**Problems DF 12 to 20**

12. If T = (1 2)(3 4)(5 6)(7 8)(9 10) determine if there is an n-cycle S with T = for some integer k.

Solution : 1->2

2-> 1 and the cycle closes

Likewise for every pair of elements.

Let’s try a composite map with application of S sequentially:

S. S Composite S^2

(1-> 2 (1-> 2 (a) 1-> 3. (using a and b except that we are applying alternatively)

2-> 3 2-> 3. (b). 3-> 5 ( using c and d similar to above)

3-> 4 3-> 4. ( c ) 5-> 7 ( using e, f, g, h)

4-> 5 4-> 5. (d). 7->9

5-> 6 5-> 6. (e). 9-> 10 ( identical steps)

6-> 7 6-> 7. (f)

7-> 8 7-> 8. (g)

8-> 9 8-> 9. (h)

9-> 10 ) 9 -> 10) (i)

Consider the example of (a b c d) (a b c d) = (a b) (c d) . Then if S = ( 1 2 3 4 5 6 7 8 9 10)

One sees that

b) T = ( 1 2) ( 3 4 5) determine if there is an n cycle S ( n >= 5) with T = for some integer k

Solution :

Suppose it is possible , let S be an n-cycle such that

If n > 5 then must fix 6, 7…but if S is an n cycle then the only way can fix these values if it fixes any value , that is = 1 Therefore we suppose that n = 5

Now since is not an n-cycle, k is not relatively prime to m. But n = 5 is prime, so 5 | k , so we must have k = 5 p do some integer p. Then = = ( = = 1 . This is a contradiction and so the assumption that S exists was invalid.

13] Element has order 2 in Sn iff its cycle decomposition is a product of commuting 2-cycles.

**Proof** :

Let n be a positive integer and that has the order 2. Let i , j be distinct integers in {1,2,3,…..n } such that . Then since , we must have Thus (i j) is a cycle in the cycle decomposition of These integers were randomly chosen and hence no cycle in the decomposition of has length more than 2. Thus can be written as a product of disjoint and commuting 2-cycles.

Now suppose that is a member of S(n) such that its cycle decomposition is a product of commuting 2-cycles , so that = (a1 b1)(a2 b2) …..(ak bk) and since each cycle commutes ,

We have =

Since is not the identity, | = 2

14] Let p be prime. Show that an element has order p in Sn iff its cycle decomposition is a product of p-commuting cycles. Show that this need not be the case if p is not prime.

**Proof**: Fix a positive integer n. Suppose that has order p. A cycle decomposition of

Where each Ti is a cycle and Ti is disjoint from Tj when ij

Now since these cycles are disjoint, they commute with each other and we can write that

1 = =

Since the original cycles are disjoint , it is clear that and are disjoint for

And so we must have = 1 for each i. This implies that the length of the cycle Ti divides p. But p is prime, so Ti is either a p-cycle or the identity. Hence is a product of commuting p-cycles.

To prove the converse, suppose that can be written as a product of commuting p-cycles for p a prime so that with each Ti being a p-cycle. Since the cycles commute, we have = 1

So | On the other hand, Since T is a p-cycle, for any positive integer t < p .

So cannot be the identity permutation. Therefore |

To prove the end part we suppose that p is not prime. For example, take p = 6 and n = 6 , then cycles

that commute.

15] Prove that the order of an element in S(n) equals the LCM of the lengths of the cycles in its cycle decomposition

**Proof:**

Let have the cycle decomposition , where each Ti is a cycle and the cycles are pairwise disjoint ( and therefore commute) . Suppose |

Then 1 = , which implies that for each i ( The Ti’s are all disjoint, and so if

So if Ti is a t-cycle , then t | n . Therefore n is the common multiple of the lengths of each cycle in the cycle decomposition of

Alternatively, if m is any common multiple of these lengths, then

And so we must have n<= m which shows that n is the least common multiple of the cycle lengths.

16] Show that if n >= m then the number of m-cycles in Sn is given by :

n(n-1)(n-2)….(n-m+1)/m

**Proof**:

We simply count the number of ways to form an m-cycle. There are n choices for the value in the first position, n-1 choices for the value in the second position, and (n-m+1) choices for the m th position.

However each cycle can be represented in m different ways, depending on the choice of the starting value. So the actual number of cycles is given by the above expression.

17] If n >= 4, then the number of permutations in Sn which are the product of two disjoint 2-cycles is n(n-1)(n-2)(n-3)/8

**Proof** :

We note that there are n(n-1)/2 ways to choose the first 2-cycles and there are (n-2)(n-3)/2 ways to choose the second 2-cycle. There are (n-2)(n-3)/2 ways to choose the second 2-cycle.However the order of the two 2-cycles does not matter and so we divide the product by 2 to get n(n-1)(n-2)(n-3)/8 possibilities.

18] Find all numbers n such that S5 contains an element of order n

**Solution** : Now n must be in the interval (1,5) and we must find an n-cycle in S5. n = 6 is also possible because of (1 2)(3 4 5) which has the order 6 but we note that a combination of longer cycles of different lengths is not possible because the underlying set has 5 elements only ( {1, 2, 3, 4, 5} . Therefore the possibilities are 1,2, 3,4,5 and 6

19] Find all numbers n such that S7 contains an element of order n.

**Solution**:

Clearly now n = 1,2…7 are the elements of the set and are valid. If contains a 2-cycle then the only other cycles that can be in the decomposition are ones that have lengths of 3, 4 and 5. The LCM of each of these is (2,3) = 6(order of 6) , (2,4) = 4 (order of 4)and (2,5) = 10 which would have an order of 10. A 3-cycle with a 4-cycle produces a permutation of order = 12. So the only possible orders are n = 1,2,3,4,5,6,7,10 and 12.

20] Find a set of generators and relations for S3

**Solution** :

The set S3 contains 1, (1 2),(1 3),(2 3), (1 2 3), and ( 1 3 2). By taking the powers of each element we see that S3 is not cyclic, so we need at least two generators. Let . Then (1 3) = , (2 3) = and (1 3 2) =

We have the relations but there is not enough information to deduce that has the order 2. So we try = 1 and now we can determine the orders of the remaining elements . So S3 = <