Problem 22

Let A be an Abelian Group and fix some k ∈ Z. Prove that the map a ->a^k is a homomorphism from A to itself

If k = -1 , prove that this homomorphism is an isomorphism ( that is an automorphism of A)

Solution:

Fix k ∈ Z., and let p: A —> A be the mapping a -> a^k.

Then for a, b ∈ A, we have p(ab) = (ab)^k = a^k b^k = p(a) p(b) ( As A is abelian)

So p is a homomorphism

In this case k = -1, we get a -> a^(-1) , p must be a bijection since it is its own inverse function

Hence a-> a^(-1) is an automorphism of A

Problem 23

Let G be a finite group which possesses an automorphism s such that s(g) = g iff g = 1. If S^2 is the identity map from G -> G , prove that G is abelian ( such an automorphism S is called the fixed point free of order 2)

Proof :

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Consider the map P:G->G given by P(x) = x^(-1) s(x). For any x, y ∈ G,

If P(x) = P(y) then

x^(-1) s(x) = y^(-1) s(y)

Rearranging,

s(y) = y x^(-1) s(x)

This further yields, y = s(s(y)) = s(y x ^(-1) s(x)) = s(y x ^(-1) x

And multiplying on the right by x ^(-1) gives y x ^(-1) = s(y x^(-1) ——(1)

Since s is fixed point free, (1) implies that yx^(-1) = 1 or x = y. Therefore P is an injection, and hence a bijection since it maps the finite set G to itself.

Therefore every x ∈ G can be written in the form x = y^(-1)s(y) for some y ∈ G

Problem 24

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Let G be a finite group and let x and y be distinct elements of order 2 in G that generate G.

Prove that G ≅ D 2n where n = | x y |

Solution:

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Let x, y be distinct elements that generate G

We know that t = x y , t x = x t ^(-1)

Let n = | x y | , the order of t

We may express any member of G uniquely in the form, x^i t ^j for some integers i,j with 0<=i<= 1 and 0<=j < n ( the representation is unique since t has order n, which implies that t, t^2,…t^(n-1) are all distinct)

Therefore |G| = 2n

Now let p : D2n -> G be given by

p(s^i r ^j) = x^i t ^ j for I, j ∈ Z with 0<=i<= 1 and 0<=j <= n-1

Since every element in D2n can be uniquely written as s^ir^j with above restrictions on I and j, the function p is well defined and since x and t satisfy the same relations in G that s and r satisfy in D2n, p must be a homomorphism

Now we show that p is a bijection. For any b ∈ G, write b = x^it^j for I ∈ {0,1} and j in {0,1…n-1}

Then if a = s^i r^j we have s(a) = b which shows that p is surjective. Since |G= |D2n| this is enough to show that p is a bijection. The function p is a bijective homomorphism , hence it is an isomorphism and D2n ≅ G

Problem 25

Let n ∈ Z +, let r and s be the usual generators of D2n and let 𝜃 = 2Pi/n

1. Prove that the matrix ❩[ cos 𝜃. -sin𝜃 is the matrix of the linear transformation which

sin𝜃. cos𝜃. ]

Rotates the x , y plane about the origin in a counter clockwise direction by 𝜃 radians

B) Prove that the map p : D2n -> GL2(R) defined on generators by w(r) = [ cos 𝜃. -sin 𝜃.

sin 𝜃. cos 𝜃.] and

w(s) = [ 0. 1]. Extends to a homomorphism of D2n into GL2(R)

1. 0]

C) Prove that the homomorphism w in part(b) is injective

Solution:

For a vector (x , y) in R2 we have

[cos 𝜃. -sin 𝜃.] [x]. = [ x cos 𝜃.-ysin 𝜃.]

[sin 𝜃. cos 𝜃.] [y] [x sin 𝜃. + y cos 𝜃.]

The distance d of this point from the origin is d = sort (x cos 𝜃.-y sin 𝜃.)^2 + (x sin 𝜃.+ y cos 𝜃.)^2

= sqrt(x^2 cos^2 𝜃. + y^2 sin^2 𝜃. + x^2 sin^2 𝜃. + y^2 cos^2 𝜃.

= sqrt(x^2 + y^2)

Which is the same distance as (x, y) is from the origin

Moreover the angle α. Between these two vectors is given by

cos α = [x(x cos 𝜃.-ysin 𝜃.) + y( x sin 𝜃. + y cos 𝜃.)]/(x^2 + y^2) = cos 𝜃. = cos (2Pi/n)

So we see that α = 2Pi/n

This shows that the image of the point (x, y) under this transform nation is the same point rotated about the origin by an angle of 𝜃.

b) Since [0 1]. Is easily seen to be a reflection across the line y = x, it is evident that w(r) and W(s) satisfy the same relations as do r and s

Namely if I is the 2 X 2 identity matrix, we have

w(r)^n = w(s)^2 = I and w(r) w(s) = W(s) W(r) ^(-1)

The latter relation comes from the fact that reflecting across the line y = x and then rotating by 𝜃. Is the same as first rotating by 2Pi-𝜃. And then reflecting across the line.

Since w(r) and w(s) satisfy the same relations as the corresponding generators of D(2n) , wee see that w extends to a homomorphism.

c) Let H denote the subgroup of GL2(R) generated by w(r) and w(s). Then the function is: D2n->H defined by restricting the codomain of w is surjective. But it is not difficult to see that |H| = 2n = |D2n| so the map is and hence w must also be injective

Problem 26

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Let I and j be the generators of Q8. Prove that the map p from Q8 to GL2(C) defined on Generators by

s(i) = [ √-1. 0]. And s(j) = [0 -1]

[ 0. -√-1]. [1 0]

Extends to a homomorphism. Prove that s is an injective

Proof:

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First, we have s(i) ^2 = [√-1. 0]. ^ (2) = [-1. 0] = -I

[ 0. -√-1] [ 0. -1]

s(j) ^2 = [0 -1] ^ 2 = [-1 0] = -I

[1. 0] [ 0 -1]

So we may take s(-1) = -I and -I commutes with all members of GL2(C)

Also, s(i) s(j) = [ √-1 0] [0 -1]. = [ 0. -√-1 ].

[. 0 -√-1]. [1. 0] [ -√-1. 0. ].

So we may let s(k) = [ 0. -√-1]

[-√-1. 0 ]

Now s(k)^2 = -I as expected

To put it all together, s(i)^2 = s(j)^2 = s(k)^2 = s(i) s(j) s(k) = s(-1)

So s(i), s(j) , s(k) and s(-1) satisfy all the the same relations as given in the presentation for Q8 and therefore s extends to a homomorphism

Lastly, consider the subgroup of GL2 (C) generated by s(i), s(j) ,s(k) and s(-1)

Very easy to see that this sub group contains 8 elements ( those named plus their inverses and the identity) . So the function obtained from s by restricting its codomain to this subgroup must be surjective. Since its domain and codomain share the same cardinality, it must also be injective . Hence s is injective.