ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 3

Professor S.J. Deng

Please remember to write down your name and GTID in the submitted homework.

Problem 1 Neftci's Book Chapter 2, P30, Exercise 3 (a), (b)

Consider a stock S_t and a plain vanilla, at the-money put option written on this stock. The option expires at time $t + \Delta$, where Δ denotes a small interval. At time t, there are only two possible ways the S_t can move. It can either go up to $S^u_{t+\Delta}$, or go down to $S^d_{t+\Delta}$. Also available to traders is risk-free borrowing and lending at annual rate r.

(a) Using the arbitrage theorem, write down a three-equation system with two states that gives the arbitrage-free values of S_t and C_t.

(b) Now plot a two-step binomial tree for S_t. Suppose at every node of the tree the markets are arbitrage-free. How many three-equation systems similar to the preceding case could then be written for the entire tree?

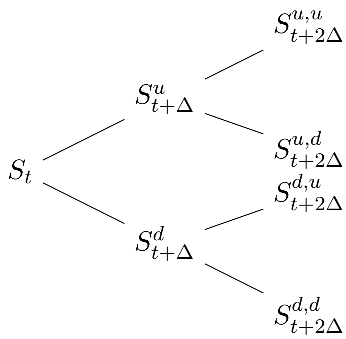
Answers:

(a)

$$\begin{bmatrix} 1 \\ S_t \\ C_t \end{bmatrix} = \begin{bmatrix} 1+\Delta r & 1+\Delta r \\ S^u_{t+\Delta} & S^d_{t+\Delta} \\ C^u_{t+\Delta} & C^d_{t+\Delta} \end{bmatrix} \begin{bmatrix} \psi_u \\ \psi_d \end{bmatrix}$$

$$\begin{aligned} 1 &= (1+\Delta r)\psi_u + (1+\Delta r)\psi_d \\ S_t &= S^u_{t+\Delta}\psi_u + S^d_{t+\Delta}\psi_d \\ C_t &= C^u_{t+\Delta}\psi_u + C^d_{t+\Delta}\psi_d \end{aligned}$$

(b)



There are 3 three-equation systems similar to the preceding case.

Problem 2 Neftci's Book Chapter 2, P31, Exercise 4

A four-step binomial tree for the price of a stock S_t is to be calculated using the up- and downticks given as follows:

$$u = 1.15$$
$$d = \frac{1}{u}$$

These up and down movements apply to one-month periods denoted by $\Delta = 1$. We have the following dynamics for S_t,

$$S_{t+\Delta}^{u} = s_t u$$
$$S_{t+\Delta}^{d} = s_t d$$

where up and down describe the two states of the world at each node.

Assume that time is measured in months and that $t = 4$ is the expiration date for a European call option C_t written on S_t. Suppose the initial price set to be $S_0 = 100$, strike price $K = 100$. The stock does not pay any dividends and its price is expected (by "market participants") to grow at an annual rate of 15%. The risk-free interest rate r is known to be constant at 5%.

(a) According to the data given above, what is the (approximate) annual volatility of S_t if this process is known to have a log-normal distribution?

(b) Calculate the four-step binomial trees for the S_t and the C_t.

(c) Calculate the arbitrage-free price C_0 of the option at time $t = 0$.

(d) Using the above setting, work out all hedging portfolios at each node of the first three period specifically, period $t = 0 \to t = 1$, period $t = 1 \to t = 2$ and period $t = 2 \to t = 3$.

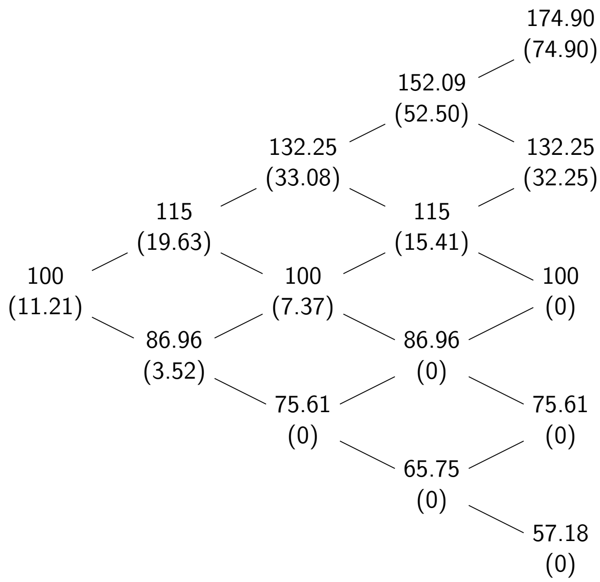
Answers:

(a)

Using year as time unit, since $u = e^{\sigma\sqrt{\Delta}}$,

$$\sigma = \frac{\ln(u)}{\sqrt{\Delta}} \approx 0.484$$

(b)

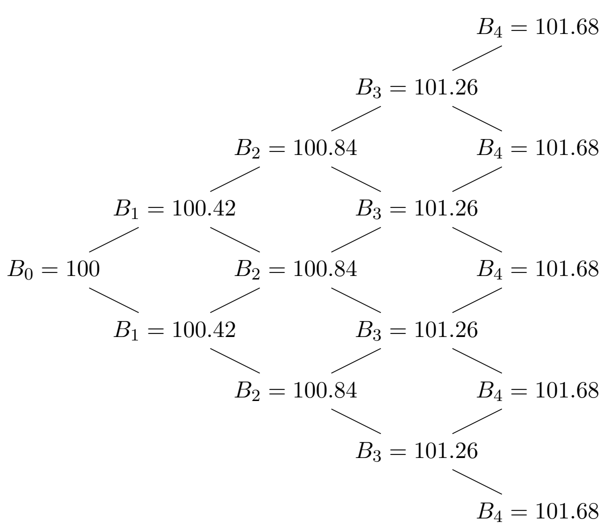


- $R = e^{r*\Delta} \approx 1.004$
- Risk neutral probability is $q = \frac{R-d}{u-d} \approx 0.48$
- Numbers on top are stock prices
- Numbers on bottom are option values

(c)

$C_0 = 11.21$ as in (b).

(d)



WLOG, assume the hedging portfolios involves a bond ($B_0 = 100$) and the stock $\theta_i = [\theta_{i,B}, \theta_{i,S}]$ using option payoffs $[C_i^u, C_i^d]^T$:

$$\begin{pmatrix} \theta_{i,B} & \theta_{i,S} \end{pmatrix} \begin{pmatrix} RB_i & RB_i \\ uS_i & dS_i \end{pmatrix} = \begin{pmatrix} C_i^u & C_i^d \end{pmatrix}$$

The hedging portfolios at each node of the first three periods are as follows:

- From $t = 0 \to t = 1$: $\theta_0 = [-0.462, 0.574]$;
- From $t = 1 \to t = 2$:
  - $\theta_1^u = [-0.718, 0.797]$;
  - $\theta_1^d = [-0.227, 0.302]$;
- From $t = 2 \to t = 3$:
  - $\theta_2^{2u} = [-0.983, 1.000]$;
  - $\theta_2^{1u1d} = [-0.472, 0.550]$;
  - $\theta_2^{2d} = [0, 0]$;

Problem 3 Neftci's Book Chapter 2, P32, Exercise 5 (Change $r = 5\%$ to $r = 0.4\%$)

You are given the following information concerning a stock denoted by S_t.

- Current value = 102.
- Annual volatility = 30%.
- You are also given the spot rate $r = 0.4\%$, which is known to be constant during the next 3 months.

It is hoped that the dynamic behavior of S_t can be approximated reasonably well by a binomial process if one assumes observation intervals of length 1 month.

(a) Consider a European call option written on S_t. The call has a strike price $K = 120$ and an expiration of 3 months. Using the S_t and the risk-free borrowing and lending B_t, construct a portfolio that replicates the option.

(b) Using the replicating portfolio, price this call.

(c) Suppose you sell, over-the-counter, 100 such calls to your customers. How would you hedge this position? Be precise.

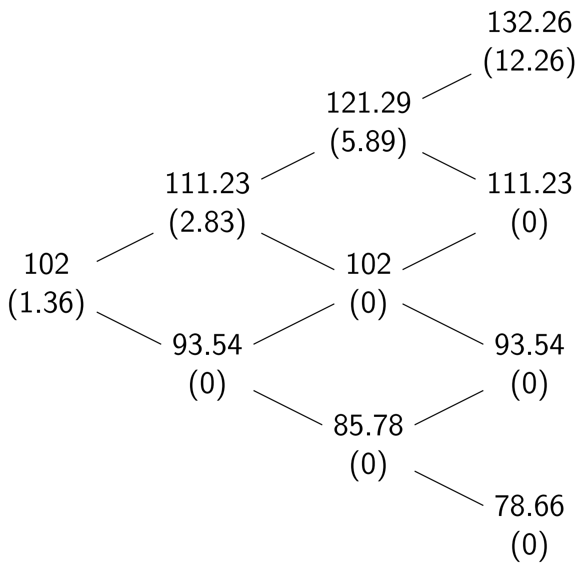
(d) Suppose the market price of this call is 5. How would you form an arbitrage portfolio?

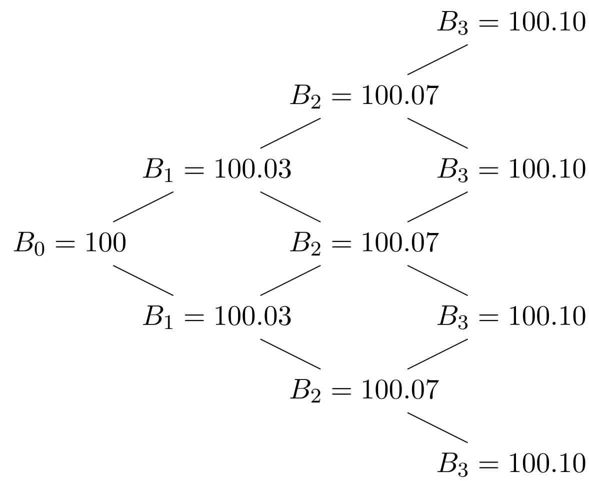
Answers:

(a)

$$u = e^{\sigma\sqrt{\Delta}} = e^{0.3\sqrt{1/12}} = 1.0905 \tag{1}$$

- $R = 1 + r * \Delta \approx 1.0003$
- Risk neutral probability is $q = \frac{R-d}{u-d} \approx 0.48$
- Numbers on top are stock prices
- Numbers on bottom are option values





- $R = 1 + r * \Delta \approx 1.0003$
- Risk neutral probability is $q = \frac{R-d}{u-d} \approx 0.48$
- Numbers on top are stock prices
- Numbers on bottom are option values
- Right graph is for bond

WLOG, assume the hedging portfolios involves a bond ($B_0 = 100$) and the stock $\theta_i = [\theta_{i,B}, \theta_{i,S}]$ using option payoffs $[C_i^u, C_i^d]^T$:

$$\begin{pmatrix} \theta_{i,B} & \theta_{i,S} \end{pmatrix} \begin{pmatrix} RB_i & RB_i \\ uS_i & dS_i \end{pmatrix} = \begin{pmatrix} C_i^u & C_i^d \end{pmatrix}$$

The hedging portfolios at each node of the first three periods are as follows:

- From $t = 2 \rightarrow t = 3$:
  - $\theta_2^{2u} = [-0.648, 0.583]$, portfolio wealth 5.887;

- $\theta_2^{1u1d} = [0, 0]$, portfolio wealth 0;
- $\theta_2^{2d} = [0, 0]$, portfolio wealth 0;

- From $t = 1 \rightarrow t = 2$:
  - $\theta_1^u = [-0.311, 0.305]$, portfolio wealth 2.827;
  - $\theta_1^d = [0, 0]$, portfolio wealth 0;

- From $t = 0 \rightarrow t = 1$: $\theta_0 = [-0.149, 0.160]$, portfolio wealth 1.357;

(b)

As in (a), replicating portfolio wealth is 1.357, by no arbitrage, $C_0 = 1.357$.

(c)

- At time 0, long 15.979 shares stock and short -14.941 bond;
- At time 1 , long 30.520 shares stock and short -31.109 bond if the stock price reach 111.23; otherwise you do not need to hedge;
- At time 2 , long 58.293 shares stock and short -64.773 bond if the stock price reach 121.29; otherwise you do not need to hedge;

(d)

If the market price of this call is 5 , sell the call , and follow the strategy in (c) to hedge the risk. The profit is $100(5 - 1.357) = \$364.3$ per call.

Problem 4, Neftci's Book Chapter 2, P32, Exercise 6

Suppose you are given the following data:

- Risk-free yearly interest rate is r = 6%.
- The stock price follows:

  $$S_{t+1} – S_t = \mu S_t + \sigma S_t \epsilon_t;$$

  where the ϵ_t is a serially uncorrelated binomial process assuming the following values:

  $$\epsilon_t = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases}$$

  The $0 < p < 1$ is a parameter.

- Volatility is 12% a year.
- The stock pays no dividends and the current stock price is 100.
- $\Delta t = 1$ year.

Now consider the following questions.

(a) Suppose μ is equal to the risk-free interest rate:

$$\mu = r$$

and that the S_t is arbitrage-free. What is the value of p?

(b) Would a $p = 1/3$ be consistent with arbitrage-free S_t?

Now suppose μ is given by:

$$\mu = r + \text{risk premium}$$

(c) What do the p and ϵ_t represent under these conditions?

(d) Is it possible to determine the value of p?

Answers:

(a)

Based on risk-neutral pricing theorem, we have $P_{\vec{F}} = \mathbb{E}^Q[\frac{\vec{F}}{1+r}]$

$$\begin{aligned} S_t &= (1 + \mu + \sigma\epsilon_t)S_{t-1} \\ S_{t-1} &= Q(\omega_1)\frac{F_{1,t}}{1+r} + Q(\omega_2)\frac{F_{2,t}}{1+r} = p\frac{(1+\mu+\sigma)S_{t-1}}{1+r} + (1-p)\frac{(1+\mu+\sigma)S_{t-1}}{1+r} \end{aligned}$$

Given $\mu = r = 0.06$, $\sigma = 0.12$ we have $p = 0.5$.

(b)

From (a), $p = 0.5$ is the only p that satisfies arbitrage-free S_t. Therefore, $p = 1/3$ is not consistent with arbitrage-free S_t.

(c)

Once the risk premium is introduced , the process is no longer in the risk-neutral measure framework. p is not the risk-neutral probability but represents an empirical or statistical value.

(d)

Since p is no longer the risk-neutral probability we can only determine the value statistically.

Problem 5, Neftci's Book Chapter 2, P32, Exercise 7

Using the data in the previous question ($r = 6\%$ and $\sigma = 12\%$), you are now asked to approximate the current value of a European call option on the stock S_t. The option has a strike price of 100, and a maturity of 200 days.

(a) Determine an appropriate time interval Δ, such that the binomial tree has 5 steps.

(b) What would be the implied u and d?

(c) What is the implied "up" probability?

(d) Determine the tree for the stock price S_t.

(e) Determine the tree for the call premium C_t.

Answers:

(a)

$T = \frac{200}{365} \approx 0.5479$, $\Delta = \frac{T}{5} = 0.1096$ (40 days)

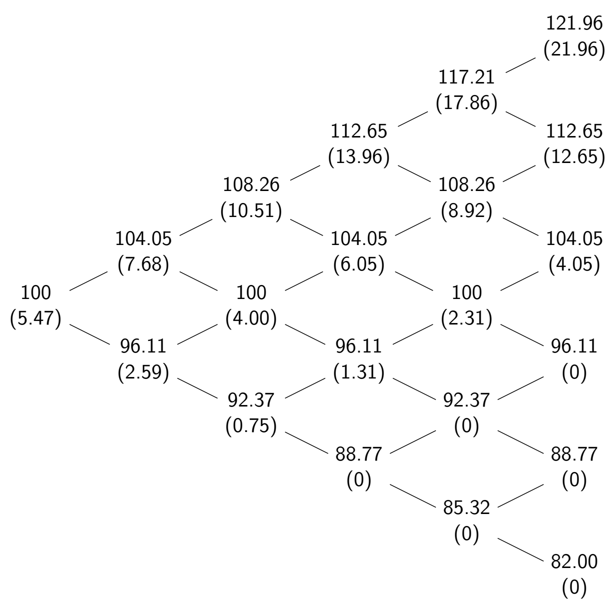
(b)

$u = e^{\sigma\sqrt{\Delta}} = e^{0.12\sqrt{40/365}} \approx 1.0405$, $d = e^{-\sigma\sqrt{\Delta}} \approx 0.9611$

(c)

$$p = \frac{R-d}{u-d} = \frac{1 + 0.06 * 40/365 - 0.9611}{1.0405 - 0.9611} \approx 0.5727$$

(d)



- Numbers on top are stock prices
- Numbers on bottom are option values

(e)

$C_0 = 5.47$ as in (d)

Problem 6 Neftci's Book Chapter 4, P63, Exercise 1

Suppose you can bet on an American presidential election in which one of the candidates is an incumbent. The market offers you the following payoffs R:

$$\begin{cases} \$1000 & \text{If incumbent wins} \\ -\$1500 & \text{If incumbent loses} \end{cases}$$

You can take either side of the bet. Let the true probability of the incumbent winning be denoted by p, $0 < p < 1$.

(a) What is the expected gain if $p = .6$?

(b) Is the value of p important for you to make a decision on this bet?

(c) Would two people taking this bet agree on their assessment of p? Which one would be correct? Can you tell?

(d) Would statistical or econometric theory help in determining the p?

(e) What weight would you put on the word of a statistician in making your decision about this bet?

(f) How much would you pay for this bet?

Answers:

(a)

The expected gain for a bet on the incumbent winning is $0.6 \times \$1,000 - 0.4 \times \$1,500 = \$0$.

(b)

Yes, the value of p is important as it determines the expected gain.

(c)

Two people taking this bet would not necessarily agree on p. Neither person would necessarily be correct since p is not observed. The assessment of p is subjective.

(d)

Yes, statistics can be employed to determine p. One could use survey sampling as in political polls to determine the true p or one could look at past data and try to estimate p historically.

(e)

The statistician's assessment of p is crucial. The assessment provides an objective, although not perfectly accurate, assessment of p.

(f)

How much one is willing to pay for this bet depends on an individual's level of risk aversion since p is not the risk-neutral probability.

Problem 7 Neftci's Book Chapter 4, P63, Exercise 2

Now place yourself exactly in the same setting as before, where the market quotes the above R. It just happens that you have a close friend who offers you the following separate bet, R^*:

$$\begin{cases} \$1500 & \text{If incumbent wins} \\ -\$1000 & \text{If incumbent loses} \end{cases}$$

Note that the random event behind this bet is the same as in R. Now consider the following:

(a) Using the R and the R^*, construct a portfolio of bets such that you get a guaranteed risk-free return (assuming that your friend or the market does not default).

(b) Is the value of the probability p important in selecting this portfolio? Do you care what the p is? Suppose you are given the R, but the payoff of R^* when the incumbent wins is an unknown to be determined.Can the above portfolio help you determine this unknown value?

(c) What role would a statistician or econometrician play in making all these decisions? Why?

Answers:

(a)

One could go long R^* and short R. The risk - free payoff is \$500 regardless of the election's outcome.

(b)

No, the value of p is not important in selecting this portfolio. The payoff of the portfolio is independent of the election outcome. Not unless one knows the portfolio **and** it's payoffs can the portfolio help determine the unknown p.

(c)

A statistician or econometrician would play no role in making these decisions since the outcome of the election does not effect the portfolio payoffs. The payoffs are independent of p.

Problem 8

Assume the dynamic behavior of stock price S, in year t satisfies the function: $S_t = S_{t-1} + B_t$. The current stock price S_0 is \$100. The continuously compounded risk free interest rate is 5%.

(a) If $B_t = \begin{cases} 10, & \text{with probability of } \frac{1}{3} \\ -10, & \text{with probability of } \frac{2}{3} \end{cases}$, what is the expected value of stock price after 10 years (S_{10}). What is the probability of $S_{10} \geq 100$?

(b) If $B_t = \begin{cases} 0.5 \times S_{t-1}, & \text{with probability of } p \\ -0.5 \times S_{t-1}, & \text{with probability of } 1-p \end{cases}$, is this economy arbitrage free? Why? A European call option is written on the stock price with strike price of 100 and expiration time of three years later. What is the arbitrage free price of the option?

(c) If the price of above call option is \$40, what dynamic arbitrage portfolio you will construct?

(d) If $B_t = \begin{cases} 0.5 \times S_{t-1}, & \text{with probability of } p_1 \\ 0, & \text{with probability of } p_2 \\ -0.5 \times S_{t-1}, & \text{with probability of } p_3 \end{cases}$, namely there are 3 possible states of the world in year 1, and B_t for $t \geq 2$ are defined the same way as those in (b). Is the market complete now? There is a financial product worth \$2 will produce payoff of \$0 in state 1 with probability p_1, \$2 in state 2 with probability p_2, and \$4 in state 3 with probability p_3, after one year from now. What is the arbitrage free price of a European call option with strike price of \$100 and expiration time of one year from now?

Answers:

Since

$$\begin{aligned} S_t &= S_{t-1} + B_t \\ &= (S_{t-2} + B_{t-1}) + B_t \\ &= (S_{t-3} + B_{t-2}) + B_{t-1} + B_t \\ &= \cdots \\ &= S_0 + \sum_{i=1}^{t} B_i \end{aligned}$$

The expected value:

$$\begin{aligned} \mathbb{E}[S_t] &= \mathbb{E}[S_0 + \sum_{i=1}^{t} B_i] \\ &= \mathbb{E}[S_0] + \sum_{i=1}^{t} \mathbb{E}[B_i] \\ &= S_0 + t\mathbb{E}[B_i] \end{aligned}$$

(a)

$$\begin{aligned} \mathbb{E}[S_{10}] &= S_0 + 10\mathbb{E}[B_i] \\ &= S_0 + 10(10 \times \frac{1}{3} + (-10) \times \frac{2}{3}) \\ &= 100 - \frac{100}{3} \approx 66.67 \end{aligned}$$

If $S_{10} \geq 100$, then $\sum_{i=1}^{10} B_i \geq 0$. Let $t^{'}$ be the times $B_i > 0$, $\sum_{i=1}^{10} B_i \geq 0 \implies t^{'} \geq 5$.

$$\begin{aligned}\mathbb{P}(S_{t=10} \geq 100) =&\mathbb{P}(t^{'} = 5) + \mathbb{P}(t^{'} = 6) + \mathbb{P}(t^{'} = 7) + \mathbb{P}(t^{'} = 8) + \mathbb{P}(t^{'} = 9) + \mathbb{P}(t^{'} = 10) \\ =&C_{10}^{5}\left(\frac{1}{3}\right)^{5}\left(\frac{2}{3}\right)^{5} + C_{10}^{6}\left(\frac{1}{3}\right)^{6}\left(\frac{2}{3}\right)^{4} + C_{10}^{7}\left(\frac{1}{3}\right)^{7}\left(\frac{2}{3}\right)^{3} \\ &+ C_{10}^{8}\left(\frac{1}{3}\right)^{8}\left(\frac{2}{3}\right)^{2} + C_{10}^{9}\left(\frac{1}{3}\right)^{9}\left(\frac{2}{3}\right)^{1} + C_{10}^{10}\left(\frac{1}{3}\right)^{10} \approx 0.213\end{aligned}$$

(b)

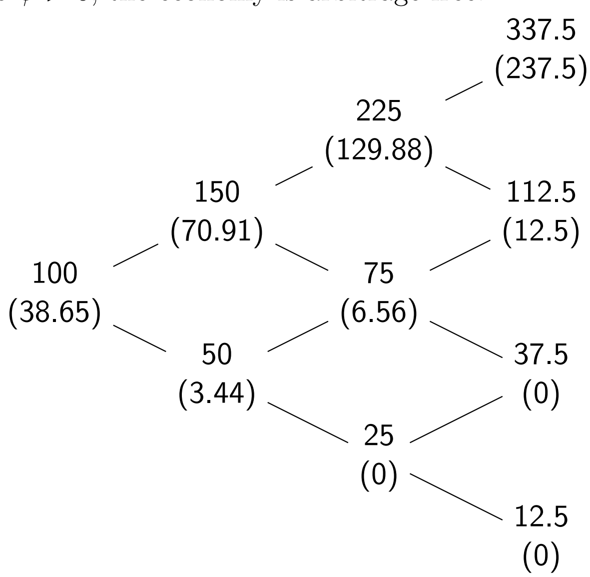
If it is arbitrage-free, then there must exist $\vec{\psi}$, such that:

$$\begin{aligned}&D\vec{\psi} = \vec{S} \\ \implies &\begin{pmatrix} e^{0.05} & e^{0.05} \\ 1.5S_0 & 0.5S_0 \end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \begin{pmatrix} 1 \\ S_0 = 100 \end{pmatrix}\end{aligned}$$

Solving the above equation, we get

$$\begin{cases}\psi_1 = \frac{e^r - d}{u-d}\frac{1}{e^r} \approx 0.5244 \\ \psi_2 = \frac{u-e^r}{u-d}\frac{1}{e^r} \approx 0.4268\end{cases}$$

Since $\vec{\psi} > 0$, the economy is arbitrage free.



- Numbers on top are stock prices
- Numbers on bottom are option values
- Arbitrage free price of the option $C_0 = 38.65$.

(c)

Since arbitrage free price of the option $C_0 = \$38.65$,

- Sell the option

- $\$40 - \$38.65 = \$1.35$ will be used as treasury bills (risk-free)
- Other $\$38.65$ will be used to build replicating portfolio as C

Replicating portfolio method $\theta D = \vec{F}$, i.e. for every node $t \in \{0,1,2\}, i \in \{0,\cdots,t+1\}$:

$$\begin{aligned}
\begin{pmatrix}\theta_1^i & \theta_2^i\end{pmatrix}\begin{pmatrix}e^{0.05} & e^{0.05}\\ S_{t+1}^i & S_{t+1}^{i+1}\end{pmatrix} &= \begin{pmatrix}C_{t+1}^i & C_{t+1}^{i+1}\end{pmatrix}\\
\begin{pmatrix}\theta_1^i & \theta_2^i\end{pmatrix} &= \begin{pmatrix}C_{t+1}^i & C_{t+1}^{i+1}\end{pmatrix}\begin{pmatrix}e^{0.05} & e^{0.05}\\ S_{t+1}^i & S_{t+1}^{i+1}\end{pmatrix}^{-1}\\
&= \begin{pmatrix}\frac{-C_{t+1}^i + 3C_{t+1}^{i+1}}{2e^r}, & \frac{C_{t+1}^i - C_{t+1}^{i+1}}{S_t^i}\end{pmatrix}
\end{aligned}$$

- Period $0 \to 1$: $(\theta_1, \theta_2) = (-28.8196, 0.6747)$
- Period $1 \to 2$:

$$(\theta_1,\theta_2) = \begin{cases}(-52.4199, 0.8222) & \text{if price goes up.}\\ (-3.1177, 0.1311) & \text{if price goes down.}\end{cases}$$

- Period $2 \to 3$:

$$(\theta_1,\theta_2) = \begin{cases}(-94.8851, 1) & \text{if price goes up 2 times.}\\ (-5.9452, 0.1667) & \text{if price goes up 1 times.}\\ (0,0) & \text{if price goes up 0 times.}\end{cases}$$

(d)

If the market is arbitrage-free, then we must have $\vec{\psi} > 0$. However, since we have 2 equations and 3 variables, $\vec{\psi} > 0$ is not guaranteed and market is in-completed.

$$\begin{pmatrix}e^{0.05} & e^{0.05} & e^{0.05}\\ 1.5S_t & S_t & 0.5S_t\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}1\\ S_t\end{pmatrix}$$

Adding the new financial product, the new payoff matrix

$$\begin{pmatrix}e^{0.05} & e^{0.05} & e^{0.05}\\ 1.5S_t & S_t & 0.5S_t\\ 0 & 2 & 4\end{pmatrix}$$

Then since $rank(D) = 2 < 3$, the economy is not arbitrage free and there is no arbitrage-free price for the option.

Problem 9

A stock has volatility $\sigma = 0.3$ and a current value of \$36. A European-style put option on this stock has a strike price of \$40 and expiration is in 5 months. The interest rate is 2% per year.

(a) Find the value of this put using a binomial lattice with $\Delta t =$ 1-month and $u = \exp(\sigma * \sqrt{\Delta t})$.

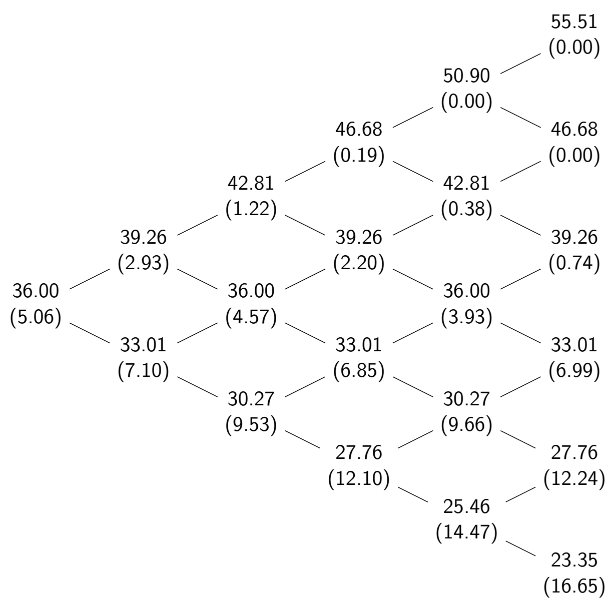
(b) Find the value of this put using a binomial lattice with $\Delta t =$ half-month and $u = \exp(\sigma * \sqrt{\Delta t})$.

(c) Are the prices obtained in (a) and (b) the same? Which one is the correct price?

Answers:

(a)

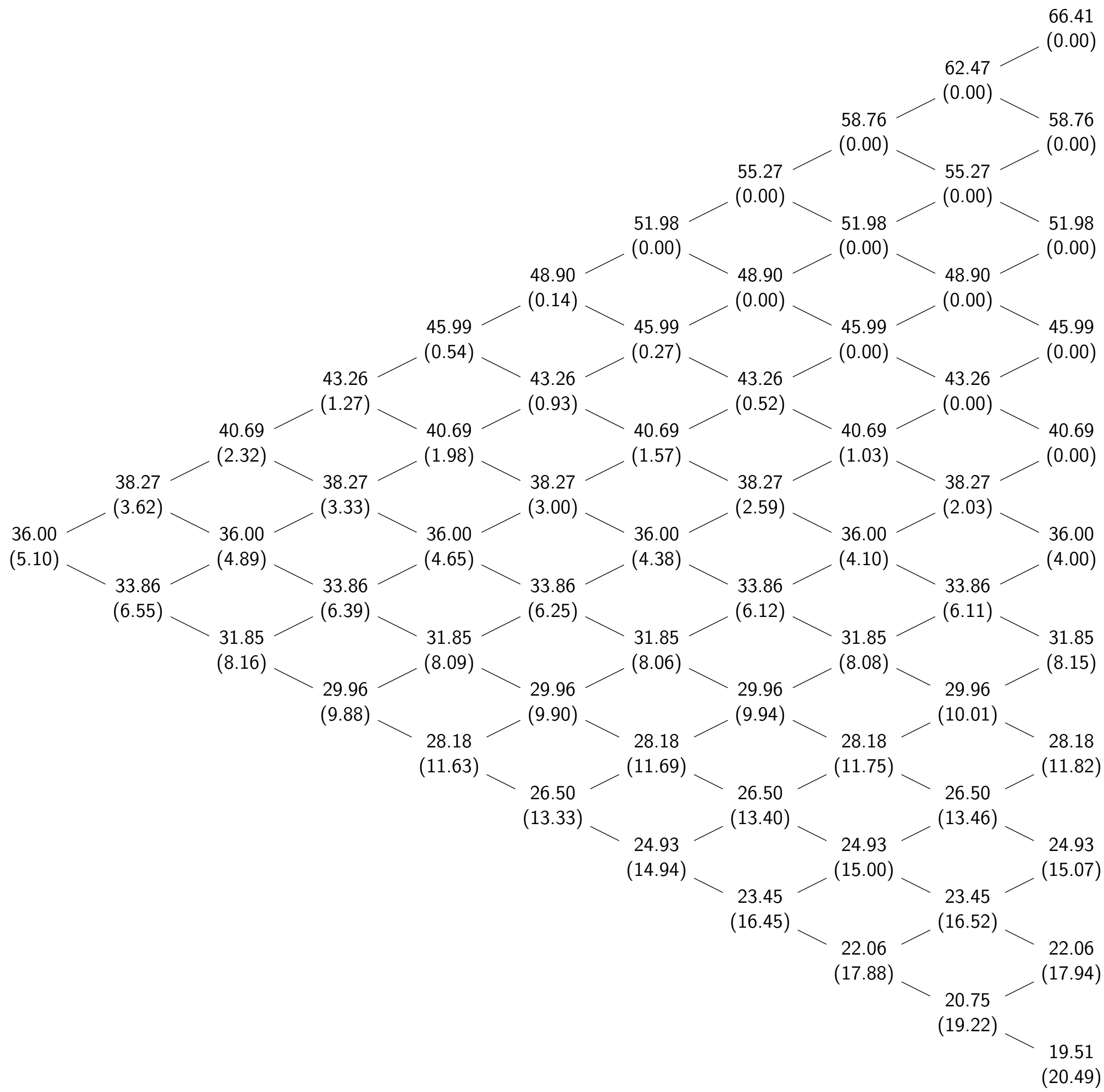
$$u = e^{\sigma\sqrt{\Delta t}} \approx 1.0905$$
$$d = \frac{1}{u} \approx 0.9170$$
$$R = e^{r\Delta t} \approx 1.00167$$
$$\psi_1 = \frac{1}{e^{r\Delta t}}\frac{R-d}{u-d} \approx 0.4872$$
$$\psi_2 = \frac{1}{e^{r\Delta t}}\frac{u-R}{u-d} \approx 0.5112$$



- Numbers on top are stock prices.
- Numbers on bottom are option values.

(b)

$$u = e^{\sigma\sqrt{\Delta t}} \approx 1.0632$$
$$d = \frac{1}{u} \approx 0.9406$$
$$R = e^{r\Delta t} \approx 1.00083$$
$$\psi_1 = \frac{1}{e^{r\Delta t}}\frac{R-d}{u-d} \approx 0.4911$$
$$\psi_2 = \frac{1}{e^{r\Delta t}}\frac{u-R}{u-d} \approx 0.5081$$



- Numbers on top are stock prices.
- Numbers on bottom are option values.

(c)

Price not the same, since (b) has smaller Δt which is closer to the continuous process. 5.1059 is closer to the real price.

Problem 10

Consider a family of European-style call options written on a non-dividend-paying stock, each option being identical except for its strike price (i.e., they all have the same expiration time). The value of the call with strike price K is denoted by $C(K)$. Prove the following three general relations using arbitrage arguments, assuming risk-free rate is non-negative:

(a) $K_2 > K_1$ implies $C(K_1) > C(K_2)$.

(b) $K_2 > K_1$ implies $K_2 - K_1 > C(K_1) - C(K_2)$.

(c) $K_3 > K_2 > K_1$ implies $C(K_2) \leq (K_3 - K_2)/(K_3 - K_1) * C(K_1) + (K_2 - K_1)/(K_3 - K_1) * C(K_3)$

Answers:

- Let $\mathbb{Q}$ denote the risk-neutral probability.
- Let T denote the expiration time.
- Let r denote the risk-free rate.

(a)

$$\begin{aligned} C_{K_1} &= \mathbb{E}^{\mathbb{Q}}[e^{-rT} \max(S_T - K_1)] \\ C_{K_2} &= \mathbb{E}^{\mathbb{Q}}[e^{-rT} \max(S_T - K_2)] \end{aligned}$$

Note that $K_2 > K_1$ implies $\max(S_T - K_2) \geq \max(S_T - K_1)$, with the inequality strict for some S_T (with such $\mathbb{Q}(S_T) > 0$)

$$C_{K_2} - C_{K_1} = \mathbb{E}^{\mathbb{Q}}[e^{-rT} (\max(S_T - K_2) - \max(S_T - K_1))] > 0$$

(b)

Note that $K_2 - K_1 \geq \max(S - K_1) - \max(S - K_2), \forall S$, with the inequality strict for some S (with such $\mathbb{Q}(S) > 0$). Since $r > 0$,

$$K_2 - K_1 > e^{-rT} (K_2 - K_1) = \mathbb{E}^{\mathbb{Q}}[e^{-rT} (K_2 - K_1)] > \mathbb{E}^{\mathbb{Q}}[e^{-rT} (\max(S_T - K_1) - \max(S_T - K_2))]$$

$$\implies K_2 - K_1 > C(K_1) - C(K_2)$$

(c)

Let $\lambda = (K_3 - K_2)/(K_3 - K_1) \in (0, 1)$, then $1 - \lambda = (K_2 - K_1)/(K_3 - K_1)$ and we have

$$\lambda K_1 + (1 - \lambda) K_3 = \frac{K_3 - K_2}{K_3 - K_1} K_1 + \frac{K_2 - K_1}{K_3 - K_1} K_3 = K_2$$

To show option price is convex in K, we first notice that $f(K) = \max(S - K)$ is convex in $K, \forall S$.

$$\lambda \max(S_T - K_1) + (1 - \lambda) \max(S_T - K_3) \geq \max(S_T - (\lambda K_1 + (1 - \lambda) K_3)) = \max(S_T - K_2)$$

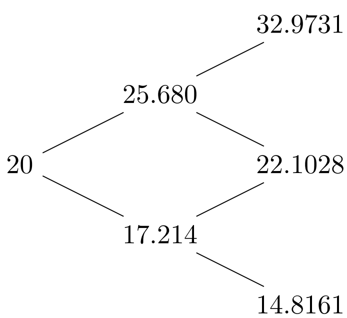
$$\begin{aligned} \frac{K_3 - K_2}{K_3 - K_1} C(K_1) + \frac{K_2 - K_1}{K_3 - K_1} C(K_3) &= \lambda C(K_1) + (1 - \lambda) C(K_3) \\ &= \lambda \mathbb{E}^{\mathbb{Q}}[e^{-rT} \max(S_T - K_1)] + (1 - \lambda) \mathbb{E}^{\mathbb{Q}}[e^{-rT} \max(S_T - K_3)] \\ &= \mathbb{E}^{\mathbb{Q}}[e^{-rT} (\lambda \max(S_T - K_1) + (1 - \lambda) \max(S_T - K_3))] \\ &\geq \mathbb{E}^{\mathbb{Q}}[e^{-rT} \max(S_T - K_2)] \\ &= C(K_2) \end{aligned}$$

Problem 11

Use a 2-period binomial tree to price an American Put Option with the following parameters: Strike Price K = 22, continuously compounding annualized risk-free rate $r_f = 5\%$.Current price $S_0 = 20$. Time to Expiration T = 2 years, each period of the tree represents one year. Given u = 1.2840, d = 0.8607.

Answers:

First, we construct the two-period binomial tree for the stock price.



The calculations for the stock prices at various nodes are as follows:

$$
\begin{aligned}
S_u &= 20 * 1.2840 = 25.680 \\
S_d &= 20 * 0.8607 = 17.214 \\
S_{uu} &= 25.68 * 1.2840 = 32.9731 \\
S_{ud} = S_{du} &= 17.214 * 1.2840 = 22.1028 \\
S_{dd} &= 17.214 * 0.8607 = 14.8161
\end{aligned}
$$

The risk-neutral probability for the stock price to go up is

$$p^* = (e^{rt} - d)/(u - d) = (e^{0.05} - 0.8607)/(1.2840 - 0.8607) = 0.4502$$

Thus, the risk-neutral probability for the stock price to go down is 0.5498

American Put Option

If the option is exercised at time 2, the value of the call would be

$$
\begin{aligned}
P_{uu} &= (22 - 32.9731)_+ = 0 \\
P_{ud} &= (22 - 22.1028)_+ = 0 \\
P_{dd} &= (22 - 14.8161)_+ = 7.1839
\end{aligned}
$$

If the option is European, then

$$
\begin{aligned}
P_u &= e^{-0.05}[0.4502P_{uu} + 0.5498P_{ud}] = 0 \\
P_d &= e^{-0.05}[0.4502P_{ud} + 0.5498P_{dd}] = 3.7571
\end{aligned}
$$

But since the option is American, we should compare P_u and P_d with the value of the option if it is exercised at time 1, which is $(22-25.680)_+ = 0$ and $(22-17.214)_+ = 4.786$, respectively. Since

$3.7571 < 4.786$, it is optimal to exercise the option at time 1 when the stock is in the down state. Thus the value of the option at time 1 is either $P_u^A = \max(0, 0) = 0$ or $P_d^A = \max(3.7571, 4.786) = 4.786$.

Finally, the value of the American put is $P^A = e^{-0.05}[0.4502(0) + 0.5498(4.786)] = 2.5030$

American Call Option(for your reference)

If the option is exercised at time 2, the value of the call would be

$$\begin{aligned} C_{uu} &= (32.9731 - 22)_+ = 10.9731 \\ C_{ud} &= (22.1028 - 22)_+ = 0.1028 \\ C_{dd} &= (14.8161 - 22)_+ = 0 \end{aligned}$$

If the option is European, then

$$\begin{aligned} C_u &= e^{-0.05}[0.4502C_{uu} + 0.5498C_{ud}] = 4.7530 \\ C_d &= e^{-0.05}[0.4502C_{ud} + 0.5498C_{dd}] = 0.0440 \end{aligned}$$

But since the option is American,we should compare C_u and C_d with the value of the option if it is exercised at time 1, which is 3.68 and 0, respectively. Since $3.68 < 4.7530$ and $0 < 0.0440$, it is not optimal to exercise the option at time 1 whether the stock is in the up or down state. Thus the value of the option at time 1 is either 4.7530 or 0.0440.

Finally, the value of the call is $C = e^{-0.05}[0.4502(4.7530) + 0.5498(0.0440)] = 2.0585$

Problem 12

Consider a stock which pays no dividend. The current stock price is \$ 62 and the annualized volatility for the stock is $\sigma = 0.20$. The annual continuously compounding risk-free rate is 2.5%. Consider a five-month option with a strike price of \$ 60. After 3 months, the purchaser will have the right to choose this option to be either an European call option or an European put option. Please use a 5-step (monthly) binomial lattice model to price this exotic option.

Note: Use $u = \exp(\sigma\sqrt{\Delta T})$ and $d = \exp(-\sigma\sqrt{\text{T}})$

Answers:

(1) Determine the value of u and d for the binomial lattice. The value for $u = \exp(\sigma\sqrt{\Delta T}) = \exp(0.2(1/2)^{0.5}) = 1.05943$. Note that $d = 1/u = 0.94390$.

(2) Determine the values for the binomial lattice for the stock price for 5 1-month periods:

0	1	2	3	4	5
62.00	65.68	69.59	73.72	78.11	82.75
	58.52	62.00	65.68	69.59	73.72
		55.24	58.52	62.00	65.68
			52.14	55.24	58.52
				49.21	52.14
					46.45

(3) Determine the appropriate risk-free rate. The interest rate per month $R = exp(0.025*1/12) = 1.00209$.

(4) Determine the risk-neutral probability q of going "up". The value for q that $q = (R-d)/(u-d) = 0.5036$.

(5) Determine the values for the call option and put option along the lattice.

0	1	2	3	4	5
4.6686	7.0172	10.1423	**13.9743**	18.2315	22.7488
	2.3054	3.876	**6.2971**	9.7137	13.7248
		0.7216	1.4359	2.8571	5.6849
			0	0	0
				0	0
					0

0	1	2	3	4	5
2.0469	0.8344	0.1797	0	0	0
	3.2858	1.5022	0.3627	0	0
		5.1091	**2.6646**	0.7322	0
			7.6107	4.6364	1.4782
				10.6604	7.8602
					13.5462

(6) Find the value of this Chooser Option. Compute the terminal value of the Chooser Option at t = 3 as the maximum of the call and put options at t = 3. From there we work backwards in the usual manner.

0	1	2	3	4	5
6.0483	7.3187	10.1423	**13.9743**		
	4.7847	4.4847	**6.2971**		
		5.1091	**2.6646**		
			7.6107		

Problem 13

A stock price is currently \$50. It is known that at the end of 2 months it will be either \$53 or \$48. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 2-month European call option with a strike price of \$49? Use no-arbitrage arguments.

Answers:

At the end of two months the value of the option will be either \$4 (if the stock price is \$53) or \$0 (if the stock price is \$48). Consider a portfolio consisting of:

$$+\Delta : shares$$
$$-1 : option$$

The value of the portfolio is either 48Δ or $53\Delta - 4$ in two months. If $48\Delta = 53\Delta - 4$, then $\Delta = 0.8$, the value of the portfolio is certain to be 38.4. For this value of Δ, the portfolio is therefore riskless. The current value of the portfolio $= 0.8 * 50 - f$, where f is the value of the option. Since the portfolio must earn the risk-free rate of interest:

$$(0.8 * 50 - f)e^{0.1*2/12} = 38.4$$

Solving the above equation, we obtain that $f = 2.23$. Therefore the value of the option is \$2.23. Alternatively, we can also solve this as follows: $u = 1.06, d = 0.96$

$$p = (e^{0.1*2/12} - 0.96)/(1.06 - 0.96) = 0.5681$$
$$f = e^{-0.1*2/12} * 0.5681 * 4 = 2.23$$

Problem 14

A stock price is currently \$25. It is known that at the end of 2 months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose $S(T)$ is the stock price at the end of 2 months. What is the value of a derivative that pays off $S(T)^2$ at this time?

Answers:

At the end of two months the value of the derivative will be either 529 (if the stock price is 23) or 729 (if the stock price is 27). Consider a portfolio consisting of:

$$+\Delta : shares$$
$$-1 : derivative$$

The value of the portfolio is either $27\Delta - 729$ or $23\Delta - 529$ in two months. If $27\Delta - 729 = 23\Delta - 529$, then $\Delta = 50$, the value of the portfolio is certain to be 621. For this value of Δ, the portfolio is therefore riskless. The current value of the portfolio $= 50 * 25 - f$, where f is the value of the derivative. Since the portfolio must earn the risk-free rate of interest:

$$(50 * 25 - f)e^{0.1*2/12} = 621$$

Solving the above equation, we obtain that $f = 639.3$. Therefore the value of the derivative is \$639.3. Alternatively, we can also solve this as follows: $u = 1.08, d = 0.92$

$$p = (e^{0.1*2/12} - 0.92)/(1.08 - 0.92) = 0.6050$$
$$f = e^{-0.1*2/12}(0.6050 * 729 + 0.3950 * 529) = 639.3$$