ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 6

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Problem 1

$$\frac{dS(t)}{S(t)} = 0.05\,dt + 0.2\,dZ(t), \qquad S(0) = 0.5$$

$$S(1) = S(0)\exp\big((\mu - \tfrac{1}{2}\sigma^2) + \sigma Z_1\big), \quad \mu = 0.05,\ \sigma = 0.2$$

$$E\big[S(1)^a\big] = S(0)^a \exp\Big(a(\mu - \tfrac{1}{2}\sigma^2) + \tfrac{1}{2}a^2\sigma^2\Big) = 1.4$$

$$S(0)^a = 0.5^a, \quad \mu - \tfrac{1}{2}\sigma^2 = 0.05 - 0.5 \cdot 0.2^2 = 0.03, \quad \tfrac{1}{2}\sigma^2 = 0.5 \cdot 0.2^2 = 0.02$$

$$0.5^a \exp(0.03a + 0.02a^2) = 1.4$$

$$a \ln 0.5 + 0.03a + 0.02a^2 = \ln 1.4$$

$$0.02a^2 + (\ln 0.5 + 0.03)a - \ln 1.4 = 0$$

$$a = \frac{-(\ln 0.5 + 0.03) \pm \sqrt{(\ln 0.5 + 0.03)^2 + 0.08 \ln 1.4}}{0.04}$$

$$a_1 \approx -0.4999, \qquad a_2 \approx 33.6572, \qquad a = a_1\ (a < 0)$$

Risk–neutral measure:

$$\frac{dS(t)}{S(t)} = r\,dt + 0.2\,dW(t), \qquad r = 0.03$$

$$S(1) = S(0)\exp\big((r - \tfrac{1}{2}\sigma^2) + \sigma W_1\big), \quad r - \tfrac{1}{2}\sigma^2 = 0.03 - 0.02 = 0.01$$

$$E^{\mathbb{Q}}[S(1)^a] = S(0)^a \exp\Big(a(r - \tfrac{1}{2}\sigma^2) + \tfrac{1}{2}a^2\sigma^2\Big) = 0.5^a \exp(0.01a + 0.02a^2)$$

$$E^{\mathbb{Q}}[S(1)^a]\big|_{a \approx -0.4999} \approx 1.4141$$

$$V_0 = e^{-r}E^{\mathbb{Q}}[S(1)^a] = e^{-0.03} \cdot 0.5^a \exp(0.01a + 0.02a^2)$$

$$V_0 \approx e^{-0.03} \cdot 1.4141 \approx 1.3723$$

Problem 2

(a) By Itô's formula,

$$
\begin{aligned}
d\Pi(t) &= dF(t, S(t)) \\
&= F_t(t, S(t))\,dt + F_s(t, S(t))\,dS(t) + \frac{1}{2}F_{ss}(t, S(t))\,d\langle S\rangle_t \\
&= \Big(F_t + rSF_s + \tfrac{1}{2}\sigma^2 S^2 F_{ss}\Big)(t, S(t))\,dt + \sigma S(t)F_s(t, S(t))\,dW(t)
\end{aligned}
$$

Using the PDE,

$$F_t + rSF_s + \tfrac{1}{2}\sigma^2 S^2 F_{ss} = rF,$$

so

$$d\Pi(t) = rF(t, S(t))\,dt + \sigma S(t)F_s(t, S(t))\,dW(t) = r\Pi(t)\,dt + g(t)\,dW(t),$$

where

$$g(t) = \sigma S(t)F_s(t, S(t))$$

(b)

$$Z(t) = \frac{\Pi(t)}{B(t)}$$

Since $B(t) = e^{rt}$, we have

$$dB(t) = rB(t)\,dt, \qquad d\big(B(t)^{-1}\big) = -rB(t)^{-1}\,dt$$

Using Itô's formula (or product rule) for $Z(t) = \Pi(t)B(t)^{-1}$,

$$
\begin{aligned}
dZ(t) &= B(t)^{-1}d\Pi(t) + \Pi(t)\,d\big(B(t)^{-1}\big) + d\Pi(t)\,d\big(B(t)^{-1}\big) \\
&= B(t)^{-1}\big(r\Pi(t)\,dt + g(t)\,dW(t)\big) + \Pi(t)\big(-rB(t)^{-1}\,dt\big),
\end{aligned}
$$

and the quadratic covariation term vanishes since B^{-1} is deterministic.

Thus

$$dZ(t) = \frac{g(t)}{B(t)}\,dW(t)$$

And then we know Z is a martingale with dynamics

$$dZ(t) = Z(t)\sigma_Z(t)\,dW(t),$$

where

$$\sigma_Z(t) = \frac{g(t)}{\Pi(t)} = \frac{\sigma S(t)F_s(t, S(t))}{F(t, S(t))}$$

Problem 3

$$\text{Payoff at } T: \quad GL(T) = \ln S(T)$$

Under the risk–neutral measure $\mathbb{Q}$,

$$\frac{dS(t)}{S(t)} = r\,dt + \sigma\,dW(t),$$

so for $0 \le t \le T$,

$$S(T) = S(t)\exp\Big((r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma\big(W(T) - W(t)\big)\Big)$$

Thus

$$\ln S(T) = \ln S(t) + (r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma\big(W(T) - W(t)\big)$$

Given $\mathcal{F}_t$, the increment $W(T) - W(t) \sim N(0, T-t)$ with mean 0, so

$$\mathbb{E}^{\mathbb{Q}}[\ln S(T) \mid \mathcal{F}_t] = \ln S(t) + (r - \tfrac{1}{2}\sigma^2)(T-t)$$

The arbitrage–free price process $\Pi(t)$ of $GL(T)$ is

$$\Pi(t) = B(t)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{\ln S(T)}{B(T)}\,\middle|\,\mathcal{F}_t\right] = e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}[\ln S(T) \mid \mathcal{F}_t]$$

Therefore

$$\boxed{\Pi(t) = e^{-r(T-t)}\left(\ln S(t) + (r - \tfrac{1}{2}\sigma^2)(T-t)\right)}$$

In particular, at time 0,

$$\Pi(0) = e^{-rT}\left(\ln S(0) + (r - \tfrac{1}{2}\sigma^2)T\right)$$

Problem 4

$$\tau = T - t$$
$$C(t) = e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}\big[(S(T)-K)^+ \mid \mathcal{F}_t\big]$$
$$S(T) = S(t)\exp\Big((r - \tfrac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\,Z\Big), \qquad Z \sim N(0,1)$$
$$d_1 = \frac{\ln\frac{S(t)}{K} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}$$
$$C(t) = S(t)N(d_1) - Ke^{-r\tau}N(d_2)$$

Problem 5

$$X = S(T)^\beta$$
$$\ln S(T) = \ln S(t) + (r - \tfrac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\,Z, \qquad Z \sim N(0,1)$$
$$\mathbb{E}^{\mathbb{Q}}\big[S(T)^\beta \mid \mathcal{F}_t\big] = S(t)^\beta \exp\Big(\beta(r - \tfrac{1}{2}\sigma^2)\tau + \tfrac{1}{2}\beta^2\sigma^2\tau\Big)$$
$$\Pi(t) = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\big[S(T)^\beta \mid \mathcal{F}_t\big] = S(t)^\beta \exp\Big((\beta-1)r\tau + \tfrac{1}{2}\beta(\beta-1)\sigma^2\tau\Big)$$

Problem 6

$$\tau = T - t$$
$$\Pi_6(t) = e^{-r\tau}\,K\,\mathbb{P}^{\mathbb{Q}}(\alpha \le S(T) \le \beta \mid \mathcal{F}_t)$$
$$\ln S(T) = \ln S(t) + (r - \tfrac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}\,Z, \qquad Z \sim N(0,1)$$
$$d_\alpha = \frac{\ln(\alpha/S(t)) - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad d_\beta = \frac{\ln(\beta/S(t)) - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$
$$\Pi_6(t) = Ke^{-r\tau}\big(N(d_\beta) - N(d_\alpha)\big)$$

Problem 7

$$X = \frac{S(T_1)}{S(T_0)}$$
$$\Pi_7(t) = e^{-r(T_1-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T_1)}{S(T_0)} \mid \mathcal{F}_t\right]$$
$$\Pi_7(t) = \begin{cases} e^{-r(T_0-t)}, & 0 \le t \le T_0, \\ \dfrac{S(t)}{S(T_0)}, & T_0 < t \le T_1. \end{cases}$$

Problem 8

$$dS_t = \alpha S_t\,dt + \sigma S_t\,d\bar{W}_1(t), \qquad dY_t = \beta Y_t\,dt + \delta Y_t\,d\bar{W}_2(t), \qquad \bar{W}_1 \perp \bar{W}_2$$

$$Z_t := S_t Y_t$$

$$dZ_t = S_t\,dY_t + Y_t\,dS_t + dS_t\,dY_t$$

$$dS_t\,dY_t = \sigma S_t \delta Y_t\,d\bar{W}_1\,d\bar{W}_2 = 0$$

$$dZ_t = (\alpha+\beta) Z_t\,dt + \sigma Z_t\,d\bar{W}_1 + \delta Z_t\,d\bar{W}_2$$

$$\exists\ \bar{W}: \quad \sigma\,d\bar{W}_1 + \delta\,d\bar{W}_2 = v\,d\bar{W}, \qquad v := \sqrt{\sigma^2+\delta^2}$$

$$\Rightarrow \quad \frac{dZ_t}{Z_t} = (\alpha+\beta)\,dt + v\,d\bar{W}_t$$

$$\text{under } \mathbb{Q}: \quad \frac{dZ_t}{Z_t} = r\,dt + v\,dW_t$$

$$Z_T = Z_t \exp\Big((r - \tfrac{1}{2}v^2)(T-t) + v\sqrt{T-t}\,Z\Big), \qquad Z \sim N(0,1)$$

$$\ln Z_T \sim N\big(m, v^2(T-t)\big), \qquad m = \ln Z_t + (r - \tfrac{1}{2}v^2)(T-t)$$

$$X = \ln\left(Z_T^2\right) = 2\ln Z_T$$

$$\mathbb{E}^{\mathbb{Q}}[X \mid Z_t = z] = 2\,\mathbb{E}^{\mathbb{Q}}[\ln Z_T \mid Z_t = z] = 2\Big(\ln z + (r - \tfrac{1}{2}v^2)(T-t)\Big)$$

$$\Pi(t,z) = e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}[X \mid Z_t = z]$$

$$\boxed{\Pi(t,z) = 2e^{-r(T-t)}\Big(\ln z + (r - \tfrac{1}{2}(\sigma^2+\delta^2))(T-t)\Big)}$$

Problem 9

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t$$

$$V_t = S_t^a$$

$$dV_t = aS_t^{a-1}dS_t + \frac{1}{2}a(a-1)S_t^{a-2}(dS_t)^2$$

$$(dS_t)^2 = \sigma^2 S_t^2\,dt$$

$$dV_t = \left(ar + \frac{1}{2}a(a-1)\sigma^2\right)S_t^a dt + a\sigma S_t^a dW_t$$

$$\text{Choose } a = -\frac{2r}{\sigma^2}: \quad ar + \frac{1}{2}a(a-1)\sigma^2 = r$$

$$\Rightarrow \quad dV_t = rV_t\,dt + a\sigma V_t\,dW_t$$

$$V_t = S_t^{-2r/\sigma^2}$$

Problem 10

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$$

$$X_t = S_t^n$$

$$dX_t = nS_t^{n-1}dS_t + \frac{1}{2}n(n-1)S_t^{n-2}(dS_t)^2$$

$$(dS_t)^2 = \sigma^2 S_t^2\,dt$$

$$dX_t = \left(n\mu + \frac{1}{2}n(n-1)\sigma^2\right)S_t^n dt + n\sigma S_t^n dW_t$$

$$d(S_t^n) = \mu_n S_t^n dt + \sigma_n S_t^n dW_t, \quad \mu_n = n\mu + \tfrac{1}{2}n(n-1)\sigma^2,\ \sigma_n = n\sigma$$