MATH-3012-D FINAL

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TOTAL POINTS

188 / 200

QUESTION 1

1 Q1 **20 / 20**

- ✓ **+ 20 pts** *Correct*
- ✓ **+ 6 pts** *(b) correct*
- ✓ **+ 4 pts** *(c) correct*
- ✓ **+ 4 pts** *(d) correct*
- ✓ **+ 4 pts** *(a) partial credit*

QUESTION 2

2 Q1 **17 / 20**

- ✓ **+ 2 pts** *(a) correct*
- ✓ **+ 3 pts** *(b) correct*
- ✓ **+ 3 pts** *(c) correct*
- ✓ **+ 3 pts** *(d) correct*
- ✓ **+ 3 pts** *(e) correct*
- ✓ **+ 3 pts** *(f) correct*

QUESTION 3

3 Q3 **20 / 20**

- ✓ **+ 6 pts** *(a) correct. gcd = 3*
- ✓ **+ 5 pts** *(b) correct. x = 7, y = -16.*
- ✓ **+ 5 pts** *(c) correct. n = 10*10+1 = 101*
- ✓ **+ 4 pts** *(d) correct.*

QUESTION 4

4 Q4 **20 / 20**

- ✓ **+ 20 pts** *Correct*
- ✓ **+ 10 pts** *(a) correct*
- ✓ **+ 4 pts** *(b) correct*
- ✓ **+ 3 pts** *(c) correct*
- ✓ **+ 1 pts** *(c) partial credit*

QUESTION 5

5 Q5 **17 / 20**

- ✓ **+ 5 pts** *(a) correct*
- ✓ **+ 7 pts** *(b) correct*
- ✓ **+ 3 pts** *(c) partial credit*
- ✓ **+ 2 pts** *(d) partial credit*

QUESTION 6

6 Q6 **15 / 20**

- ✓ **+ 6 pts** *(a) correct*
- ✓ **+ 3 pts** *(c) correct*
- ✓ **+ 3 pts** *(d) correct*
- ✓ **+ 3 pts** *(e) correct*

QUESTION 7

7 Q7 **19 / 20**

- ✓ **+ 20 pts** *Correct*
- **- 1** *Point adjustment*

1 Why? -1 here

QUESTION 8

8 Q8 **20 / 20**

- ✓ **+ 20 pts** *Correct*

QUESTION 9

9 Q9 **20 / 20**

(a)

✓ **+ 5 pts** *two of three/ or other partial*

✓ **+ 2 pts** *partial*

(b)

✓ **+ 4 pts** *correct*

(c)

✓ **+ 7 pts** *No proof/why clique number is 4/ or other partial*

+ 2 *Point adjustment*

QUESTION 10

10 Q10 **20 / 20**

(a)

✓ **+ 9 pts** *correct*

(b)

✓ **+ 8 pts** *correct*

(c)

✓ **+ 1 pts** *partial*

+ 2 *Point adjustment*

Question 1. (20 points)

(Answers should be a sum/difference of a no more than 3 combination numbers).

(a) Find the number of integer solutions of the system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 50 \\ x_i \geq 3i \text{ for } i = 1, 2, 3, 4 \end{cases}$$

(b) Find the number of integer solutions of the system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 48 \\ x_i \geq 2 \text{ for } i = 1, 2, 3, 4, \text{ and } x_4 \leq 16 \end{cases}$$ $X_4 > 16 \rightarrow X_4 \geq 17$

(c) There are unlimited supplies of balls of colors A, B, C, and D. How many are there different ways to choose 45 balls from the above so that the number of balls of each color A, B, C is even (note: 0 is even). (Hint: What parity of the number of D balls then?)

(d) Find the numerical value of $\binom{2/3}{3}$

(which is "2/3 choose 3". The answer should be a rational similar to $-5/217$).

(a) $\begin{cases} X_1+X_2+X_3+X_4=50 \\ X_1 \geq 3, X_2 \geq 6, X_3 \geq 9, X_4 \geq 12 \end{cases} \rightarrow \begin{cases} X_1+3+X_2+6+X_3+9+X_4+12=50 \\ X_i \geq 0 \end{cases} \rightarrow \begin{cases} X_1+X_2+X_3+X_4=20 \\ X_i \geq 0 \end{cases}$

$\binom{20+4-1}{20} = \boxed{\binom{23}{3}}$

(b) $\begin{cases} X_1+2+X_2+2+X_3+2+X_4+2=48 \\ X_i \geq 0 \end{cases} - \begin{cases} X_1+17+X_2+2+X_3+2+X_4+2=48 \\ X_i \geq 0 \end{cases}$

$\begin{cases} X_1+X_2+X_3+X_4=40 \\ X_i \geq 0 \end{cases} - \begin{cases} X_1+X_2+X_3+X_4=25 \\ X_i \geq 0 \end{cases}$

$\binom{40+3}{3} = \binom{43}{3}$ $\quad \binom{25+3}{3} = \binom{28}{3}$ $\quad \boxed{\binom{43}{3} - \binom{28}{3}}$

(c) $f_{A,B,C} = 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$

$f_D = x + x^3 + x^5 + \cdots = \frac{x}{1-x^2}$

Coef(x^{45}) in $\frac{x}{(1-x^2)^4} \rightarrow \frac{1}{(1-x^2)^4} = \sum \binom{3+n}{n} x^{2n} \rightarrow x \sum \binom{3+n}{n} x^{2n} = \sum \binom{3+n}{n} x^{2n+1}$

@ $n=22$, coef $= \boxed{\binom{25}{3}}$

(d) $\frac{(2/3)(-1/3)(-4/3)}{3 \cdot 2 \cdot 1} = \frac{8}{27 \cdot 3 \cdot 2 \cdot 1} = \boxed{\frac{4}{81}}$

1 Q1 **20 / 20**

✓ **+ 20 pts** *Correct*

✓ **+ 6 pts** *(b) correct*

✓ **+ 4 pts** *(c) correct*

✓ **+ 4 pts** *(d) correct*

✓ **+ 4 pts** *(a) partial credit*

Question 2. (20 points). Box your answers. No work is needed, except for (c), (d) and (g), where short explanations are needed.

How many 15-letter strings can be formed using the 26 letters of the alphabet, with the following restrictions. (Questions are independent)

(a) Repetition is allowed.
(b) Repetition is not allowed.
(c) Repetition is not allowed; A or B (or both) must be present.
(d) There are exactly four A and two B, and repetition is allowed.
(e) There are exactly four A, four B, three C, two D, one E and one F.
(f) Among the strings in (e) how many have E next to F?
(g) How many functions $f : \{a, b, c, d, x, y, z\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ have that

$$f(a) \leq f(b) \leq f(c) \leq (d)?$$

(a) $\boxed{26^{15}}$

(b) $P(26,15) = \boxed{\frac{26!}{11!}}$

(c) $P(26,15) - P(24,15) = \boxed{\frac{26!}{11!} - \frac{24!}{9!}}$ → complement of when neither A or B is present

(d) $\boxed{\binom{15}{4}\binom{11}{2} 24^9}$ → choose 4 spots for A, then 2 spots for B, and then the rest 9 spots have 24 options each

(e) $\boxed{\frac{15!}{4!4!3!2!}}$

(f) $\frac{2 \cdot 14!}{4!4!3!2!} = \boxed{\frac{14!}{4!4!3!}}$

(g) $\boxed{8^7 - \binom{8}{4} 8^3}$ → complement of when $f(a) > f(b) > f(c) > f(d)$

2 Q1 **17 / 20**

- ✓ **+ 2 pts** *(a) correct*
- ✓ **+ 3 pts** *(b) correct*
- ✓ **+ 3 pts** *(c) correct*
- ✓ **+ 3 pts** *(d) correct*
- ✓ **+ 3 pts** *(e) correct*
- ✓ **+ 3 pts** *(f) correct*

Question 3. (20 points) **Answers alone will get no credit. Show work. Follow instruction.**

(a) Use the **Euclidean algorithm** to find $d = gcd(117, 51)$.

(b) Use the **extended Euclidean algorithm** to find a pair of integers x, y such that $117x + 51y = d$.

(c) Find the least positive integer n having the following property: Among any n integers one can choose a collection of 11 such that the difference between any two in the collection is a multiple of 10.

(d) If a, b are integers and $400a + 17b = 7$. Show that b and 400 are co-prime. (Hint: From the equation, what are the possible values of gcd(400, 17)?)

(a) $117 = 2 \cdot 51 + 15$

$51 = 3 \cdot 15 + 6$

$15 = 2 \cdot 6 + 3$

$6 = 2 \cdot 3 + 0$

$\boxed{gcd = 3}$

(b) $3 = 15 - 2 \cdot 6 = 15 - 2(51 - 3 \cdot 15)$

$= 7 \cdot 15 - 2 \cdot 51 = 7(117 - 2 \cdot 51) - 2 \cdot 51$

$= 7 \cdot 117 - 16 \cdot 51$

$\boxed{(x, y) = (7, -16)}$

(c) $\left\lceil \frac{x}{10} \right\rceil = 11 \rightarrow x = 10 \cdot 10 + 1 = \boxed{101}$

(d) We can say $d = gcd(400, b)$

Given that a linear combination of 400 and b makes 7,

we can also say $d \mid 7$.

Therefore, d must be either 1 or 7.

$d \mid 400$ and $d \mid b$, however $7 \nmid 400$, so d must be 1.

Since $gcd(400, b) = 1$, they are co-prime.

3 Q3 **20 / 20**

✓ **+ 6 pts** *(a) correct. gcd = 3*

✓ **+ 5 pts** *(b) correct. x = 7, y = -16.*

✓ **+ 5 pts** *(c) correct. n = 10*10+1 = 101*

✓ **+ 4 pts** *(d) correct.*

Question 4. (20 points)

(a) Solve the recurrence equation

$$a_n = 10a_{n-1} - 25a_{n-2}, \quad a_0 = 3, a_1 = 5.$$

(b) Find the general solution for the recurrence relation $b_n - 5b_{n-1} + 6b_{n-2} = 4n - 12$.

(c) Find the general solution for the recurrence relation $c_n + 3c_{n-1} + 3c_{n-2} + c_{n-3} = 0$
(Use the binomial formula to factorize the characteristic polynomial.)

(d) How many are there lattice paths from point $(0,0)$ to point $(5,5)$ which are never above the line $y = x$ (which is the line passing through $(0,0)$ and has slope 1).

(a) $x^2 - 10x + 25 = 0$

$(x-5)^2 = 0$

$a_n = 5^n(s + tn)$

$3 = 5^0(s+0) = s$

$5 = 5^1(3+t)$

$t = -2$

$\boxed{a_n = 5^n(3-2n)}$

(b) $p_n = An + B$

$An + B - 5(A(n-1)+B) + 6(A(n-2)+B) = 4n-12$

$An + B - 5(An - A + B) + 6(An - 2A + B) = 4n - 12$

$n(A - 5A + 6A) + (B - 5B + 6B) + 5A - 12A = 4n - 12$

$2An + 2B - 7A = 4n - 12$

$A = 2$

$B = 1$

$x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

$\boxed{b_n = s\cdot 3^n + t\cdot 2^n + 2n + 1}$

(c) $x^3 + 3x^2 + 3x + 1 = 0$ $\rightarrow$ $\boxed{c_n = (-1)^n(k_0 n^2 + k_1 n + k_2)}$

$(x+1)^3 = 0$

(d)

0,0 2,2

111

RRRUUU
RRURUU
RRUURU
RURURU
RURRUU

$\frac{1}{5+1}\binom{10}{5} = \frac{1}{\cancel{6}} \cdot \frac{\cancel{10}\cdot\cancel{9}\cdot\cancel{8}\cdot 7\cdot\cancel{6}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 3\cdot 2\cdot 7 = 6\cdot 7 = \boxed{42}$

4 Q4 **20 / 20**

✓ **+ 20 pts** *Correct*

✓ **+ 10 pts** *(a) correct*

✓ **+ 4 pts** *(b) correct*

✓ **+ 3 pts** *(c) correct*

✓ **+ 1 pts** *(c) partial credit*

Question 5. (20 points) For this problem, you can use the following approximated values

$$\frac{3!}{e} \cong 2.20; \quad \frac{4!}{e} \cong 8.82; \quad \frac{5!}{e} \cong 44.14; \quad \frac{6!}{e} \cong 264.87; \quad \frac{7!}{e} \cong 1854.11; \quad \frac{8!}{e} \cong 14832.89$$

(a) How many are there bijective maps $f : \{a_1, a_2, \ldots, a_8\} \to \{b_1, b_2, \ldots, b_8\}$ satisfying the condition $f(a_i) \neq b_i$ for all $i = 1, 2, \ldots, 8$. Answer should be an exact number, like 320.

(b) Among $1, 2, \ldots, 1000$ how many are there numbers not divisible by any of 4, 5, and 6?

(c) How many are there functions from a set A of 10 elements to the set $\{1, 2, 3, 4, 5, x, y\}$ where each of $1, 2, 3, 4, 5$ is in the image.

(d) Find the generating function of (a_n), given by $a_n = 2\binom{-5}{n} - 5$.

(a) $d_8 = \boxed{14833}$

$8!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!}\right) =$

$\frac{8!}{2!} - \frac{8!}{3!} + \frac{8!}{4!} - \frac{8!}{5!} + \frac{8!}{6!} - \frac{8!}{7!} + 1 =$

(b) $N_0 - N_1 + N_2 - N_3$

$1000 - \left(\left\lfloor\frac{1000}{4}\right\rfloor + \left\lfloor\frac{1000}{5}\right\rfloor + \left\lfloor\frac{1000}{6}\right\rfloor\right) + \left(\left\lfloor\frac{1000}{20}\right\rfloor + \left\lfloor\frac{1000}{30}\right\rfloor + \left\lfloor\frac{1000}{24}\right\rfloor\right)$

$- \left(\left\lfloor\frac{1000}{120}\right\rfloor\right) = 1000 - (250 + 200 + 166) + (50 + 33 + 41) - (8)$

$= 1000 - (616) + (124) - (8)$

$= 384 + 116 = \boxed{500}$

(c) $N_0 - N_1 + N_2 - N_3 + N_4 - N_5$

$\boxed{7^{10} - \binom{5}{1}6^{10} + \binom{5}{2}5^{10} - \binom{5}{3}4^{10} + \binom{5}{4}3^{10} - \binom{5}{5}2^{10}}$

(d) $2\sum_{n=0}^{\infty}\binom{-5}{n}x^n - 5\sum_{n=0}^{\infty}x^n$

$2(1+x)^{-5} - 5 \cdot \frac{1}{1-x} = \boxed{\frac{2}{(1+x)^5} - \frac{5}{1-x}}$

5 Q5 **17 / 20**

- ✓ **+ 5 pts** *(a) correct*
- ✓ **+ 7 pts** *(b) correct*
- ✓ **+ 3 pts** *(c) partial credit*
- ✓ **+ 2 pts** *(d) partial credit*

Question 6. (20 points)

(a) Let a_n be the number of non-negative integer solutions of $x + 2y + 5z = n$. Find the generating function for the sequence (a_n). Answer should be a quotient of two polynomials.

(b) Let f_n be the number of string of length n consisting of letters A, B, C with the restriction: At most four A's and the number of B is even. Write down either the generating function, or the exponential generating function for the sequence (f_n). (You have to determine which one to choose. Your answer should not contain any infinite sum).

(c) Let c_n be the number of partitions of non-negative integer n where summand 2 occurs either one, two, or three times. Write down the generating function of (c_n).

(d) Let d_n be the number of partitions of non-negative n with distinct summands. Find the generating function of (d_n).

(e) Find the coefficient of x^{41} in $(x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^{10}$.

(a) $$\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^5} = \boxed{\frac{1}{(1-x)(1-x^2)(1-x^5)} = gf}$$

(b) $$f_A = 1+x+x^2+x^3+x^4 = \frac{1-x^5}{1-x} \qquad f_B = 1+x^2+x^4+\dots = \frac{1}{1-x^2} \qquad f_C = \frac{1}{1-x}$$

$$\boxed{gf = \frac{1-x^5}{(1-x)^2(1-x^2)}}$$

(c) $$2 \to x^{2\cdot 1} + x^{2\cdot 2} + x^{2\cdot 3} = \frac{1-x^{2\cdot 4}}{1-x^2} - 1 = \frac{1-x^8-1+x^2}{1-x^2} = \frac{x^2(1-x^6)}{1-x^2}$$

$$\boxed{gf = x^2(1-x^6)\prod_{n=1}^{\infty}\frac{1}{1-x^n}}$$

(d) $$1 \to x^{1\cdot 0} + x^{1\cdot 1} = 1+x$$
$$2 \to x^{2\cdot 0} + x^{2\cdot 1} = 1+x^2$$
$$\vdots$$
$$\to \boxed{gf = \prod_{n=1}^{\infty}(1+x^n)}$$

(e) $$\frac{1-x^9}{1-x} - x^2 - x - 1 = \frac{1-x^9-x^2(1-x)-x(1-x)-1(1-x)}{1-x} = \frac{\cancel{1-x^9-x^2+x^3-x+x^2-1+x}}{1-x} = \frac{x^3-x^9}{1-x} = \frac{x^3(1-x^6)}{1-x}$$

$$\left(\frac{x^3(1-x^6)}{(1-x)}\right)^{10} = \frac{x^{30}(1-x^6)^{10}}{(1-x)^{10}}$$

$$\text{coef}(x^{41}) = \text{coef}(x^{11}) \text{ in } \frac{(1-x^6)^{10}}{(1-x)^{10}} \to \binom{9+n}{n}x^n - \binom{9+n}{n}\binom{10}{1}x^{6+n} + \text{H.O.T} \quad \boxed{\binom{20}{9} - \binom{14}{5}\binom{10}{1}}$$

$$\binom{20}{9} - \binom{14}{5}\binom{10}{1}$$

6 Q6 **15 / 20**

✓ **+ 6 pts** *(a) correct*

✓ **+ 3 pts** *(c) correct*

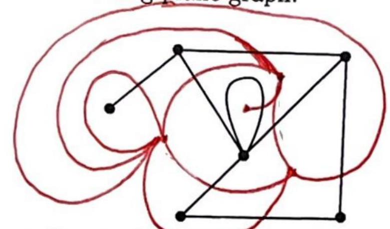
✓ **+ 3 pts** *(d) correct*

✓ **+ 3 pts** *(e) correct*

(20 points)

Question 7.

(a) Draw the dual multigraph of the following plane graph.



(Superimpose the dual graph on the graph, using red color for the dual.)

(b) The **degrees of faces** of a connected plane multigraph G, listed in decreasing order, are $6, 6, 5, 5, x, 4, 4, 4, 4, 3$, where x is unknown. Find the number of faces, the number of edges, and the number of vertices of G.

(c) A tree T has vertices of degrees 1, 2, or 5 only. Suppose T has 302 leaves. How many vertices of degree 5 does T have? With these data, can you determine the number of vertices of the tree? (Clearly justify your answer. Answer alone has no credit)

(b) $\sum \deg(\text{faces}) = 2|E| \rightarrow x = 5$

$6+6+5+5+5+4+4+4+4+3 = 2|E|$

$12+15+16+3 = 2|E|$

$27+19 = 2|E|$

$46 = 2|E|$

$E = 23$

$v - e + f = 2$

$v - 23 + 10 = 2$

$v = 13 + 2 = 15$

$\boxed{F = 10 \quad V = 15 \quad E = 23}$

(c) $d_1 + d_2 + d_5 = e + 1$

$d_1 + 2d_2 + 5d_5 = 2e$

$d_1 - 3d_5 = 2$

$302 - 3d_5 = 2$

$3d_5 = 300$

$d_5 = 100$

$d_5 = 100$

You cannot find the number of vertices in the tree because we are unable to determine the number of vertices with degree 2 using what is given

7 Q7 **19 / 20**

✓ **+ 20 pts** *Correct*

- 1 *Point adjustment*

1 Why? -1 here

Question 8. (20 points. **Show work!**) Recall that the hypercube Q_4 is the graph (V, E) where V is the set of all binary sequences of length 4, and two vertices are adjacent if and only if they differ in exactly one position. It is known that Q_4 has 32 edges.

Order the vertices of Q_4 so that the first 5 vertices are

$$v_1 = (0,0,0,0), \qquad v_2 = (0,0,1,0), \qquad v_3 = (1,1,0,0), \qquad v_4 = (0,1,1,0), \qquad v_5 = (1,1,1,1)$$

(a) Let A be the adjacency matrix of Q_4 with respect to this order. Find $A_{2,4}$, $A_{1,5}$, and $(A^4)_{1,5}$. Clearly explain how you get the answers.

(b) Draw the subgraph induced on $\{v_1, v_2, v_3, v_4\}$, clearly marking the vertices (i.e. which one is v_1, which one is v_2, etc).

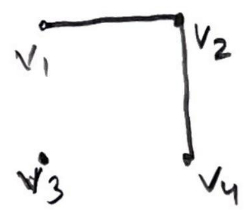
(c) Let G be the result of removing 5 edges from the complete graph K_{10}. Using Ore's theorem show that G has a Hamiltonian cycle.

(a) $A_{2,4} = 1 \rightarrow$ they are adjacent since the 2^{nd} value can be flipped to go from one to another $[(0,\underline{0},1,0) \rightarrow (0,\underline{1},1,0)]$

$A_{1,5} = 0 \rightarrow$ there is no way to go from v_1 to v_5 in one step since the shortest path is 4.

$(A^4)_{1,5} = 24 \rightarrow$ need to flip every value in v_1, so there is $4!$ ways to change $(0,0,0,0)$ to $(1,1,1,1)$

(b)



(c) Consider edge ab, such that it is removed. There are also four other edges, which we can consider removed. Outside of this, there are atleast $8+8 = 16$ more edges in the graph. Since a is connected to 9 vertices, but subtract one since ab is removed (and same thing for b). From this 16, there are atleast $6+6 = 12$ that are not removed. considering $\deg(a) + \deg(b) \geq 12 > 10$, we can say this graph has a Hamiltonian cycle by Ore's theorem.

8 Q8 **20 / 20**

✓ **+ 20 pts** *Correct*

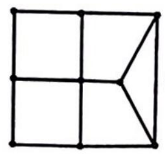
Question 9. (20 points. **Show your work. Answers alone have no credit.**)

(a) Assume the chromatic polynomial of a loop-free graph G is

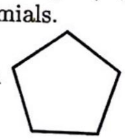
$$P(G,k) = (k-1)(k-2)(k-3)(k-4)[(k-1)^5+1]$$

Find the number of vertices, the chromatic number, and the number of 5-colorings of G. (For the number of $\cancel{6}$5-coloring, the answer can be of the form similar to $4*6*(2+5^6)$.)

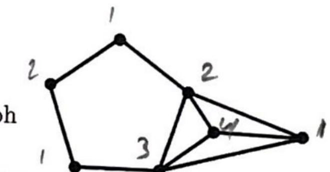
(b) In order to calculate the chromatic polynomial of the following graph G, one first removes the edge e to get graph G_e and then identify the two vertices of e to get graph G'_e.



Draw G_e and G'_e, and write down a formula expressing $P(G,k)$ in terms of $P(G_e,k)$ and $P(G'_e,k)$. You don't have to calculate the chromatic polynomials.

(c) Given: the chromatic polynomial of the pentagon  is $k(k-1)(k-2)(k^3-4k^2+6k-4)$.

Calculate the chromatic polynomial AND the clique number of the graph

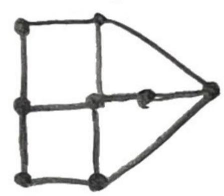


(a) $|V| = 8$ because $P(G,k)$ was found with Ge and Ge', so the degree is 1 more than the number of vertices.

$\chi = 5$ because that is the smallest value that makes $P(G,k)$ greater than 0

$P(G,5) = 4\cdot3\cdot2\cdot1(4^5+1) = 24(4^5+1)$

(b)



$P(G,K) = P(G_e,k) - P(G_e',k)$

(c)

P[] P[]

$$\frac{P[\;]\,P[\;]}{k^{(2)}} = \frac{\cancel{k(k-1)}(k-2)(k^3-4k^2+6k-4)\cdot k(k-1)(k-2)(k-3)}{\cancel{k(k-1)}}$$

$\boxed{P(G,k) = k(k-1)(k-2)^2(k-3)(k^3-4k^2+6k-4)}$

Clique # = 4

9 Q9 **20 / 20**

(a)

✓ **+ 5 pts** *two of three/ or other partial*

✓ **+ 2 pts** *partial*

(b)

✓ **+ 4 pts** *correct*

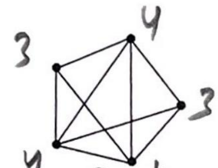
(c)

✓ **+ 7 pts** *No proof/why clique number is 4/ or other partial*

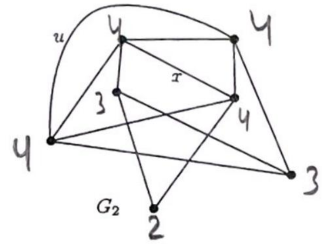
+ 2 *Point adjustment*

Question 10. (20 points.) **Show work! Answers alone have no credit.**)

Recall that $K_{n,m}$ is the complete bipartite graph with n vertices of group 1 and m vertices of group 2. Let G_1 and G_2 be the graphs below.



G_1



G_2

(a) Which of the graphs $K_{2,6}, G_1, G_2$ have an Euler circuit? Euler trail?

(b) Which of the graphs $K_{2,6}, G_1, G_2$ are planar? In each case either exhibit a planar embedding or a subgraph homeomorphic to K_5 or $K_{3,3}$. (Hint for G_2: What happens if you remove edges u and x?)

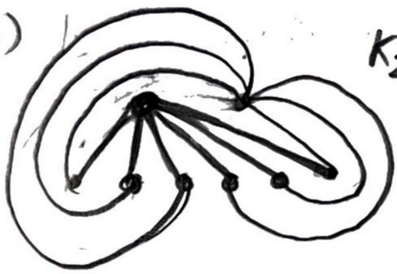
(c) How many are there collections A of edges of the complete graph K_{10} having the following property: If one removes A from K_{10} then one gets a tree.

(a) $K_{2,6}$ → Euler Circuit: Yes because all vertices have even degree
Euler Trail: No because there aren't two vertices with odd degree

G_1 → Euler Circuit: No because not all vertices have even degree
Euler Trail: Yes because two vertices have odd degree

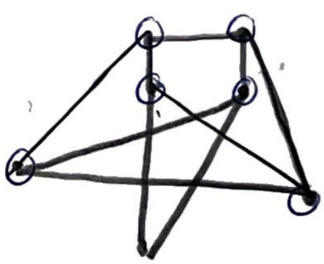
G_2 → Euler Circuit: No because not all vertices have even degree
Euler Trail: Yes because two vertices have odd degree

(b)



$K_{2,6}$ is planar

G_1 is planar



Homeomorphic to $K_{3,3}$ since there are 6 vertices with degree 3 in this graph (removing u and x)

(c) $e - v + 1 \rightarrow \binom{10}{2} - 10 + 1 = \frac{10 \times 9}{2} - 10 + 1 = 45 - 10 + 1 = 36$

$\binom{45}{36} = \boxed{\binom{45}{9}}$

10 Q10 **20 / 20**

(a)

✓ **+ 9 pts** *correct*

(b)

✓ **+ 8 pts** *correct*

(c)

✓ **+ 1 pts** *partial*

+ 2 *Point adjustment*

Vidit Pokharna

MATH 3012D S2023 Final Exam

- Please have your id on the desk.
- You need to spell out the formulas $P(n,k), d_n$ (derangement number), $Surj(n,j)$, etc. For example, instead of $P(10,4)$ you should write $\frac{10!}{6!}$ in the final answer.
  The only exception is the combination number $\binom{n}{k}$. However, when you are asked for the numerical value, then instead of, say, $\binom{7}{3}$ you should write the numerical value, which is 35. You will not be asked to do complicated arithmetics.
- No books, calculators/computers are permitted. **A two-sided standard-sized formula sheet is allowed.** Print your name on the formula sheet. The formula sheet can contain only formulas (but, say, not solutions to problems of review sheets). Scratch papers will be provided. All you can have on your desk: pens (pencils, black and red), your ID (don't forget!), and formula sheet.
- Work on papers, scan and submit to Gradescope, then turn in your exam booklet together with the formula sheet. 15% penalty for not matching pages with Questions when submitting to Gradescope.
  The last 5 minutes are for scanning and uploading your work. Have your cellphone fully charged and practice color scanning and uploading beforehand. Anybody who writes on exam papers during the scanning period will get a 0 for the exam/quiz.
- Please refrain from disturbing your fellow students. If you finish early and want to submit early: please quietly collect your belongings, come to the instructor's desk, scan and upload your work.
- Box you answers except for proof questions.
- **Strict: Work in detail appropriate for this course level must be shown in order to earn full or partial credit for a problem. Exceptions in which no work need be shown are identified as such. Answers without sufficient supporting work will receive NO CREDIT! Follow instruction**
- Strict: No redundant solutions. Cross out anything not relevant to the solutions. If you give two answers/solutions to the same questions, the worse one will be graded. A wrong statement/formula in a correct solution might get penalized.
- All students are expected to comply with the Georgia Tech Honor Code. Any evidence of cheating or other violations of the Georgia Tech Honor Code will be submitted directly to the Dean of Students.
- You need a black pen/pencil and a red pen/pencil (to draw paths, cycles, dual graphs etc).

Some useful formulas:

(1) $$\prod_{i=1}^{10}\frac{1}{1-x^i} = 1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+30x^9+42x^{10}+\dots$$

(2) $$\prod_{i=1}^{10}\frac{1}{1-x^{2i-1}} = 1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+\dots$$

(3) $$\prod_{i=1}^{10}\frac{1}{1-x^{2i}} = 1+x^2+2x^4+3x^6+5x^8+7x^{10}+\dots$$