Math 6635 Homework #3

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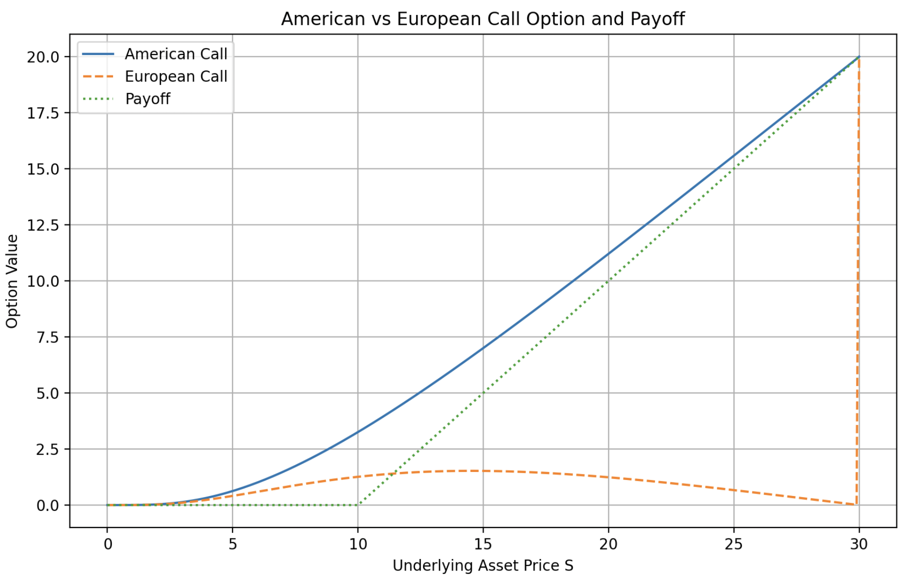
1. To solve the untransformed Black-Scholes equation for an American option, reverse time by setting $\tau = T - t$ and discretize the PDE using a uniform mesh with asset price step size ΔS and time step size $\Delta\tau$. V_j^n approximates the option value $V(j\Delta S, n\Delta\tau)$. Now apply the Crank-Nicolson method, which averages the explicit and implicit finite difference schemes, resulting in a tridiagonal linear system of the form $AV^{n+1} = BV^n$, where matrices A and B are derived using central difference approximations for the first and second spatial derivatives. Since American options allow early exercise, we must enforce the constraint $V_j^{n+1} \geq$ payoff, which transforms the problem into a LCP. To solve this, use PSOR, which iteratively solves the linear system while projecting each updated value to satisfy the early exercise condition.

   - $\alpha_j = \frac{1}{4}\Delta\tau\left[\sigma^2 j^2 - rj\right]$
   - $\beta_j = -\frac{1}{2}\Delta\tau\left[\sigma^2 j^2 + r\right]$
   - $\gamma_j = \frac{1}{4}\Delta\tau\left[\sigma^2 j^2 + rj\right]$
   - $AV^{n+1} = BV^n$
   - $V_j^{n+1} \geq P_j$
   - $V_j^{n+1} \leftarrow \max(V_j^{n+1}, P_j)$

S	American Put	European Put
0.0	10.000000	10.000000
2.0	8.000000	7.578344
4.0	6.000000	5.669973
6.0	4.000000	3.692186
8.0	2.000010	1.877007
10.0	0.679621	0.647064
12.0	0.200805	0.192960
14.0	0.053209	0.051426
16.0	0.000000	0.000000

Table 1: American and European Put Option Values

2. (a)

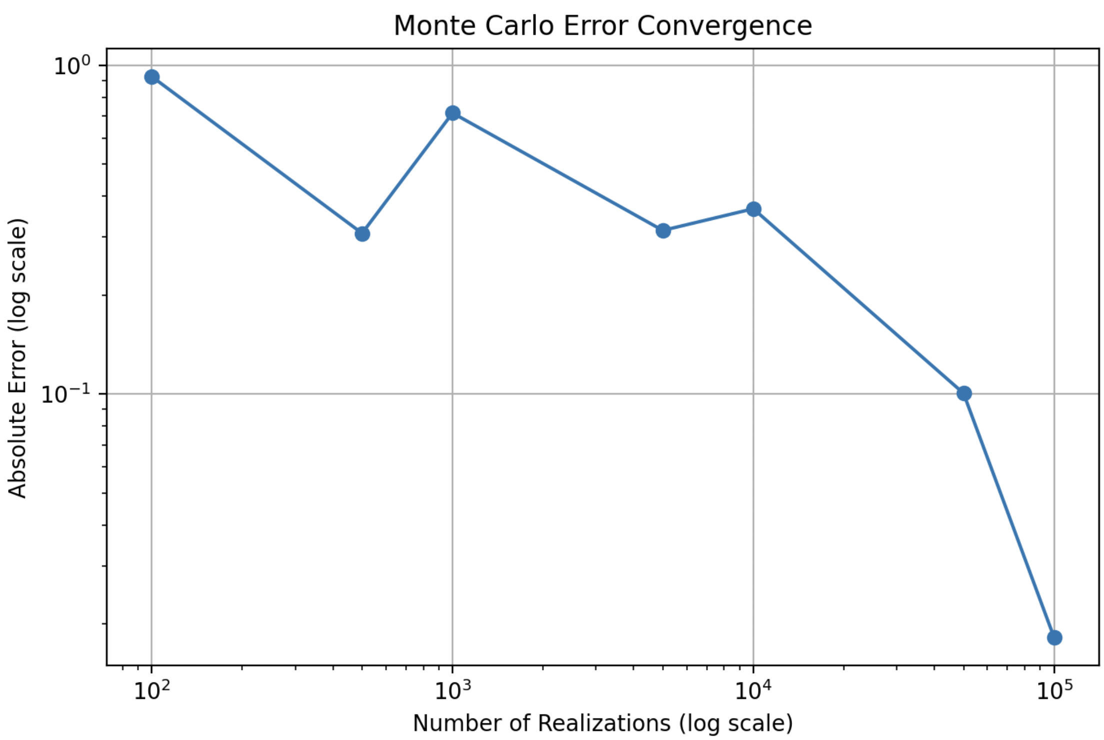


(b)

2.5. I used Monte Carlo simulation to compute the price of a European call option with parameters $S_0 = 100$, $E = 100$, $r = 0.05$, $\sigma = 0.2$, and maturity $T = 1$, evaluated at time $t = 0$. The simulation involved generating standard normal random variables to model the risk-neutral terminal stock price S_T, and computing the discounted payoff $\max(S_T - E, 0)$ to estimate the call price. The Monte Carlo estimates were then compared to the exact price obtained from the Black-Scholes formula, which yields a theoretical value of approximately 10.45058.

I performed simulations with increasing numbers of realizations: 100, 500, 1,000, 5,000, 10,000, 50,000, and 100,000. As expected, the accuracy of the Monte Carlo method improved with more samples. For instance, with only 100 simulations, the estimated price was 11.37407 with an error of 0.92349. At 10,000 samples, the error decreased to 0.36600, and by 100,000 samples, the estimate was very close to the Black-Scholes value, with an error of just 0.01819.

From the results, I observe that to reduce the error by a factor of $\frac{1}{2}$, the number of simulations must be increased by approximately a factor of 4. This is consistent with the theoretical convergence rate of Monte Carlo methods, where the standard error is proportional to $1/\sqrt{N}$. This relationship is also clearly visible in the log-log plot of error versus number of realizations. The simulation confirms that Monte Carlo methods, while computationally intensive, provide accurate estimates of option prices when a sufficiently large number of paths is used.



3. Using the parameters from 2b $\rightarrow r = 0.25$, $D_0 = 0.2$, $\sigma = 0.8$, and $T = 1$, I computed the binomial factors as $u = 1.08329$, $d = 0.92312$, and $p = 0.48313$. The option values were then computed at three different underlying asset prices: $S = 10$, $S = 20$, and $S = 25$, yielding American call option prices of 2.82524, 10.45230, and 15.05767, respectively.

   When comparing these values to those obtained from the Crank-Nicolson method in Problem (2b), I found that the binomial results are consistent and closely aligned, particularly at higher asset prices where early exercise becomes less advantageous. This confirms that the binomial method with early exercise logic correctly captures the valuation of American-style options under dividend-paying conditions.

4. The stock price falls from S just before the dividend date to $(1 - dy)S$ immediately after. This creates a discontinuity in the asset price evolution that must be reflected in the tree.

   To modify the binomial method accordingly, adjust the stock price nodes at and after the dividend time t_d. Specifically, during the tree construction phase, for any time step that occurs after the dividend date, shift the corresponding nodes downward by the dividend amount. This can be done by first constructing the standard binomial tree as usual and then, at time t_d, applying a multiplicative drop of $(1 - dy)$ to all stock price nodes in that and subsequent layers.

   Alternatively, the drop can be incorporated directly into the tree construction: when calculating the asset prices, reduce the stock value by $dy \cdot S$ at node levels corresponding to t_d and propagate this reduction through the remaining steps of the tree.

   The discrete dividend only affects the asset price evolution, not the probability or payoff formulas. Therefore, the values of u, d, and p remain unchanged, and the backward induction for option values is performed as usual using the modified stock prices.