ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 5

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Please remember to write down your name and GTID in the submitted homework.

Problem 1 Neftci's Book Chapter 9, P162, Exercise 1

Let W_t be a Wiener process defined over $[0, T]$ and consider the integral:

$$\int_0^t W_s^2 dW_s$$

Use the subdivision of $[0, t]$:

$$t_0, t_1, \cdots, t_{n-1}, t_n$$

in the following:

(a) Write the approximation of the above integral as three different Riemann sums.

(b) Write the integral in discrete time using an Ito sum.

(c) Calculate the expectation of the three Riemann sums.

(d) Calculate the expectation of the Ito sum.

Answers:

(a)

Three possible choices are left, right, and middle of partition as follows:

1) $\sum_{i=1}^n W_i^2 (W_i - W_{i-1})$

2) $\sum_{i=1}^n W_{i-1}^2 (W_i - W_{i-1})$

3) $\sum_{i=1}^n \left(\frac{W_i + W_{i-1}}{2}\right)^2 (W_i - W_{i-1})$

(b)

Ito sum is represented by integral 2)

(c)

The expectation of Ito integral, number 2), is zero since W_{i-1}^2 is independent of $W_i - W_{i-1}$:

$$\mathbb{E}\left[\sum_{i=1}^{n} W_{i-1}^2 (W_i - W_{i-1})\right] = \mathbb{E}\left[\sum_{i=1}^{n} W_{i-1}^2 \mathbb{E}\left[W_i - W_{i-1} | I_{i-1}\right]\right] = \mathbb{E}\left[\sum_{i=1}^{n} W_{i-1}^2 * 0\right] = 0$$

Integral 1):

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} W_i^2 (W_i - W_{i-1})\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} W_i^3 - W_i^2 W_{i-1}\right] \\
&= \sum_{i=1}^{n} \left(\mathbb{E}\left[W_i^3\right] - \mathbb{E}\left[W_i^2 W_{i-1}\right]\right) \\
&= -\sum_{i=1}^{n} \mathbb{E}\left[W_i^2 W_{i-1}\right] \\
&= -\sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}\left[W_i^2 W_{i-1} | I_{i-1}\right]\right] \\
&= -\sum_{i=1}^{n} \mathbb{E}\left[W_{i-1} \mathbb{E}\left[W_i^2 | I_{i-1}\right]\right] \\
&= -\sum_{i=1}^{n} \mathbb{E}\left[W_{i-1} \mathbb{E}\left[W_{i-1}^2 + 2W_{i-1}(W_i - W_{i-1}) + (W_i - W_{i-1})^2 | I_{i-1}\right]\right] \\
&= -\sum_{i=1}^{n} \mathbb{E}\left[W_{i-1}(W_{i-1}^2 + t_i - t_{i-1})\right] \\
&= -\sum_{i=1}^{n} \left(\mathbb{E}\left[W_{i-1}^3\right] + \mathbb{E}\left[W_{i-1}\right](t_i - t_{i-1})\right) \\
&= 0
\end{aligned}$$

As for integral 3:

$$\left(\frac{W_i + W_{i-1}}{2}\right)^2 = \frac{1}{4}(W_i^2 + W_{i-1}^2) + \frac{1}{2} W_i W_{i-1}$$

$$\mathbb{E}\left[\sum_{i=1}^{n}\left(\frac{W_i+W_{i-1}}{2}\right)^2(W_i-W_{i-1})\right]$$
$$=\frac{1}{4}(0+0)+\frac{1}{2}\mathbb{E}\left[\sum_{i=1}^{n}W_iW_{i-1}(W_i-W_{i-1})\right]$$
$$=\frac{1}{2}\mathbb{E}\left[\sum_{i=1}^{n}W_i^2W_{i-1}\right]-\frac{1}{2}\mathbb{E}\left[\sum_{i=1}^{n}W_iW_{i-1}^2\right]$$
$$=\frac{1}{2}\mathbb{E}\left[\sum_{i=1}^{n}(W_{i-1}^2+2W_{i-1}(W_i-W_{i-1})+(W_i-W_{i-1})^2)W_{i-1}\right]-\frac{1}{2}\mathbb{E}\left[\sum_{i=1}^{n}(W_{i-1}+(W_i-W_{i-1}))W_{i-1}^2\right]$$
$$=0-0=0$$

(d)

The Ito integral has zero expectation as was noted above.

Problem 2 Neftci's Book Chapter 9, P162, Exercise 2

Show that given

$$t_0, t_1, \cdots, t_{n-1}, t_n$$

and

$$W_{t_0}, W_{t_1}, \cdots, W_{t_{n-1}}, W_{t_n}$$

we can always write:

$$\sum_{j=1}^{n}\left[t_jW_{t_j}-t_{j-1}W_{t_{j-1}}\right]=\sum_{j=1}^{n}\left[t_j\left(W_{t_j}-W_{t_{j-1}}\right)\right]+\sum_{j=1}^{n}\left[\left(t_j-t_{j-1}\right)W_{t_{j-1}}\right]$$

How is this different from the standard formula for the differentiation of products:

$$d(uv)=(du)v+u(dv)$$

Answers:

$$\begin{aligned}\sum_{j=1}^{n}\left[t_jW_{t_j}-t_{j-1}W_{t_{j-1}}\right]&=\sum_{j=1}^{n}\left[t_jW_{t_j}-t_jW_{t_{j-1}}+t_jW_{t_{j-1}}-t_{j-1}W_{t_{j-1}}\right]\\&=\sum_{j=1}^{n}\left[t_jW_{t_j}-t_jW_{t_{j-1}}\right]+\sum_{j=1}^{n}\left[t_jW_{t_{j-1}}-t_{j-1}W_{t_{j-1}}\right]\\&=\sum_{j=1}^{n}\left[t_j\left(W_{t_j}-W_{t_{j-1}}\right)\right]+\sum_{j=1}^{n}\left[\left(t_j-t_{j-1}\right)W_{t_{j-1}}\right]\end{aligned}$$

Problem 3 Neftci's Book Chapter 9, P228, Exercise 3

Now use this information to show that:

$$\int_0^t s dW_s = tW_t - \int_0^t W_s ds$$

Answers:

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left[t_j W_{t_j} - t_{j-1}W_{t_{j-1}}\right] = \lim_{n\to\infty}\sum_{j=1}^{n}\left[t_j\left(W_{t_j} - W_{t_{j-1}}\right)\right] + \lim_{n\to\infty}\sum_{j=1}^{n}\left[\left(t_j - t_{j-1}\right)W_{t_{j-1}}\right]$$

$$\Longrightarrow tW_t = \int_0^t s dW_s + \int_0^t W_s ds$$

Problem 4 Neftci's Book Chapter 6, P110, Exercise 4

You are given the following representation of a martingale process $M_t \equiv M(X_t)$:

$$M(X_T) = M(X_0) + \int_0^T g(t, X_t) dW_t$$

where the equality holds, given the sequence of information sets $\{I_t\}$. The underlying process X_t is known to follow the SDE:

$$dX_t = \mu d_t + \sigma dW_t$$

Determine the $g(\cdot)$ in the above representation for the case where $M(\cdot)$ is given by:

(a) $M(X_T) = W_T$.

(b) $M(X_T) = W_T^2$.

(c) $M(X_T) = e^{W_T}$.

Answers:

$$dX_t = \mu d_t + \sigma dW_t$$

Thus $X_t = \mu t + \sigma W_t$ and $W_t = \frac{X_t - \mu t}{\sigma}$.

$$M(X_t) = M(X_0) + \int_0^t g(s, X_s) dW_s$$

The RHS is always a martingale as $g(t, X_t)$ is adapted to the filtration generated by W_t. The Ito integral is a martingale and the initial term is constant and therefore a martingale. In all cases, the left hand side is also a martingale. Determine g for the following:

(a)

By $M(X_T) = W_T$, we have $M(X_t) = W_t$ and

$$M(X_t) = \int_0^t 1 dW_s$$

Hence $g(t, X_t) = 1$.

(b)

By $M(X_T) = W_T^2$, we have $M(X_t) = W_t^2 - t + T$ (recall $W_t^2 - t$ is a martingale) and

$$dM(X_t) = 2W_t dW_t$$

Hence $g(t, X_t) = 2W_t = 2\frac{X_t - \mu t}{\sigma}$.

(c)

By $M(X_T) = e^{W_T}$, we have $M(X_t) = e^{W_t - \frac{t}{2}} e^{\frac{T}{2}}$ (recall $e^{W_t - \frac{t}{2}}$ is a martingale) and

$$dM(X_t) = e^{W_t + \frac{T-t}{2}} dW_t$$

Hence $g(t, X_t) = e^{W_t + \frac{T-t}{2}} = e^{\frac{X_t - \mu t}{\sigma} + \frac{T-t}{2}}$.

Problem 5 Neftci's Book Chapter 10, P177, Exercise 1

Differentiate the following functions with respect to the Wiener process W_t and, if applicable, with respect to t.

(a) $f(W_t) = W_t^2$

(b) $f(W_t) = \sqrt{W_t}$

(c) $f(W_t) = e^{\left(W_t^2\right)}$

(d) $f(W_t, t) = e^{\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)}$

(e) $f(W_t) = e^{\sigma W_t}$

(f) $g = \int_0^t W_s ds$

Answers:

(a)

$$df(W_t) = dt + 2W_t dW_t$$

(b)

$$df(W_t) = -\frac{1}{8} W_t^{-\frac{3}{2}} dt + \frac{1}{2} W_t^{-\frac{1}{2}} dW_t$$

(c)

$$df(W_t) = (e^{(W_t^2)} + 2W_t^2 e^{(W_t^2)})dt + 2W_t e^{(W_t^2)} dW_t$$

(d)

$$df(W_t, t) = \sigma e^{(\sigma W_t - \frac{1}{2}\sigma^2 t)} dW_t$$

(e)

$$df(W_t) = \frac{\sigma^2}{2} e^{\sigma W_t} dt + \sigma e^{\sigma W_t} dW_t$$

(f)

$$dg(t) = W_t dt$$

Problem 6 Neftci's Book Chapter 10, P177, Exercise 2

Suppose the $W_t^i, i = 1, 2$ are two Wiener processes. Use Ito's Lemma in obtaining appropriate stochastic differential equations for the following transformations.

(a) $X_t = (W_t^1)^4$

(b) $X_t = (W_t^1 + W_t^2)^2$

(c) $X_t = t^2 + e^{W_t^2}$

(d) $X_t = e^{t^2 + W_t^2}$

Answers:

(a)

$$dX_t = 6(W_t^1)^2 dt + 4(W_t^1)^3 dW_t^1$$

(b)

$$\frac{\partial X_t}{\partial W_t^1} = 2W_t^1 + 2W_t^2 \qquad \frac{\partial^2 X_t}{\partial (W_t^1)^2} = 2$$
$$\frac{\partial X_t}{\partial W_t^2} = 2W_t^1 + 2W_t^2 \qquad \frac{\partial^2 X_t}{\partial (W_t^1)^2} = 2 \qquad \frac{\partial^2 X_t}{\partial W_t^1 W_t^2} = 2$$

$$\begin{aligned} dX_t &= \frac{1}{2}(2+2)dt + 2(W_t^1 + W_t^2)dW_t^1 + 2(W_t^1 + W_t^2)dW_t^2 + 2d < W_t^1, W_t^2 > \\ &= 2dt + 2(W_t^1 + W_t^2)dW_t^1 + 2(W_t^1 + W_t^2)dW_t^2 + 2d < W_t^1, W_t^2 > \end{aligned}$$

Remark: if W_t^1 and W_t^2 has correlation ρ, $d < W_t^1, W_t^2 >= \rho dt$

(c)

$$dX_t = \left(\frac{1}{2}e^{W_t^2} + 2t\right)dt + e^{W_t^2}dW_t^2$$

(d)

$$dX_t = \left(\frac{1}{2}e^{t^2+W_t^2} + 2te^{t^2+W_t^2}\right)dt + e^{t^2+W_t^2}dW_t^2$$

Problem 7 Neftci's Book Chapter 10, P178, Exercise 3

Let W_t be a Wiener process. Consider the geometric process S_t again:

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

(a) Calculate dS_t.

(b) What is the "expected rate of change" of S_t?

(c) If the exponential term in the definition of S_t did not contain the term $-\frac{1}{2}\sigma^2$, what would be the dS_t? What would then be the expected change in S_t?

Answers:

(a)

$$\begin{aligned}
dS_t &= S_0\left(\left(\mu - \frac{1}{2}\sigma^2\right)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t} + \frac{1}{2}\sigma^2 e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}\right)dt + S_0\sigma e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}dW_t \\
&= S_0\mu e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}dt + S_0\sigma e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}dW_t \\
&= \mu S_t dt + \sigma S_t dW_t
\end{aligned}$$

(b)

The "expected rate of change" of S_t is μ

(c)

Without the term $-\frac{1}{2}\sigma^2$,

$$\begin{aligned}
dS_t &= S_0\left(\mu e^{\mu t+\sigma W_t} + \frac{1}{2}\sigma^2 e^{\mu t+\sigma W_t}\right)dt + S_0\sigma e^{\mu t+\sigma W_t}dW_t \\
&= S_0\left(\mu + \frac{1}{2}\sigma^2\right)e^{\mu t+\sigma W_t}dt + S_0\sigma e^{\mu t+\sigma W_t}dW_t \\
&= \left(\mu + \frac{1}{2}\sigma^2\right)S_t dt + \sigma S_t dW_t.
\end{aligned}$$

The "expected rate of change" of S_t is $\mu + \frac{1}{2}\sigma^2$.

Problem 8 Neftci's Book Chapter 15, P267, Exercise 1

The chooser option is an exotic option that gives the holder the right to choose, at some future date, between a call and a put written on the same underlying asset. Let the T be the expiration date, S_t be the stock price, K the strike price. If we buy the chooser option at time t, we can choose between call or put with strike K, written on S_t. At time t the value of the call is

$$C\left(S_t, t\right) = e^{-r(T-t)}\mathbb{E}\left[\max\left(S_T - K, 0\right) \mid I_t\right]$$

whereas the value of the put is:

$$P\left(S_t, t\right) = e^{-r(T-t)}\mathbb{E}\left[\max\left(K - S_T, 0\right) \mid I_t\right]$$

and thus, at time t, the chooser option is worth:

$$H\left(S_t, t\right) = \max\left[C\left(S_t, t\right), P\left(S_t, t\right)\right]$$

(a) Using these, show that:

$$C\left(S_t, t\right) - P\left(S_t, t\right) = S_t - Ke^{-r(T-t)}$$

Does this remind you of a well-known parity condition?

(b) Next, show that the value of the chooser option at time t is given by

$$\begin{aligned} H\left(S_t, t\right) = \max\Big[C\left(S_t, t\right), C\left(S_t, t\right) - S_t \\ +Ke^{-r(T-t)}\Big] \end{aligned}$$

(c) Consequently, show that the option price at time zero will be given by

$$\begin{aligned} H\left(S_0, 0\right) = C\left(S_0, 0\right) + e^{-rT}\mathbb{E} \\ \left[\max\left[K - S_0 e^{rT} e^{\sigma W_t - \frac{1}{2}\sigma^2 t}, 0\right]\right] \end{aligned}$$

where S_0 is the underlying price observed at time zero.

(d) Now comes the point where you use the Girsanov theorem. How can you exploit the Girsanov theorem and evaluate the expectation in the above formula easily?

(e) Write the final formula for the chooser option.

Answers:

(a)

$$\begin{aligned} C\left(t, S_t\right) - P\left(t, S_t\right) &= e^{-r(T-t)}\mathbb{E}\left[\max\left(S_T - K, 0\right) \mid I_t\right) - e^{-r(T-t)}\mathbb{E}\left[\max\left(K - S_T, 0\right) \mid I_t\right] \\ &= e^{-r(T-t)}\mathbb{E}\left[\max\left(S_T - K, 0\right) - \max\left(K - S_T, 0\right) \mid I_t\right] \\ &= e^{-r(T-t)}\mathbb{E}\left[S_T - K \mid I_t\right] \\ &= \mathbb{E}\left[e^{-r(T-t)} S_T\right] - e^{-r(T-t)} K \\ &= S(t) - e^{-r(T-t)} K. \end{aligned}$$

This is a the put-call parity.

(b)

Plug in the result of (a) into:

$$\begin{aligned} H(t, S_t) &= \max\left[C(t, S_t), P(t, S_t)\right] \\ &= \max\left[C(t, S_t), C(t, S_t) + e^{-r(T-t)}K - S_t\right]. \end{aligned}$$

(c)

By part (b)

$$\begin{aligned} H(t, S_t) &= \max\left[C(t, S_t), C(t, S_t) + e^{-r(T-t)}K - S_t\right] \\ &= C(t, S_t) + \max\left[0, e^{-r(T-t)}K - S_t\right] \end{aligned}$$

so

$$\begin{aligned} H(0, S) &= \mathbb{E}\left[e^{-rt}\left[C(t, S_t) + \max\left[0, e^{-r(T-t)}K - S_t\right]\right]\right] \\ &= \mathbb{E}\left[e^{-rt}C(t, S_t) + e^{-rt}\max\left[0, e^{-r(T-t)}K - S_t\right]\right] \\ &= C(0, S) + e^{-rt}\mathbb{E}\left[\max\left[0, e^{-r(T-t)}K - S_t\right]\right] \\ &= C(0, S) + e^{-rt}e^{-r(T-t)}\mathbb{E}\left[\max\left[0, K - e^{r(T-t)}Se^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W_t}\right]\right] \\ &= C(0, S) + e^{-rT}\mathbb{E}\left[\max\left[0, K - Se^{rT}e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\right]\right]. \end{aligned}$$

(d)

The Girsanov theorem says that under the risk-neutral measure, the discounted process is a martingale, thus $e^{-rT}\mathbb{E}\left[\max\left[0, K - Se^{rT}e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\right]\right]$ can be treated as a price of an asset, which is a put option with strike $Ke^{-r(T-t)}$ and maturity t. So

$$H(0, S) = C(0, S, K, T) + P\left(0, S, Ke^{-r(T-t)}, t\right).$$

(e)

Use the Black-Scholes formula:

$$H(0, S) = S\left(N(d_1) - N\left(-\bar{d}_1\right)\right) + Ke^{-r(t-T)}\left(N\left(-\bar{d}_2\right) - N(d_2)\right)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \\ \bar{d}_1 &= \frac{\ln\left(\frac{S}{K}\right) + rT + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} \\ \bar{d}_2 &= \bar{d}_1 - \sigma\sqrt{t}. \end{aligned}$$

Problem 9

Two non-dividend-paying assets have the following price processes:

$$\frac{dS_1(t)}{S_1(t)} = 0.08dt + 0.2dZ(t)$$
$$\frac{dS_2(t)}{S_2(t)} = 0.0925dt - 0.25dZ(t),$$

where $Z(t)$ is a standard Brownian motion. An investor is to synthesize the risk-free asset by allocating \$1000 between these two assets.

Determine the amount to be invested in the first asset, S_1.

Answers:

Observe that

$$\begin{cases} dS_1(t) = 0.08S_1(t)dt + 0.2S_1(t)dZ(t) \\ dS_2(t) = 0.0925S_2(t)dt - 0.25S_2(t)dZ(t) \end{cases}.$$

If we long Δ_1 unit of S_1, then we must long $\Delta_2 = \frac{0.2S_1}{0.25S_2}\Delta_1 = \frac{0.8S_1}{S_2}\Delta_1$ shares of S_2 so that has no $dZ(t)$ term.

$$\Delta_1 S_1 + \Delta_2 S_2 = \Delta_1 S_1 + \frac{0.8S_1}{S_2}\Delta_1 S_2 = 1.8\Delta_1 S_1 = 1000$$

The amount to be invested in the first asset is

$$\Delta_1 S_1 = 555.56$$

Remarks:

- The two expected rates of return, 0.08 and 0.0925, are not used in determining the proportion. They are needed for determining r.
- The proportion is independent of t.

Problem 10

Consider an arbitrage-free securities market model, in which the risk-free interest rate is constant. There are two non-dividend-paying stocks whose price processes are

$$S_1(t) = S_1(0)e^{0.1t+0.2Z(t)}$$
$$S_2(t) = S_2(0)e^{0.125t+0.3Z(t)}$$

Where $Z(t)$ is a standard Brownian motion and $t \geq 0$. Determine the continuously compounded risk-free interest rate.

Answers:

Note that the two assets are driven by the same BM $Z(t)$.

Let $dS(t) = \mu S(t)dt + \sigma S(t)dZ(t)$. Then

$$S(t) = S(0)\exp\left[\left(\mu - 0.5\sigma^2\right)t + \sigma Z(t)\right].$$

Thus,

- for stock 1, $\sigma_1 = 0.2$ and $\mu_1 = 0.1 + 0.5 \times 0.2^2 = 0.12$.
- for stock 2, $\sigma_2 = 0.3$ and $\mu_2 = 0.125 + 0.5 \times 0.3^2 = 0.17$

Because of the no-arbitrage constraint, (at each point of time) the market price of risks $\lambda = \frac{\mu - r}{\sigma}$ of the two stocks must be equal:

$$\begin{aligned}\lambda &= \frac{0.12 - r}{0.2} = \frac{0.17 - r}{0.3} \\ r &= 0.02\end{aligned}$$

Problem 11

$X(t)$ is an Ornstein-Uhlenbeck process defined by

$$dX(t) = 2(4 - X(t))dt + 8dZ(t)$$

where $Z(t)$ is a standard Brownian motion.

Let $Y(t) = \frac{1}{X(t)}$. You are given that $dY(t) = \alpha(Y(t))dt + \beta(Y(t))dZ(t)$ for some functions and $\alpha(y)$ and $\beta(y)$.

Determine $\alpha\left(\frac{1}{2}\right)$.

Answers:

By Ito's lemma,

$$\begin{aligned}\alpha(Y(t)) &= \frac{\partial Y}{\partial X} \times 2(4 - X(t)) + \frac{1}{2}\frac{\partial^2 Y}{\partial X^2} \times 8^2 \\ &= -\frac{1}{X(t)^2} \times 2(4 - X(t)) + \frac{1}{2}\frac{2}{X(t)^3} \times 8^2\end{aligned}$$

As $Y(t) = \frac{1}{2}$, $X(t) = 2$.

$$\alpha\left(\frac{1}{2}\right) = -\frac{1}{2^2} \times 2(4-2) + \frac{1}{2}\frac{2}{2^3} \times 8^2 = 7$$

Problem 12

Prove the following process is a martingale:

$$Z_t = \exp\left(\int_0^t f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds\right)$$

where f is a continuous function from $[0, T]$ to R.

Answers:

Denote $X_t = \int_0^t f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds$, then $dX_t = f(t)dB_t - \frac{1}{2}f^2(t)dt$. Let $Z_t = F(X_t) = \exp(X_t)$ Using Ito's lemma.

$$\begin{aligned} dZ_t = dF(X_t) &= \left[0 + \exp(X_t) \times \left(-\frac{1}{2}f^2(t)\right) + \frac{1}{2}\exp(X_t)f^2(t)\right] \cdot dt + \exp(X_t)\, f(t)dB_t \\ &= \exp(X_t)\, f(t)dB_t = f(t)Z_t dB_t \end{aligned}$$

The drift term is zero. $\therefore Z_t$ is a martingale.