Binomial Theorem in Probability Problem

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Problem 1 : Probability of even number of heads

A coin with $p\{h\} = p = 1 - q$ is tossed n times. Show that the probability that the number of heads is even equals $\frac{1}{2}[1 + (q - p)^n]$

Solution

P_{2k} = probability of the number of heads being even
X_i = event of 2i even tosses, $i = 1, 2, 3...$
The set of of the number of heads being even is

$$\{2k\ heads\} = X_0 \cup X_1 \cup ...X_k \tag{1}$$

The probability of 2i even tosses is

$$P(X_i) = \binom{n}{2i} p^{2i} q^{n-2i} \tag{2}$$

So the total probability of all events with even number of heads is

$$P_{2k} = P(\bigcup_{i=0}^{k} X_i) = \sum_{i=0}^{i=k} P(X_i) = \sum_{i=0}^{i=k} \binom{n}{2i} p^{2i} q^{n-2i} \tag{3}$$

What we got in the right side is all the binomial coefficients with p to an even power. That is the result of

$$\sum_{i=0}^{i=k} \binom{n}{2i} p^{2i} q^{n-2i} = \frac{1}{2}[\sum_{i=0}^{i=k} \binom{n}{i} p^i q^{n-i} + \sum_{i=0}^{i=k} (-1)^i \binom{n}{i} p^i q^{n-i}] \tag{4}$$

We divide by two to remove the overcount on the even coefficients. Now what is between the brackets is the binomial decomposition of $(q + p)^n$ and $(q - p)^n$

$$P_{2k} = \frac{1}{2}[(q + p)^n + (q - p)^n] \tag{5}$$

But $q + p = 1$, so we get the desired result:

$$P_{2k} = \frac{1}{2}[1 + (q - p)^n] \tag{6}$$