Cauchy Sequence

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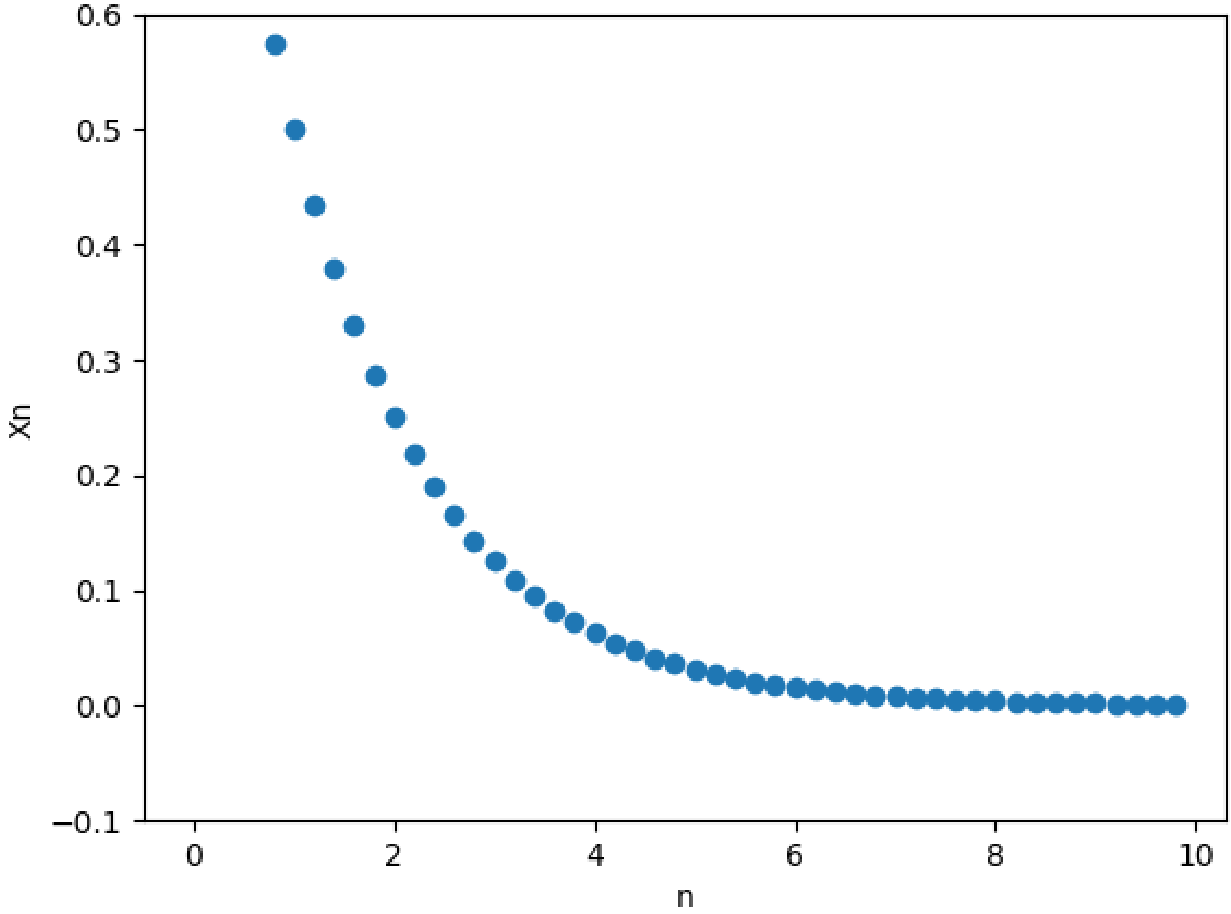

Figure 1: Representation for the Cauchy Sequence $\{\frac{1}{2^x}\}$

Definition. Cauchy Sequence

A sequence $\{a_n\}$ is called a **Cauchy Sequence** if for every $\epsilon > 0$ there is a natural number N such that, for all m and n,

$$if\ m, n > N\ then\ |a_n - a_m| < \epsilon$$

(This condition is usually written $\lim_{m,n\to+\infty} |a_m - a_n| = 0$)

Theorem 1. *A sequence $\{a_n\}$ converges if and only if it is a Cauchy sequence.*

Proof. The first part of the proof is satisfied by the Bolzano-Weierstrass Theorem (every bounded sequence has a convergent subsequence). What is needed to prove the converse assertion is that every Cauchy sequence $\{a_n\}$ is bounded. If we take $\epsilon = 1$ in the definition of a Cauchy sequence we find that there is some N such that

$$|a_m - a_n| < 1\ for\ m, n > N$$

In particular, this means that

$$|a_m - a_{N+1}| < 1\ for\ all\ m > N$$

Thus $\{a_m : m > N\}$ is bounded; since there are only finitely many other a_is such that the whole sequence is bounded. Suppose that a subsequence of a Cauchy sequence converges. Taking into consideration that the difference between the elements of a Cauchy sequence is very small and some subsequence of it converges, then for every $\epsilon > 0$ there is a natural number N such that, for all m and n,

$$if\ m, n > N\ then\ |a_n - a_m| < \epsilon$$

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