The quaternion group Q8 = { 1,-1,i,-i,j,-j,k,-k }

Products computed as follows:

1.a = a.1 = a for all a Q8

(-1).(-1) = 1, (-1).a = a.(-1) = -a for all a Q8

i . i = j . j = k . k = -1

i . j = k , j . i = - k

j . k = i , k . j = - i

k . i = j , i . k = -j

Q8 is a non-abelian group of order 8

1] Compute the order of each of the elements in Q8

**Solution:**

Order of each element of Q8 : 1 has order 1 and -1 has the order 2. Since = = = -1 we see that i , j and k each have order 4 and since = = = -1 we know that

-i, -j and -k each have order 4 as well.

2]Group of tables for S3, D8 and Q8

**S3**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | 1 | (1 2) | ( 1 3) | (2 3) | (1 2 3) | (1 3 2) |
|  | 1 | (1 2) | (1 3) | (2 3) | (1 2 3) | ( 1 3 2) |
| ( 1 2) | ( 1 2) | 1 | ( 1 3 2) | ( 1 2 3) | (2 3) | (1 3) |
| ( 1 3) | ( 1 3) | ( 1 2 3) | 1 | ( 1 3 2) | ( 1 2) | ( 2 3) |
| ( 2 3) | ( 2 3) | ( 1 3 2) | (1 2 3) | 1 | ( 1 3) | ( 1 2) |
| ( 1 2 3) | ( 1 2 3) | ( 1 3) | (2 3) | ( 1 2) | (1 3 2) | 1 |

**D8**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | 1 | r | r^2 | r^3 | s | sr | sr^2 | sr^3 |
| 1 | 1 | r | r^2 | r^3 | s | sr | sr^2 | sr^3 |
| r | r | r^2 | r^3 | 1 | sr^3 | s | sr | sr^2 |
| r^2 | r^2 | r^3 | 1 | r | sr^2 | sr^3 | s | Sr |
|  |  |  |  |  |  |  |  |  |
| r^3 | r^3 | 1 | r | r^2 | sr | sr^2 | sr^3 | S |
| s | s | sr | sr^2 | sr^3 | s^2 | r | r^2 | r^3 |
| sr | sr | sr^2 | sr^3 | s | r^3 | 1 | r | r^2 |
| sr^2 | sr^2 | sr^3 | s | Sr | r^2 | r^3 | 1 | R |
| sr^3 | sr^3 | s | sr | Sr^2 | r | r^2 | r^3 | 1 |

**Q8**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | 1 | -1 | i | -i | j | -j | k | -k |
| 1 | 1 | -1 | i | -i | j | -j | k | -k |
| -1 | -1 | 1 | -i | i | -j | j | -k | k |
| i | i | -i | -1 | 1 | k | -k | -j | j |
| -i | -i | i | 1 | -1 | -k | k | j | -j |
| j | j | -j | -k | k | -1 | 1 | i | -i |
| -j | -j | j | k | -k | 1 | -1 | -i | I |
| k | k | -k | j | -j | -i | i | -1 | 1 |
| -k | -k | k | -j | j | i | -i | 1 | -1 |

3] Find a set of generators and relations for Q8

**Solution**

One presentation is = < -1, i, j, k | = = = i j k = -1 >

Where -1 commutes with the other elements of

Let’s get into this with some depth :

We want to find a subset of such that we generate all the other elements as products of the generators. Obviously a one element set is found to be insufficient.

But now consider { i, j } . We get k because i j = k. Also = - 1 and i j = k so the set can now be extended to { -1, i, j , k } . Now multiply these elements with -1 = and we get all the elements of . Thus { i, j } is a generating set.

Now we had prior knowledge that the generating set was generated by the following relations in the group { i.e. = = = -1 and i j k = - 1 } but suppose we need to find relations that do not involve anything besides { i, j } and we would like to say is the group generated by i and j such that some relation holds for i and j i.e. = { i, j | some conditions on i, j i.e.

We reiterate the relations :

= ------(1)

Multiplying the second equation with k we get i j = k so we might as well define k : = i j

Thus (1) is equivalent to :

The second equation is redundant – it’s same as (

So we get where (2) is equivalent to (1).

But -1 is still non trivial element.

So instead of writing

So we get the conditions : = 1, = , = -------(3)

So we get, j i = i

Now we can get a set of generators with the relations : = { i, j | = 1, , j i = i