Entropy as Expected Value

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Proposition. *If x and y are two random variables with respective densities $a(x)$ and $b(y)$, then*

$$\mathbf{E}\{\ln a(x)\} \geq \mathbf{E}\{\ln b(x)\}$$

Equality holds iff $a(x) = b(x)$

Proof. Suppose we have x and y two random variables with their respective densities $a(x)$ and $b(y)$ such that $z = b(x)/a(x)$. We apply the inequality $\ln z \leq z - 1$ and obtain

$$\ln b(x) - \ln a(x) = \ln \frac{b(x)}{a(x)} \leq \frac{b(x)}{a(x)} - 1$$

We multiply this by a(x) and integrate to obtain

$$\int_{-\infty}^{\infty} a(x)[\ln b(x) - \ln a(x)]dx \leq \int_{-\infty}^{\infty} [\ln b(x) - \ln a(x)]dx = 0$$

The right side is 0 because the functions $a(x)$ and $b(y)$ are densities by assumption. If we continue, we obtain

$$\int_{-\infty}^{\infty} a(x) \ln b(x) - a(x) \ln a(x) dx \leq 0$$

This gets us to the result

$$\int_{-\infty}^{\infty} a(x) \ln b(x) dx \leq \int_{-\infty}^{\infty} a(x) \ln a(x) dx$$

Then we can conclude that

$$\mathbf{E}\{\ln a(x)\} \geq \mathbf{E}\{\ln b(x)\}$$

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