

# ISyE/Math 6759 Stochastic Processes in Finance – I

## Homework Set 6

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Please remember to write down your name and GTID in the submitted homework.

### Problem 1

Assume the Black-Scholes framework. For  $t \geq 0$ , let  $S(t)$  be the time- $t$  price of a non-dividend-paying stock. You are given:

1.  $S(0) = 0.5$
2. The stock price process is  $\frac{dS(t)}{S(t)} = 0.05dt + 0.2dZ(t)$ , where  $Z(t)$  is a standard Brownian motion.
3.  $\mathbb{E}[S(1)^a] = 1.4$ , where  $a$  is a negative constant.
4. The continuously compounded risk-free interest rate is 3%. Consider a contingent claim that pays  $S(1)^a$  at time 1.

Calculate the time-0 price of the contingent claim.

### Answers:

To find the price, we need to first determine the negative constant  $a$ . From (2), we know that true stock price process is a geometric Brownian motion with  $\mu = 0.05$  and  $\sigma = 0.2$ .

By the moment-generating function formula for a normal random variable, the expected value of the contingent claim at time  $T$  is

$$\mathbb{E}[S(T)^a] = S(0)^a \exp \left\{ \left[ a\mu + \frac{1}{2}a(a-1)\sigma^2 \right] T \right\}.$$

**Remarks:** We can attain the same result by showing  $S(t)^a$  also follows a GBM as in Problem 17.

Substituting  $T = 1$ ,  $S(0) = 0.5$ ,  $\mu = 0.05$  and  $\sigma = 0.2$  into the equation above,

$$0.5^a \exp \left[ 0.05a + \frac{1}{2}a(a-1)(0.2)^2 \right] = 1.4$$

$$a \ln 0.5 + 0.05a + 0.02a(a-1) = \ln 1.4$$

$$0.02a^2 + (0.03 + \ln 0.5)a - \ln 1.4 = 0$$

$$a = -0.49985 \text{ or } 33.66 \text{ (rejected)}$$

Then the time-0 price of the contingent claim is  $\mathbb{E}^{\mathbb{Q}}[e^{-r}S(1)^a]$  where  $\frac{dS(t)}{S(t)} = 0.03dt + 0.2d\tilde{Z}(t)$ ,  $\tilde{Z}(t)$  is a standard Brownian motion under risk neutral probability.

$$\mathbb{E}^{\mathbb{Q}}[e^{-r}S(1)^a] = e^{-r}S(0)^a \exp\left\{\left[ar + \frac{1}{2}a(a-1)\sigma^2\right]\right\} = 1.372$$

## Problem 2 Exercise 6.1

Consider the standard Black-Scholes model and a  $T$ -claim  $\mathcal{X}$  of the form  $\mathcal{X} = \Phi(S(T))$ . Denote the corresponding arbitrage free price process by  $\Pi(t)$ .

- (a) Show that, under the martingale measure  $\mathbb{Q}$ ,  $\Pi(t)$  has a local rate of return equal to the short rate of interest  $r$ . In other words show that  $\Pi(t)$  has a differential of the form

$$d\Pi(t) = r \cdot \Pi(t)dt + g(t)dW(t)$$

Hint: Use the  $\mathbb{Q}$ -dynamics of  $S$  together with the fact that  $F$  satisfies the pricing PDE.

- (b) Show that, under the martingale measure  $\mathbb{Q}$ , the process  $Z(t) = \frac{\Pi(t)}{B(t)}$  is a martingale. More precisely, show that the stochastic differential for  $Z$  has zero drift term, i.e. it is of the form

$$dZ(t) = Z(t)\sigma_Z(t)dW(t)$$

Determine also the diffusion process  $\sigma_Z(t)$  (in terms of the pricing function  $F$  and its derivatives).

## Answers:

(a)

We use the definition:  $\Pi(t) = F(t, S(t))$ , where  $F(t, S)$  satisfies the Black-Scholes PDE:

$$F_t + rxF_x + \frac{1}{2}\sigma^2x^2F_{xx} - rF = 0,$$

and under the risk-neutral measure  $\mathbb{Q}$ , the dynamics of  $S$  :

$$dS = rSdt + \sigma SdW$$

Apply Ito's lemma to  $\Pi$  :

$$\begin{aligned} d\Pi(t) &= F_tdt + F_xdS + \frac{1}{2}F_{xx}(dS)^2 \\ &= F_tdt + F_x(rSdt + \sigma SdW) + \frac{1}{2}F_{xx}\sigma^2S^2dt \\ &= \left(F_t + rSF_x + \frac{1}{2}F_{xx}\sigma^2S^2\right)dt + \sigma SF_xdW \\ &= rFdt + \sigma SF_xdW \\ &= r\Pi dt + \sigma SF_xdW. \end{aligned}$$

(b)

Assume interest rate is constant,  $B(t) = e^{rt}$ ,  $Z(t) = \frac{\Pi(t)}{B(t)} = e^{-rt}\Pi(t)$ . Apply the Ito's Lemma:

$$\begin{aligned} dZ &= \Pi(-r)e^{-rt}dt + e^{-rt}d\Pi \\ &= -r\Pi e^{-rt}dt + e^{-rt}(r\Pi dt + \sigma S F_x dW) \\ &= e^{-rt}\sigma S F_x dW \\ &= e^{-rt}\Pi \frac{\sigma S F_x}{F} dW \\ &= Z\sigma_Z dW, \end{aligned}$$

so  $\sigma_Z = \frac{\sigma S F_x}{F}$ .

### Problem 3 Exercise 6.2

Consider the standard Black-Scholes model. An innovative company, F&H INC, has produced the derivative "the Golden Logarithm", henceforth abbreviated as the  $GL$ . The holder of a  $GL$  with maturity time  $T$ , denoted as  $GL(T)$ , will, at time  $T$ , obtain the sum  $\ln S(T)$ . Note that if  $S(T) < 1$  this means that the holder has to pay a positive amount to F&H INC. Determine the arbitrage free price process for the  $GL(T)$ .

#### Answers:

By Feynman-Kac formula, the solution of the PDE  $\Pi(t) = F(t, S(t))$  is:

$$\Pi(t) = \mathbb{E} \left[ e^{-r(T-t)} \ln S(T) \mid S(t) = S \right],$$

where  $dS = rSdt + \sigma SdW$ . Let  $X = \ln S$ , apply Ito's lemma:

$$dX = \left( r - \frac{\sigma^2}{2} \right) dt + \sigma dW,$$

then  $X(T) - X(t) \sim N \left( \left( r - \frac{\sigma^2}{2} \right) (T-t), \sigma^2(T-t) \right)$ , so

$$\begin{aligned} \Pi(t) &= \mathbb{E} \left[ e^{-r(T-t)} X(T) \mid X(t) = \ln S(t) \right] \\ &= e^{-r(T-t)} \mathbb{E} [X(T) - X(t) + X(t) \mid X(t) = \ln S(t)] \\ &= e^{-r(T-t)} \left( \left( r - \frac{\sigma^2}{2} \right) (T-t) + \ln S(t) \right). \end{aligned}$$

### Problem 4 Exercise 6.3

Consider the standard Black-Scholes model. Derive the Black-Scholes formula for the European call option.

## Answers:

Let the price of European call option be  $F(t, S)$ , by Feynman-Kac formula,

$$F = \mathbb{E} \left[ e^{-r(T-t)} (S(T) - K)^+ \mid S(t) = S \right],$$

let  $X(t) = \ln S(t)$ , then  $(X(T) - X(t) + \ln S) \sim N \left( \left( r - \frac{\sigma^2}{2} \right) (T - t) + \ln S, \sigma^2 (T - t) \right)$ ,

$$\begin{aligned} F &= \mathbb{E} \left[ e^{-r(T-t)} \left( e^{X(T)} - K \right)^+ \mid X(t) = \ln S \right] \\ &= e^{-r(T-t)} \mathbb{E} \left[ \left( e^{X(T) - X(t) + X(t)} - K \right)^+ \mid X(t) = \ln S \right] \\ &= e^{-r(T-t)} \mathbb{E} \left[ \left( e^{(X(T) - X(t)) + \ln S} - K \right)^+ \right] \end{aligned}$$

Now let  $\bar{\mu} = \left( r - \frac{\sigma^2}{2} \right) (T - t) + \ln S$  and  $\bar{\sigma} = \sigma \sqrt{T - t}$ . We calculate  $\mathbb{E} [e^X - K]^+$  where  $X \sim N(\bar{\mu}, \bar{\sigma}^2)$ ,  $Y = \frac{X - \bar{\mu}}{\bar{\sigma}} \sim N(0, 1)$  :

$$\begin{aligned} \mathbb{E} \left[ (e^X - K)^+ \right] &= \int_{-\infty}^{\infty} (e^x - K)^+ \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} e^{-\frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2}} dx \\ &= \int_{\frac{\ln K - \bar{\mu}}{\bar{\sigma}}}^{\infty} e^{\bar{\sigma}y + \bar{\mu}} \frac{e^{-\frac{y^2}{2}}}{\bar{\sigma}\sqrt{2\pi}} d(\bar{\sigma}y + \bar{\mu}) - \int_{\frac{\ln K - \bar{\mu}}{\bar{\sigma}}}^{\infty} K \frac{1}{\bar{\sigma}\sqrt{2\pi}} e^{-\frac{y^2}{2}} d(\bar{\sigma}y + \bar{\mu}) \\ &= e^{\bar{\mu}} \int_{\frac{\ln K - \bar{\mu}}{\bar{\sigma}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\bar{\sigma})^2 - \bar{\sigma}^2}{2}} dy - KN(d_2) \\ &= e^{\bar{\mu} + \frac{\bar{\sigma}^2}{2}} \int_{\frac{\ln K - \bar{\mu}}{\bar{\sigma}} - \bar{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\bar{y}^2}{2}} d\bar{y} - KN(d_2) \\ &= e^{\bar{\mu} + \frac{\bar{\sigma}^2}{2}} N(d_1) - KN(d_2), \end{aligned}$$

where  $d_2 = -\frac{\ln K - \bar{\mu}}{\bar{\sigma}}$ ,  $d_1 = d_2 + \bar{\sigma}$ , so

$$\begin{aligned} F &= e^{-r(T-t)} \left[ e^{\ln S + r(T-t)} N(d_1) - KN(d_2) \right] \\ &= SN(d_1) - e^{-r(T-t)} KN(d_2) \end{aligned}$$

and  $d_2 = -\frac{\ln K - \left( r - \frac{\sigma^2}{2} \right) (T-t) - \ln S}{\sigma \sqrt{T-t}}$ , and  $d_1 = d_2 + \sigma \sqrt{T-t}$

## Problem 5 Exercise 6.4

Consider the standard Black-Scholes model. Derive the arbitrage free price process for the  $T$ -claim  $\mathcal{X}$  where  $\mathcal{X}$  is given by  $\mathcal{X} = \{S(T)\}^\beta$ . Here  $\beta$  is a known constant.

## Answers:

Again we find the price process  $F(t, S)$  by:

$$F(t, S) = \mathbb{E} \left[ e^{-r(T-t)} \{S(T)\}^\beta \mid S(t) = S \right]$$

Let  $Y = S^\beta$ , and apply Ito's lemma:

$$\begin{aligned}
dY &= \beta S^{\beta-1} dS + \frac{1}{2} \beta(\beta-1) S^{\beta-2} (dS)^2 \\
&= \beta S^{\beta-1} (rSdt + \sigma SdW) + \frac{1}{2} \beta(\beta-1) S^{\beta-2} (\sigma^2 S^2) dt \\
&= \left( r\beta S^\beta + \frac{1}{2} \sigma^2 \beta(\beta-1) S^\beta \right) dt + \sigma \beta S^\beta dW \\
&= \left( r\beta + \frac{1}{2} \sigma^2 \beta(\beta-1) \right) Y dt + \sigma \beta Y dW.
\end{aligned}$$

Let  $Z = \ln Y$ , then

$$\begin{aligned}
dZ &= \left( r\beta + \frac{1}{2} \sigma^2 \beta(\beta-1) - \frac{1}{2} \sigma^2 \beta^2 \right) dt + \sigma \beta dW \\
&= \left( r\beta - \frac{1}{2} \sigma^2 \beta \right) dt + \sigma \beta dW.
\end{aligned}$$

So  $Z(T) - Z(t) \sim N \left( \left( r\beta - \frac{1}{2} \sigma^2 \beta \right) (T-t), \sigma^2 \beta^2 (T-t) \right)$ .

$$\begin{aligned}
F(t, S) &= e^{-r(T-t)} \mathbb{E} \left[ e^{Z(T)} \mid Z(t) = \ln S^\beta \right] \\
&= e^{-r(T-t)} \mathbb{E} \left[ e^{Z(T)-Z(t)+Z(t)} \mid Z(t) = \ln S^\beta \right] \\
&= e^{-r(T-t)} e^{\ln S^\beta} \mathbb{E} \left[ e^{Z(T)-Z(t)} \right] \\
&= e^{-r(T-t)} S^\beta e^{\left( r\beta - \frac{1}{2} \sigma^2 \beta \right) (T-t) + \frac{1}{2} \sigma^2 \beta^2 (T-t)} \\
&= S^\beta e^{\left[ r(\beta-1) + \frac{1}{2} \beta(\beta-1) \sigma^2 \right] (T-t)}.
\end{aligned}$$

## Problem 6 Exercise 6.5

A so-called binary option is a claim which pays a certain amount if the stock price at a certain date falls within some pre-specified interval. Otherwise nothing will be paid out. Consider a binary option which pays  $K$  SEK to the holder at date  $T$  if the stock price at time  $T$  is in the interval  $[\alpha, \beta]$ . Determine the arbitrage free price. The pricing formula will involve the standard Gaussian cumulative distribution function  $N$ .

## Answers:

The arbitrage-free price  $F(t, S)$  is:

$$F(t, S) = \mathbb{E} \left[ e^{-r(T-t)} K \times 1_{S(T) \in [\alpha, \beta]} \right]$$

Let  $Z = \ln S$ ,  $Z(T) - Z(t) \sim N \left( \left( r - \frac{\sigma^2}{2} \right) (T-t), \sigma^2 (T-t) \right)$ , and  $\mathbb{P}(\cdot)$  denote probability under risk-neutral measure, then

$$\begin{aligned}
F(t, S) &= K \times e^{-r(T-t)} \mathbb{P}(\alpha < S(T) < \beta \mid S(t) = S) \\
&= K \times e^{-r(T-t)} \mathbb{P}(\ln \alpha < Z(T) < \ln \beta \mid Z(t) = \ln S) \\
&= K \times e^{-r(T-t)} \mathbb{P}(\ln \alpha - \ln S < Z(T) - Z(t) < \ln \beta - \ln S) \\
&= K \times e^{-r(T-t)} \mathbb{P} \left( \frac{\ln \alpha - \ln S - \left( r - \frac{\sigma^2}{2} \right) (T-t)}{\sigma \sqrt{T-t}} < \frac{[Z(T) - Z(t)] - \left( r - \frac{\sigma^2}{2} \right) (T-t)}{\sigma \sqrt{T-t}} < \frac{\ln \beta - \ln S - \left( r - \frac{\sigma^2}{2} \right) (T-t)}{\sigma \sqrt{T-t}} \right)
\end{aligned}$$

Let  $N(\cdot)$  denote the standard normal distribution function and  $d(x) = \frac{\ln x - \ln S - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$ , then

$$F(t, S) = K \times e^{-r(T-t)} [N(d(\beta)) - N(d(\alpha))].$$

## Problem 7 Exercise 6.6

Consider the standard Black-Scholes model. Derive the arbitrage free price process for the claim  $\mathcal{X}$  where  $\mathcal{X}$  is given by  $\mathcal{X} = \frac{S(T_1)}{S(T_0)}$ . The times  $T_0$  and  $T_1$  are given and the claim is paid out at time  $T_1$ .

### Answers:

We have  $S(t) = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$ ,  $\chi = \frac{S(T_1)}{S(T_0)} = e^{\left(r - \frac{\sigma^2}{2}\right)(T_1 - T_0) + \sigma(W(T_1) - W(T_0))}$ .

The price process

$$\begin{aligned} F(t, S) &= \mathbb{E} \left[ e^{-r(T-t)} \chi \mid S(t) = S \right] \\ &= e^{-r(T_1-t)} \mathbb{E} \left[ e^{\left(r - \frac{\sigma^2}{2}\right)(T_1 - T_0) + \sigma(W(T_1) - W(T_0))} \mid S(t) = S \right]. \end{aligned}$$

For  $t \in [0, T_0]$ ,  $W(T_1) - W(T_0) \sim N(0, T_1 - T_0)$  and is independent of  $\mathcal{F}_t$ , so

$$\begin{aligned} F(t, S) &= e^{-r(T_1-t)} \mathbb{E} \left[ e^{\left(r - \frac{\sigma^2}{2}\right)(T_1 - T_0) + \sigma(W(T_1) - W(T_0))} \right] \\ &= e^{-r(T_1-t)} e^{\left(r - \frac{\sigma^2}{2}\right)(T_1 - T_0)} e^{\frac{1}{2}\sigma^2(T_1 - T_0)} \\ &= e^{-r(T_1-t) + r(T_1 - T_0)} \\ &= e^{-r(T_0-t)}. \end{aligned}$$

## Problem 8 Exercise 6.7

Consider the American corporation ACME INC. The price process  $\{S_t : 0 \leq t \leq T\}$  for ACME is denoted in US \$ and has the  $\mathbb{P}$ -dynamics.

$$dS_t = \alpha S_t dt + \sigma S_t d\bar{W}_{1,t},$$

where  $\alpha$  and  $\sigma$  are known constants. The currency ratio SEK/US \$ at time  $t$  is denoted by  $Y_t$  and  $Y_t$  has the dynamics

$$dY_t = \beta Y_t dt + \delta Y_t d\bar{W}_{2,t},$$

where  $\bar{W}_2$  is independent of  $\bar{W}_1$ . The broker firm F&H has invented the derivative "Euler". The holder of a  $T$ -Euler will, at the time of maturity  $T$ , obtain the payoff

$$\mathcal{X} = \ln [\{Z(T)\}^2]$$

in SEK.  $Z(t)$  denotes the price of the ACME stock at time  $t$  in SEK.

Compute the arbitrage free price (in SEK) at time  $t$  of a  $T$ -Euler, given that the price (in SEK) of the ACME stock is  $z$ . The Swedish short rate is denoted by  $r$ .

## Answers:

$Z = SY$ , by Ito's lemma,

$$\begin{aligned} dZ &= SdY + YdS + dYdS \\ &= S(\beta Ydt + \delta YdW_2) + Y(\alpha Sdt + \sigma SdW_1) + 0 \\ &= (\beta + \alpha)Zdt + \sigma Zd\bar{W}_1 + \delta Zd\bar{W}_2. \end{aligned}$$

Note that  $Z$  is traded asset in SEK. Under no arbitrage condition, we have

$$dZ = rZdt + \sigma ZdW_1 + \delta ZdW_2.$$

where  $W_1$  and  $W_2$  are Brownian motions under risk neutral measure  $\mathbb{Q}$  (because the SEK risk free rate is  $r$ ).

$X = \ln Z^2 = 2 \ln Z$ , by Ito's lemma, under the risk neutral measure  $\mathbb{Q}$

$$dX = [2r - \sigma^2 - \delta^2] dt + 2\sigma dW_1 + 2\delta dW_2.$$

Notice that  $X(t)$  is multivariate normally distributed, then the price process

$$\begin{aligned} F(t, s, y) &= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[X(T) \mid X(t) = 2 \ln sy] \\ &= e^{-r(T-t)} ((2r - \sigma^2 - \delta^2)(T - t) + 2 \ln sy). \end{aligned}$$

## Problem 9

Show that  $S_t^{-2r/\sigma^2}$  can be the price of a traded derivative security at time  $t$ , where  $S_t$  denotes the price of the underlying asset at  $t$ .

## Answers:

If  $f = S^{-\frac{2r}{\sigma^2}}$  then

$$\begin{aligned} \frac{\partial f}{\partial S} &= -\frac{2r}{\sigma^2} S^{-\frac{2r}{\sigma^2}-1} \\ \frac{\partial^2 f}{\partial S^2} &= \left(\frac{2r}{\sigma^2}\right) \left(\frac{2r}{\sigma^2} + 1\right) S^{-\frac{2r}{\sigma^2}-2} \\ \frac{\partial f}{\partial t} &= 0 \\ \frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} &= rS^{-\frac{2r}{\sigma^2}} = rf \end{aligned}$$

This shows that the Black-Scholes-Merton equation is satisfied.  $S_t^{-\frac{2r}{\sigma^2}}$  could therefore be the price of a traded security (whose underlying asset  $S_t$  follows a GBM with percentage variance  $\sigma$ ).

## Problem 10

Suppose the price process of a stock, denoted by  $S_t$ , follows a geometric Brownian motion with expected return  $\mu$  and volatility  $\sigma$ :

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

What is the process followed by the variable  $(S_t)^n$ ? Show that  $(S_t)^n$  is a geometric Brownian motion.

## Answers:

If  $G(S, t) = S^n$  then

$$\begin{aligned}\frac{\partial G}{\partial t} &= 0 \\ \frac{\partial G}{\partial S} &= nS^{n-1} \\ \frac{\partial^2 G}{\partial S^2} &= n(n-1)S^{n-2}\end{aligned}$$

Using Itô's lemma:

$$dG = \left[ \mu nG + \frac{1}{2}n(n-1)\sigma^2 G \right] dt + \sigma nG dz$$

This shows that  $G = S^n$  follows geometric Brownian motion  $dG = \mu_G G dt + \sigma_G G dz$  where the expected return is

$$\mu_G = \mu n + \frac{1}{2}n(n-1)\sigma^2$$

and the volatility is

$$\sigma_G = n\sigma$$

## Problem 11

Prices of stock A and stock B follow geometric Brownian motions. Changes in any short interval of time are uncorrelated with each other. Does the value of a portfolio consisting of one share of stock A and one share of stock B follow geometric Brownian motion process? Explain your answer.

## Answers:

- Denote  $S_t^A, \mu_A$  and  $\sigma_A$  as the stock price, expected return and volatility for stock A.
- Denote  $S_t^B, \mu_B$  and  $\sigma_B$  as the stock price, expected return and volatility for stock B.

Since each of the two stocks follows geometric Brownian motion,

$$\begin{aligned}dS_t^A &= \mu_A S_t^A dt + \sigma_A S_t^A dW_t^A \\ dS_t^B &= \mu_B S_t^B dt + \sigma_B S_t^B dW_t^B\end{aligned}$$

where  $W_t^A$  and  $W_t^B$  are independent standard BM.

$$d(S_t^A + S_t^B) = (\mu_A S_t^A + \mu_B S_t^B) dt + \sigma_A S_t^A dW_t^A + \sigma_B S_t^B dW_t^B$$

This cannot be written as

$$d(S_t^A + S_t^B) = \mu (S_t^A + S_t^B) dt + \sigma (S_t^A + S_t^B) dW_t$$

for any constants  $\mu$  and  $\sigma$ , and standard BM  $W_t$ . (Neither the drift term nor the stochastic term correspond.) Hence the value of the portfolio does not follow geometric Brownian motion.

## Problem 12

Let  $\{r_t : 0 \leq t \leq T\}$  denote the interest rate process. The time- $t$  price of a zero-coupon bond paying \$1 at maturity time  $T$  is equal to  $E_t^Q[e^{-\int_t^T r(s)ds}]$ .

- (1) The stochastic process of the short-rate  $r(t)$  is given by

$$dr(t) = (0.008 - 0.1r(t))dt + 0.05dZ(t)$$

where  $Z(t)$  is a standard Brownian motion under the physical probability measure.

- (2) The short-rate process under the risk-neutral probability measure is given by

$$dr(t) = (0.013 - 0.1r(t))dt + 0.05d\tilde{Z}(t),$$

Where  $\tilde{Z}(t)$  is a standard Brownian motion under the risk-neutral probability measure.

- (3) For  $t \leq T$ , let  $P(r, t, T)$  be the price at time  $t$  of a zero-coupon bond that pays \$1 at time  $T$ , if the short-rate at time  $t$  is  $r$ . The price of each zero-coupon bond follows an Ito process:

$$\frac{dP(r(t), t, T)}{P(r(t), t, T)} = \alpha(r(t), t, T)dt - q(r(t), t, T)dZ(t), t \leq T.$$

Calculate  $\alpha(0.04, 2, 5)$ .

## Answers:

The Vasicek-Model short rate process is given by

$$dr(t) = \kappa(\mu - r(t))dt + \sigma d\tilde{Z}(t).$$

From (1) and (2) we can interpret:

- $\kappa = 0.1$
- $\mu = 0.13$
- $\sigma = 0.05$
- Market price of risk  $\lambda = \frac{(0.008 - 0.1r(t)) - (0.013 - 0.1r(t))}{0.05} = -0.1$
- $dZ(t) = d\tilde{Z}(t) - \lambda dt = d\tilde{Z}(t) + 0.1dt$

The bond price process satisfying fundamental PDE

$$P_t + P_r \kappa(\mu - r(t)) + \frac{1}{2} P_{rr} \sigma^2 - r(t)P = 0.$$

has affine term structure solution is given by

$$\begin{aligned} P(r(t), t, T) &= e^{A(T-t) + B(T-t)r(t)} \\ B(T-t) &= \frac{1}{\kappa} \left( e^{-\kappa(T-t)} - 1 \right) = 10 \left( e^{-0.1(T-t)} - 1 \right) \\ A(T-t) &= -\frac{\sigma^2}{4\kappa} (B(T-t))^2 - \left( \mu - \frac{\sigma^2}{2\kappa^2} \right) (T-t + B(T-t)) \end{aligned}$$

Applying Ito's lemma to the bond price and using the fundamental PDE to obtain the dynamics

$$\begin{aligned}\frac{dP(r(t), t, T)}{P(r(t), t, T)} &= \sigma B(T - t) d\tilde{Z}(t) + r(t) dt \\ &= \sigma B(T - t) dZ(t) + (r(t) - \lambda \sigma B(T - t)) dt\end{aligned}$$

Hence we have

$$\begin{aligned}q(r(t), t, T) &= -\sigma B(T - t) \\ \alpha(r(t), t, T) &= r(t) - \lambda \sigma B(T - t) \\ &= r(t) - 0.1 \times 0.05 \times 10 \left( e^{-0.1(T-t)} - 1 \right) \\ \alpha(0.04, 2, 5) &= 0.04 - 0.1 \times 0.05 \times 10 \left( e^{-0.1(5-2)} - 1 \right) \approx 0.05296\end{aligned}$$

**Remarks:** In this problem, the market price of risk (risk premium) is a constant so both the short rates in physical world and risk-neutral world have Vasicek dynamics.

## Problem 13

Suppose that  $x_t$  is the yield to maturity with continuous compounding on a zero-coupon bond that pays off \$1 at time  $T$ . Assume that  $x_t$  follows the process

$$dx_t = a(x_0 - x_t) dt + sx_t dZ_t$$

where  $a, x_0$ , and  $s$  are positive constants and  $Z_t$  is a Wiener process. What type of stochastic process is the bond price process?

## Answers:

$x$  is the yield to maturity, satisfying

$$B = e^{-x(T-t)}$$

the partial derivatives are

$$\begin{aligned}\frac{\partial B}{\partial t} &= xe^{-x(T-t)} = xB \\ \frac{\partial B}{\partial x} &= -(T-t)e^{-x(T-t)} = -(T-t)B \\ \frac{\partial^2 B}{\partial x^2} &= (T-t)^2 e^{-x(T-t)} = (T-t)^2 B\end{aligned}$$

By Itô's lemma:

$$\begin{aligned}dB &= \left[ \frac{\partial B}{\partial x} a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2} \frac{\partial^2 B}{\partial x^2} s^2 x^2 \right] dt + \frac{\partial B}{\partial x} s x dz \\ &= \left[ -a(x_0 - x)(T-t) + x + \frac{1}{2} s^2 x^2 (T-t)^2 \right] B dt - sx(T-t)B dz\end{aligned}$$

## Problem 14 Neftci's Book Chapter 15, P268, Exercise 2

In this exercise we work with the Black-Scholes setting applied to foreign currency denominated assets. We will see a different use of Girsanov theorem. [For more details see Musiela and Rutkowski (1997).] Let  $r, f$  denote the domestic and the foreign risk-free rates. Let  $S_t$  be the exchange rate, that is, the price of one unit of foreign currency in terms of domestic currency. Assume a geometric process for the dynamics of  $S_t$  :

$$dS_t = (r - f)S_t dt + \sigma S_t dW_t$$

(a) Show that

$$S_t = S_0 e^{(r-f-\frac{1}{2}\sigma^2)t + \sigma W_t}$$

where  $W_t$  is a Wiener process under probability  $\mathbb{P}$ .

(b) Is the process

$$\frac{S_t e^{ft}}{S_0 e^{rt}} = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

a martingale under measure  $\mathbb{P}$ ?

(c) Let  $\mathbb{Q}$  be the probability

$$\mathbb{Q}(A) = \int_A e^{\sigma W_T - \frac{1}{2}\sigma^2 T} dP$$

What does Girsanov theorem imply about the process,  $W_t - \sigma t$ , under  $\mathbb{Q}$ ?

(d) Show, using Ito's Lemma, that

$$dZ_t = Z_t [(f - r + \sigma^2) dt - \sigma dW_t]$$

where  $Z_t = 1/S_t$ .

(e) Under which probability measure is the process  $Z_t e^{rt}/e^{ft}$  a martingale?

(f) Can  $\mathbb{Q}$  be the arbitrage-free probability measure for the foreign economy in the sense that the price of a financial instrument is equal to its expected future payoff under  $\mathbb{Q}$  in that economy?

## Answers:

(a)

$S_t = f(t, W) = S_0 e^{(r-f-\frac{\sigma^2}{2})t + \sigma W}$ , apply Ito's lemma:

$$\begin{aligned} dS_t &= f_t dt + f_x dW + \frac{1}{2} f_{xx} dt \\ &= (r - f)S_t dt + \sigma S_t dW \end{aligned}$$

(b)

$f(t, W) = e^{\sigma W - \frac{1}{2}\sigma^2 t}$ , apply Ito's lemma:

$$\begin{aligned} df &= f_t dt + f_x dW + \frac{1}{2} f_{xx} dt \\ &= -\frac{1}{2}\sigma^2 f dt + \sigma f dW + \frac{1}{2}\sigma^2 f dt \\ &= \sigma f dW \end{aligned}$$

so  $f = e^{\sigma W - \frac{1}{2}\sigma^2 t}$  is a martingale.

(c)

Let  $X_t = \sigma W_t$ . The Girsanov theorem states that for standard Brownian motion  $W_t$  under measure  $\mathbb{P}$ ,

$$W_t - [W, X]_t = W_t - \sigma[W, W]_t = W_t - \sigma t$$

is a martingale under  $\mathbb{Q}$  defined by  $\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi_T = e^{\sigma W_T - \frac{1}{2}\sigma^2 T}$ .

The process  $W_t - \sigma t$  is a martingale under  $\mathbb{Q}$ .

(d)

Apply Ito's formula:

$$\begin{aligned} dZ &= -\frac{1}{S^2} dS + \frac{1}{2} \frac{2}{S^3} (dS)^2 \\ &= -\frac{1}{S^2} ((r - f)S dt + \sigma S dW) + \frac{1}{S^3} \sigma^2 S^2 dt \\ &= Z \left( -(r - f) dt - \sigma dW + \sigma^2 dt \right) \\ &= Z \left( (f - r + \sigma^2) dt - \sigma dW_t \right). \end{aligned}$$

(e)

Let  $X_t = e^{(r-f)t} Z_t$ , By Ito's lemma,

$$dX_t = \sigma^2 X_t dt - \sigma X_t dW_t.$$

We want to construct a probability measure  $\tilde{\mathbb{P}}$  such that  $X_t$  is a martingale under  $\tilde{\mathbb{P}}$ :

$$dX_t = -\sigma X_t d\tilde{W}_t,$$

where  $\tilde{W}_t$  is a BM under  $\tilde{\mathbb{P}}$ . Hence

$$d\tilde{W}_t = dW_t - \sigma dt.$$

Note that  $e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$  is a  $\mathbb{P}$  martingale starting with value 1, by Girsanov theorem, the equivalent martingale measure is defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \xi_T = e^{\int_0^T \sigma dW_t - \frac{1}{2} \int_0^T \sigma^2 dt} = e^{\sigma W_T - \frac{1}{2}\sigma^2 T}$$

so  $Z_t e^{rt} / e^{ft}$  is a martingale under  $\tilde{\mathbb{P}}$ .

(f)

Yes, because it is an equivalent martingale measure for  $Z_t e^{(r-f)t}$  (discounted foreign currency process).