Math 6635 Homework #1

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February 17, 2025

1. (a)

$$
\begin{aligned}
f(S) &= AS \\
df(S) &= Ads \\
df(S) &= A(\sigma S dX + \mu S dt) \\
df(S) &= A\sigma S dX + A\mu S dt
\end{aligned}
$$

(b)

$$
\begin{aligned}
f(S) &= S^n \\
df(S) &= f'(S)ds + \frac{1}{2}f''(S)(dS)^2 \\
f'(S) &= nS^{n-1} \\
f''(S) &= n(n-1)S^{n-2} \\
df(S) &= nS^{n-1}(\sigma S dX + \mu S dt) + \frac{1}{2}n(n-1)S^{n-2}(\sigma^2 S^2 dt) \\
df(S) &= nS^n \sigma dX + nS^n \mu dt + \frac{1}{2}n(n-1)S^n\sigma^2 dt \\
df(S) &= nS^n \sigma dX + \left(n\mu + \frac{1}{2}n(n-1)\sigma^2\right) S^n dt
\end{aligned}
$$

2.

$$Y = ln(S(t))$$
$$dY = \frac{\partial(\ln(S))}{\partial S}dS + \frac{1}{2}\frac{\partial^2(\ln(S))}{\partial S^2}(dS)^2$$
$$\frac{\partial(\ln(S))}{\partial S} = \frac{1}{S}, \quad \frac{\partial^2(\ln(S))}{\partial S^2} = \frac{-1}{S^2}$$
$$dY = \frac{1}{S}(\sigma S dX + \mu S dt) - \frac{1}{2S^2}(\mu^2 S^2 dt)$$
$$dY = \mu dt + \sigma dX - \frac{1}{2}\sigma^2 dt$$
$$dY = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dX$$

$$Y(t) = \ln(S_0) + (\mu - \frac{1}{2}\sigma^2)dt + \sigma X(t), \quad X(t) \sim N(0,t)$$
$$S(t) = e^{\ln(S_0) + (\mu - \frac{1}{2}\sigma^2)dt + \sigma X(t)}$$
$$S(t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)dt + \sigma X(t)}$$

$$Y(t) = \ln(S(t)) \sim N\left(\ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right)$$
$$S(t) \sim \text{Lognormal}\left(\ln(S_0) + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right)$$
$$f_{S(t)}(s) = \frac{1}{s\sigma\sqrt{2\pi t}} e^{\left(\frac{\left(\ln s - \ln S_0 - (\mu - \frac{1}{2}\sigma^2)t\right)^2}{2\sigma^2 t}\right)}$$

3.

$$
\begin{aligned}
dS_i &= \sigma_i S_i \, dX_i + \mu_i S_i \, dt, \quad i = 1, 2, \ldots, n \\
df &= \sum_{i=1}^{n} \frac{\partial f}{\partial S_i} dS_i + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial S_i^2} (dS_i)^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{\partial^2 f}{\partial S_i \partial S_j} dS_i dS_j \\
(dS_i)^2 &= (\sigma_i S_i)^2 dt = \sigma_i^2 S_i^2 \, dt \\
dS_i dS_j &= \sigma_i S_i \, \sigma_j S_j \, dX_i dX_j = \sigma_i S_i \, \sigma_j S_j \, \rho_{ij} dt
\end{aligned}
$$

(a) First-order terms:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial S_i} dS_i = \sum_{i=1}^{n} \frac{\partial f}{\partial S_i} \left(\sigma_i S_i \, dX_i + \mu_i S_i \, dt \right)$$

(b) Second-order terms for $(dS_i)^2$:

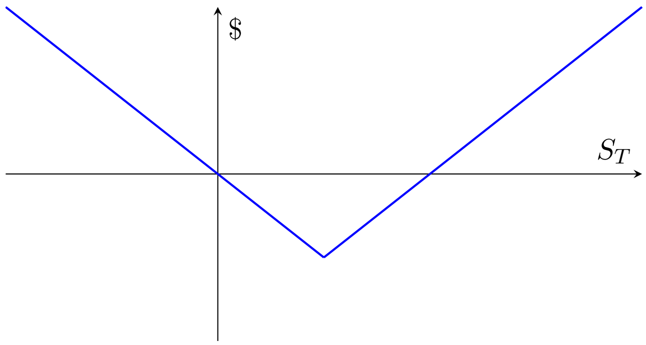
$$\frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial S_i^2} (dS_i)^2 = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial S_i^2} \sigma_i^2 S_i^2 \, dt$$

(c) Cross terms $dS_i dS_j$:

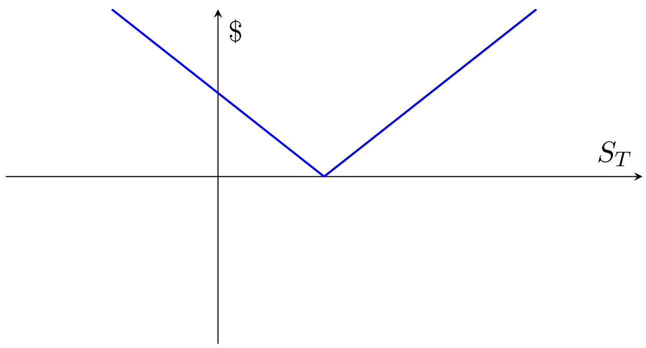
$$\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{\partial^2 f}{\partial S_i \partial S_j} dS_i dS_j = \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{\partial^2 f}{\partial S_i \partial S_j} \sigma_i S_i \, \sigma_j S_j \, \rho_{ij} dt$$

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial S_i} \left(\sigma_i S_i \, dX_i + \mu_i S_i \, dt \right) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial S_i^2} \sigma_i^2 S_i^2 \, dt + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{\partial^2 f}{\partial S_i \partial S_j} \sigma_i S_i \, \sigma_j S_j \, \rho_{ij} dt$$

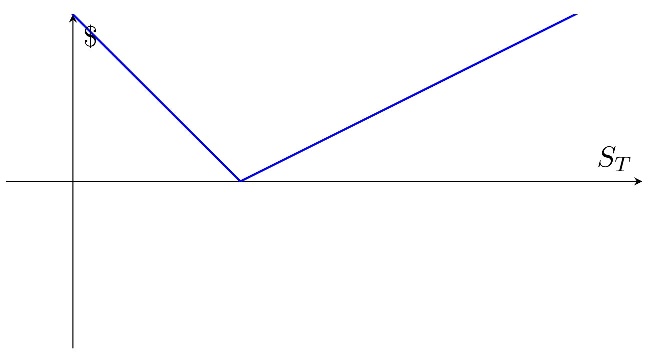
4. Assuming no premiums:



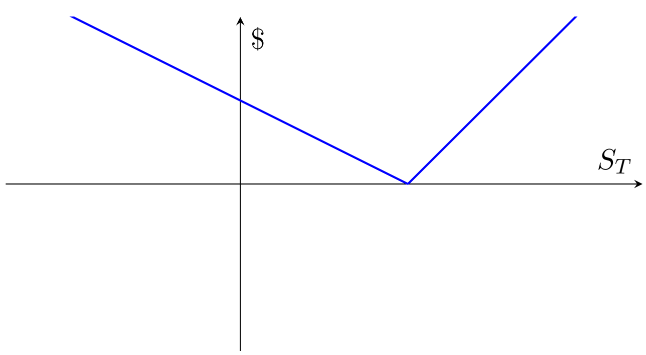
(a)



(b)

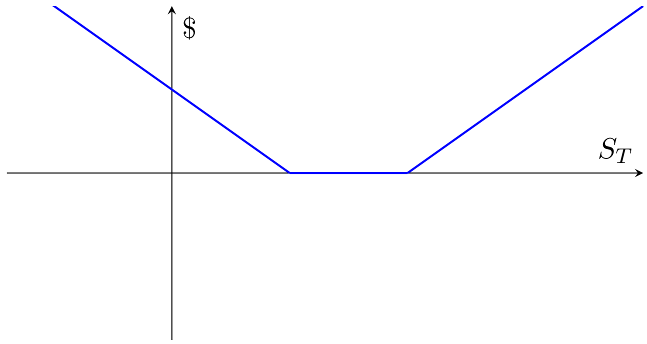


(c)

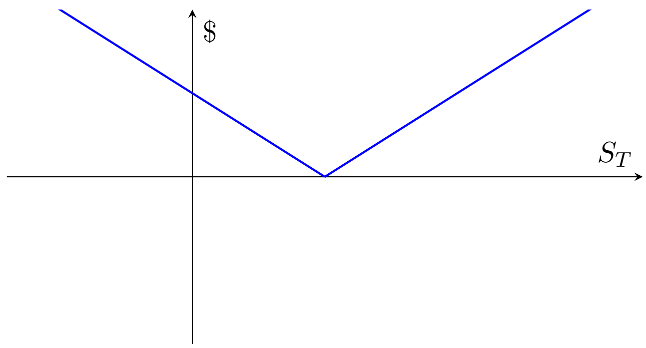


(d)

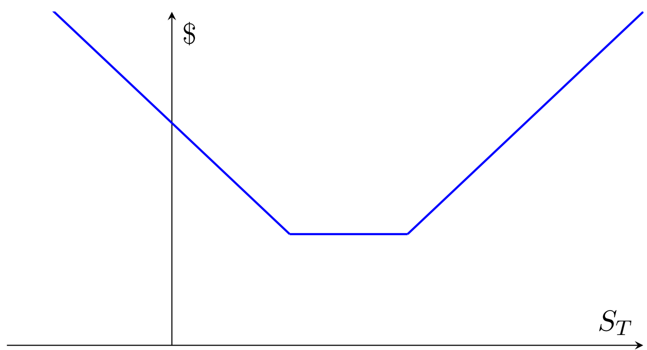
(e) i.



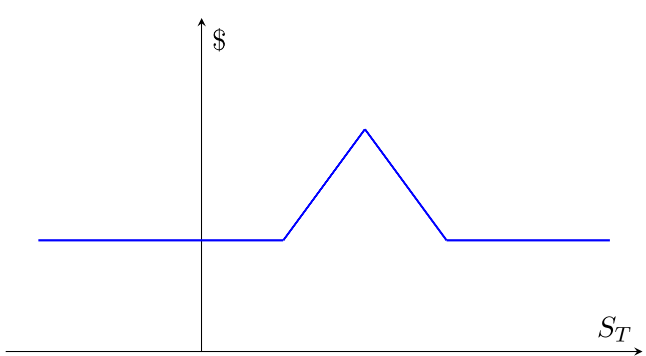
ii.



iii.



(f)



5. (a) To prove $C \leq S$, consider a European call option with a payoff at expiry of $\max(S_T - K, 0)$, where S_T is the stock price at expiry and K is the strike price. The maximum payoff occurs when $K = 0$, giving a value of S_T. Thus, the call option's value at any time cannot exceed the stock price. If $C > S$, an arbitrage opportunity arises by selling the overpriced call for C, buying the stock for S, and locking in a risk-free profit. Therefore, the no-arbitrage condition implies $C \leq S$.

   (b) To prove $C \geq S - Ke^{-r(T-t)}$, consider the no-arbitrage principle. At expiry, the call option payoff is $\max(S_T - K, 0)$. Before expiry, the call price C must reflect the time value of money and cannot be less than the intrinsic value $S - Ke^{-r(T-t)}$. If $C < S - Ke^{-r(T-t)}$, an arbitrage opportunity arises by buying the call for C, shorting the stock for S, and investing $Ke^{-r(T-t)}$ at the risk-free rate to cover the strike price at maturity. This strategy locks in a risk-free profit. Hence, the no-arbitrage condition implies $C \geq S - Ke^{-r(T-t)}$.

   (c) To prove $0 \leq C(S, t; K_1) - C(S, t; K_2) \leq K_2 - K_1$ for $K_1 < K_2$, consider two European call options differing only in strike prices K_1 and K_2. The call with strike K_1 always has a higher intrinsic value than the one with K_2, as $\max(S_T - K_1, 0) \geq \max(S_T - K_2, 0)$, ensuring $C(S, t; K_1) \geq C(S, t; K_2)$. The maximum difference in payoff is $(K_2 - K_1)$, which occurs if $S_T \geq K_2$. Thus, the no-arbitrage condition implies $0 \leq C(S, t; K_1) - C(S, t; K_2) \leq K_2 - K_1$.

   (d) To prove $C(S, t; T_1) \leq C(S, t; T_2)$ for $T_1 < T_2$, consider two European call options with the same strike price K but different expiry times T_1 and T_2. Assume $C(S, t; T_1) > C(S, t; T_2)$. A trader could long the call with T_2 and short the call with T_1. At time T_1, the shorter option expires, leaving the trader with the longer option. This creates a risk-free profit, violating the no-arbitrage principle. Thus, the value of the call with T_2 must be greater than or equal to the one with T_1, ensuring $C(S, t; T_1) \leq C(S, t; T_2)$.

6. To determine the value of the call option C, we set up a hedged portfolio consisting of δ shares of the stock and one short call option. The portfolio is constructed to be risk-neutral, meaning its value remains the same regardless of whether the stock price goes up or down. The stock price today is $S = 100$, and tomorrow it will be either 101 with probability p or 99 with probability $1 - p$. The call option has a strike price $K = 100$ and expires tomorrow.

   The portfolio value at expiry is:

$$V_T = \delta S_T - \text{Call Payoff}$$

   For the two possible values of S_T, to eliminate risk:

$$\delta = \frac{dV}{dS} = \frac{C_{\text{up}} - C_{\text{down}}}{S_{\text{up}} - S_{\text{down}}}$$

   Solve for δ:

$$\delta = \frac{1 - 0}{101 - 99} = 0.5$$

   At expiry, the portfolio value is the expected value of V_T, given the probabilities p and $1 - p$:

$$\begin{aligned} V &= p \cdot (\delta \cdot 101 - 1) + (1 - p) \cdot (\delta \cdot 99) \\ 0.5 \cdot 100 - C &= p \cdot (0.5 \cdot 101 - 1) + (1 - p) \cdot (0.5 \cdot 99) \\ 50 - C &= p \cdot (50.5 - 1) + (1 - p) \cdot 49.5 \\ 50 - C &= p \cdot 49.5 + (1 - p) \cdot 49.5 \\ 50 - C &= 49.5 \\ C &= 0.5 \end{aligned}$$

   The value of the call option is: $\boxed{C = 0.5}$

7. (a)

$$C = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where:

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\frac{\partial C}{\partial \sigma} = SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T-t}.$$

As $N'(d) > 0$ and $\frac{\partial d_1}{\partial \sigma} > \frac{\partial d_2}{\partial \sigma} > 0$:

$$\frac{\partial C}{\partial \sigma} > 0 \quad \text{for any } t < T$$

(b)

$$C = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

$$\frac{\partial C}{\partial S} = \frac{\partial}{\partial S}\left(SN(d_1) - Ke^{-r(T-t)}N(d_2)\right)$$

$$\frac{\partial C}{\partial S} = N(d_1)$$

8.

$$
\begin{aligned}
C &= Se^{-qT}N(d_1) - Ee^{-rT}N(d_2) \\
P &= Ee^{-rT}N(-d_2) - Se^{-qT}N(-d_1) \\
C - P &= \left[Se^{-qT}N(d_1) - Ee^{-rT}N(d_2)\right] - \left[Ee^{-rT}N(-d_2) - Se^{-qT}N(-d_1)\right] \\
C - P &= \left[Se^{-qT}N(d_1) + Se^{-qT}N(-d_1)\right] - \left[Ee^{-rT}N(-d_2) + Ee^{-rT}N(d_2)\right] \\
C - P &= Se^{-qT}(N(d_1) + N(-d_1)) - Ee^{-rT}(N(d_2) + N(-d_2)) \\
C - P &= Se^{-qT} - Ee^{-rT}
\end{aligned}
$$

9.

$$
\begin{aligned}
C &= Se^{-qT}N(d_1) - Ee^{-rT}N(d_2) \\
\Delta_C &= \frac{\partial C}{\partial S} = e^{-qT}N(d_1)
\end{aligned}
$$

10.

Put Option Valuation for Continuous Dividend Yield: $P = Ee^{-rT}N(-d_2) - Se^{-qT}N(-d_1)$

$d_1 = \frac{\ln(S/E)+(r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$

Put Option Valuation for Discrete Dividend Yield: $P = Ee^{-rT}N(-d_2) - (S - D)N(-d_1)$

The standard put-call parity equation $C - P = Se^{-qT} - Ee^{-rT}$ must be modified in the discrete dividend case because the stock price will drop by D:

$$C - P = (S - D) - Ee^{-rT}$$

Both continuous and discrete dividends increase the value of a put option. This is because dividends lower the future expected stock price, making put options (which benefit from falling stock prices) more valuable. A stock price decrease directly benefits the put option holder, as puts gain intrinsic value when the underlying asset price declines. For continuous dividends, the effect is gradual and spreads over time. For discrete dividends, the effect is immediate when the dividend is paid.