ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 2

Vidit Pokharna, 903772087

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Problem 1
Let $S \sim U[0,1]$ be the true, unknown value. If we bid b, the bid is accepted iff $b \geq S$. Upon the news of the bid, the value becomes $2S$. Thus, conditional on acceptance, the payoff is $2S - b$; otherwise it is 0.

Thus, the expected payoff is

$$\mathbb{E}[\pi(b)] = \int_0^1 (2s - b)\, \mathbf{1}\{s \leq b\}\, ds = \int_0^b (2s - b)\, ds = \left[s^2 - bs\right]_0^b = b^2 - b^2 = 0, \qquad \text{for } b \in [0,1]$$

If $b > 1$, acceptance is certain and

$$\mathbb{E}[\pi(b)] = \int_0^1 (2s - b)\, ds = 1 - b < 0$$

Therefore, the maximum expected payoff is 0, achieved by a bid $b \in [0,1]$ (e.g., bid any number ≤ 1). Bidding more than 1 yields a negative expected payoff.

Problem 2
The first non-one outcome can be classified as $X \in \{2,3,4,5,6\}$. For any $j \in \{2,\ldots,6\}$,

$$\mathbb{P}(X = j) = \sum_{k=0}^{\infty} \mathbb{P}(\underbrace{1,\ldots,1}_{k \text{ times}}, j) = \sum_{k=0}^{\infty} \left(\frac{1}{6}\right)^k \left(\frac{1}{6}\right) = \frac{1}{6} \cdot \frac{1}{1 - \frac{1}{6}} = \frac{1}{5}$$

Thus X is uniform on $\{2,3,4,5,6\}$. The expected payoff is

$$\mathbb{E}[X] = \frac{2+3+4+5+6}{5} = \frac{20}{5} = 4$$

Problem 3
Let D be the event that the chosen coin is the two–headed coin, and H_{10} be the event of getting 10 heads in a row.

$$\mathbb{P}(D) = \frac{1}{1000}, \qquad \mathbb{P}(H_{10} \mid D) = 1, \qquad \mathbb{P}(H_{10} \mid D^c) = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$$

$$\mathbb{P}(D \mid H_{10}) = \frac{\mathbb{P}(H_{10} \mid D)\,\mathbb{P}(D)}{\mathbb{P}(H_{10} \mid D)\,\mathbb{P}(D) + \mathbb{P}(H_{10} \mid D^c)\,\mathbb{P}(D^c)} = \frac{\frac{1}{1000}}{\frac{1}{1000} + \frac{999}{1000} \cdot \frac{1}{1024}} = \frac{1024}{2023} \approx 0.506.$$

It is slightly more likely than not that we picked the *two–headed* coin (posterior probability $1024/2023 \approx 50.6\%$).

Problem 4
Let the four cards be W (water), E (earth), F (fire), and I (wind). In a random permutation, look only at the relative order of the three cards $\{W, E, F\}$. The game ends when either F appears or when the second of $\{W, E\}$ appears. Thus you win iff F comes *after* both W and E, i.e., F is last among $\{W, E, F\}$.

The position of the wind card is irrelevant. All $3! = 6$ orders of $\{W, E, F\}$ are equally likely, and in exactly two of them ($W-E-F$ and $E-W-F$) is F last. Therefore

$$\mathbb{P}(\text{win}) = \frac{2}{6} = \frac{1}{3}$$

Problem 5
B is "infected", A is "test is positive"

- $\mathbb{P}(B) = 0.1$
- $\mathbb{P}(A \mid B) = 0.95$
- $\mathbb{P}(A^c \mid B^c) = 0.99$
- $\mathbb{P}(A \mid B^c) = 0.01$

R is the event "in three independent tests we see 2 positives and 1 negative". Then:

$$\mathbb{P}(R \mid B) = \binom{3}{2}(0.95)^2(0.05), \qquad \mathbb{P}(R \mid B^c) = \binom{3}{2}(0.01)^2(0.99)$$

$$\mathbb{P}(B \mid R) = \frac{\mathbb{P}(B)\,\mathbb{P}(R \mid B)}{\mathbb{P}(B)\,\mathbb{P}(R \mid B) + \mathbb{P}(B^c)\,\mathbb{P}(R \mid B^c)} = \frac{0.1 \cdot 3 \cdot 0.95^2 \cdot 0.05}{0.1 \cdot 3 \cdot 0.95^2 \cdot 0.05 + 0.9 \cdot 3 \cdot 0.01^2 \cdot 0.99} \approx 0.9806$$

$$\boxed{\mathbb{P}(\text{infected} \mid \text{two positives, one negative}) \approx 98.1\%}$$

Problem 6
Let X be the number of people who choose this cinema in a day.

$$X \sim \text{Bin}(n = 1600, p = \tfrac{3}{4}), \qquad \mu = np = 1200, \quad \sigma = \sqrt{np(1-p)} = \sqrt{300}$$

"Over 200 empty seats" means $q - X \geq 201$ ($X \leq q - 201$). We want the largest q such that

$$\mathbb{P}(X \leq q - 201) \leq 0.1$$

$$\mathbb{P}(X \leq t) \approx \Phi\left(\frac{t + 0.5 - \mu}{\sigma}\right) = 0.1, \qquad z_{0.1} = -1.2816$$

$$t + 0.5 = \mu + z_{0.1}\sigma = 1200 - 1.2816\sqrt{300} \approx 1177.78,$$

so $t \approx 1177$ (largest integer with CDF ≤ 0.1).

Therefore

$$q = t + 201 = 1177 + 201 = \boxed{1378}$$

Problem 7
Payoffs (price+dividend) next period:

$$x_A = (124, 71), \qquad x_B = (83, 61), \qquad x_C = (92, 160)$$

$q = (q_1, q_2)$ is the state–price vector so that $p_i = q \cdot x_i$

(a) $p_A = 100,\ p_B = 70$ to solve for q:

$$\begin{cases} 124q_1 + 71q_2 = 100 \\ 83q_1 + 61q_2 = 70 \end{cases} \quad \Rightarrow \quad q_1 = \frac{93790}{138693} \approx 0.67624, \quad q_2 = \frac{380}{1671} \approx 0.22741$$

Then the implied no–arbitrage price of C is

$$p_C^* = q \cdot x_C = 92q_1 + 160q_2 = \frac{54920}{557} \approx 98.6$$

Since the quoted price is 180, the three "current" prices are *not* arbitrage–free (security C is overpriced).

(b) Replicate C with A and B:

$$\alpha x_A + \beta x_B = x_C \ \Rightarrow \ \alpha = \frac{92 \cdot 61 - 160 \cdot 83}{124 \cdot 61 - 83 \cdot 71} = -\frac{7668}{1671} \approx -4.589,$$

$$\beta = \frac{124 \cdot 160 - 92 \cdot 71}{124 \cdot 61 - 83 \cdot 71} = \frac{13308}{1671} \approx 7.964$$

Cost of the replicating portfolio:

$$100\alpha + 70\beta = \frac{54920}{557} \approx 98.6$$

Arbitrage: *short 1 unit of* C and *take the replicating position* (α of A, β of B). Initial profit $= 180 - 98.600 \approx \$81.40$, and state payoffs net to 0.

(c) All arbitrage–free price triples are

$$(p_A, p_B, p_C) = \big(124q_1 + 71q_2,\ 83q_1 + 61q_2,\ 92q_1 + 160q_2\big), \qquad q_1, q_2 > 0$$

Using the q above (which makes A and B priced at 100 and 70), the arbitrage–free price of C is $p_C^* = \dfrac{54920}{557} \approx 98.600$.

Additionally, $q_1 + q_2 = \dfrac{1510}{1671}$, so the risk–free rate is $r = \dfrac{1}{q_1 + q_2} - 1 = \dfrac{161}{1510} \approx 10.66\%$.

(d) The delivery price F_0 that makes the contract worth 0 today is

$$0 = q_1(80 - F_0) + q_2(60 - F_0) \;\Rightarrow\; F_0 = \frac{q_1 \cdot 80 + q_2 \cdot 60}{q_1 + q_2} = \mathbb{E}_Q[S_1^B] = \frac{11320}{151} \approx \boxed{74.967}$$

(e) European put on C with $K = 125$
Payoff: $(125 - S_1^C)^+ = (35,\ 0)$

Price:

$$P_0 = q_1 \cdot 35 + q_2 \cdot 0 = \boxed{\frac{39\,550}{1671} \approx 23.668}.$$

Problem 8

$$x_A = (120, 70, 80), \qquad x_B = (80, 60, 50), \qquad x_C = (90, 150, 190), \qquad x_D = (30, 20, 30)$$

$q = (q_1, q_2, q_3)$ is the state–price vector. No–arbitrage requires for $i \in \{A, B, C\}$:

$$p_i = q \cdot x_i, \qquad \text{with } (p_A, p_B, p_C) = (100, 70, 180)$$

Solving

$$\begin{pmatrix} 120 & 70 & 80 \\ 80 & 60 & 50 \\ 90 & 150 & 190 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 180 \end{pmatrix} \quad \Rightarrow \quad q_1 = \frac{5}{19}, \quad q_2 = \frac{94}{247}, \quad q_3 = \frac{129}{247}$$

$$p_D = q \cdot x_D = 30q_1 + 20q_2 + 30q_3 = \frac{7700}{247} \approx \boxed{31.174}$$

$$\alpha x_A + \beta x_B + \gamma x_C = x_D : \qquad \alpha = \frac{101}{247}, \quad \beta = -\frac{6}{19}, \quad \gamma = \frac{17}{247}$$

$$100\alpha + 70\beta + 180\gamma = \frac{7700}{247} \approx 31.174 = p_D$$

Problem 9
Let $X_i = \mathbf{1}\{i\text{th toss is H}\}$, so $X_i \overset{iid}{\sim} \text{Bernoulli}(1/2)$ and $S_n = \sum_{i=1}^n X_i$ is the number of heads in the first n tosses.
By the Strong Law of Large Numbers,

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{S_n}{n} = \frac{1}{2}\right) = 1.$$

Equivalently, the limiting proportion of H (and of T) equals 1/2 with probability 1, and the set of sequences for which the limit differs from 1/2 has probability 0.

(a) $\mathbb{P}(\text{proportion} = 1/2) = 1$

(b) $\mathbb{P}(\text{not } 1/2) = 0$

Problem 10
Let there be $n = 100$ strings (so $2n$ ends). Pair the $2n$ ends uniformly at random. Label the ends $1, 2, \ldots, 2n$ so that the two ends of string k are $2k-1$ and $2k$.

Follow a loop as you alternate between "tied–together" ends and "the other end of the same string." Every loop corresponds to a cycle of a permutation on $\{1, \ldots, 2n\}$, and the smallest label in each cycle must be *odd* (since if $2k$ is in the cycle, so is $2k-1 < 2k$).

For each odd i, let X_i be the indicator that i is the smallest label in its loop. Then $L = \sum_{k=1}^n X_{2k-1}$. Starting from end i and tracing its loop until we first hit an element of $\{1, 2, \ldots, i\}$, by symmetry this first hit is equally likely to be any of these i ends; hence

$$\mathbb{P}(X_i = 1) = \frac{1}{i}, \qquad (i \text{ odd})$$

$$\mathbb{E}[L] = \sum_{k=1}^{n} \mathbb{E}[X_{2k-1}] = \sum_{k=1}^{n} \frac{1}{2k-1} = H_{2n} - \frac{1}{2}H_n$$

$$\boxed{\mathbb{E}[L] = \sum_{k=1}^{100} \frac{1}{2k-1} = H_{200} - \tfrac{1}{2}H_{100} \approx 3.28434}$$

Problem 11

(a) Let the six chambers to be fired in order be positions $1, 2, \ldots, 6$. With one bullet, its position is uniform on $\{1, \ldots, 6\}$. Player 1 fires at positions $1, 3, 5$; Player 2 at $2, 4, 6$. Thus each dies with probability $3/6 = 1/2$, so you are indifferent.

$$\boxed{\mathbb{P}(\text{survive if first}) = \tfrac{1}{2} = \mathbb{P}(\text{survive if second})}$$

(b) Each shot kills with probability $1/6$, independently. The game ends at the first "success". The probability the first success occurs on Player 2's turn is

$$\sum_{k=0}^{\infty} \left(\tfrac{25}{36}\right)^k \left(\tfrac{5}{6}\right)\left(\tfrac{1}{6}\right) = \frac{5/36}{1 - 25/36} = \frac{5}{11}$$

Hence

$$\boxed{\mathbb{P}(\text{survive if first}) = \tfrac{5}{11} \approx 0.4545, \quad \mathbb{P}(\text{survive if second}) = \tfrac{6}{11} \approx 0.5455}$$

So preference is to go *second*.

(c) The opponent went first and survived. Without spinning, the next chamber is one specific position among the remaining 5; conditional on the first being empty, there are 2 bullets among those 5 positions, so

$$\mathbb{P}(\text{die if no spin}) = \tfrac{2}{5} = 0.4$$

If you spin, each shot has $\mathbb{P}(\text{die}) = 2/6 = 1/3 \approx 0.333$.

$$\boxed{\text{Spin the barrel}}$$

(d) The two-bullet block can start at positions $2, 3, 4, 5$ (four equally likely cases); only if it starts at 2 is the next chamber a bullet. Thus,

$$\Pr(\text{die if no spin}) = \tfrac{1}{4} = 0.25, \qquad \Pr(\text{die if spin}) = \tfrac{2}{6} = \tfrac{1}{3} \approx 0.333$$

$$\boxed{\text{Do not spin the barrel}}$$