ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 4

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Please remember to write down your name and GTID in the submitted homework.

Problem 1 Neftci's Book Chapter 6, P110, Exercise 2

Consider the random variable:

$$X_n = \sum_{i=1}^{n} B_i$$

where each B_i is obtained as a result of the toss of a fair coin:

$$B_i = \begin{cases} 1 & \text{Head} \\ -1 & \text{Tail} \end{cases}$$

We let $n = 4$ and consider X_4.

(a) Calculate the $\mathbb{E}[X_4|I_1]$, $\mathbb{E}[X_4|I_2]$, $\mathbb{E}[X_4|I_4]$.

(b) Let $Z_i = \mathbb{E}[X_4|I_i]$. Is $Z_i, i = 1, \cdots, 4$ a martingale?

(c) Now define:

$$V_i = B_i + \sqrt{i}$$

and

$$\tilde{X}_n = \sum_{i=1}^{n} V_i$$

Is $\tilde{X}_n$ a martingale?

(d) Can you convert $\tilde{X}_n$ into a martingale by an appropriate transformation?

(e) Can you convert $\tilde{X}_n$ into a martingale by changing the probabilities associated with a coin toss?

Answers:

(a)

$\mathbb{E}[X_4|I_1] = X_1$, $\mathbb{E}[X_4|I_2] = X_2$, $\mathbb{E}[X_4|I_4] = X_4$

(b)

$Z_i, i = 1, \cdots, 4$ is a martingale because

- $Z_i = \mathbb{E}[X_4|I_i], i = 1, \cdots, 4$ is known given I_i
- $\mathbb{E}[|Z_i|] \leq 4 < \infty$
- For $\mathbb{E}[Z_4|I_i] = \mathbb{E}[X_4|I_i] = Z_i, i = 1, \cdots, 4$

(c)

No. Note that information generated by $\tilde{X}_i$ and information generated by X_i are the same I_i

$$\begin{aligned}
\mathbb{E}[\tilde{X}_{i+k}|I_i] &= \mathbb{E}[\tilde{X}_i + \sum_{j=i+1}^{i+k} V_j|I_i] \\
&= \sum_{j=i+1}^{i+k} \mathbb{E}[V_j|I_i] + \tilde{X}_i \\
&= \sum_{j=i+1}^{i+k} \mathbb{E}[B_j + \sqrt{j}|I_i] + \tilde{X}_i \\
&= \sum_{j=i+1}^{i+k} \sqrt{j} + \tilde{X}_i
\end{aligned}$$

Thus $\tilde{X}$ is not a martingale.

(d)

Yes subtracting the deterministic component $\sqrt{i}$ from V_i. Then X_i is a martingale.

(e)

No.

$$\begin{aligned}
\mathbb{E}[\tilde{X}_{i+k}|I_i] &= \sum_{j=i+1}^{i+k} \mathbb{E}[B_j + \sqrt{j}|I_i] + \tilde{X}_i \\
&= \sum_{j=i+1}^{i+k} (\sqrt{j} + 2p - 1) + \tilde{X}_i
\end{aligned}$$

Since p is constant and the term $\sum_{j=i+1}^{i+k}(\sqrt{j} + 2p - 1)$ is non-constant(not always 0). $\tilde{X}$ is not a martingale.

Problem 2 Neftci's Book Chapter 6, P110, Exercise 3

Let W_t be a Wiener process and t denote the time. Are the following stochastic processes martingales?

(a) $X_t = 2W_t + t$

(b) $X_t = W_t^2$

(c) $X_t = W_t^2 - 2\int_0^t sW_s ds$ (This question is optional)

Answers:

(a)

No.

$$\begin{aligned}
\mathbb{E}[X_{t+s}|I_t] &= \mathbb{E}[2W_{t+s} + t + s|I_t] \\
&= 2\mathbb{E}[W_t + (W_{t+s} - W_t)|I_t] + t + s \\
&= 2W_t + t + s \neq 2W_t + t
\end{aligned}$$

(b)

No.

$$\begin{aligned}
\mathbb{E}[X_{t+s}|I_t] &= \mathbb{E}[W_{t+s}^2|I_t] \\
&= \mathbb{E}[W_t^2 + 2W_t(W_{t+s} - W_t) + (W_{t+s} - W_t)^2|I_t] \\
&= W_t^2 + s \neq W_t^2
\end{aligned}$$

(c)

No.

$$\begin{aligned}
\mathbb{E}[X_{t+s}|I_t] &= \mathbb{E}[W_{t+s}^2 - 2\int_0^{t+s} uW_u du|I_t] \\
&= \mathbb{E}[W_t^2 + 2W_t(W_{t+s} - W_t) + (W_{t+s} - W_t)^2 - 2\int_0^t uW_u du - 2\int_t^{t+s} uW_u du|I_t] \\
&= W_t^2 - 2\int_0^t uW_u du + s - 2\mathbb{E}[\int_t^{t+s} uW_u du|I_t] \\
&= W_t^2 - 2\int_0^t uW_u du + s - 2\int_t^{t+s} u\mathbb{E}[W_u|I_t] du \\
&= W_t^2 - 2\int_0^t uW_u du + s - 2\int_t^{t+s} uW_t du \\
&= W_t^2 - 2\int_0^t uW_u du + s - 2W_t\int_t^{t+s} u du \\
&= W_t^2 - 2\int_0^t uW_u du + s - 2W_t(s^2 + 2st) \\
&\neq W_t^2 - 2\int_0^t uW_u du
\end{aligned}$$

Problem 3 Brownian motion

Suppose the standard deviation of continuously compounded annual return of stock AAA is 10%. Assume that the stock return follows a Brownian motion. What is the standard deviation of continuously compounded four-year return of stock AAA?

Hint: consider the property of independent and stationary increments.

Answers:

WLOG, denote the stock price $S_t = \sigma W_t \sim N(0, \sigma^2 t)$, and σW_1 has standard deviation 10%. Then σW_4 has standard deviation 20%.

Problem 4 Martingale

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ be a filtered probability space and $Y_n, n \geq 0$, a sequence of absolutely integrable random variables adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$. Assume that there exist real numbers $u_n, v_n, n \geq 0$, such that

$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = u_n Y_n + v_n.$$

Find two real sequences a_n and b_n, $n \geq 0$, so that the sequence of random variables $M_n := a_n Y_n + b_n, n > 1$, be martingale w.r.t. the same filtration.

Answers:

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}[a_{n+1}Y_{n+1} + b_{n+1}|\mathcal{F}_n] \\
&= a_{n+1}u_n Y_n + a_{n+1}v_n + b_{n+1} \\
&= a_n Y_n + b_n.
\end{aligned}$$

Thus

$$\begin{cases} a_{n+1} = \frac{a_n}{u_n} \\ b_{n+1} = b_n - a_{n+1}v_n \end{cases}.$$

One solution is

$$\begin{cases} a_{n+1} = \frac{1}{\prod_{k=1}^{n} u_k} & a_0 = 1 \\ b_{n+1} = -\sum_{k=1}^{n} a_{k+1}v_k & b_0 = 0 \end{cases}.$$

Problem 5 Neftci's Book Chapter 13, P228, Exercise 2

Suppose X_t is a geometric Brownian motion process with drift μ and diffusion parameter σ. When would the $e^{-rt}X_t$ be a martingale? That is, when would the following equality hold:

$$\mathbb{E}\left[X_t \mid X_s, s < t\right] = e^{r(t-s)}X_s$$

Answers:

We have X_t is a GBM, so $dX_t = \mu X_t dt + \sigma X_t dW_t$.

Let $Y_t = \ln X_t$, then $dY_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$, and $X_t = e^{Y_t}$. In order for $e^{-rt}X_t$ to be martingale, we need:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[e^{-rt}X_t \mid \mathcal{F}_s\right] &= e^{-rt}\mathbb{E}^{\mathbb{P}}\left[X_t \mid \mathcal{F}_s\right] \\
&= e^{-rt}\mathbb{E}^{\mathbb{P}}\left[e^{Y_t} \mid \mathcal{F}_s\right] \\
&= e^{-rt}\mathbb{E}^{\mathbb{P}}\left[e^{Y_t - Y_s + Y_s} \mid \mathcal{F}_s\right] \\
&= e^{-rt}e^{Y_s}\mathbb{E}^{\mathbb{P}}\left[e^{Y_t - Y_s} \mid \mathcal{F}_s\right]
\end{aligned}$$

where $Y_t - Y_s \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)(t-s), \sigma^2(t-s)\right)$. The moment generating function gives:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[e^{Y_t - Y_s} \mid \mathcal{F}_s\right] &= e^{\left(\mu - \frac{1}{2}\sigma^2\right)(t-s) + \frac{1}{2}\sigma^2(t-s)} \\
&= e^{\mu(t-s)}
\end{aligned}$$

so

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[e^{-rt}X_t \mid \mathcal{F}_s\right] &= e^{-rt}X_s e^{\mu(t-s)} \\
&= e^{-rs}X_s e^{\mu(t-s) - rt + rs} \\
&= e^{-rs}X_s e^{(\mu - r)(t-s)}
\end{aligned}$$

for any t and s. So we need $\mu - r = 0$, or $\mu = r$ in order to make $e^{-rt}X_t$ a martingale.

Problem 6 Neftci's Book Chapter 13, P228, Exercise 3

Consider

$$Z_t = e^{-rt}X_t$$

where X_t is an exponential Wiener process:

$$X_t = e^{W_t}$$

(a) Calculate the expected value of the increment (consider $Z(t_2) - Z(t_1)$ or $dZ(t)$).

(b) Is Z_t a martingale?

(c) Calculate $\mathbb{E}[Z_t]$. How would you change the definition of X_t to make Z_t a martingale?

(d) How would $\mathbb{E}[Z_t]$ then change?

Answers:

(a)

We have $Z_t = e^{-rt}e^{W_t} = e^{W_t - rt}$. Define $f(t,x) = e^{x-rt}$, $f_t = -re^{x-rt}$, $f_x = e^{x-rt}$, $f_{xx} = e^{x-rt}$, so

$$\begin{aligned}
dZ_t &= -re^{W_t - rt}dt + e^{W_t - rt}dW_t + \frac{1}{2}e^{W_t - rt}dt \\
&= \left(-r + \frac{1}{2}\right) Z_t dt + Z_t dW_t
\end{aligned}$$

so the expectation of increment is:

$$\left(-r + \frac{1}{2}\right) Z_t dt$$

(b)

Z_t is not a martingale unless $r = \frac{1}{2}$.

(c)

$$\begin{aligned}\mathbb{E}\left[Z_t\right] &= \mathbb{E}\left[e^{W_t - rt}\right] \\ &= e^{-rt}\mathbb{E}\left[e^{W_t}\right]\end{aligned}$$

where $W_t \sim N(0,t)$, so

$$\begin{aligned}\mathbb{E}\left[Z_t\right] &= e^{-rt}e^{\frac{1}{2}t} \\ &= e^{\left(\frac{1}{2}-r\right)t}.\end{aligned}$$

let $X_t = e^{cW_t}$, then

$$\begin{aligned}\mathbb{E}\left[Z_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[e^{-rt}e^{cW_t} \mid \mathcal{F}_s\right] \\ &= e^{-rt}\mathbb{E}\left[e^{cW_t - cW_s + cW_s} \mid \mathcal{F}_s\right] \\ &= e^{-rt}e^{cW_s}e^{\frac{1}{2}c^2(t-s)} \\ &= e^{-rs}e^{cW_s}e^{\frac{1}{2}c^2(t-s)-rt+rs} \\ &= Z_s e^{(t-s)\left(\frac{1}{2}c^2 - r\right)}\end{aligned}$$

by setting $\frac{1}{2}c^2 - r = 0$,or $c = \sqrt{2r}$, we can make Z_t a martingale, then $X_t = e^{\sqrt{2r}W_t}$.

(d)

Since Z_t is a martingale, $\mathbb{E}\left[Z_t\right] = Z_0 = e^{-r0}e^{\sqrt{2r}0} = 1$.

Problem 7 Neftci's Book Chapter 14, P252, Exercise 2

Assume that the return R_t of a stock has the following log-normal distribution for fixed t:

$$\log\left(R_t\right) \equiv X \sim \mathcal{N}\left(\mu, \sigma^2\right)$$

Suppose we let the density of $\log\left(R_t\right)$ be denoted by $f\left(x\right)$ and hypothesize that $\mu = 0.17$. We further estimate the variance as $\sigma^2 = 0.09$.

(a) Find a function $\xi\left(x\right)$ such that $\xi\left(x\right) \cdot f\left(x\right)$ is a new probability density of $\log(R_t)$ (or, X), and under this density, X has a mean equal to the risk-free rate $r = 0.05$.

(b) Find a $\xi\left(x\right)$ such that X has mean zero.

(c) Under which probability is it "easier" to calculate $\mathbb{E}\left[X^2\right]$?

(d) Is the variance different under these probabilities?

Answers:

For simplicity let $X = \ln R \sim N\left(\mu, \sigma^2\right)$.

(a)

We want: $\mathbb{E}[X] = \int_{\mathbb{R}} xf(x)dx = \int_{\mathbb{R}} x\xi(x)f(x) = r$. The simplist way is to make $\xi(x)f(x)$ a density of $N\left(r, \sigma^2\right)$, or $\xi(x)f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-r)^2}{2\sigma^2}}$, then

$$\begin{aligned}\xi(x) &= \frac{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-r)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}} \\ &= e^{-\frac{(x-r)^2-(x-\mu)^2}{2\sigma^2}}.\end{aligned}$$

For the special case $r = 0.05$, $\mu = 0.17$, $\sigma^2 = 0.09$

$$\xi(x) = e^{-\frac{(x-0.05)^2-(x-0.17)^2}{2(0.09)^2}}.$$

(b)

Let $r = 0$ in (a), then

$$\xi(x) = e^{-\frac{(x)^2-(x-0.17)^2}{2(0.09)^2}}.$$

(c)

It's easier under the probability in (b) because $\mathbb{E}\left[X^2\right] = \mathrm{Var}(X) + \mathbb{E}[X]^2 = \mathrm{Var}(X) = \sigma^2$.

(d)

No, the variances are the same. Both transformations adjusted the mean but not the variance.

Problem 8

Let M_n represents a symmetric random walk (i.e., a sum of n i.i.d. Bernoulli random variable taking value 1 or -1 with probability $1/2$) and let τ be a bounded stopping time, there is a constant $C < \infty$, such that $\mathrm{P}(\tau \leq C) = 1$

(a) What is the definition of martingale?

(b) Show that $M_n^2 - n$ is a martingale.

(c) Explain why one shall get $\mathbb{E}\left[M_\tau^2\right] = \mathbb{E}[\tau]$. (This problem is challenging. You don't have to solve it.)

Answers:

(a)

A stochastic process $\{S_t : t \in [0, \infty]\}$ is a martingale with respect to a filtration $\{I_t : t \in [0, \infty]\}$ if for all $t \geq 0$ the following properties hold:

- S_t is known given I_t (namely, S_t is I_t-adapted).

- $\mathbb{E}[|S_t|] < +\infty$ (Integrable, which implies that unconditional forecast is finite for all t).
- $\mathbb{E}_t[S_T] \equiv \mathbb{E}[S_T|I_t] = S_t$ for all $t < T$.

(b)

Let $X_i, i \in \{1, 2, \cdots\}$ denote the i.i.d rvs. We have

$$\begin{aligned} \mathbb{E}[X_i] &= 0 \\ Var(X_i) &= 1 \\ Cov(X_i, X_j) &= 0, \quad i \neq j \end{aligned}$$

$M_t^2 - t$ is a martingale because

- M_t^2 is known given I_t, $M_t^2 - t$ is I_t-adapted).
- $\mathbb{E}[|M_t^2 - t|] \leq \mathbb{E}[M_t^2] + t \leq t^2 + t < +\infty$.
- For $s < t$,

$$\begin{aligned} \mathbb{E}[M_t^2 - t|I_s] &= \mathbb{E}[M_t^2|I_s] - t \\ &= \mathbb{E}\left[\left(M_s + \sum_{i=s+1}^{t} X_i\right)^2 \middle| I_s\right] - t \\ &= M_s^2 + 2M_s\mathbb{E}\left[\sum_{i=s+1}^{t} X_i \middle| I_s\right] + \mathbb{E}\left[\left(\sum_{i=s+1}^{t} X_i\right)^2 \middle| I_s\right] - t \\ &= M_s^2 + 0 + (t - s) - t = M_s^2 - s \end{aligned}$$

(c)

For simplicity, let $X_t \equiv M_t^2 - t$ and $X_0 = 0$. First we show that $Y_t = X_{t\wedge\tau}$ is a martingale, where $t \wedge \tau \equiv \min(t, \tau)$. Let $1_{\{\cdot\}}$ be the indicator function. Notice that

$$\begin{aligned} Y_t = X_{t\wedge\tau} &= \sum_{i=0}^{t-1} 1_{\{\tau=i\}} X_i + 1_{\{\tau\geq t\}} X_t \\ &= \sum_{i=0}^{t-1} \left(1_{\{\tau\geq i\}} - 1_{\{\tau\geq i+1\}}\right) X_i + 1_{\{\tau\geq t\}} X_t - 1_{\{\tau\geq 0\}} X_0 \\ &= \sum_{i=1}^{t} 1_{\{\tau\geq i\}} X_i - \sum_{i=0}^{t-1} 1_{\{\tau\geq i+1\}} X_i \\ &= \sum_{i=1}^{t} 1_{\{\tau\geq i\}} (X_i - X_{i-1}) \end{aligned}$$

For $s < i \leq t$, by definition of stopping time (discrete time), $\{\tau \leq i\}$ is I_i measurable, thus $\{\tau \geq i\} = \{\tau \leq i-1\}^C$ is I_{i-1} measurable.

$$\begin{aligned}
\mathbb{E}[Y_t|I_s] &= Y_s + \mathbb{E}\left[\sum_{i=s+1}^{t} 1_{\{\tau\geq i\}}(X_i - X_{i-1})|I_s]\right] \\
&= Y_s + \mathbb{E}\left[\sum_{i=s+1}^{t} 1_{\{\tau\geq i\}}(X_i - X_{i-1})|I_s\right] \\
&= Y_s + \mathbb{E}\left[\sum_{i=s+1}^{t} \mathbb{E}[1_{\{\tau\geq i\}}(X_i - X_{i-1})|I_{i-1}]\middle| I_s\right] \\
&= Y_s + \mathbb{E}\left[\sum_{i=s+1}^{t} 1_{\{\tau\geq i\}}\mathbb{E}[(X_i - X_{i-1})|I_{i-1}]\middle| I_s\right] \\
&= Y_s + \mathbb{E}\left[\sum_{i=s+1}^{t} 1_{\{\tau\geq i\}}0\middle| I_s\right] \\
&= Y_s
\end{aligned}$$

Y_t is a martingale process and thus $\mathbb{E}[Y_C] = \mathbb{E}[Y_0]$. Since $\tau < C$, we have:

$$\begin{aligned}
&\begin{cases} Y_C = X_{\tau\wedge C} = X_\tau \\ Y_0 = X_{\tau\wedge 0} = X_0 \end{cases} \\
&\mathbb{E}[Y_C] = \mathbb{E}[Y_0] \\
\Longrightarrow\ &\mathbb{E}[X_\tau] = \mathbb{E}[M_\tau^2 - \tau] = \mathbb{E}[X_0] = 0 \\
\Longrightarrow\ &\mathbb{E}\left[M_\tau^2\right] = \mathbb{E}[\tau]
\end{aligned}$$

Problem 9 (Prob 3.46): (Quant Job Interview Question)

If W_t is a standard Brownian motion, is W_t^3 a martingale?

Answers:

There are some further technical considerations, but in general if such a process is a martingale then the stochastic differential equation it satisfies will only have a diffusion (dW_t) term. The technical considerations are only necessary if this condition is satisfied, so we begin by applying Itô's formula to $X_t = W_t^3$. This gives

$$\begin{aligned}
dX_t &= 3W_t^2 dW_t + 3W_t\,(dW_t)^2 \\
&= 3W_t dt + 3W_t^2 dW_t.
\end{aligned}$$

Since there is a drift term ($3W_t$) we can conclude that W_t^3 is not a martingale.

Problem 10 Neftci's Book Chapter 3, P75, Exercise 8

Consider the function:

$$f(x) = x^3$$

(a) Take the integral and calculate

$$\int_0^1 f(x)dx$$

(b) Now consider splitting the interval $[0,1]$ into 4 pieces, where you choose the x_i. They may or may not be equally spaced. Calculate the following sums numerically:

$$\sum_{i=1}^{4} f(x_i)(x_i - x_{i-1})$$
$$\sum_{i=1}^{4} f(x_{i-1})(x_i - x_{i-1})$$

(c) What are the differences between these two sums and how well do they approximate the true value of the integral?

Answers:

(a)

$$\int_0^1 x^3 dx = \frac{1}{4}$$

(b)

Choose $x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$ then:

$$\sum_{i=1}^{4} f(x_i)(x_i - x_{i-1}) = \frac{3}{8}$$
$$\sum_{i=1}^{4} f(x_{i-1})(x_i - x_{i-1}) = \frac{9}{64}$$

(c)

The true value lies between the two "endpoints"

Problem 11 Neftci's Book Chapter 3, P75, Exercise 9

Consider the function:

$$f(x) = \begin{cases} x sin\left(\frac{\pi}{x}\right) & 0 < x < 1 \\ 0 & x = 0 \end{cases}$$

(a) Take the integral and calculate

$$\int_0^1 f(x)dx$$

(b) Again, split the interval $[0, 1]$ into 4 pieces,

$$0 = x_0 < x_1 < x_2 < x_3 < x_4 = 1$$

by choosing the x_i numerically. Calculate the following sums:

$$\sum_{i=1}^{4} f(x_i)(x_i - x_{i-1})$$
$$\sum_{i=1}^{4} f(x_{i-1})(x_i - x_{i-1})$$

(c) How do the above sums approximate the true integral?

(d) Comment on the level of accuracy of the above approximations to the integral and explain why.

Answers:

(a)

The integral has no closed form solution. It can be simplified using successive applications of integration by parts.

$$\int_0^1 f(x)dx = \frac{\pi^2}{2} \int_0^\pi \frac{\sin(t)}{t} dt - \frac{\pi}{2} - \frac{\pi^3}{4}$$

The integral $\int_0^\pi \frac{\sin(t)}{t} dt \approx 1.8519$ can be evaluated numerically with the following command.

```
import numpy as np
from scipy import integrate

def f(x):
    return np.sin(x) / x

print(np.pi**2 / 2 * integrate.quadrature(f, 0, np.pi)[0] - np.pi / 2- np.pi**3 / 4)
```

Thus, the entire expression is approximately

$$\int_0^1 f(x)dx = \frac{\pi^2}{2} \int_0^\pi \frac{\sin(t)}{t} dt - \frac{\pi}{2} - \frac{\pi^3}{4} \approx -0.18342$$

(b)

Approximation of $\int_0^1 f(x)dx$: Choose the same evenly spaced partition of with $\frac{1}{4}$ for the mesh size. Since the intensity of the fluctuations increases as x approaches zero, placing a finer grid near zero could improve performance. Usually, for a given number of nodes, placing more nodes in regions where the function fluctuates more intensely increases the accuracy of the approximation. Let $x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$. The terms $x_i - x_{i-1}, \forall i = 1, \cdots, 4$ evaluate to $\frac{1}{4}$. Therefore, with

$f(x_i) = x_i \left(\sin\left(\frac{\pi}{x_i}\right)\right)$, the sum becomes

$$\sum_{i=1}^{4} f(x_i)(x_i - x_{i-1}) = \frac{1}{4}[0 + 0 - \frac{3\sqrt{3}}{8} + 0] = -\frac{3\sqrt{3}}{31} \approx -0.16238$$
$$\sum_{i=1}^{4} f(x_{i-1})(x_i - x_{i-1}) = \frac{1}{4}[0 + 0 - \frac{3\sqrt{3}}{8} + 0] = -\frac{3\sqrt{3}}{31} \approx -0.16238$$

(c)

The sums do not accurately approximate the integral.

(d)

The sums do not approximate the integral accurately since the function oscillates rapidly(unbounded total variation).