ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 3

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Problem 1

(a) Using state prices q_u, q_d (price today of \$1 delivered in up/down at $t + \Delta$) the three pricing equations are:

$$\text{Bond} = \frac{1}{1 + r\Delta} = q_u + q_d$$

$$\text{Stock} = S_t = q_u S^u_{t+\Delta} + q_d S^d_{t+\Delta}$$

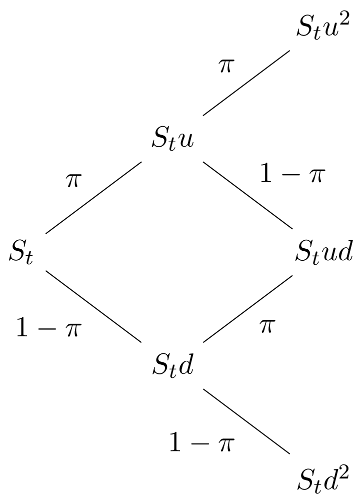
$$\text{Put} = C_t = q_u \max(S_t - S^u_{t+\Delta}, 0) + q_d \max(S_t - S^d_{t+\Delta}, 0)$$

Because the strike is S_t (because it is an at-the-money put option), $\max(S_t - S^u_{t+\Delta}, 0) = 0$ and $\max(S_t - S^d_{t+\Delta}, 0) = S_t - S^d_{t+\Delta}$. Thus,

$$C_t = q_d \left(S_t - S^d_{t+\Delta}\right)$$

Equivalently, in risk-neutral probabilities $\pi_u = (1 + r\Delta)q_u,\ \pi_d = (1 + r\Delta)q_d$,

$$\begin{aligned}
&\pi_u + \pi_d = 1 \\
&S_t = \frac{1}{1 + r\Delta}\left(\pi_u S^u_{t+\Delta} + \pi_d S^d_{t+\Delta}\right) \\
&C_t = \frac{1}{1 + r\Delta}\left(\pi_u \cdot 0 + \pi_d \left(S_t - S^d_{t+\Delta}\right)\right)
\end{aligned}$$



(b)

In a recombining two-step tree there are three nonterminal (branching) nodes where you can write the same kind of two-state, three-equation system: the root at time t, and the two nodes at time $t+\Delta$ (the up node and the down node). Therefore, the number of such systems for the entire tree is $\boxed{3}$.

Problem 2

(a)

$$u = e^{\sigma\sqrt{\Delta t}}, \quad \Delta t = 1/12$$

$$\sigma = \frac{\ln u}{\sqrt{\Delta t}} = \frac{\ln(1.15)}{\sqrt{1/12}} = \ln(1.15)\sqrt{12} \approx 0.484$$

$$\boxed{\sigma \approx 48.4\% \text{ per annum}}$$

(b) Stock tree (S)

- $t = 0 : [100]$
- $t = 1 : [86.9565,\ 115.0000]$
- $t = 2 : [75.6144,\ 100.0000,\ 132.2500]$
- $t = 3 : [65.7516,\ 86.9565,\ 115.0000,\ 152.0875]$
- $t = 4 : [57.1753,\ 75.6144,\ 100.0000,\ 132.2500,\ 174.9006]$

Call option (C) with terminal payoffs $C_{4,k} = \max(S_{4,k} - 100, 0)$ and backward induction

$$C_{n,k} = \frac{p\,C_{n+1,k+1} + (1-p)\,C_{n+1,k}}{R}$$

- $t = 4 : [0,\ 0,\ 0,\ 32.2500,\ 74.9006]$
- $t = 3 : [0,\ 0,\ 15.4158,\ 52.5033]$
- $t = 2 : [0,\ 7.3689,\ 33.0799]$
- $t = 1 : [3.5224,\ 19.6283]$
- $t = 0 : [11.2065]$

(c) $C_0 \approx 11.2065$

(d) At each node (n, k), replicate the option over the next month with

$$\Delta_{n,k} = \frac{C_{n+1,k+1} - C_{n+1,k}}{S_{n+1,k+1} - S_{n+1,k}}, \qquad B_{n,k} = \frac{S_{n+1,k+1}C_{n+1,k} - S_{n+1,k}C_{n+1,k+1}}{(S_{n+1,k+1} - S_{n+1,k})\,R}$$

(Here B is the cash/bond holding today; negative B means borrow)

Period $t = 0 \to 1$ (root):

- $\Delta_{0,0} = 0.5743, \quad B_{0,0} = -46.2255$

Period $t = 1 \to 2$:

- Up-node: $\Delta_{1,1} = 0.7972, \quad B_{1,1} = -72.0542$
- Down-node: $\Delta_{1,0} = 0.3022, \quad B_{1,0} = -22.7543$

Period $t = 2 \to 3$:

- Up-up: $\Delta_{2,2} = 1.0000, \quad B_{2,2} = -99.1701$
- Up-down (= down-up): $\Delta_{2,1} = 0.5497, \quad B_{2,1} = -47.6022$
- Down-down: $\Delta_{2,0} = 0.0000, \quad B_{2,0} = 0.0000$

Problem 3

(a) This is the following given information:

$\Delta t = \frac{1}{12}, \quad r = 0.004, \quad \sigma = 0.3, \quad S_0 = 102, \quad K = 120$

- $u = e^{\sigma\sqrt{\Delta t}} = e^{0.3\cdot\sqrt{\frac{1}{12}}} = 1.09046$
- $d = \frac{1}{u} = 0.91704$
- $R = e^{r\Delta t} = e^{0.004\cdot\frac{1}{12}} = 1.00033$
- $p = \frac{R-d}{u-d} = \frac{1.00033-0.91704}{1.09046-0.91704} = 0.48027$

Stock Price Tree:

Step 0: $S_0 = 102$
Step 1: $S_u = 111.23, \quad S_d = 93.54$
Step 2: $S_{uu} = 121.29, \quad S_{ud} = 102, \quad S_{dd} = 85.78$
Step 3: $S_{uuu} = 132.26, \quad S_{uud} = 111.23, \quad S_{udd} = 93.54, \quad S_{ddd} = 78.66$

Payoffs at Maturity:

$C_{uuu} = \max(132.26 - 120, 0) = 12.26$
$C_{uud} = \max(111.23 - 120, 0) = 0$
$C_{udd} = \max(93.54 - 120, 0) = 0$
$C_{ddd} = \max(78.66 - 120, 0) = 0$

(b) $C_{uu} = (\frac{1}{R})(p \cdot C_{uuu} + (1-p)\cdot C_{uud}) = \frac{1}{1.00033}\cdot(0.48027\cdot 12.26) = 5.89$
$C_{ud} = (\frac{1}{R})(p \cdot C_{uud} + (1-p)\cdot C_{udd}) = 0$
$C_{dd} = (\frac{1}{R})(p \cdot C_{udd} + (1-p)\cdot C_{uud}) = 0$
$C_{u} = (\frac{1}{R})(p \cdot C_{uu} + (1-p)\cdot C_{ud}) = \frac{1}{1.00033}\cdot(0.48027\cdot 5.89) = 2.83$
$C_{d} = (\frac{1}{R})(p \cdot C_{ud} + (1-p)\cdot C_{dd}) = 0$
$C_{0} = (\frac{1}{R})(p \cdot C_{u} + (1-p)\cdot C_{d}) = \frac{1}{1.00033}\cdot(0.48027\cdot 2.83) = 1.36$

The arbitrage-free call price is $\boxed{\$1.36}$

(c) Consider Δ as the number of stock units and B as the bond holding. We need to solve $\Delta S_u + BR = C_u$ and $\Delta S_d + BR = C_d$.

- $S_u = 111.23$
- $S_d = 93.54$
- $C_u = 2.83$
- $C_d = 0$
- $\Delta = \frac{C_u - C_d}{S_u - S_d} = 0.15997$
- $B = \frac{C_u - \Delta S_u}{R} = -14.95852$

To replicate the portfolio, $\Delta \approx 0.16$ shares and one must borrow about \$14.96

To hedge 100 shares, one must hold $100\Delta = -15.997$ or short ~ 16 shares and one must lend about $100B \approx \$1496.85$ at the risk free rate

(d) Model price is 1.36, market price 5

Since market price > fair value:

- Strategy: Sell the call at 5, buy the replicating portfolio for 1.36
- Result: Lock in risk free arbitrage profit ≈ 3.64 per option

If we do this for N options, profit scales with N.

Problem 4

(a)

$$\frac{\mathbb{E}[S_{t+1} \mid S_t]}{S_t} = 1 + \mu + \sigma\mathbb{E}[\varepsilon_t] = 1 + r + \sigma(2p - 1) \overset{\text{no-arb}}{=} 1 + r,$$

thus $2p - 1 = 0 \Rightarrow \boxed{p = \frac{1}{2}}$

(b) No. With $\mu = r$ the martingale (no arbitrage) condition forces $p = \frac{1}{2}$. At $p = \frac{1}{3}$, $\mathbb{E}[\varepsilon_t] = 2p - 1 = -\frac{1}{3} \neq 0$, so the expected gross return would be $1 + r - \sigma/3$, violating the martingale requirement.

(c) ε_t is the Bernoulli shock (± 1) and p is the objective up probability that embeds the asset's risk premium λ. For pricing, we pass to the risk-neutral measure with probability

$$\tilde{p} = \frac{(1 + r) - d}{u - d} = \frac{(1 + r) - (1 + \mu - \sigma)}{(1 + \mu + \sigma) - (1 + \mu - \sigma)} = \frac{\sigma - \lambda}{2\sigma} = \frac{1}{2} - \frac{\lambda}{2\sigma},$$

so discounted S_t is a martingale under $\tilde{p}$.

(d) No. Arbitrage-free pricing determines only the risk-neutral probability $\tilde{p}$, not the physical p. To identify p we need extra information (historical data, a model for λ, etc.).

Problem 5
This is the following given information:

$r = 0.06, \quad \sigma = 0.12, \quad S_0 = 100, \quad K = 100, \quad T = 200$

(a) $N = 5 \Rightarrow \Delta t_d = T/N = 200/5 = 40$
$\Delta t = 40/365 \approx 0.1095890411$

(b) $u = e^{\sigma\sqrt{\Delta t}} = e^{0.12\cdot\sqrt{0.109589}} = 1.040524668$
$d = \frac{1}{u} = \frac{1}{1.040524668} = 0.96105362069$

(c) $R = e^{r\Delta t} = e^{0.06\cdot 0.10958904109} = 1.006597007488620$
$p = \frac{R-d}{u-d} = \frac{1.006597007488620-0.96105362069}{1.040524668-0.96105362069} = 0.57308149747$

(d) Step 0: [100.00000]

Step 1: [96.10536, 104.05247]

Step 2: [92.36241, 100.00000, 108.26916]

Step 3: [88.76522, 96.10536, 104.05247, 112.65673]

Step 4: [85.30814, 92.36241, 100.00000, 108.26916, 117.22211]

Step 5: [81.98570, 88.76522, 96.10536, 104.05247, 112.65673, 121.97249]

(e) Terminal $n = 5$: payoffs $\max(S - 100, 0)$
[0.00000, 0.00000, 0.00000, 4.05247, 12.65673, 21.97249]

Backward induction:

Step 4: [0.00000, 0.00000, 2.30718, 8.92454, 17.87748]

Step 3: [0.000000, 1.31353, 6.05949, 13.96319]

Step 2: [0.74783, 4.00692, 10.51956]

Step 1: [2.59841, 7.68847]

Step 0: [5.479282]

So the arbitrage-free value of the call today is $C_0 \approx \boxed{\$5.48}$

Problem 6

(a) Let $R =$ taking the pro-incumbent side:

- $\mathbb{E}[R] = 1000p - 1500(1-p) = 2500p - 1500$
- $\mathbb{E}[-R] = 1500 - 2500p$

$$\mathbb{E}[R] = 2500(0.6) - 1500 = 0$$

So it's a fair bet at $p = 0.6$ (for either side).

(b) Yes. Break-even is $p = 0.6$.

- If $p > 0.6$: take the pro-incumbent side (positive EV $= 2500p - 1500$)
- If $p < 0.6$: take the anti-incumbent side (positive EV $= 1500 - 2500p$)

(c) They might not agree as p is the real probability, but nobody actually knows it. With just one election you can't really tell who was 'right' ahead of time. You'd only be able to judge over a bunch of similar events, where you can see how well people's calls line up with reality.

(d) Yes—polls and fundamentals models provide an estimate $\hat{p}$ with uncertainty (CIs, model error, priors). Will be useful but not definitive.

(e) Use their historical calibration and uncertainty: weight more if their forecasts have low Brier scores/strong past performance. Formally, combine prior with their $\hat{p}$ via Bayesian updating, weighting by precision.

(f) If I must pick a side now and I'm risk-neutral:

- Pro-incumbent side fair price: $V^{+} = 2500p - 1500$
- Anti-incumbent side fair price: $V^{-} = 1500 - 2500p$

Choose the larger; the value of the opportunity to choose the side is

$$\boxed{V = 2500\,|p - 0.6|}$$

Problem 7

(a) Let R be the market bet (+1000 if win, −1500 if lose) and R^* be your friend's bet (+1500 if win, −1000 if lose).

Hold a units of R and b units of R^*. Set the two state payoffs equal:

- If WIN: $1000a + 1500b$
- If LOSE: $-1500a - 1000b$

Require WIN = LOSE:

$$1000a + 1500b = -1500a - 1000b \implies 2500a + 2500b = 0 \implies b = -a$$

Pick $a = -1,\ b = +1$ (short one R, long one R^*)
Payoff if WIN: $-1000 + 1500 = +500$
Payoff if LOSE: $+1500 - 1000 = +500$

So long 1 of R^* and short 1 of R yields a guaranteed +$500 no matter what happens.

If these bets require upfront prices Π and Π^*, the initial cost is $-\Pi + \Pi^*$. It's an arbitrage as you collect cash today and still lock in +$500 later. If the cost is positive, you've created a risk-free investment paying $500 at election time for price $\Pi^* - \Pi$; its implied return is $500/(\Pi^* - \Pi)$.

(b) No need to know p. To set up the arbitrage portfolio, you don't need the true probability p of the incumbent winning. All you use are the market prices and the payoffs. That's why arbitrage pricing works: it's independent of subjective probabilities.

(c) Arbitrage arguments tell you relative prices and help you find mispricings, but they don't tell you the actual true probability p. Statisticians or econometricians try to estimate p from data. This matters because even if the market prices are internally consistent (no arbitrage), we still might want to know whether the market odds line up with your our view of the real chance of the event. That's how to decide whether to actually place the bet or not.

Problem 8

(a) One-step mean: $\mathbb{E}[B] = 10 \cdot \frac{1}{3} - 10 \cdot \frac{2}{3} = -\frac{10}{3}$

After 10 years:

$$\mathbb{E}[S_{10}] = 100 + 10\,\mathbb{E}[B] = 100 + 10\left(-\tfrac{10}{3}\right) = \tfrac{200}{3} \approx 66.67$$

Let $U \sim \mathrm{Bin}(10, \frac{1}{3})$ be the number of "up" years. Then:

$$S_{10} = 100 + 10(U - (10 - U)) = 20U, \quad S_{10} \geq 100 \iff U \geq 5.$$

$$\Pr(S_{10} \geq 100) = \sum_{k=5}^{10} \binom{10}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{10-k} \approx \boxed{0.2131}$$

(b) Now $S_t = S_{t-1}(1 \pm 0.5) \Rightarrow u = 1.5,\ d = 0.5$

no-arbitrage $\iff$ there exists a risk-neutral $q \in (0,1)$ such that the discounted stock is a martingale:

$$S_{t-1} = e^{-r}\mathbb{E}[S_t | S_{t-1}] = e^{-r} S_{t-1}[qu + (1-q)d]$$

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.05} - 0.5}{1.0} = 0.55127 \in (0,1),$$

so the economy $\boxed{\text{is arbitrage-free}}$.

European call, $K = 100$, maturity $T = 3$ (years). With three annual steps:

- Terminal prices: $100\{0.5^3, 1.5 \cdot 0.5^2,\ 1.5^2 \cdot 0.5,\ 1.5^3\} = \{12.5, 37.5, 112.5, 337.5\}$
- Terminal payoffs: $\{0, 0, 12.5, 237.5\}$

Risk-neutral price:

$$C_0 = e^{-3r} \sum_{k=0}^{3} \binom{3}{k} q^k (1-q)^{3-k} \max(100u^k d^{3-k} - 100, 0) \approx \boxed{38.648}$$

(c) Short the call at 40 and buy the replicating portfolio (for one call) and rebalance each year.

Initial replicating positions (from binomial deltas):

$$\Delta_0 = 0.67466, \qquad B_0 = -28.81833$$

Interpretation: buy 0.67466 shares and borrow \$28.8183

Rebalance if the stock goes up/down:

- At $t = 1$:
  - Up node: $\Delta = 0.82215,\ B = -52.41874$

– Down node: $\Delta = 0.13110,\ B = -3.11757$

- At $t = 2$:
  - UU: $\Delta = 1.00000,\ B = -95.12294$
  - UD: $\Delta = 0.16667,\ B = -5.94518$
  - DD: $\Delta = 0,\ B = 0$

This portfolio replicates the option's payoff, so the short is perfectly hedged and we lock in the profit $40 - 38.648 = \$1.352$ per option today.

(d) Year-1 has three states: $+0.5S_{t-1}$ with prob p_1, 0 with prob p_2, $-0.5S_{t-1}$ with prob p_3. For $t \geq 2$ it's as in (b).

- Is the market complete at $t = 1$ with just stock + bond? No. There are 3 states but only 2 securities $\Rightarrow$ incomplete; many payoffs (e.g., $(50, 0, 0)$ for a 1-year $K = 100$ call) aren't spanned uniquely.
- Also is a mention of an extra product paying $(0, 2, 4)$ next year and (apparently) priced at \$2 today. Note $(0, 2, 4) = 6 \cdot (1, 1, 1) - 0.04 \cdot (150, 100, 50)$, i.e., it's a linear combination of the bond and the stock, so it does not add a new state—the market is still incomplete. Its no-arb price implied by stock + bond would be

$$6e^{-0.05} - 0.04 \cdot 100 \approx 1.707$$

  If it trades at \$2, that's an arbitrage (short the product, buy the replicating combo).
- Bounds for the 1-year $K = 100$ call with only stock + bond (no extra, or treating it consistently): Let state prices be $\phi_u, \phi_m, \phi_d \geq 0$ with

$$\phi_u + \phi_m + \phi_d = e^{-0.05}, \qquad 150\phi_u + 100\phi_m + 50\phi_d = 100$$

  Then $C_0 = 50\phi_u$ lies between

$$\boxed{4.88 \leq C_0 \leq 26.22}$$

  (Min at $\phi_d = 0 \Rightarrow \phi_u = 2(1 - e^{-0.05})$; max at $\phi_m = 0 \Rightarrow \phi_u = e^{-0.05} - \frac{1}{2}\phi_d$ with $\phi_d = (3e^{-0.05} - 2)/2$)

Problem 9

(a)

$$u = 1.090463, \quad d = 0.917042, \quad R = 1.001668, \quad p = 0.487981$$

Backward-induct the terminal payoffs $P_T = \max(K - S, 0) \Rightarrow$

$$\boxed{P_0 \approx 5.0554}$$

(b)

$$u = 1.063151, \quad d = 0.940600, \quad R = 1.000834, \quad p = 0.491498.$$

Backward induction $\Rightarrow$

$$\boxed{P_0 \approx 5.1040}$$

(c) They're close but not identical. By refining the grid (more steps), the binomial price converges to the continuous-time Black–Scholes price. For these inputs, the BS put is

$$\boxed{P_{\text{BS}} \approx 5.1097},$$

so the 10-step value from part (b) is closer and thus the better approximation.

Problem 10
Let $C(K)$ be the price today of a European call with strike K. For $K_2 > K_1$ define the bull spread $B := C(K_1) - C(K_2)$.

(a) At maturity,
$$\max(S_T - K_1, 0) - \max(S_T - K_2, 0) \geq 0 \qquad \forall\, S_T$$
Thus the payoff of $C(K_1)$ dominates that of $C(K_2)$ state-by-state, so by no-arbitrage
$$C(K_1) \geq C(K_2)$$
(and strictly if $r > 0$ and there is positive probability that $S_T > K_1$)

(b) The bull-spread payoff at T is
$$B_T = \begin{cases} 0 & S_T \leq K_1 \\ S_T - K_1 & K_1 < S_T \leq K_2 \\ K_2 - K_1 & S_T \geq K_2 \end{cases}$$
Thus $0 \leq B_T \leq K_2 - K_1$ in every state. Pricing today gives
$$0 \leq B = C(K_1) - C(K_2) \leq \mathrm{PV}(K_2 - K_1) \leq K_2 - K_1$$
because $r \geq 0$ implies the present value of a sure $K_2 - K_1$ at T is at most $K_2 - K_1$. Therefore:
$$K_2 - K_1 \geq C(K_1) - C(K_2)$$

(c) Take $K_3 > K_2 > K_1$ and set
$$\alpha = \frac{K_3 - K_2}{K_3 - K_1}, \qquad 1 - \alpha = \frac{K_2 - K_1}{K_3 - K_1}$$
Consider the butterfly: long α calls at K_1, long $1-\alpha$ calls at K_3, short one call at K_2. Its payoff at T is
$$\alpha \max(S_T - K_1, 0) + (1-\alpha)\max(S_T - K_3, 0) - \max(S_T - K_2, 0) \geq 0 \qquad \forall\, S_T$$
No-arbitrage then gives us
$$\alpha C(K_1) + (1-\alpha) C(K_3) - C(K_2) \geq 0$$
$$\Rightarrow C(K_2) \leq \frac{K_3 - K_2}{K_3 - K_1} C(K_1) + \frac{K_2 - K_1}{K_3 - K_1} C(K_3)$$

Problem 11
This is the following given information:

$S_0 = 20,\ K = 22,\ r = 5\%,\ T = 2,\ \Delta t = 1,\ u = 1.2840,\ d = 0.8607$

- $R = e^{r\Delta t} = e^{0.05} = 1.051271$
- $p = \dfrac{R-d}{u-d} = 0.450203$

$$S_{1,u} = S_0 u = 25.680, \qquad S_{1,d} = S_0 d = 17.214,$$
$$S_{2,uu} = S_0 u^2 = 32.9731, \quad S_{2,ud} = S_0 ud = 22.1028, \quad S_{2,dd} = S_0 d^2 = 14.8161$$

American put P (early exercise allowed). Terminal payoffs:

$$P_{2,uu} = 0, \quad P_{2,ud} = 0, \quad P_{2,dd} = 22 - 14.8161 = 7.1839$$

Step back to $t = 1$ (continuation $= \frac{(1-p)\,\text{down} + p\,\text{up}}{R}$; then max with intrinsic):

$$\begin{aligned} &\text{At } S_{1,d}: \ \text{cont} = \frac{0.549797 \cdot 7.1839}{1.051271} = 3.76,\ \text{intrinsic} = 22 - 17.214 = 4.786 \Rightarrow P_{1,d} = 4.786, \\ &\text{At } S_{1,u}: \ \text{cont} = 0,\ \text{intrinsic} = 0 \Rightarrow P_{1,u} = 0. \end{aligned}$$

Price at $t = 0$:

$$\boxed{P_0 = \frac{(1-p)P_{1,d} + pP_{1,u}}{R} = \frac{0.549797 \cdot 4.786}{1.051271} = 2.5030}$$

(Optimal early exercise at the down node $t = 1$)

American call C (no dividends $\Rightarrow$ never optimal to exercise early). Terminal payoffs:

$$C_{2,uu} = 32.9731 - 22 = 10.9731, \quad C_{2,ud} = 0.1028, \quad C_{2,dd} = 0$$

Back one step:

$$\begin{aligned} &\text{At } S_{1,d}: \ \text{cont} = \frac{p \cdot 0.1028}{R} = 0.0440 \Rightarrow C_{1,d} = 0.0440 \\ &\text{At } S_{1,u}: \ \text{cont} = \frac{(1-p) \cdot 0.1028 + p \cdot 10.9731}{R} = 4.7529\ (> 25.680 - 22 = 3.68) \Rightarrow C_{1,u} = 4.7529 \end{aligned}$$

Price at $t = 0$:

$$\boxed{C_0 = \frac{(1-p)C_{1,d} + pC_{1,u}}{R} = \frac{0.549797 \cdot 0.0440 + 0.450203 \cdot 4.7529}{1.051271} = 2.0585}$$

Summary: American put $P_0 \approx 2.5030$ (exercise at $t = 1$ if down), American call $C_0 \approx 2.0585$ (equals European value since no dividends).

Problem 12
This is the following given information:

$S_0 = 62,\ K = 60,\ \sigma = 0.20,\ r = 0.025,\ T = 5/12,\ N = 5,\ \Delta t = T/N = 1/12$

- $u = e^{\sigma\sqrt{\Delta t}} = e^{0.20\sqrt{1/12}} = 1.0583$
- $d = \frac{1}{u} = 0.9450$
- $R = e^{r\Delta t} = e^{0.025/12} = 1.0021$
- $p = \frac{R-d}{u-d} = 0.5072$

At $t = 5$ (5 months), $S_{5,j} = S_0 u^j d^{5-j},\ j = 0, \ldots, 5$

At maturity, call payoff $C_{5,j} = \max(S_{5,j} - K, 0)$, put payoff $P_{5,j} = \max(K - S_{5,j}, 0)$

At $t = 3$ months (step 3), holder chooses call or put. Thus continuation value is

$$V_{3,j} = \max\left(C_{3,j},\ P_{3,j}\right)$$

For $n = 4, 2, 1, 0$ compute

$$V_{n,j} = \frac{p\, V_{n+1,j+1} + (1-p)\, V_{n+1,j}}{R}$$

After rolling back to $t = 0$, $\boxed{V_0 \approx 4.52}$ is the arbitrage-free value of the option.

Problem 13
This is the following given information:

$S_0 = 50,\ K = 49,\ T = \frac{2}{12} = \frac{1}{6},\ r = 0.10$

Possible prices at T: $S_u = 53,\ S_d = 48$

- $u = \frac{53}{50} = 1.06$
- $d = \frac{48}{50} = 0.96$
- $R = e^{rT} = e^{0.10 \cdot \frac{1}{6}} = 1.01682$
- $p = \frac{R-d}{u-d} = \frac{1.01682-0.96}{1.06-0.96} = 0.5682$

$$C_u = \max(53-49, 0) = 4, \quad C_d = \max(48-49, 0) = 0$$

$$C_0 = \frac{1}{R}\big(pC_u + (1-p)C_d\big) = \frac{1}{1.01682}(0.5682 \cdot 4 + 0.4318 \cdot 0)$$

$$C_0 = \frac{2.2727}{1.01682} = 2.235$$

The no-arbitrage price of the European call option is about $\boxed{C_0 \approx \$2.24}$.

Problem 14
This is the following given information:

$S_0 = 25,\ T = \frac{2}{12} = \frac{1}{6},\ r = 0.10$

Possible stock prices at T: $S_u = 27,\ S_d = 23$

Derivative payoff: $X_T = S(T)^2$

- $u = \frac{27}{25} = 1.08$
- $d = \frac{23}{25} = 0.92$
- $R = e^{rT} = e^{0.10 \cdot \frac{1}{6}} = e^{0.01667} = 1.0168$
- $p = \frac{R-d}{u-d} = \frac{1.0168-0.92}{1.08-0.92} = \frac{0.0968}{0.16} = 0.605$

$$X_u = S_u^2 = 27^2 = 729, \quad X_d = S_d^2 = 23^2 = 529$$

$$X_0 = \frac{1}{R}\Big(pX_u + (1-p)X_d\Big) = \frac{1}{1.0168}\Big(0.605 \cdot 729 + 0.395 \cdot 529\Big)$$

$$= \frac{1}{1.0168}(441.345 + 209.155) = \frac{650.5}{1.0168} = 639.2$$

The no-arbitrage value of the derivative today is about $\boxed{X_0 \approx \$639.20}$.