Combinatorial Proof

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13-08-2019

Theorem. *For m,n,k non-negative integers* $\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m}=\binom{n}{m}2^{n-m}$

Proof. In terms of combinations, $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ is the sum of the nth row (counting from 0) of the binomial coefficients in the Pascal's triangle (Figure 1).

Let's take first $m = 1$ for which $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ is a more simple case. What does this mean?

On the left-hand side, it is similar to the triangle being $n-1$ in size. This results in 2^{n-1} combinations. Multiplying this by k gives

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

Increasing the number of rows omitted to $m \leq n$ would result in $\binom{n}{m}$ binomial coefficients in the Pascal triangle of size $n-m$. This is equal to the right-hand side of the equations. This means that the equation is valid and

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}$$

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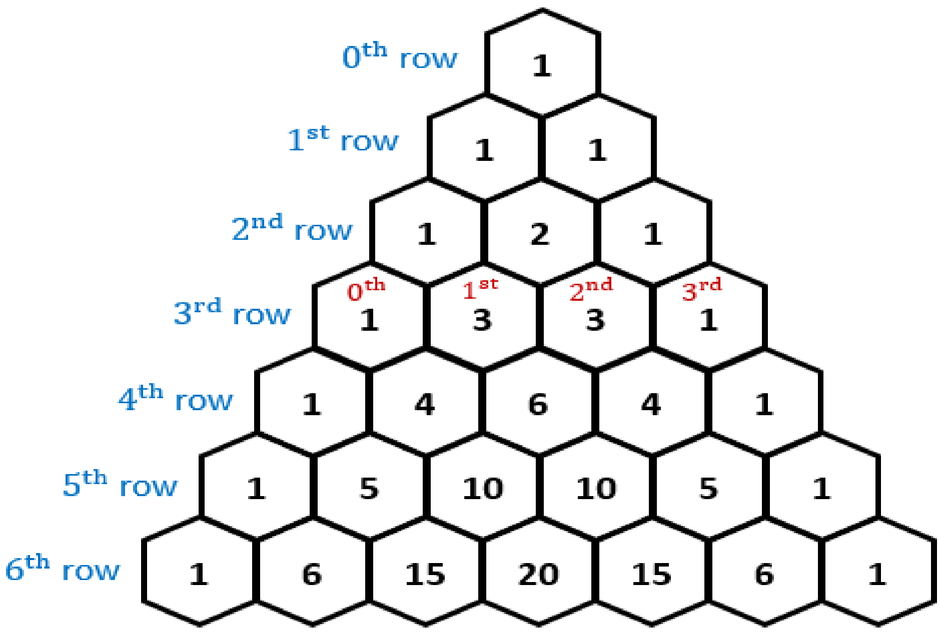

Figure 1: Representation of Pascal's triangle