# ISyE/Math 6759 Stochastic Processes in Finance – I
# Homework Set 4

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**Problem 1**

(a) $\mathbb{E}[X_4 \mid I_i] = \mathbb{E}\left[\sum_{k=1}^{4} B_k \,\middle|\, I_i\right] = \sum_{k=1}^{i} B_k + \sum_{k=i+1}^{4} \mathbb{E}[B_k \mid I_i]$

All $B_k$ are independent, with no change given past information. They also have a mean of 0. Thus, $\mathbb{E}[B_k \mid I_i] = 0$, for $k > i$. Therefore $\mathbb{E}[X_4 \mid I_i] = \sum_{k=1}^{i} B_k = X_i$.

$$\mathbb{E}[X_4 \mid I_1] = B_1 = X_1$$
$$\mathbb{E}[X_4 \mid I_2] = B_1 + B_2 = X_2$$
$$\mathbb{E}[X_4 \mid I_4] = B_1 + B_2 + B_3 + B_4 = X_4$$

(b) $Z_i = \mathbb{E}[X_4 \mid I_i] = X_i$

$\mathbb{E}[Z_{i+1} \mid I_i] = \mathbb{E}[X_{i+1} \mid I_i] = \mathbb{E}[X_i + B_{i+1} \mid I_i] = X_i + \mathbb{E}[B_{i+1} \mid I_i] = X_i = Z_i$

$Z_i$ is a martingale

(c) $\tilde{X}_n = \sum_{i=1}^{n}(B_i + \sqrt{i}) = X_n + \sum_{i=1}^{n} \sqrt{i}$

$\mathbb{E}\left[\tilde{X}_{n+1} \mid I_n\right] = \mathbb{E}[X_{n+1} \mid I_n] + \sum_{i=1}^{n+1} \sqrt{i} = X_n + \sum_{i=1}^{n} \sqrt{i} + \sqrt{n+1} \neq \tilde{X}_n$

$\tilde{X}_n$ is not a martingale

(d) $\tilde{X}'_n = \tilde{X}_n - \sum_{i=1}^{n} \sqrt{i}$

$\tilde{X}'_n = X_n$, which is a martingale

(e) We can convert the coin to be biased: $\mathbb{P}[B_i = 1] = p_i, \quad \mathbb{P}[B_i = -1] = 1 - p_i$

Need to set $\mathbb{E}[V_i] = 0$ for martingale increments:
$\mathbb{E}[V_i] = \mathbb{E}[B_i] + \sqrt{i} = (2p_i - 1) + \sqrt{i} = 0 \quad \Rightarrow \quad p_i = \frac{1}{2}(1 - \sqrt{i})$

But since $p_i$ must be in [0,1], this only works for $\sqrt{i} \leq 1$ So beyond $i = 1$, no valid probability makes it a true martingale.

Thus:

- Theoretically, we need $p_i = (1-\sqrt{i})/2$
- Practically, since that's not a valid probability, we cannot achieve a martingale by re-weighting fair coin probabilities for $i > 1$

**Problem 2**

(a) $\mathbb{E}[X_t \mid \mathcal{F}_s] = 2\,\mathbb{E}[W_t \mid \mathcal{F}_s] + t = 2W_s + t \neq 2W_s + s = X_s$, $\boxed{\text{not a martingale}}$

(b) $d(W_t^2) = 2W_t dW_t + dt \Rightarrow \mathbb{E}[d(W_t^2) \mid \mathcal{F}_s] = dt$
$\mathbb{E}[W_t^2 \mid \mathcal{F}_s] = W_s^2 + (t-s) \neq W_s^2 = X_s$, $\boxed{\text{not a martingale}}$

(c) $dX_t = d(W_t^2) - 2tW_t dt = (2W_t dW_t + dt) - 2tW_t dt = 2W_t dW_t + (1 - 2tW_t)dt$
$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s + (t-s) - W_s(t^2 - s^2) \neq X_s$, $\boxed{\text{not a martingale}}$

**Problem 3**

Brownian motion $(W_t)$ has a variance of $\text{Var}(R_t) = \sigma^2 T$
$\text{SD}(R_4) = \sigma\sqrt{T} = 0.1 \cdot \sqrt{4} = 0.2 = \boxed{20\%}$

**Problem 4**

$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = u_n Y_n + v_n$
Need to find $M_n = a_n Y_n + b_n$ such that $M_n$ is martingale:

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = a_{n+1}\,\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] + b_{n+1} = a_{n+1}\,(u_n Y_n + v_n) + b_{n+1} \stackrel{!}{=} a_n Y_n + b_n$$

$$a_{n+1}u_n = a_n, \quad a_{n+1}v_n + b_{n+1} = b_n$$

Assuming $u_n \neq 0$,

$$a_n = a_0 \prod_{j=0}^{n-1} \frac{1}{u_j}, \quad b_n = b_0 - \sum_{k=0}^{n-1} \Big( \prod_{j=k+1}^{n-1} \frac{1}{u_j} \Big) v_k$$

One convenient choice (take $a_0 = 1, , b_0 = 0$) is

$$\boxed{a_n = \prod_{j=0}^{n-1} \frac{1}{u_j}, \quad b_n = -\sum_{k=0}^{n-1} \Big( \prod_{j=k+1}^{n-1} \frac{1}{u_j} \Big) v_k}$$

and then $M_n = a_n Y_n + b_n$ is a martingale

## Problem 5

$dX_t = \mu X_t dt + \sigma X_t dW_t$

The discounted process $e^{-rt}X_t$ is a martingale iff its drift $\mu$ is zero

$d\left(e^{-rt}X_t\right) = e^{-rt}X_t\big[(\mu - r)dt + \sigma dW_t\big] \Rightarrow \boxed{\mu = r}$

## Problem 6

(a) $dX_t = X_t dW_t + \frac{1}{2}X_t dt$

$$\begin{aligned} dZ_t &= d(e^{-rt}X_t) = e^{-rt}dX_t + X_t d(e^{-rt}) \\ &= e^{-rt}X_t dW_t + (\tfrac{1}{2} - r)e^{-rt}X_t dt \\ &= Z_t dW_t + (\tfrac{1}{2} - r)Z_t dt \end{aligned}$$

$$Z_{t_2} - Z_{t_1} = \underbrace{\int_{t_1}^{t_2} Z_s dW_s}_{\text{mean } 0} + (\tfrac{1}{2} - r)\int_{t_1}^{t_2} Z_s ds$$

$$\boxed{\mathbb{E}[Z_{t_2} - Z_{t_1} \mid \mathcal{F}_{t_1}] = (\tfrac{1}{2} - r)\,\mathbb{E}\left[\int_{t_1}^{t_2} Z_s, ds \middle| \mathcal{F}_{t_1}\right]}$$

(b) From the SDE above, the drift is $(\frac{1}{2} - r)Z_t$.

Thus $Z_t$ is a martingale iff the drift vanishes: $r = \frac{1}{2}$. Otherwise, it is not a martingale.

(c) Since $W_t \sim N(0,t), \mathbb{E}[e^{W_t}] = e^{t/2}$. With $(X_0 = e^{W_0} = 1)$:

$$\boxed{\mathbb{E}[Z_t] = e^{-rt}\mathbb{E}[e^{W_t}] = e^{(\frac{1}{2}-r)t}}$$

This is constant in $t$ (thus $Z_t$ is a martingale) only if $r = \frac{1}{2}$. If $r \neq \frac{1}{2}$, we can change the definition of $X_t$ to remove the drift in $Z_t$. Take $X_t := \exp\big(W_t + (r - \frac{1}{2})t\big)$. Then

$$Z_t = e^{-rt}X_t = \exp\big(W_t - \tfrac{1}{2}t\big)$$

so $Z_t$ is a martingale for any $r$.

(d) With the modified definition $X_t = \exp\big(W_t + (r - \frac{1}{2})t\big)$, we have $Z_t = e^{-rt}X_t = \exp\big(W_t - \frac{1}{2}t\big)$ which is the standard exponential martingale.

Therefore, $\mathbb{E}[Z_t] = \mathbb{E}\left[e^{W_t - \frac{1}{2}t}\right] = e^{-\frac{1}{2}t}\,\mathbb{E}\big[e^{W_t}\big] = e^{-\frac{1}{2}t}\,e^{\frac{1}{2}t} = 1$. Thus, after this modification, $Z_t$ becomes a martingale and its expectation remains constant over time:

$$\boxed{\mathbb{E}[Z_t] = \mathbb{E}[Z_0] = 1}$$

**Problem 7**

Let $Y := \log(R_t) \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 0.17$ and $\sigma^2 = 0.09$ ($\sigma = 0.3$)

(a) The goal is $\mu + \sigma^2\theta = r$. Thus,

$$\theta_r = \frac{r - \mu}{\sigma^2} = \frac{0.05 - 0.17}{0.09} = -\frac{4}{3}$$

Therefore

$$\boxed{\xi_r(y) = \exp\left(\frac{r-\mu}{\sigma^2}\,y - \psi\left(\frac{r-\mu}{\sigma^2}\right)\right) = \exp\big(-\tfrac{4}{3}\,y - \psi(-\tfrac{4}{3})\big)}$$

with

$$\psi(-\tfrac{4}{3}) = \mu(-\tfrac{4}{3}) + \tfrac{1}{2}\sigma^2(\tfrac{16}{9}) = -0.226\overline{6} + 0.08 = -0.146\overline{6}$$

(b) The goal is $\mu + \sigma^2\theta = 0$. Thus,

$$\theta_0 = -\frac{\mu}{\sigma^2} = -\frac{0.17}{0.09} = -1.888\overline{8}, \qquad \boxed{\xi_0(y) = \exp\Big(-\frac{\mu}{\sigma^2}\,y - \psi\Big(-\frac{\mu}{\sigma^2}\Big)\Big)}\,.$$

with

$$\psi(\theta_0) = \mu\theta_0 + \tfrac{1}{2}\sigma^2\theta_0^2 = -0.321\overline{1} + 0.160\overline{5} = -0.160\overline{5}$$

(c) Under $\tilde{\mathbb{P}}_{\theta_0}$ we have $Y \sim \mathcal{N}(0, \sigma^2)$, thus

$$\boxed{\mathbb{E}_{\tilde{\mathbb{P}}_{\theta_0}}[Y^2] = \sigma^2 = 0.09}$$

(d) No, the variance stays the same.

Changing the probability measure just shifts the mean of $\log(R_t)$. It doesn't stretch or shrink the distribution. Since $\log(R_t)$ is normally distributed, the variance $\sigma^2 = 0.09$ remains unchanged under all these measures.

**Problem 8**

(a) A process $(Y_n, \mathcal{F}_n)$ is a martingale if

   (a) $Y_n$ is $\mathcal{F}_n$-adapted and $\mathbb{E}[|Y_n|] < \infty$ for all $n$

   (b) the martingale property holds:

$$\mathbb{E}[\, Y_{n+1} \mid \mathcal{F}_n \,] = Y_n \quad \text{a.s. for all } n$$

(b)

$$M_{n+1} = M_n + X_{n+1}, \qquad M_{n+1}^2 = M_n^2 + 2M_n X_{n+1} + X_{n+1}^2$$

Taking conditional expectation given $\mathcal{F}_n$ (given $\mathbb{E}[X_{n+1}] = 0,\ \mathbb{E}[X_{n+1}^2] = 1$):

$$\mathbb{E}\big[M_{n+1}^2 \mid \mathcal{F}_n\big] = M_n^2 + 2M_n \underbrace{\mathbb{E}[X_{n+1}]}_{0} + \underbrace{\mathbb{E}[X_{n+1}^2]}_{1} = M_n^2 + 1$$

Thus

$$\mathbb{E}\big[M_{n+1}^2 - (n+1) \mid \mathcal{F}_n\big] = \big(M_n^2 + 1\big) - (n+1) = M_n^2 - n,$$

so $\big(M_n^2 - n\big)_{n \geq 0}$ is a martingale with respect to $\mathcal{F}_n$.

(c) Since $\tau$ is bounded, the Optional Stopping Theorem applies to the martingale $(M_n^2 - n)$:

$$\mathbb{E}\big[M_\tau^2 - \tau\big] = \mathbb{E}\big[M_0^2 - 0\big]$$

With $M_0 = 0$:

$$\boxed{\mathbb{E}\big[M_\tau^2\big] = \mathbb{E}[\tau]}$$

**Problem 9**

No, $W_t^3$ is not a martingale.

Using Itô's lemma: $d(W_t^3) = 3W_t^2\, dW_t + 3W_t\, dt$

The drift term $3W_t\, dt$ means the conditional expectation changes over time:

$$\mathbb{E}[W_t^3 \mid \mathcal{F}_s] = W_s^3 + 3W_s(t-s)$$

so it's not constant, thus not a martingale.

However, by removing the drift term: $W_t^3 - 3\int_0^t W_s\, ds$ is a martingale.

**Problem 10**

(a)

$$f(x) = x^3, \quad \int_0^1 f(x)\,dx = \int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4} = 0.25$$

(b) Split $[0,1]$ into 4 equal parts: $x_0 = 0,\ x_1 = 0.25,\ x_2 = 0.5,\ x_3 = 0.75,\ x_4 = 1$

Each interval width: $\Delta x = 0.25$

Right sum:

$$\sum_{i=1}^{4} f(x_i)(x_i - x_{i-1}) = 0.25[(0.25)^3 + (0.5)^3 + (0.75)^3 + (1)^3]$$

$$= 0.25(0.0156 + 0.125 + 0.4219 + 1) = 0.25(1.5625) = 0.3906$$

Left sum:

$$\sum_{i=1}^{4} f(x_{i-1})(x_i - x_{i-1}) = 0.25[(0)^3 + (0.25)^3 + (0.5)^3 + (0.75)^3]$$

$$= 0.25(0 + 0.0156 + 0.125 + 0.4219) = 0.25(0.5625) = 0.1406$$

(c) The true integral is 0.25

- Left sum = 0.1406: underestimates
- Right sum = 0.3906: overestimates

The difference between them is $0.3906 - 0.1406 = 0.25$, which shows the bounds around the true value 0.25.

**Problem 11**

(a)

$$f(x) = \begin{cases} x\sin\left(\frac{\pi}{x}\right) & 0 < x < 1 \\ 0 & x = 0 \end{cases}$$

$u = \frac{\pi}{x}$, so $du = -\frac{\pi}{x^2}dx \Rightarrow dx = -\frac{x^2}{\pi}du$. Then $x = \frac{\pi}{u}$:

$$\int_0^1 x\sin\left(\frac{\pi}{x}\right)dx = \int_\infty^\pi \frac{\pi}{u}\sin(u)\left(-\frac{\pi}{u^2}\right)du = \pi^2\int_\pi^\infty \frac{\sin(u)}{u^3}du$$

$$\boxed{\int_0^1 f(x)\,dx \approx -0.1834}$$

(b) Split $[0,1]$ into 4 equal parts:

$$x_0 = 0,\ x_1 = 0.25,\ x_2 = 0.5,\ x_3 = 0.75,\ x_4 = 1, \quad \Delta x = 0.25$$

$$\begin{aligned} f(0.25) &= 0.25\sin(4\pi) = 0 \\ f(0.5) &= 0.5\sin(2\pi) = 0 \\ f(0.75) &= 0.75\sin\left(\tfrac{4\pi}{3}\right) \approx -0.6495 \\ f(1) &= \sin(\pi) = 0 \end{aligned}$$

Right sum:

$$\sum f(x_i)(x_i - x_{i-1}) = 0.25(0 + 0 - 0.6495 + 0) = -0.1624$$

Left sum:

$$\sum f(x_{i-1})(x_i - x_{i-1}) = 0.25(0 + 0 + 0 - 0.6495) = -0.1624$$

(c) Both sums ($-0.1624$) are close to the true value ($-0.1834$), slightly underestimating it. They capture the general trend but miss some oscillations near $x = 0$.

(d) Near $x = 0$, the function oscillates really fast, so using only a few big intervals misses a lot of that movement. That's why the Riemann sums aren't super accurate. If smaller steps near $x = 0$ are utilized, the estimate would get much closer to $-0.1834$.