ON POWER VALUES OF SUM OF DIVISORS FUNCTION IN ARITHMETIC PROGRESSIONS

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Abstract. Let $a \geq 1, b \geq 0$ and $k \geq 2$ be any given integers. It has been proven that there exist infinitely many natural numbers m such that sum of divisors of m is a perfect kth power. We try to generalize this result when the values of m belong to any given infinite arithmetic progression $an + b$. We prove that there exist infinitely many natural numbers n such that sum of divisors of $an + 1$ is a perfect kth power. We also prove that, either sum of divisors of $an + b$ is not a perfect kth power for any natural number n or sum of divisors of $an + b$ is a perfect kth power for infinitely many natural numbers n.

1. Introduction

Sum of divisors function (denoted by σ) is an important function in number theory. Perfect numbers, natural numbers n satisfying $\sigma(n) = 2n$, have been studied as early as 300 BC. Euclid showed in [E] that $\frac{q(q+1)}{2}$ is an even perfect number whenever q is a prime number of the form $2^p - 1$, for natural number p. Euler (in [LE]) proved that all even perfect numbers are of this form. It is still not known whether there are infinitely many perfect numbers and whether there exists an odd perfect number.

The problem of finding natural numbers n such that $\sigma(n)$ is a perfect kth power has been considered by 17th century mathematicians like Fermat, Wallis, and Frenicle. Fermat in 1657, asked two problems: (i) Find a cube $m = n^3$ for which $\sigma(m)$ is a perfect square (ii) Find a square $m = n^2$ for which $\sigma(m)$ is a perfect cube (see Chapter 2, p.54 of [D66] for detailed history).

In [B12] (see also [F12], [Z18]) it has been proven that for any natural number $k \geq 2$, $\sigma(m)$ is a perfect kth power for infinitely many natural numbers m. There are several sequences in Neil Sloane's online encyclopedia of integer sequences [S] dedicated to this problem, namely A006532 ($\sigma(m)$ is perfect square), A020477 ($\sigma(m)$ is perfect cube), A019422 ($\sigma(m)$ is perfect fourth power), A019423 ($\sigma(m)$ is perfect fifth power), A019424 ($\sigma(m)$ is

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perfect sixth power), A048257 ($\sigma(m)$ is perfect seventh power) and A048258 ($\sigma(m)$ is perfect eighth power).

In this paper, we investigate this problem when the values of m belong to any given infinite arithmetic progression $\{an + b : n \in \mathbb{N}\}$ (here $\mathbb{N} = \{1, 2, 3, \ldots\}$). We study the following problem.

Q. *For any given integers $a \geq 1, b \geq 0, k \geq 2$ do there exist infinitely many natural numbers n such that $\sigma(an + b)$ is a perfect kth power?*

In the next section, we give an answer to the above problem when $b = 1$. We prove the following theorem.

Theorem 1. *For any natural number a and $k \geq 2$ there exist infinitely many natural numbers n such that $\sigma(an + 1)$ is a perfect kth power.*

For general b, we prove the following theorem.

Theorem 2. *For any given integers $a \geq 1, b \geq 0$ and $k \geq 2$, either $\sigma(an+b)$ is not a perfect kth power for any natural number n or $\sigma(an+b)$ is a perfect kth power for infinitely many natural numbers n.*

In the final section, we give few open questions for further research.

2. Proofs of Theorem 1 and Theorem 2

We require some lemmas in order to prove Theorem 1 and Theorem 2. We will use a group theoretic result mentioned in [A94] (Theorem 2 of [A94]).

Theorem(Alford et al.) 3. *If G is a finite abelian group in which the maximal order of an element is m, then in any sequence of at least $m(1 + \log(|G|/m))$ (not necessarily distinct) elements of G, there is a nonempty subsequence whose product is the identity.*

Using the above theorem we can prove this lemma.

Lemma 1. *Let $k \geq 2$ be any natural number and P be any nonempty finite set of primes. Let A be any finite set of natural numbers such that all the prime factors of all elements of A belong to P. If $|A| \geq k|P| \log k + 1$ then there exists a nonempty subset $A' \subset A$ such that $\prod_{n \in A'} n$ is a perfect kth power.*

Proof. Let $n_1, \ldots, n_{|A|}$ be distinct elements of A, and let $p_1, \ldots, p_{|P|}$ be distinct elements of P. Let $G = (\mathbb{Z}/k\mathbb{Z})^{|P|}$. As all the prime factors of

n_i are in P there exist nonnegative integers $a_{i,1}, \ldots, a_{i,|P|}$ such that $n_i = p_1^{a_{i,1}} \cdots p_{|P|}^{a_{i,|P|}}$. Now define $e_i \in G$ in the following way:

$$e_i = (a_{i,1} \bmod k, \ldots, a_{i,|P|} \bmod k),$$

for $1 \leq i \leq |A|$. Now for group G, we have that the maximal order(m) of group equal to k, and number of elements in the group is equal to $|G| = k^{|P|}$. As

$$\begin{aligned} m(1 + \log(|G|/m)) &= k(1 + \log(k^{|P|-1})) \\ &= k|P|\log k + k - k\log k \\ &\leq k|P|\log k + 1 \\ &\leq |A|, \end{aligned}$$

we have, number of elements in the sequence e_i, $|A| \geq m(1 + \log(|G|/m))$, therefore we can apply above theorem. From Theorem 3 (Theorem 2 of [A94]), there exists a nonempty subsequence $e_{m_1}, \ldots, e_{m_s}$ such that

$$\sum_{i=1}^{s} e_{m_i} = (0, \ldots, 0).$$

Let $A' = \{n_{m_1}, \ldots, n_{m_s}\} \subset A$ then it follows from definition of e_i and $\sum_{i=1}^{s} e_{m_i} = (0, \ldots, 0)$ that all the exponents in the prime factorization $\prod_{n \in A'} n$ are multiples of k, which implies that $\prod_{n \in A'} n$ is a perfect kth power. □

Let ϕ denote Euler's totient function. Using the fact that the infinite product over primes $\prod_p \left(1 - \frac{1}{p}\right)^{-1}$ diverges we prove the following lemma.

Lemma 2. *For any given natural numbers $a, k \geq 2$ there exists a natural number b which is relatively prime to a, and satisfying the inequality*

$$\frac{b}{\phi(b)} > (k \log k + 1)\phi(a).$$

Proof. As the product over primes $\prod_{p \nmid a} \left(1 - \frac{1}{p}\right)^{-1}$ diverges, there exist primes $p_1, \ldots, p_t$ with $p_i \nmid a$ for $1 \leq i \leq t$ and

$$\prod_{i=1}^{t} \left(1 - \frac{1}{p_i}\right)^{-1} > (k \log k + 1)\phi(a).$$

Let $b = \prod_{i=1}^{t} p_i$, as $p_i \nmid a$ for $1 \leq i \leq t$ we can conclude that b is relatively prime to a. Now,

$$\frac{b}{\phi(b)} = \prod_{i=1}^{t} \left(\frac{p_i}{p_i - 1}\right) = \prod_{i=1}^{t} \left(1 - \frac{1}{p_i}\right)^{-1} > (k \log k + 1)\phi(a),$$

which proves the lemma. □

Using Lemma 1 and Lemma 2, we can prove the following lemma.

Lemma 3. *For any natural numbers a and $k \geq 2$ there exists a positive real number $x_0(a)$ such that for all $x \geq x_0(a)$ there exists a natural number n such that all the prime factors of $an+1$ are in the interval $(\sqrt{x}, x]$ and $\sigma(an+1)$ is a perfect kth power.*

Proof. Given natural numbers a and $k \geq 2$, from Lemma 2, there exists a natural number b relatively prime to a such that

$$\frac{b}{\phi(b)} > (k \log k + 1)\phi(a).$$

For any $x \geq b^2$, define

$$A = \{p+1 : p \text{ prime}, \sqrt{x} < p \leq x, p \equiv 1 \bmod a, p \equiv -1 \bmod b\},$$

and

$$P = \left\{p : p \text{ prime}, \, p \leq \frac{x+1}{b}\right\}.$$

If $p+1 \in A$ then as $p \equiv -1 \bmod b$ we have $p+1 = b(\frac{p+1}{b})$, as $b \leq \frac{x+1}{b}$ and $\frac{p+1}{b} \leq \frac{x+1}{b}$, all prime factors of $p+1$ are less than or equal to $\frac{x+1}{b}$. Therefore, all prime factors of $p+1$ lie in the set P. From Prime Number Theorem and Prime Number Theorem for Arithmetic progressions we have $|A| \sim \frac{x}{\phi(a)\phi(b)\log x}$ and $|P| \sim \frac{x}{b \log x}$, which implies $k|P|\log k + 1 \sim \frac{kx \log k}{b \log x}$.

As $\frac{b}{\phi(b)} > (k \log k+1)\phi(a)$ we have $\frac{1}{\phi(a)\phi(b)} > \frac{k \log k}{b}$. Since, $|A| \sim \frac{x}{\phi(a)\phi(b)\log x}$ and $k|P|\log k+1 \sim \frac{kx \log k}{b \log x}$, we must have $|A| \geq k|P|\log k+1$ for sufficiently large $x \geq x_0(a)$, hence we can apply Lemma 1 for $x \geq x_0(a)$. Therefore, for $x \geq x_0(a)$, there exists a nonempty subset $A' \subset A$ such that

$$\prod_{p+1 \in A'} (p+1) = \sigma\left(\prod_{p+1 \in A'} p\right)$$

is a perfect kth power.

Since, $p+1 \in A'$ implies $p \equiv 1 \bmod a$ we have $\prod_{p+1\in A'} p \equiv 1 \bmod a$. Hence, $\prod_{p+1\in A'} p$ is of the form $an+1$ whose sum of divisors is a perfect kth power and all prime divisors of $\left(\prod_{p+1\in A'} p\right)$ lie in the interval $(\sqrt{x}, x]$. □

We can now prove Theorem 1 and Theorem 2 using Lemma 3.

2.1. **Proof of Theorem 1.**

Proof. Assume for a contradiction that there are only finitely many n for which $\sigma(an+1)$ is a perfect kth power, then there exists an $x_1 > 0$ such that $\sigma(an+1)$ is not a perfect kth power for any $n \geq x_1$. This implies that there cannot be any $an+1$ all of whose all prime factors lie in the range $(\sqrt{x}, x]$

for $x = \max\{x_0(a), (ax_1+1)^2\}$ and whose sum of divisors is a perfect kth power, contradicting Lemma 3. □

2.2. **Proof of Theorem 2.**

Proof. If $\sigma(an+b)$ is not a perfect kth power for all natural numbers n, there is nothing to prove. If there exists a natural number n' such that $\sigma(an'+b)$ is perfect kth power then define a sequence x_n with $x_1 = \max\{(an'+b)^2, x_0(a)\}$, $x_{n+1} = x_n^2$ for $n \geq 1$.

From Lemma 3, for all natural numbers m there exists n_m such that $\sigma(an_m+1)$ is a perfect kth power and all prime factors of an_m+1 lie in the interval $(\sqrt{x_m}, x_m]$. Since, $(\sqrt{x_m}, x_m]$ are disjoint intervals for $m \in \mathbb{N}$, natural numbers an_m+1 along with $an'+b$ are pairwise relatively prime and hence distinct.

Define $s_m = (an'+b)(an_m+1)$ for all natural numbers m. We can observe that s_m is of the form $an+b$ and

$$\sigma(s_m) = \sigma(an'+b)\sigma(an_m+1)$$

is a perfect kth power as product of perfect kth powers is also a perfect kth power. Therefore, if there exists a natural number n' such that sum of divisors of $an'+b$ is a perfect kth power then there exist infinitely many natural numbers n such that sum of divisors of $an+b$ is a perfect kth power. □

3. Future prospects

We ask five questions Q1 to Q5 in this section.

Given a, b and $k \geq 2$, Theorem 2 leaves us with two possibilities, either $\sigma(an+b)$ is perfect kth power for infinitely many n or $\sigma(an+b)$ is never a perfect kth power. This implies that if there exists a natural number n such that $\sigma(an+b)$ is perfect kth power then there are infinitely many n such that $\sigma(an+b)$ is a perfect kth power. So we ask the following question.

Q1. *Do there exist integers $a \geq 1, b \geq 0, k \geq 2$ such that $\sigma(an+b)$ is not a perfect kth number for any natural number n?*

If the answer to Q1 is no, then from Theorem 2, it follows that for all integers $a \geq 1, b \geq 0, k \geq 2$ there exist infinitely many n such that $\sigma(an+b)$ is a perfect kth power.

For natural numbers a, $k \geq 2$ and integer b such that $0 \leq b \leq a-1$, let $N(x; a, b, k)$ denote number of natural numbers $n \leq x$ such that $n \equiv a \mod b$ and $\sigma(n)$ is perfect kth power.

Q2. *Given natural numbers $a, 0 \leq b_1, b_2 \leq a-1, k \geq 2$ such that $N(x; a, b_1, k) > 0$, $N(x; a, b_2, k) > 0$ for some real number x. Is it true that*

$$\liminf_{x\to\infty} \frac{N(x; a, b_1, k)}{N(x; a, b_2, k)} > 0?$$

We will now ask a stronger version of the above question. Are perfect kth power values of $\sigma(n)$ equally distributed among residue classes modulo a?

Q3. *Given natural numbers $a, 0 \leq b_1, b_2 \leq a-1, k \geq 2$, is it true that there exists a real number x such that $N(x; a, b_1, k) > 0$, $N(x; a, b_2, k) > 0$ and*

$$\lim_{x\to\infty} \frac{N(x; a, b_1, k)}{N(x; a, b_2, k)} = 1?$$

Numerical evidence suggests that Q3 may not always be true, for example when we consider perfect square values of $\sigma(n)$ for $n \leq x$ and residue classes $n \equiv 0 \bmod 7$ and $n \equiv 1 \bmod 7$, we have the following data.

x	$N(x; 7, 0, 2)$	$N(x; 7, 1, 2)$	$\frac{N(x;7,0,2)}{N(x;7,1,2)}$
100	2	2	1.0000...
1000	16	5	3.2000...
10000	68	28	2.4285...
100000	307	147	2.0884...
1000000	1508	748	2.1299...
10000000	7562	3811	1.9842...
100000000	37652	18882	1.9940...

TABLE 1. Comparison of $N(x; 7, 0, 2)$ and $N(x; 7, 1, 2)$

Table 1 suggests that, there might be larger number of natural numbers n of the form $7k$ compared to $7k+1$ for which $\sigma(n)$ is a perfect square. We ask the following question.

Q4. *Is it true that*

$$\limsup_{n\to\infty} \frac{N(x; 7, 0, 2)}{N(x; 7, 1, 2)} > 1?$$

If the answer for question Q4 is yes, then the answer for Q3 is no.

Till now, we have been dealing with linear polynomials $an+b$. A natural generalization would be the following question.

Q5. *Let $p(n)$ be any given polynomial with nonnegative integer coefficients and $k \geq 2$ be any natural number. Is $\sigma(p(n))$ perfect kth power for infinitely many n?*

Question 2.5 (when $p(n) = n^l$), Conjecture 2.6 (when $p(n) = n^2$), Conjecture 2.7 (when $p(n) = n^3$) of [B12] are special cases of the above question. In Neil Sloane's online encyclopedia of integer sequences [S] we find few sequences related to this problem, namely, A008847 ($\sigma(n^2)$ is a perfect square, $p(n) = n^2$, $k = 2$), A008850 ($\sigma(n^2)$ is a perfect cube, $p(n) = n^2$, $k = 3$) and A008849 ($\sigma(n^3)$ is a perfect square, $p(n) = n^3$, $k = 2$).

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