ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 1

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Problem 1

(a) The marginal distributions are:

$$\mathbb{P}(X=1) = 0.20 + 0.40 = 0.60, \quad \mathbb{P}(X=0) = 0.15 + 0.25 = 0.40$$
$$\mathbb{P}(Y=1) = 0.20 + 0.15 = 0.35, \quad \mathbb{P}(Y=0) = 0.4 + 0.25 = 0.65$$

(b)

$$\mathbb{P}(X=1) \cdot \mathbb{P}(Y=1) = 0.60 \times 0.35 = 0.21 \quad \neq \quad \mathbb{P}(X=1, Y=1) = 0.20$$

Thus, X and Y are not independent.

(c) Expectations of X and Y:

$$\mathbb{E}[X] = 1 \cdot \mathbb{P}(X=1) + 0 \cdot \mathbb{P}(X=0) = 0.60$$
$$\mathbb{E}[Y] = 1 \cdot \mathbb{P}(Y=1) + 0 \cdot \mathbb{P}(Y=0) = 0.35$$

(d) Conditional distribution $\mathbb{P}(X \mid Y=1)$:

$$\mathbb{P}(X=1 \mid Y=1) = \frac{\mathbb{P}(X=1, Y=1)}{\mathbb{P}(Y=1)} = \frac{0.20}{0.35} = \frac{4}{7} \approx 0.5714$$

$$\mathbb{P}(X=0 \mid Y=1) = \frac{\mathbb{P}(X=0, Y=1)}{\mathbb{P}(Y=1)} = \frac{0.15}{0.35} = \frac{3}{7} \approx 0.4286$$

(e)

$$\mathbb{E}[X \mid Y=1] = 1 \cdot \mathbb{P}(X=1 \mid Y=1) + 0 \cdot \mathbb{P}(X=0 \mid Y=1) = \frac{4}{7} \approx 0.5714$$

$$\begin{aligned}
\mathrm{Var}(X \mid Y=1) &= \sum_{x \in \{0,1\}} (x - \tfrac{4}{7})^2 \, P(X=x \mid Y=1) \\
&= (0 - \tfrac{4}{7})^2 \cdot \tfrac{3}{7} + (1 - \tfrac{4}{7})^2 \cdot \tfrac{4}{7} \\
&= \tfrac{16}{49} \cdot \tfrac{3}{7} + \tfrac{9}{49} \cdot \tfrac{4}{7} \\
&= \tfrac{48}{343} + \tfrac{36}{343} = \tfrac{84}{343} \\
&= \tfrac{12}{49} \approx 0.2449
\end{aligned}$$

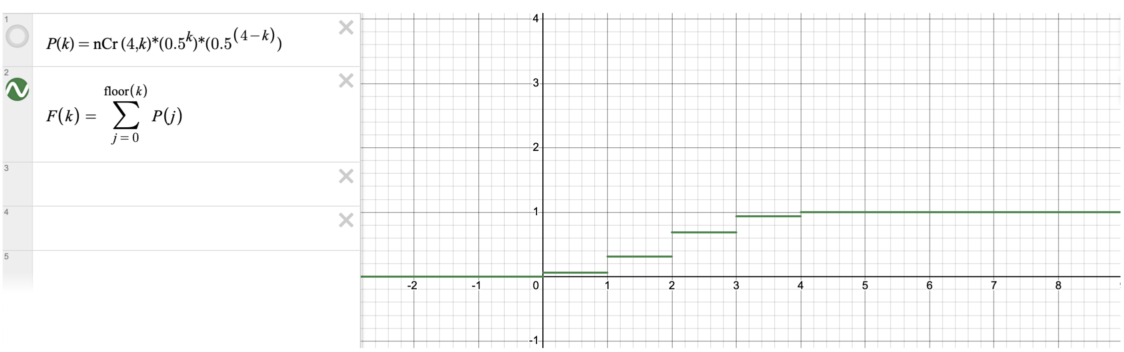
Problem 2

(a)

$$\mathbb{P}(X_4 > k) = 1 - \mathbb{P}(X_4 \leq k) = 1 - \sum_{j=0}^{k} \binom{4}{j} p^j (1-p)^{4-j}$$

$$\begin{aligned}
&\mathbb{P}(X_4 > 0) = 1 - (1-p)^4 \\
&\mathbb{P}(X_4 > 1) = 1 - \left[(1-p)^4 + 4p(1-p)^3\right] \\
&\mathbb{P}(X_4 > 2) = 1 - \left[(1-p)^4 + 4p(1-p)^3 + 6p^2(1-p)^2\right] \\
&\mathbb{P}(X_4 > 4) = 0
\end{aligned}$$

$$\begin{aligned}
&\mathbb{P}(X_4 > 0) = \frac{15}{16} = 0.9375 \\
&\mathbb{P}(X_4 > 1) = \frac{11}{16} = 0.6875 \\
&\mathbb{P}(X_4 > 2) = \frac{5}{16} = 0.3125 \\
&\mathbb{P}(X_4 > 4) = 0
\end{aligned}$$



$$F_{X_4}(k) = \mathbb{P}(X_4 \leq k) = \begin{cases} 0, & k < 0 \\ \frac{1}{16}, & 0 \leq k < 1 \\ \frac{5}{16}, & 1 \leq k < 2 \\ \frac{11}{16}, & 2 \leq k < 3 \\ \frac{15}{16}, & 3 \leq k < 4 \\ 1, & k \geq 4 \end{cases}$$

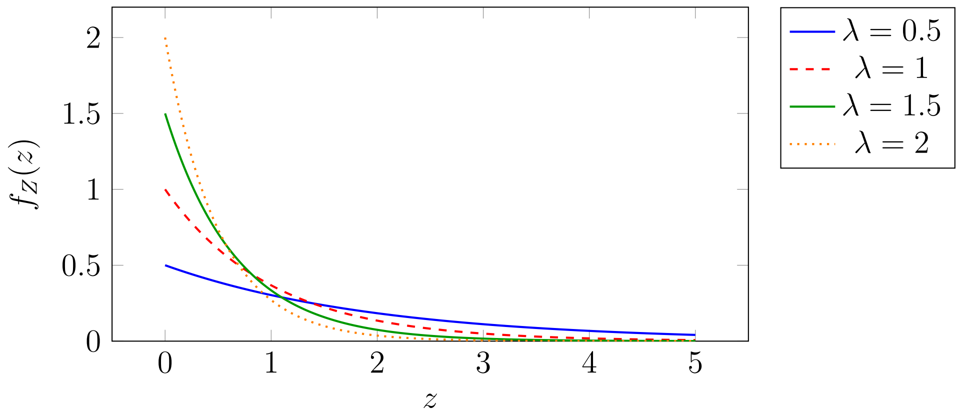
(b) For $n = 3$: $\mathbb{E}[X_3] = 3p$ and $\mathbb{V}(X_3) = 3p(1-p)$

When $p = \frac{1}{2}$: $\mathbb{E}[X_3] = \frac{3}{2}$ and $\mathbb{V}(X_3) = \frac{3}{4}$

Problem 3

(a)

$$f_Z(z) = F_Z'(z) = \begin{cases} \lambda e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$



(b)

$$\mathbb{E}[Z] = \int_0^\infty \mathbb{P}(Z > z)\, dz = \int_0^\infty e^{-\lambda z}\, dz = (-\frac{1}{\lambda}) \cdot e^{-\lambda z}\Big|_0^\infty = \frac{1}{\lambda}$$

(c)

$$\mathbb{V}(Z) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

(d) Let $Z_1, Z_2 \overset{\text{iid}}{\sim} \exp(\lambda)$ and $S = Z_1 + Z_2$. By convolution,

$$f_S(s) = \begin{cases} \lambda e^{-\lambda x}\, \lambda e^{-\lambda(s-x)} \;\; = \lambda^2 s\, e^{-\lambda s} & s \geq 0 \\ 0 & s < 0 \end{cases}$$

$$F_S(s) = \mathbb{P}(S \leq s) = \begin{cases} 0 & s < 0 \\ \displaystyle\int_0^s \lambda^2 u\, e^{-\lambda u}\, du & s \geq 0 \end{cases}$$

$$= \begin{cases} 0 & s < 0 \\ \Big[-\lambda u e^{-\lambda u}\Big]_0^s + \displaystyle\int_0^s \lambda e^{-\lambda u}\, du & s \geq 0 \end{cases}$$

$$= \begin{cases} 0 & s < 0 \\ -\lambda s e^{-\lambda s} + \Big[-e^{-\lambda u}\Big]_0^s & s \geq 0 \end{cases}$$

$$= \begin{cases} 0 & s < 0 \\ 1 - e^{-\lambda s}(1 + \lambda s) & s \geq 0 \end{cases}$$

(e)

$$\mathbb{E}[S] = \mathbb{E}\left[\sum_{i=1}^{m} Z_i\right] = \sum_{i=1}^{m} \mathbb{E}[Z_i] = \mathbb{E}[Z_1] + \mathbb{E}[Z_2] = \frac{2}{\lambda}$$

$$\mathbb{V}(S) = \mathbb{V}\left(\sum_{i=1}^{m} Z_i\right) = \sum_{i=1}^{m} \mathbb{V}(Z_i) = \mathbb{V}[Z_1] + \mathbb{V}[Z_2] = \frac{2}{\lambda^2}$$

Problem 4

(a)

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$e^{\lambda} \cdot e^{-\lambda} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$1 = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=0}^{\infty} p(k)$$

(b)

$$\begin{aligned}\mathbb{E}[Z] &= \sum_{k=0}^{\infty} kp(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{k \cdot \lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{k \cdot \lambda^k}{k!} = e^{-\lambda}\lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\ &= e^{-\lambda}\lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda}\lambda e^{\lambda} = \lambda\end{aligned}$$

$$\begin{aligned}\mathbb{E}[Z^2] &= \sum_{k=0}^{\infty} k^2 p(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{k^2 \cdot \lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{[k(k-1)+k] \cdot \lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=1}^{\infty} \frac{k(k-1) \cdot \lambda^k}{k!} + e^{-\lambda} \sum_{k=1}^{\infty} \frac{k \cdot \lambda^k}{k!} = e^{-\lambda}\lambda^2 \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} + e^{-\lambda}\lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\ &= e^{-\lambda}\lambda^2 e^{\lambda} + e^{-\lambda}\lambda e^{\lambda} = \lambda^2 + \lambda\end{aligned}$$

$$\mathbb{V}[Z] = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Problem 5

(a)

- $\mathbb{P}(coin = HH, down = H) = \frac{2}{5} \cdot 1 = \frac{2}{5}$
- $\mathbb{P}(coin = TT, down = H) = \frac{1}{5} \cdot 0 = 0$
- $\mathbb{P}(coin = HT, down = H) = \frac{2}{5} \cdot \frac{1}{2} = \frac{1}{5}$

$$\mathbb{P}(down = H) = \frac{2}{5} + 0 + \frac{1}{5} = \frac{3}{5}$$

(b)

- $\mathbb{P}(up = H \mid coin = HH) = 1$
- $\mathbb{P}(up = H \mid coin = TT) = 0$
- $\mathbb{P}(up = H \mid coin = HT) = \frac{1}{2}$

$$\begin{aligned}\mathbb{P}(up = H) &= \mathbb{P}(coin = HH) \cdot \mathbb{P}(up = H \mid coin = HH)\\ &+ \mathbb{P}(coin = TT) \cdot \mathbb{P}(up = H \mid coin = TT)\\ &+ \mathbb{P}(coin = HT) \cdot \mathbb{P}(up = H \mid coin = HT)\\ &= \frac{2}{5} \cdot 1 + \frac{1}{5} \cdot 0 + \frac{2}{5} \cdot \frac{1}{2} = \frac{3}{5}\end{aligned}$$

$$\mathbb{P}(coin = HH \mid up = H) = \frac{\mathbb{P}(coin = HH, up = H)}{\mathbb{P}(up = H)} = \frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}$$

(c)

- $\mathbb{P}(coin = HH \mid up = H) = \frac{2}{3}$
- $\mathbb{P}(coin = TT \mid up = H) = \frac{\mathbb{P}(coin=TT) \cdot \mathbb{P}(up=H|coin=TT)}{\mathbb{P}(up=H)} = \frac{\frac{1}{5} \cdot 0}{\frac{3}{5}} = 0$
- $\mathbb{P}(coin = HT \mid up = H) = \frac{\mathbb{P}(coin=HT) \cdot \mathbb{P}(up=H|coin=HT)}{\mathbb{P}(up=H)} = \frac{\frac{2}{5} \cdot \frac{1}{2}}{\frac{3}{5}} = \frac{1}{3}$

- $\mathbb{P}(down = H \mid coin = HH) = 1$
- $\mathbb{P}(down = H \mid coin = TT) = 0$
- $\mathbb{P}(down = H \mid coin = HT) = \frac{1}{2}$

$$\mathbb{P}(down = H \ \ next\ toss \mid seen\ up = H) = \frac{2}{3} \cdot 1 + 0 \cdot 0 + \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

(d)

$$\mathbb{P}(down = H \mid up = H) = \frac{\mathbb{P}(down = H, up = H)}{\mathbb{P}(up = H)} = \frac{\mathbb{P}(coin = HH)}{\mathbb{P}(up = H)} = \frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}$$

Problem 6

(a) My reasoning begins with looking backwards from the last roll, where we can define x as the outcome of the current roll:

- If we have reached the 3rd roll, we must take the value regardless of the outcome $\Rightarrow V_3 = \mathbb{E}[x] = 3.5$
- After seeing the value of x, we can compare it to 3.5. Stop here if $x \geq 4$. The value of the roll $\Rightarrow V_2 = \mathbb{E}[\max(x, V_3)] = (\frac{1}{6})(3 \cdot 3.5 + 4 + 5 + 6) = 4.25$
- After seeing the value of x, we can compare it to 4.25. Stop here if $x \geq 5$.

So our strategy here is that after roll 1, we should stop if we get an outcome of 5 or 6. After roll 2, we should stop if we get an outcome of 4, 5 or 6.

The resulting payoff $\Rightarrow V_1 = \mathbb{E}[\max(x, V_1)] = (\frac{1}{6})(4 \cdot 4.25 + 5 + 6) = \frac{14}{3} \approx 4.67$

(b) For many repeated plays (using law of large numbers), the average payoff will approach the expectation under optimal play: $\boxed{4.67}$

(c) We can consider the three rolls X_1, X_2, X_3, and $M = \max(X_1, X_2, X_3)$. Each $X_i \sim$ uniform$\{1, \cdots, 6\}$ i.i.d.

$$\mathbb{P}(X_i \leq m) = \frac{m}{6}$$

$$\mathbb{P}(M \leq m) = \mathbb{P}(X_1 \leq m) \cdot \mathbb{P}(X_2 \leq m) \cdot \mathbb{P}(X_3 \leq m) = \left(\frac{m}{6}\right)^3$$

$$\mathbb{P}(M = m) = \mathbb{P}(M \leq m) - \mathbb{P}(M \leq m - 1) = \left(\frac{m}{6}\right)^3 - \left(\frac{m-1}{6}\right)^3 = \frac{m^3 - (m-1)^3}{6^3}$$

$$\mathbb{E}[M] = \sum_{m=1}^{6} m\,\mathbb{P}(M = m) = \frac{1}{216} \cdot (1\cdot1 + 2\cdot7 + 3\cdot19 + 4\cdot37 + 5\cdot61 + 6\cdot91) = \frac{119}{24} \approx 4.96$$

Thus, the amended game has the higher expected payoff, as $4.96 > 4.67$.

Problem 7

Let:

- C: prize is behind your chosen door
- K: prize is behind a specific other unopened door k
- E: the host opens some other door $h \neq C$ and shows it's empty

Before the host opens a door, the chosen door has the prize with probability $\frac{1}{10} = 0.1$. After the host opens one empty door (not the chosen one) and offers a switch, the probability of staying becomes:

$$\mathbb{P}(C \mid E) = \frac{\mathbb{P}(E \mid C)\,\mathbb{P}(C)}{\mathbb{P}(E)}$$

where,

- $\mathbb{P}(E \mid C) = 1/9$ (host picks one of the 9 empty doors)
- $\mathbb{P}(E \mid K) = 1/8$ for any $K \neq C$ and $k \neq h$ (host picks one of 8 empty doors)

$$\mathbb{P}(E) = \underbrace{\tfrac{1}{9} \cdot \tfrac{1}{10}}_{\text{prize at } C} + \underbrace{8 \cdot \left(\tfrac{1}{8} \cdot \tfrac{1}{10}\right)}_{\text{prize at one of the 8 remaining others}} = \tfrac{1}{90} + \tfrac{1}{10} = \tfrac{1}{9}$$

$$\mathbb{P}(C \mid E) = \frac{\frac{1}{9} \cdot \frac{1}{10}}{\frac{1}{9}} = \boxed{\tfrac{1}{10} = 0.1}$$

The probability of switching can be determined using:

$$\mathbb{P}(K \mid E) = \frac{\mathbb{P}(E \mid K)\,\mathbb{P}(K)}{\mathbb{P}(E)} = \frac{\frac{1}{8} \cdot \frac{1}{10}}{\frac{1}{9}} = \boxed{\tfrac{9}{80} \approx 0.1125}$$

So after the host opens one empty door, the probability of staying and earning the prize stays at 1/10, while each of the 8 other unopened doors has probability 9/80. Thus switching leads to a more successful outcomes, slightly better than staying with the original choice.

Problem 8

Let X and Y be the arrival times (in hours after 12 pm) of Mr. Buffet and Mr. Zhao. Assume X and Y are independent and uniformly distributed on $[0, 1]$. The waiting time of the first arriver as per the second's arrival is:

$$T = |X - Y|, \qquad 0 \leq T \leq 1$$

To find the distribution function of T, we need to find:

$$F_T(t) = \Pr(T \leq t) = \Pr(|X - Y| \leq t)$$

The pair (X, Y) is uniformly distributed over the unit square $[0, 1] \times [0, 1]$. The event $\{|X - Y| \leq t\}$ corresponds to the region within distance t of the diagonal $X = Y$. The complement $\{|X - Y| > t\}$ consists of two right triangles in opposite corners of the square, each with side length $1 - t$ and hence area $\frac{1}{2}(1 - t)^2$. The total excluded area is therefore $(1 - t)^2$. Since the area of the whole square is 1, we know:

$$F_T(t) = 1 - (1 - t)^2, \qquad 0 \leq t \leq 1$$

Thus the distribution function is

$$F_T(t) = \begin{cases} 0 & t < 0 \\ 2t - t^2 & 0 \leq t \leq 1 \\ 1 & t \geq 1 \end{cases}$$

Consequently, the pdf of T can be found by differentiation:

$$f_T(t) = F_T'(t) = 2 - 2t, \qquad 0 \leq t \leq 1$$

Problem 9

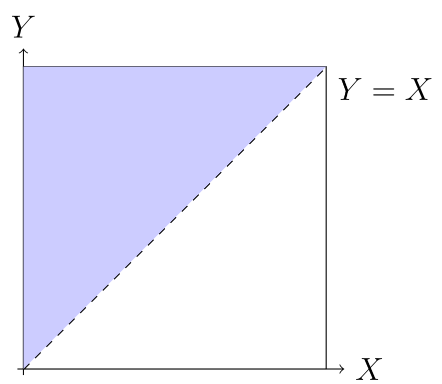
We can consider X and Y to be two points on the line from 0 to 1 that can be considered as breaking points (such that $0 < X < Y < 1$) for the three pieces A, B, C. Thus, the values of each are: $A = X$, $B = Y - X$, and $C = 1 - Y$.

To ensure a triangle can be formed from these pieces, we can determine using the logic that for three positive lengths a, b, c to form a triangle, they must satisfy: $a+b > c$, $b+c > a$, $c+a > b$. We have a unit stick, so $a + b + c = 1$. Plugging this in, we get:

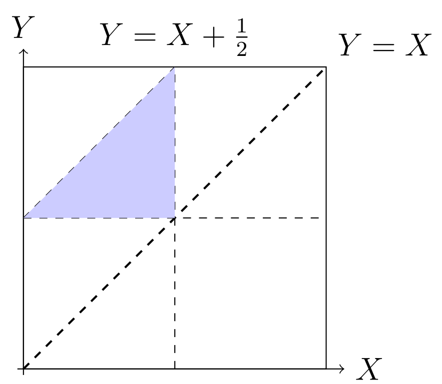
- $a + b > c \iff 1 - c > c \iff c < \frac{1}{2}$
- $b + c > a \iff 1 - a > a \iff a < \frac{1}{2}$
- $c + a > b \iff 1 - b > b \iff b < \frac{1}{2}$

Therefore, we know $A, B, C < \frac{1}{2} \iff X < \frac{1}{2}, Y - X < \frac{1}{2}, 1 - Y < \frac{1}{2}$

The total area for the overall condition $0 < X < Y < 1$ is below, with an area of $\frac{1}{2}$:



The total area for the required condition $0 < X < Y < 1$ is below, with an area of $\frac{1}{8}$:



$$\mathbb{P}(\text{triangle}) = \frac{\text{favorable area}}{\text{total area}} = \frac{\frac{1}{8}}{\frac{1}{2}} = \boxed{\frac{1}{4}}$$

Problem 10

1.

$$1 = \int_0^1 \int_0^y Cxy^2\,dx\,dy = C\int_0^1 y^2\left[\tfrac{x^2}{2}\right]_0^y dy = C\int_0^1 \tfrac{y^4}{2}\,dy = C\cdot\left[\tfrac{y^5}{10}\right]_0^1 = C\cdot\tfrac{1}{10} \;\Rightarrow\; C = 10$$

2. No. The support is the triangle $\{0 \le x \le y \le 1\}$ (not a rectangle), so $f_{X,Y}(x,y)$ cannot factor as $f_X(x)f_Y(y)$ on a product set. (e.g. $f_{X,Y}(x,y) = 10xy^2 \neq f_X(x)f_Y(y)$)

3. (a)

$$f_X(x) = \int_x^1 10xy^2\,dy = 10x\left[\tfrac{y^3}{3}\right]_x^1 = \frac{10}{3}x(1-x^3), \quad 0 \le x \le 1$$

$$\mathbb{E}[X] = \int_0^1 x f_X(x)\,dx = \frac{10}{3}\int_0^1 (x^2 - x^5)dx = \frac{10}{3}\left(\frac{1}{3} - \frac{1}{6}\right) = \boxed{\frac{5}{9}}$$

(b)

$$f_Y(y) = \int_0^y 10xy^2\,dx = 10y^2\left[\tfrac{x^2}{2}\right]_0^y = 5y^4, \quad 0 \le y \le 1$$

$$\mathbb{E}[Y] = \int_0^1 y\,f_Y(y)\,dy = 5\int_0^1 y^5 dy = \boxed{\frac{5}{6}}$$

(c)

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{10xy^2}{5y^4} = \frac{2x}{y^2}, \quad 0 \le x \le y \le 1$$

$$\mathbb{E}[X \mid Y = y] = \int_0^y x\,\frac{2x}{y^2}\,dx = \frac{2}{y^2}\left[\tfrac{x^3}{3}\right]_0^y = \boxed{\frac{2y}{3}}$$

(d)

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)} = \frac{3y^2}{1-x^3}, \quad x \le y \le 1$$

$$\mathbb{E}[Y \mid X = x] = \int_x^1 y\,\frac{3y^2}{1-x^3}\,dy = \frac{3}{1-x^3}\left[\tfrac{y^4}{4}\right]_x^1 = \boxed{\frac{3}{4}\cdot\frac{1-x^4}{1-x^3}}$$

$$= \frac{3}{4}\left(1 + \frac{x^3}{1+x+x^2}\right), \quad 0 \le x < 1$$

Problem 11

$$\begin{aligned}
\mathbb{E}[(X-c)^2] &= \mathbb{E}[X^2] - 2c\,\mathbb{E}[X] + c^2 \\
Q(c) &= c^2 - 2\,\mathbb{E}[X]\,c + \mathbb{E}[X^2] \\
\frac{dQ}{dc} &= 2c - 2\,\mathbb{E}[X] \quad \Rightarrow \quad c = \mathbb{E}[X]
\end{aligned}$$

Thus, the mean is the best constant predictor of X in the mean squared error sense.

Problem 12

(a)

$$1 = \int_0^\infty cxe^{-k^2x^2}\,dx \overset{t=kx}{=} \int_0^\infty \frac{c}{k^2}\,te^{-t^2}\,dt = \frac{c}{k^2}\cdot\frac{1}{2} \;\Rightarrow\; \boxed{c = 2k^2}$$

(b)

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty x\,f(x)\,dx = c\int_0^\infty x^2e^{-k^2x^2}dx \overset{t=kx}{=} \frac{c}{k^3}\int_0^\infty t^2e^{-t^2}dt = \frac{c}{k^3}\cdot\frac{\sqrt{\pi}}{4} = \boxed{\frac{\sqrt{\pi}}{2k}} \\
\mathbb{E}[X^2] &= \int_0^\infty x^2f(x)\,dx = c\int_0^\infty x^3e^{-k^2x^2}dx \overset{t=kx}{=} \frac{c}{k^4}\int_0^\infty t^3e^{-t^2}dt = \frac{c}{k^4}\cdot\frac{1}{2} = \frac{1}{k^2} \\
\mathbb{V}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{k^2} - \left(\frac{\sqrt{\pi}}{2k}\right)^2 = \boxed{\frac{1-\pi/4}{k^2}}
\end{aligned}$$

Problem 13

Let A's payoff be positive when A wins and negative when B wins.

- Both red: prob 1/4, A gets +3
- Both blue: prob 1/4, A gets +1
- Mismatch: prob 1/2, A gets −2

Expected payoff to A:

$$\mathbb{E}[\text{A payoff}] = \tfrac{1}{4}(3) + \tfrac{1}{4}(1) + \tfrac{1}{2}(-2) = \tfrac{3}{4} + \tfrac{1}{4} - 1 = 0$$

Thus, the game is fair under the "randomly present" rule where neither A nor B has an advantage.

Problem 14

- C_F: drew a fair coin
- C_{2H}: drew the two-headed coin
- D: observed 10 heads in 10 tosses

$$\begin{aligned} P(C_{2H} \mid D) &= \frac{P(D \mid C_{2H})P(C_{2H})}{P(D \mid C_{2H})P(C_{2H}) + P(D \mid C_F)P(C_F)} = \frac{1 \cdot \frac{1}{100}}{1 \cdot \frac{1}{100} + (\frac{1}{2})^{10} \cdot \frac{99}{100}} \\ &= \frac{1024}{1024 + 99} = \frac{1024}{1123} \approx \boxed{0.911} \end{aligned}$$

Problem 15

Each set of 4 distinct cards can appear in any of the 4! permutations with equal probability. Only one of these is strictly ascending.

$$\mathbb{P}(A < B < C < D) = \frac{1}{4!} = \frac{1}{24} \approx 0.04167$$

This does not depend on deck size (50, 100, …) as long as there are no ties.

$$\mathbb{E}[\text{payoff}] = 10 \cdot \frac{1}{24} - 1 \cdot \frac{23}{24} = \frac{10 - 23}{24} = -\frac{13}{24} \approx -\$0.54$$

So you shouldn't play regardless of the deck size. The fair prize to make expected payout \$0 would be \$23 (since $x \cdot \frac{1}{24} = \frac{23}{24} \Rightarrow x = 23$).

Problem 16

$p_A = \frac{1}{3},\ p_B = \frac{2}{3},\ p_C = 1$

If A deliberately misses, then B shoots at C. With probability $\frac{2}{3}$, B hits C and we are left with A vs B, with A shooting first. In this duel, A's survival probability is

$$S_{AB} = p_A + (1 - p_A)(1 - p_B)S_{AB} = \tfrac{1}{3} + \tfrac{2}{9}S_{AB} \quad \Rightarrow \quad S_{AB} = \tfrac{3}{7}$$

Thus in this branch, A's chance is $\frac{2}{3} \cdot \frac{3}{7} = \frac{2}{7}$.

If B misses C (probability $\frac{1}{3}$), then C kills B and A faces C with A shooting first, so A's chance is $\frac{1}{3}$. This branch contributes $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

$$P_{\text{miss}} = \tfrac{2}{7} + \tfrac{1}{9} = \tfrac{25}{63} \approx 0.397$$

If A shoots at C: with probability $\frac{1}{3}$ he hits, leaving A vs B but B shoots next, so A's chance is $(1 - p_B)S_{AB} = \frac{1}{3} \cdot \frac{3}{7} = \frac{1}{7}$. If A misses C (prob $\frac{2}{3}$), then B shoots C; if B hits $(\frac{2}{3})$ A faces B with A first $(\frac{3}{7})$; if B misses $(\frac{1}{3})$, C kills B and A faces C with A first $(\frac{1}{3})$. Thus,

$$P_{A \to C} = \tfrac{1}{3} \cdot \tfrac{1}{7} + \tfrac{2}{3}\big(\tfrac{2}{3} \cdot \tfrac{3}{7} + \tfrac{1}{3} \cdot \tfrac{1}{3}\big) = \tfrac{59}{189} \approx 0.312$$

If A shoots at B: with probability $\frac{1}{3}$ he hits, but then C kills A immediately. With probability $\frac{2}{3}$ he misses, reducing to the same branch as above when B shoots at C. Thus,

$$P_{A \to B} = \tfrac{2}{3}\big(\tfrac{2}{3} \cdot \tfrac{3}{7} + \tfrac{1}{3} \cdot \tfrac{1}{3}\big) = \tfrac{50}{189} \approx 0.265$$

Hence, the best strategy is to deliberately miss the first shot, giving survival probability

$$\boxed{\tfrac{25}{63} \approx 39.7\%}$$

Problem 17

For each couple, the number of children is geometric with success "girl" $p = \frac{1}{2}$.

- Probability they have exactly k children (first $k-1$ boys then a girl) is $(\frac{1}{2})^k$
- Boys per such family $= k-1$. So the expected number of boys per family is:

$$\begin{aligned}
\mathbb{E}[\text{boys}] = \sum_{k\geq 1}(k-1)\left(\frac{1}{2}\right)^k &= \sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k - \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \\
&= \frac{\left(\frac{1}{2}\right)}{(1-\left(\frac{1}{2}\right))^2} - \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = \frac{\left(\frac{1}{2}\right)^2}{(1-\frac{1}{2})^2} \\
&= 1
\end{aligned}$$

- Girls per family is always 1 (they stop at the first girl)

Thus, in expectation each family contributes 1 boy and 1 girl, so across the whole society the gender ratio converges to 50% girls, 50% boys.

Problem 18

After shot 2, we have exactly 1 make and 1 miss. For every following shot, the probability of a make is the ratio of makes so far to shots so far. This is exactly the same as the problem with a red ball (for "make") and a blue ball (for "miss"), such that:

- Every shot you add another "ball" of whichever result occurred
- The probability of the next shot is proportional to how many balls of that color are already in the urn

Now, imagine simulating n shots in total:

- Start with 1 red, 1 blue
- After $n-2$ additional draws, we end up with some total of red balls (say k) and blue balls $(n-k)$

We know that every sequence of results with exactly k makes and $n-k$ misses has the same probability. (this is because at each step, when you multiply out all the conditional probabilities, the product simplifies in such a way that the order of makes and misses doesn't matter — only the counts)
So if you fix k, all $\binom{n-2}{k-1}$ sequences are equally probable. And when you add them up, the total probability for "exactly k makes in n shots" turns out to be the same for every possible k.
After 100 shots (starting with 1 make and 1 miss already locked in), the possible final totals of makes are $1, 2, 3, \ldots, 99$. Each one of these totals is equally likely, because the order of the successes and failures doesn't matter in this urn process. There are 99 such possible totals. So each has probability

$$\boxed{\frac{1}{99}}$$