ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 2

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Please remember to write your name and GTID in the homework submitted.

Problem 1

You are bidding for a firm whose unknown true value is uniformly distributed between 0 and 1. Although you do not know the true value S of the firm, you do know that as soon as people learn that you have made a bid this news will cause the value to double to $2S$. Your bid, however, will be accepted only if it is at least as the original value of the firm. How do you bid so as to maximize your expected payoff?

Answers:

(a)

Assume x to be the bid price. If our bid is accepted ($x > S$), the payoff will be $2S - x$; otherwise the payoff will be zero.

S is uniformly distributed between 0 and 1. Then the expected payoff under bid price x is

$$\begin{aligned}\mathbb{E}[\text{payoff}] &= \int_0^1 (2s - x)1_{\{x>s\}} ds \\ &= \begin{cases} \int_0^x (2s - x)ds = 0 & 0 < x \leq 1 \\ \int_0^1 (2s - x)ds = 1 - x < 0 & x > 1 \end{cases}\end{aligned}$$

Therefore, our optimized strategy is to bid at price no higher than 1. With any bid price x below or equal to 1, our expected payoff is zero. If we choose a large $x(> 1)$, the probability of successful bid is high, but the expected payoff will not change. And we will get the firm for sure if we bid at 1.

Problem 2

You and I are to play a game. You roll a die until a number other than a one appears. When such a number appears for the first time, I pay you the same number of dollars as there are dots on the upturned face of the die, and the game ends. What is the expected payoff to this game?

Answers:

The probability distribution:

Result	1	2	3	4	5	6
Payoff	Try again	2	3	4	5	6
Probability	1/6	1/6	1/6	1/6	1/6	1/6

The expectation of payoff never changes no matter how many times you die.

$$\begin{aligned} \mathbb{E}[\text{payoff}] &= \frac{6+5+4+3+2+\mathbb{E}[\text{payoff}]}{6} \\ \implies \mathbb{E}[\text{payoff}] &= 4 \end{aligned}$$

Problem 3

You have a large jar containing 999 fair pennies and one two-headed penny. Suppose you pick one coin out of the jar and flip it 10 times and get all heads. What is your opinion on the type of penny you picked up?

Answers:

Assume:

- $A =$ I pick a fair penny, $\mathbb{P}(A) = 0.999$
- $B =$ I flipped 10 times and get all heads, $\mathbb{P}(B|A) = \left(\frac{1}{2}\right)^{10}, \mathbb{P}(B|\neg A) = 1$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A) * \mathbb{P}(B|A)}{\mathbb{P}(A) * \mathbb{P}(B|A) + \mathbb{P}(\neg A) * \mathbb{P}(B|\neg A)} = \frac{0.00097 * 0.999}{0.00097 * 0.999 + 1 * 0.001} = 0.4938$$

The probability that I pick a fair penny is 0.4938, while the probability that I pick a two-headed penny is 0.5062.

Problem 4

Four cards are shuffled and placed face down in front of you. Their faces (hidden) display the four elements: water, earth, wind, and fire. You are to turn the cards over one card at a time until you either win or lose. You win if both the water and earth cards are turned over. You lose if the fire card is turned over. What is the probability of winning?

Answers:

(a)

Assume:

- $A =$ First two cards turned over are water and earth (4 situations)

- B = First three cards turned over are wind, water and earth and the last card is not wind (4 situations)

$$\mathbb{P}(\text{win}) = \mathbb{P}(A) + \mathbb{P}(B) = \frac{2*2}{4!} + \frac{2*2}{4!} = \frac{1}{3}$$

Problem 5

Assume that 10% of the population is infected by a virus. The virus-test machine is known to be imperfect. Errors can happen when people are tested for virus infection.

Assume:

- A = test result is positive, and A^c means negative.
- B = this person is infected
- $\mathbb{P}(A|B) = 0.95, \mathbb{P}(A^c|B) = 0.05$
- $\mathbb{P}(A|B^c) = 0.01, \mathbb{P}(A^c|B^c) = 0.99$

Now someone received 3 independent tests, and the results turned out to be positive twice and negative once. What is the probability that the person is actually infected?

You may find this formula useful:

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(B_i) * \mathbb{P}(A|B_i)}{\sum_j \mathbb{P}(B_j) * \mathbb{P}(A|B_j)}$$

Answers:

Assume: C = someone did 3 times of test independently, and turns out to be positive twice and negative once

$$\begin{aligned}
\mathbb{P}(B|C) &= \frac{\mathbb{P}(B) * \mathbb{P}(C|B)}{\mathbb{P}(B) * \mathbb{P}(C|B) + \mathbb{P}(\neg B) * \mathbb{P}(C|\neg B)} \\
\mathbb{P}(B) &= 0.1 \\
\mathbb{P}(C|B) * \mathbb{P}(B) &= 3 * (\mathbb{P}(A|B))^2 * \mathbb{P}(\neg A|B) * \mathbb{P}(B) = 3 * 0.95^2 * 0.05 * 0.1 = 0.0135375 \\
\mathbb{P}(C|\neg B) * \mathbb{P}(\neg B) &= 3 * (\mathbb{P}(A|\neg B))^2 * \mathbb{P}(\neg A|\neg B) * \mathbb{P}(\neg B) = 3 * 0.01^2 * 0.99 * 0.9 = 0.0002673
\end{aligned}$$

We have

$$\mathbb{P}(B|C) = \frac{0.0135375}{0.0135375 + 0.0002673} = 0.9806$$

Problem 6

A new cinema is under construction and the total number of the seats is to be determined. Assume that in this area, 1600 people will go to cinema everyday and for the probability for each person to choose this cinema is $\frac{3}{4}$. Following are the requirements for determining the seat quantity q:

1. q is as large as possible.
2. The probability of "there are over 200 empty seats in one day" is no more than 0.1

Answers:

Assume:

$$I_i = \begin{cases} 1 & \text{if person } i \text{ comes to the cinema} \\ 0 & \text{if person } i \text{ does not come} \end{cases}$$
$$X = number of people come to the cinema in a single day$$

$$\mathbb{P}(I_i) = \frac{3}{4}$$
$$\mathbb{E}[I_i] = \mathbb{P}(I_i) * 1 + (1 - \mathbb{P}(I_i)) * 0 = \frac{3}{4}$$
$$Var[I_i] = \mathbb{P}(I_i) * (1 - \mathbb{P}(I_i)) = \frac{3}{16}$$

Let $n = 1600$, we have $X = \sum_{i=1}^{1600} I_i$, by Central Limit Theorem,

$$X \sim N(\frac{3}{4} * n, \frac{3}{16} * n) = N(1200, 300)$$

According to the second requirement that the probability of "there are over 200 empty seats in one day" is no more than 0.1, we get

$$\mathbb{P}(q - X \geq 200) \leq 0.1$$
$$\mathbb{P}(X \leq q - 200) \leq 0.1$$
$$\Phi\left(\frac{q - 200 - 1200}{\sqrt{300}}\right) \leq 0.1$$
$$q \leq \sqrt{300} * Z_{0.1} + 1400 = \sqrt{300} * (-1.28) + 1400 = 1377.8$$

The largest $q = 1377$.

Problem 7 Neftci's Book Chapter 2, P30, Exercise 2

In an economy there are two states of the world and four assets. You are given the following prices for three of these securities in different states of the world:

	Price		Dividends	
	State 1	State 2	State 1	State 2
Security A	120	70	4	1
Security B	80	60	3	1
Security C	90	150	2	10

"Current" prices for A, B, C are 100, 70, and 180, respectively.

(a) Are the "current" prices of the three securities arbitrage-free?

(b) If not, what type of arbitrage portfolio should one form?

(c) Determine a set of arbitrage-free prices for securities A, B, and C.

(d) Suppose we introduce a fourth security, which is a one-period futures contract written on B. What is its price?

(e) Suppose a put option with strike price $K = 125$ is written on C. The option expires in period 2. What is its arbitrage free price?

Answers:

(a)

$$D = \begin{pmatrix} 124 & 71 \\ 83 & 61 \\ 92 & 160 \end{pmatrix}, \qquad \vec{S_0} = \begin{pmatrix} 100 \\ 70 \\ 180 \end{pmatrix}$$

We cannot find a $\vec{\psi} \in \mathbb{R}^2_{++}$ such that $D\vec{\psi} = \vec{S_0}$. There is arbitrage opportunity. (Theorem 1)

(b)

Construct a portfolio θ containing A and B, so that the payoff matrix of the portfolio θ equals to that of Asset C.

$$D_1 = \begin{pmatrix} 124 & 71 \\ 83 & 61 \end{pmatrix}, \qquad \theta D_1 = \begin{pmatrix} 92 & 160 \end{pmatrix}$$
$$\Longrightarrow \theta = \begin{pmatrix} 92 & 160 \end{pmatrix} D_1^{-1} = \begin{pmatrix} -4.5889 & 7.9641 \end{pmatrix}$$

The current price of portfolio

$$P = \begin{pmatrix} -4.5889 & 7.9641 \end{pmatrix} \begin{pmatrix} 100 \\ 70 \end{pmatrix} = 98.6$$

Noticed that the price is lower than the current price of C, we could short C and long the portfolio to get negative net commitment today and 0 profits in the future, which is Type II arbitrage.

Specifically, we short 1 asset C; short 4.5889 asset A and buy 7.9641 asset B. The cost today is -81.4, and profits in the future will be 0.

(c)

Based on what we have calculated in (b), we know that $\vec{S_0'} = \begin{pmatrix} 100 \\ 70 \\ 98.6 \end{pmatrix}$ is a possible arbitrage-free price for asset A, B and C. However, there are also other possible prices. We can construct any $\vec{S_0} = \begin{pmatrix} P_a \\ P_b \\ P_c \end{pmatrix}$, which satisfies the following requirements:

$$D\vec{\psi} = \vec{S_0} \quad \vec{\psi} \in \mathbb{R}^2_{++} \quad and \quad |\vec{\psi}| = \frac{1}{1+r}$$

(d)

- Step 1: Find $\vec{\psi} \in \mathbb{R}^2_{++}$, satisfies the requirement in (c)
- Step 2: Price the future contract, so that the current value of the contract equal to zero. Suppose X to be the strike price, and we can get the future payoff vector $\vec{F} = [83-X, 61-X]$. Given $\vec{F}\vec{\psi} = 0$, we can get the arbitrage-free X.

(e)

For the put option with strike price $K = 125$, the payoff is $\vec{F} = [33, 0]$.

Then, the arbitrage-free price $P = \vec{F}\vec{\psi}$ using the same $\vec{\psi}$ as in (d).

Problem 8 (State-Price Vector)

Now we consider an economy of three states and four assets with prices in different states as follows:

	Price		
	State 1	**State 2**	**State 3**
Security A	120	70	80
Security B	80	60	50
Security C	90	150	190
Security D	30	20	30

"Current" price for A, B, C are 100, 70 and 180, respectively.

Question: Calculate the no arbitrage price of D and give the replicating portfolio.

Remark: As you have noticed, this problem is similar to Problem 7. I put this problem here to help you become very comfortable with portfolio replication and matrix calculation.

Answers:

Let the replicating portfolio for asset D be $\theta = \{\theta_a, \theta_b, \theta_c\}$.

$$\theta D = \vec{F}$$

$$\begin{bmatrix}\theta_a & \theta_b & \theta_c\end{bmatrix}\begin{bmatrix}120 & 70 & 80\\ 80 & 60 & 50\\ 90 & 150 & 190\end{bmatrix} = \begin{bmatrix}30 & 20 & 30\end{bmatrix}$$

$$\begin{bmatrix}\theta_a & \theta_b & \theta_c\end{bmatrix} = \begin{bmatrix}0.4089 & -0.3158 & 0.0688\end{bmatrix}$$

$$P = \begin{bmatrix}0.4089 & -0.3158 & 0.0688\end{bmatrix}\begin{bmatrix}100\\ 70\\ 180\end{bmatrix} = 31.17$$

Problem 9

Let H (Heads) and T (Tails) denote the two outcomes of a random experiment of tossing a fair coin. Suppose I toss the coin infinite many times and divide the outcomes (which are infinite sequences of Heads and Tails) into two types of events:

(a) the portion of H or T is exactly one half (e.g. $HTHTHTHT\cdots$ or $HHTTHHTT\cdots$)

(b) the portion of H or T is not one half (i.e. the complement of event (a). e.g. $HTTHTTHTT\cdots$).

What are the probabilities for events (a) and (b), respectively?

Answers:

According to Law of Large Number (LLN), sample mean converges to true expectation as sample size n goes to infinity. In this particular problem, infinite trials will end up in strictly $\frac{1}{2}$ Heads and $\frac{1}{2}$ Tails for each and every such trail. In other words, $\mathbb{P}(a) = 1$ and $\mathbb{P}(b) = 0$.

Problem 10

Suppose there are 100 strings, each of the string, of course, has 2 ends. Then you randomly choose 2 of the ends and tie them together. Each end will be tied only once and the process repeats until there are no free ends left. (i.e. It will lead to 100 randomly chosen pairs of tied ends.) Let L be the number of resulting loops. What is $\mathbb{E}[L]$?

Answers:

Suppose you start with n ropes. You pick two free ends and tie them together:

- If you happened to pick two ends of the same rope, you've added one additional loop (which you can set aside, since you'll never pick it now on), and have $n-1$ ropes
- If you happened to pick ends of different ropes, you've added no loop, and effectively replaced the two ropes with a longer rope, so you have $n-1$ ropes in this case too.

Of the $\binom{2n}{n}$ ways of choosing two ends, n of them result in the first case, so the first case has probability $\frac{n}{2n(2n-1)/2} = \frac{1}{2n-1}$. So the expected number of loops you add in the first step, when you start with n ropes, is

$$\frac{1}{2n-1} * 1 + 0 = \frac{1}{2n-1}$$

After this, you start over with $n-1$ ropes. Since what happens in the first step and later are independent, the expected number of loops for next step is $\frac{1}{2n-3}$, so the total number of expected loops is just to sum up each case's expected loops:

$$\mathbb{E}[\text{Total loops}] = \frac{1}{2n-1} + \frac{1}{2n-3} + \frac{1}{2n-5} + \cdots + \frac{1}{3} + 1$$

In particular, for n=100, the answer is roughly 3.28.

Problem 11 Russian Roulettes

(a) Suppose two players play a traditional Russian Roulettes game. One bullet is put into a 6-revolver and the barrel is randomly spun so that there is equal chance for each chamber to be under the hammer. Two players take turns to pull the trigger against themselves until one kills him or herself. Under such rules, would you rather go first or second? What's the probability of survival to go first?

(b) Now the rule is changed, the barrel gets spun after each shot. Now would you rather go first or second? What's the probability of survival for each choice?

(c) Suppose two bullets instead of one are randomly put into the chamber. Your opponent went first and survived the shot. Now you are given the chance to spin the barrel. Should you do it or not?

(d) What if the two bullets are consecutively put into the chamber and the barrel is spun (i.e. two bullets are next to each other), should you spin again after your opponent survived the first round?

Answers:

(a)

Many people have the wrong impression that the first person has higher probability of loss. After all, the first player has a 1/6 chance of getting killed in the first round before the second player starts. Unfortunately, this is one of the few times that intuition is wrong. Once the barrel is spun, the position of the bullet is fixed. If you go first, you lose if and only if the bullet is in chamber 1, 3 and 5. So the probability that you lose is the same as the second player, 1/2. In that sense, whether to go first or second does not matter.

(b)

The difference is that each run now becomes independent. Assume that the first player's probability of losing is p, then the second player's probability of losing is $1-p$. Let's condition the probability on the first person's first trigger pull. He has 1/6 probability of losing in this run. Otherwise, he essentially becomes the second player in the game with new (conditional) probability of losing $1-p$. That situation happens with probability 5/6. So $p = 1 * 1/6 + (1-p) * 5/6 \implies p = 6/11$. Obviously the first player has higher probability of losing than the second. So you should choose to go second.

(c)

If you spin the barrel, the probability that you will lose in this round is 2/6. If you don't spin the barrel, there are only 5 chambers left and your probability of losing in this round (conditioned on that your opponent survived) is 2/5. So you should spin the barrel.

(d)

Now we have to condition our probability on the fact that the positions of the two bullets are consecutive. Let's label the empty chambers as 1, 2, 3 and 4; label the ones with bullets 5 and 6.

Since your opponent survived the first round, the possible position he encountered is 1, 2, 3 or 4 with equal probability. With 1/4 chance, the next one is a bullet (the position was 4). So if you don't spin, the chance of survival is 3/4. If you spin the barrel, each position has equal probability of being chosen, and your chance of survival is only 2/3. So you should not spin the barrel.