M)}] = 0$

(d) $E\left[\int_0^t W_s^2 dW_s\right] = 0$

Using Ito on ($f(W) = W^3/3$):

$$d(W_t^3/3) = W_t^2 dW_t + W_t dt \quad \Rightarrow \quad \int_0^t W_s^2 dW_s = \frac{W_t^3}{3} - \int_0^t W_s ds$$

whose expectation is (0).

Problem 2

$$\begin{aligned} t_jW_{t_j} - t_{j-1}W_{t_{j-1}} &= t_jW_{t_j} - t_jW_{t_{j-1}} \cdot t_jW_{t_{j-1}} - t_{j-1}W_{t_{j-1}} \\ &= t_j\big(W_{t_j} - W_{t_{j-1}}\big) \cdot (t_j - t_{j-1})W_{t_{j-1}} \\ tW_t - t_0W_{t_0} &= \sum_{j=1}^{n} t_j\Delta W_j \cdot \sum_{j=1}^{n} W_{t_{j-1}}\Delta t_j \end{aligned}$$

In the limit as the mesh goes to zero, this converges to the stochastic product rule:

$$d(tW_t) = tdW_t + W_tdt$$

The difference from the classical rule $d(uv) = (du)v + u(dv)$ is that here W_t is not differentiable, but because t has finite variation, no additional Ito correction term appears.

Problem 3

$$tW_t = \sum_{j=1}^{n} t_j\Delta W_j + \sum_{j=1}^{n} W_{t_{j-1}}\Delta t_j$$

$$tW_t = \int_0^t sdW_s + \int_0^t W_sds$$

$$\boxed{\int_0^t sdW_s = tW_t - \int_0^t W_sds}$$

Problem 4

(a)

$$W_T = W_0 + \int_0^T 1dW_t \quad \Rightarrow \quad g(t, X_t) = 1$$

(b) Ito:

$$d(W_t^2 - t) = 2W_tdW_t \quad \Rightarrow \quad g(t, X_t) = 2W_t$$

(c) Ito:

$$d\left(e^{W_t - t/2}\right) = e^{W_t - t/2}dW_t \Rightarrow g(t, X_t) = e^{W_t - t/2}$$

Problem 5

(a) $df = 2W_t dW_t + 1dt$

(b) $df = \frac{1}{2\sqrt{W_t}} dW_t - \frac{1}{8W_t^{3/2}} dt$

(c) $df = 2W_t e^{W_t^2} dW_t + (2W_t^2 + 1)e^{W_t^2} dt$

(d) $df = \sigma f dW_t$

(e) $df = \sigma f, dW_t + \frac{1}{2}\sigma^2 f dt$

(f) $dg(t) = W_t dt$

Problem 6

(a) $dX_t = 4(W_t^1)^3 dW_t^1 + 6(W_t^1)^2 dt$

(b) Let $Y_t = W_t^1 + W_t^2$:
$dX_t = 2Y_t(dW_t^1 + dW_t^2) + 2dt$

(c) $dX_t = 2t, dt + (2W_t^2 + 1)e^{W_t^2} dt + 2W_t e^{W_t^2} dW_t$

(d) $dX_t = X_t \left[\left(2t + \frac{1}{2}(W_t^2 + 1)\right) dt + W_t dW_t \right]$

Problem 7

(a) $dS_t = \mu S_t dt + \sigma S_t dW_t$

(b) $E[dS_t \mid \mathcal{F}_t] = \mu S_t dt$

(c) $\tilde{S}_t = S_0 e^{\mu t + \sigma W_t} \Rightarrow d\tilde{S}_t = (\mu + \frac{1}{2}\sigma^2)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t$

Problem 8

(a) $C-P = e^{-r(T-t)}E[(S_T-K)-(K-S_T) \mid \mathcal{I}_t] = e^{-r(T-t)}E[S_T-K \mid \mathcal{I}_t] = S_t - Ke^{-r(T-t)}$

(b) $H = \max(C,P) = \max\big(C,\ C-(C-P)\big) = \max\big(C,\ C-S_t+Ke^{-r(T-t)}\big)$.

(c) $H_0 = C(S_0,0)\cdot e^{-rt}E\left[(Ke^{-r(T-t)}-S_t)^+\right]$ with $S_t = S_0\exp\big((r-\frac{1}{2}\sigma^2)t+\sigma W_t\big)$

(d) Can use the Girsanov theorem to change to a new measure $\mathbb{Q}^*$ under which $\widetilde{W}_t = W_t - \theta t$ is a Brownian motion for a suitable constant θ. Choosing θ so that the drift of the exponential term is adjusted, the random variable inside

$$E\big[(Ke^{-r(T-t)}-S_t)^+\big]$$

becomes the payoff of a standard European put on a lognormal asset under $\mathbb{Q}^*$. Thus this expectation can be evaluated directly by the usual Black-Scholes put formula.

(e) $H_0 = C_{\mathrm{BS}}(S_0,K,T)\cdot P_{\mathrm{BS}}\big(S_0,\ Ke^{-r(T-t)},\ t\big)$

Problem 9

Want portfolio weight w in S_1 so diffusion term vanishes:

$$0.2w - 0.25(1-w) = 0 \Rightarrow w = \frac{5}{9}$$

For capital 1000:

$$\boxed{\text{Invest } 555.56 \text{ in } S_1}$$

Problem 10

$$0.1 = r + 0.2\lambda, \qquad 0.125 = r + 0.3\lambda$$

$$0.025 = 0.1\lambda \Rightarrow \lambda = 0.25$$

$$r = 0.1 - 0.05 = 0.05$$

$$\boxed{r = 5\%}$$

Problem 11

$$dX = 2(4 - X)dt + 8dZ$$

Let $Y = 1/X$:

$$y'(x) = -\frac{1}{x^2}, \qquad y''(x) = \frac{2}{x^3}$$

$$dY = -\frac{2(4-X)}{X^2}dt - \frac{8}{X^2}dZ + \frac{64}{X^3}dt$$

$$\alpha(x) = -\frac{2(4-x)}{x^2} + \frac{64}{x^3}$$

At $Y = 1/2 \Rightarrow X = 2$:

$$\alpha(1/2) = -\frac{2(2)}{4} + \frac{64}{8} = -1 + 8 = 7$$

$$\boxed{\alpha(1/2) = 7}$$

Problem 12

$$Z_t = \exp\left(\int_0^t f(s)dB_s - \frac{1}{2}\int_0^t f(s)^2 ds\right)$$

$$X_t = \int_0^t f(s)dB_s, \qquad [X]_t = \int_0^t f(s)^2 ds$$

Set $Y_t = X_t - \frac{1}{2}[X]_t$. Ito on $Z_t = e^{Y_t}$:

$$dY_t = f(t)dB_t - \tfrac{1}{2}f(t)^2 dt, \qquad d[Y]_t = f(t)^2 dt$$

$$dZ_t = Z_t dY_t + \tfrac{1}{2}Z_t d[Y]_t = Z_t f(t) dB_t$$

No drift term $\rightarrow$ martingale

$$\boxed{Z_t \text{ is a martingale}}$$