Motion in two Dimension

If two co-ordinates are required to specify the position of the object in space changes w.r.t. time then it is called two dimensional motion.

In such a motion, the object moves in a plane.

Projectile Motion

A particle thrown in the space which moves under the effect of gravity alone is called projectile and its motion is called 'projectile motion'.

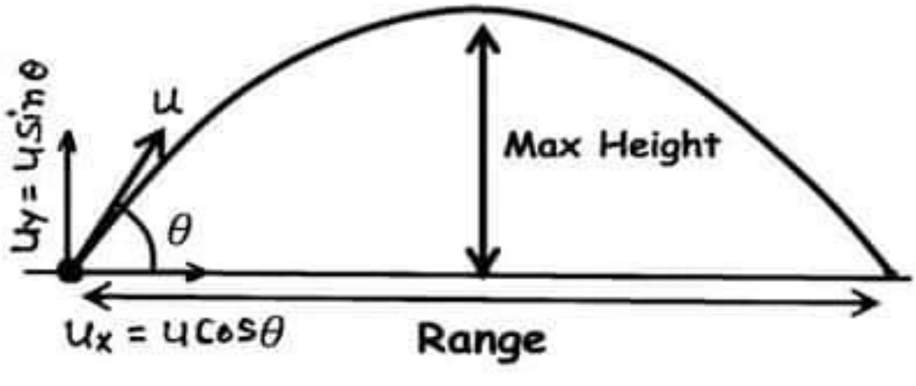
Example – (i) A bomb released from an aeroplane
(ii) An arrow released from bow

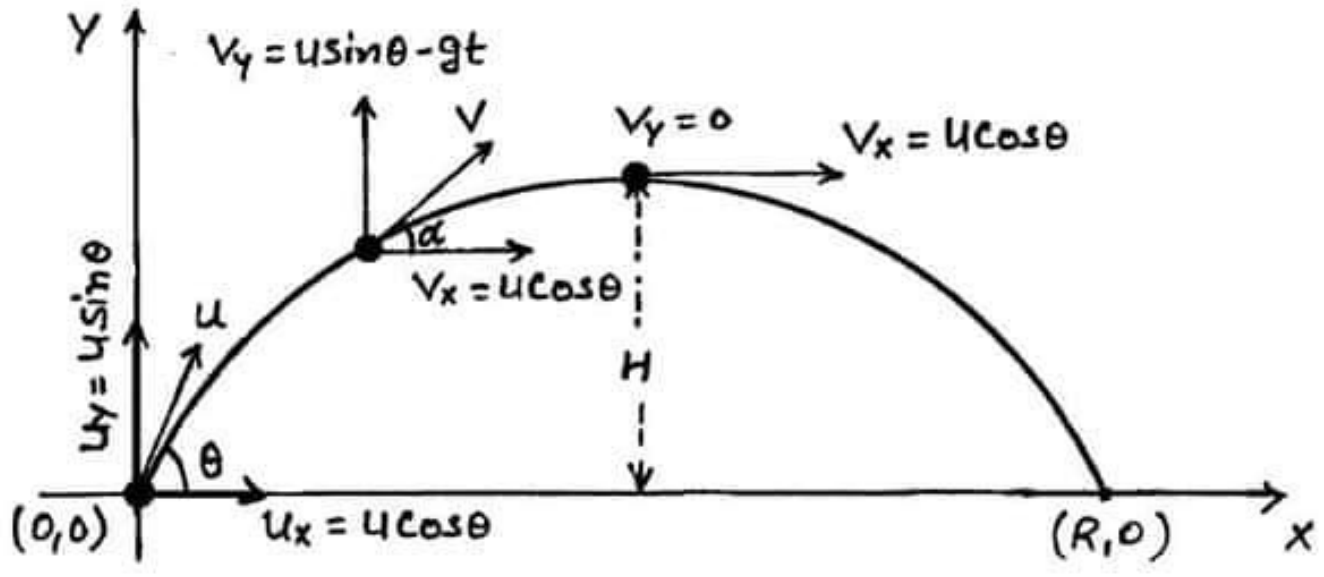
Note –

(i) Projectile motion is an example of constant acceleration.
(ii) If a constant acceleration is given to particle in oblique direction with initial velocity, then the resultant path is parabolic.

Ground to Ground Projection

Projectile Motion = Horizontal Motion + Verticle motion





Horizontal Motion

→ Initial Velocity $u_x = u\cos\theta$

→ Acceleration $a_x = 0$ (Neglect air resistance)
Therefore horizontal velocity remains unchanged.
Horizontal velocity at any instant $u_x = u\cos\theta$

→ At any time t, x coordinate or displacement along x-direction is

$$x = u_x t \quad \text{or} \quad x = (u\cos\theta)\times t$$

Verticle Motion

It is the motion under the effect of gravity, so that as particle moves upwards its verticle speed decreases. i.e. $a_y = -g$

→ Verticle speed at time t, $v_y = u_y - gt$

→ Displacement in verticle direction in time t,

$$y = u_y t - \frac{1}{2}gt^2 \quad \text{or} \quad y = (u\sin\theta)t - \frac{1}{2}gt^2$$

Here y is the 'height' of the particle above the ground.

Net Motion

Net initial velocity $\vec{u} = u_x\hat{i} + u_y\hat{j}$

$$\vec{u} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$$

Net acceleration $\vec{a} = a_x\hat{i} + a_y\hat{j}$

$$\vec{a} = -g\hat{j} \qquad [\because a_x = 0]$$

Coordinate of particle at time t

$$x = u_x t \quad \text{and} \quad y = u_y t - \tfrac{1}{2}gt^2$$

Net Displacement in time t is $\boxed{s = \sqrt{x^2+y^2}}$

Velocity of particle at time t

$$\vec{v} = v_x\hat{i} + v_y\hat{j} \Rightarrow \vec{v} = u_x\hat{i} + (u_y - gt)\hat{j}$$

or $$\vec{v} = u\cos\theta\,\hat{i} + (u\sin\theta - gt)\hat{j}$$

Magnitude of velocity $\boxed{v = \sqrt{v_x^2 + v_y^2}}$

If angle of velocity $\vec{v}$ from the ground is α, then

$$\boxed{\tan\alpha = \frac{v_y}{v_x}} \qquad \tan\alpha = \frac{u\sin\theta - gt}{u\cos\theta}$$

Time of Flight (T)

At time $t = T$, the particle will be at ground again i.e. displacement along Y-axis becomes zero.

$$\because\ y = (u\sin\theta)t - \tfrac{1}{2}gt^2 \qquad \therefore\ 0 = (u\sin\theta)T - \tfrac{1}{2}gT^2$$

or $$\boxed{T = \frac{2u\sin\theta}{g}} \qquad \boxed{T = \frac{2u_y}{g}}$$

$$\boxed{\text{Assending time} = \text{Desending time} = \frac{T}{2} = \frac{u\sin\theta}{g}}$$

At time $\frac{T}{2}$, particle attains maximum height of its trajectory.

Horizontal Range (R)

It is the displacement of the particle along x-direction during its complete flight.

$\because \quad x = (u\cos\theta)\,t \qquad \therefore \quad R = (u\cos\theta)\,T$

or $\quad R = u\cos\theta \times \frac{2u\sin\theta}{g} = \frac{2u^2\sin\theta\cos\theta}{g}$

$$\boxed{R = \frac{u^2\sin 2\theta}{g}} \Rightarrow \boxed{R = \frac{2u_x u_y}{g}}$$

Maximum Height attained (H)

At maximum height verticle velocity becomes zero. At this instant y coordinate is its maximum height.

$\because \quad v_y^2 = u_y^2 - 2gy \qquad \therefore \quad 0 = u_y^2 - 2gH$

$$H = \frac{u_y^2}{2g} \Rightarrow \boxed{H = \frac{u^2\sin^2\theta}{2g}}$$

$$\boxed{H = \frac{u_y^2}{2g}}$$

Equation of Trajectory

We know that $x = (u\cos\theta)t$ and $y = (u\sin\theta)t - \frac{1}{2}gt^2$ on eliminating t from these two equations, we get

$$y = (u\sin\theta)\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

or $$\boxed{y = x\tan\theta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}}$$

This is the equation of parabola. $[y = ax - bx^2]$

Energy of Projectile

When a projectile moves upward its Kinetic energy decreases, potential energy increases but the total energy always remains constant.

Kinetic energy $= \frac{1}{2}m(u\cos\theta)^2 = \frac{1}{2}mu^2\cos^2\theta$

Potential energy $= mgH = mg\frac{u^2\sin^2\theta}{2g}$

$= \frac{1}{2}mu^2\sin^2\theta$

Total Energy = Kinetic energy + Potential energy

$= \frac{1}{2}mu^2\cos^2\theta + \frac{1}{2}mu^2\sin^2\theta$

$= \frac{1}{2}mu^2$ = Energy at the point of projection

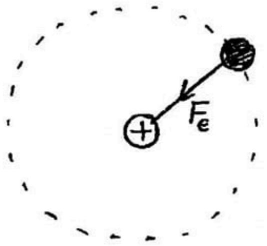
This is in accordance with the Law of conservation of energy.

Circular Motion

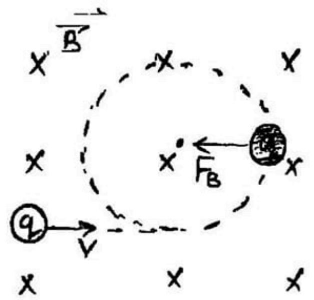
When a particle moves in a plane such that its distance from a fixed point remains constant then its motion is called as circular motion with respect to the fixed point.

The fixed point is called centre and the distance is called radius of circular path.

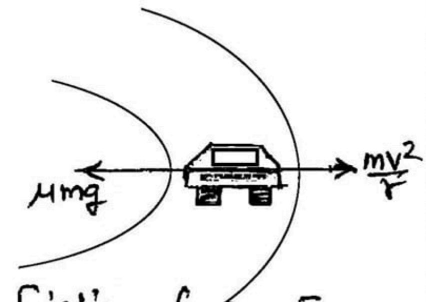
Note- Circular motion के लिए Centre की ओर एक Force होना जरूरी है, जिसे Centripetal force कहते हैं। अलग-अलग Conditions में अलग-अलग type के force (जैसे Electric force, Magnetic force, friction force, Reaction force, tension force etc.) इसे Provide कर सकते हैं।



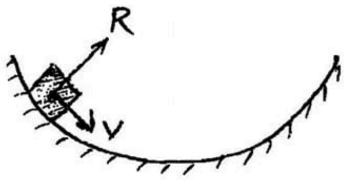
$F_e = F_c$



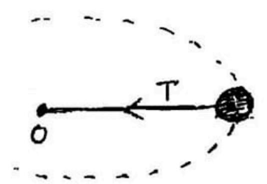
$F_B = F_c$



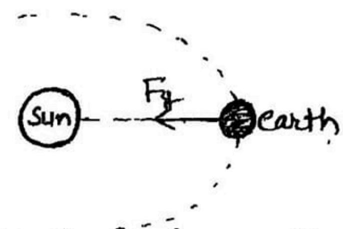
friction force = F_c



Reaction force = F_c



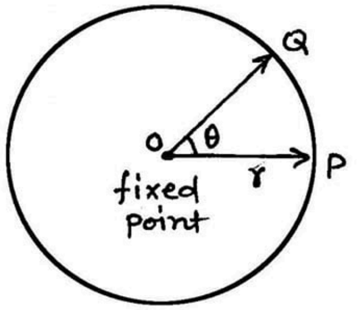
Tension force = F_c



Gravitational force = F_c

Angular Displacement (θ) -

Angle traced by position vector of a particle moving w.r.t. some fixed point is called angular displacement.



$$\boxed{\text{Angular displacement } \theta = \frac{\text{Arc } PQ}{r}}$$

* Its direction is perpendicular to plane of rotation and given by right hand screw rule.
* It is dimensionless and has SI unit 'Radian'

$$2\pi \text{ radian} = 360^\circ = 1 \text{ revolution.}$$

Note - small angular displacement $\vec{d\theta}$ is a vector quantity but large angular displacement θ is scalar quantity.

Frequency (n) -

Number of revolutions describes by particle per second is its frequency.
Its unit is revolution per sec. (r.p.s.)

Time Period (T) -

It is time taken by particle to complete one revolution.

$$\boxed{T = \frac{1}{n}}$$

Angular Velocity (ω) -

It is defined as the rate of change of angular displacement of moving object.

$$\text{Angular Velocity} = \frac{\text{Angle traced}}{\text{Time taken}}$$

$$\boxed{\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}}$$

* Its SI unit is **radian/sec.**
* It is an axial vector quantity. Its direction is perpendicular to the plane of rotation and along the axis according to right hand screw rule.

Relation between Linear and Angular Velocity -

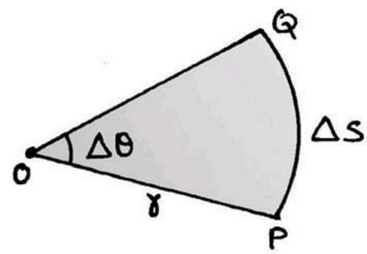
Angle $= \dfrac{\text{Arc}}{\text{Radius}}$ or $\Delta\theta = \dfrac{\Delta s}{r}$

or $\Delta s = r\,\Delta\theta$

$\therefore \dfrac{\Delta s}{\Delta t} = \dfrac{r\Delta\theta}{\Delta t}$ if $\Delta t \to 0$ then $\dfrac{ds}{dt} = r\dfrac{d\theta}{dt}$

or $\boxed{v = r\omega}$

In vector form $\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$



direction of $\vec{v}$ is according to right hand thumb rule.

Average Angular Velocity (ω_{av}) -

$$\omega_{av} = \frac{\text{total angle of rotation}}{\text{total time taken}}$$

$$\boxed{\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}}$$

where θ_1 and θ_2 are angular position of the particle at instant t_1 and t_2.

Instantaneous Angular Velocity (ω) -

It is the angular velocity at a particular instant.

$$\boxed{\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}} \quad \text{or} \quad \boxed{\vec{\omega} = \frac{d\vec{\theta}}{dt}}$$

Angular Acceleration (α)

The rate of change of angular velocity is called angular acceleration.

$$\boxed{\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}}$$

or

$$\boxed{\vec{\alpha} = \frac{d\vec{\omega}}{dt}}$$

* It is an axial vector quantity. Its direction is along the axis according to the '**Right hand Rule**'.
* Its SI Unit is **radian/sec²**.

Note- अगर Particle की speed बढ़ रही है तो Angular velocity भी बढ़ेगी और α की Direction ω की Direction में ही होगी।

अगर Particle की Speed कम हो रही है तो Angular velocity भी कम होगी और α की direction, ω की direction के Opposite होगी।

Relation between Angular and linear Acceleration

We Know that $\vec{v} = \vec{\omega} \times \vec{r}$

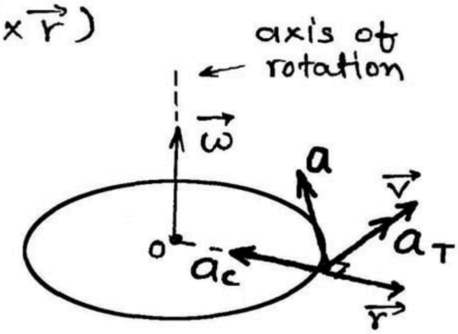
Here $\vec{v}$ is a tangential vector, $\vec{\omega}$ is a axial vector and $\vec{r}$ is a radial vector. These three vectors are mutually perpendicular.

but $\vec{a} = \frac{d\vec{v}}{dt}$ $\Rightarrow$ $\vec{a} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$

or $\vec{a} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$

$$\boxed{\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}}$$

$$\boxed{\vec{a} = \vec{a}_T + \vec{a}_c}$$



Here $\vec{a}_T$ is tangential acceleration and $\vec{a}_c$ is centripetal acceleration.

> Hence $\vec{a}_T$ and $\vec{a}_c$ are two components of net linear acceleration.

Tangential Acceleration ($\vec{a}_T$)-

$\vec{a}_T = \vec{\alpha} \times \vec{r}$, its direction is parallel to velocity.

As $\vec{\omega}$ and $\vec{\alpha}$ both are parallel and along the axis so that $\vec{v}$ and $\vec{a}_T$ are also parallel and along the tengential direction.

Magnitude of tangential acceleration is

$a_T = \alpha r \sin 90^\circ$ $\Rightarrow$ $\boxed{a_T = \alpha r}$

As $\vec{a}_T$ is along the direction of motion (along $\vec{v}$) so that $\vec{a}_T$ is responsible for change in speed of particle. Its magnitude is rate of change of speed of the particle. On circular path with constant speed tangential acceleration is zero.

Note -

(i) अगर Particle की speed बढ़ रही है तो a_T positive होगा और इसकी Direction Velocity के along होगी।

[$\vec{V}\uparrow$ then $\vec{a}_T = +ve$ and is along $\vec{V}$]

(ii) अगर Particle की Speed कम हो रही है तो a_T Negative होगा और इसकी Direction Velocity के opposite होगी।

[$\vec{V}\downarrow$ then $\vec{a}_T = -ve$ and is opposite to $\vec{V}$]

(iii) If $\vec{V} = \text{Const}$ then $a_T = 0$

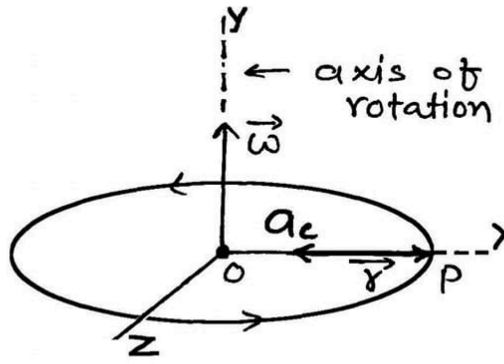
Centripetal Acceleration (a_c) -

$$\vec{a}_c = \vec{\omega} \times \vec{V} \quad \Rightarrow \quad \vec{a}_c = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Let $\vec{r}$ is in $\hat{i}$ direction and $\vec{\omega}$ is in $\hat{j}$ direction, then direction of $\vec{a}_c$ is along

$$\hat{j} \times (\hat{j} \times \hat{i}) = \hat{j} \times (-\hat{k}) = -\hat{i}$$

opposite direction of $\vec{r}$ i.e. from P to O and it is centripetal direction.



Magnitude of centripetal acceleration is

$$\boxed{a_c = \omega V = \frac{V^2}{r} = \omega^2 r} \Rightarrow \boxed{\vec{a}_c = \frac{V^2}{r}(-\hat{r})}$$

Note -

(i) $\vec{a_c}$ is always perpendicular to $\vec{v}$ or displacement, so the work done by centripetal force is always zero.

$$\boxed{W_c = 0}$$

(ii) Circular motion के लिए a_c होना जरूरी है लेकिन α और a_T का होना जरूरी नहीं है।

(iii) On any curved path (Not necessarily circular one) the acceleration of the particle has two components a_T and a_c in two mutually perpendicular directions.

Component of $\vec{a}$ along $\vec{v}$ is $\vec{a_T}$ and perpendicular to $\vec{v}$ is $\vec{a_c}$.

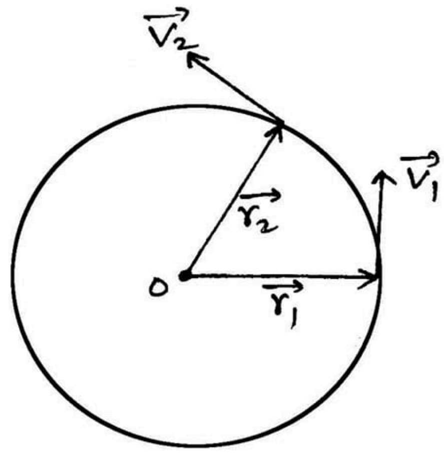
Net Linear Acceleration -

As $a_T \perp a_c$ so that

by $\boxed{\vec{a} = \vec{a_T} + \vec{a_c}}$ $\Rightarrow$ $\boxed{|\vec{a}| = \sqrt{a_T^2 + a_c^2}}$

Uniform Circular Motion -

When a particle moves in a circle at a constant speed then the motion is said to be a uniform circular motion.



In this motion, position vector keep changing continuously.

Speed is constant, so that

$\boxed{\vec{a_T} = 0}$ $\boxed{\alpha = 0}$

Acceleration of particle $\vec{a} = \vec{a_c} = \vec{\omega} \times \vec{V}$

or $a = \omega V$

$$\therefore \quad \boxed{a = \frac{V^2}{r} = \omega^2 r = \text{Centripetal acceleration}}$$

Due to centripetal acceleration the velocity of the particle keeps on changing the direction i.e. the particle is **accelerated**.

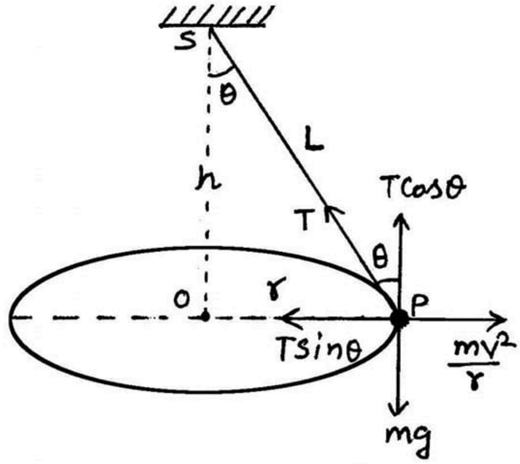
Note - Important difference between the projectile motion and circular motion is that in circular motion the magnitude of acceleration remains constant but the direction continuously changes.

while in projectile motion both the magnitude and direction of acceleration (g) remains constant.

Motion in Horizontal Circle

Conical Pendulum-

It consist of a body attached to a string of length L, such that it can revolved a horizontal circle with uniform speed. The string traces out a cone in the space.



Forces acting on the bob are -

(i) Tension in string $= T$

(ii) Weight of bob $= mg$

From the figure

$$T\sin\theta = \frac{mv^2}{r} \quad \text{and} \quad T\cos\theta = mg$$

Hence $$\tan\theta = \frac{mv^2/r}{mg} = \frac{v^2}{rg}$$

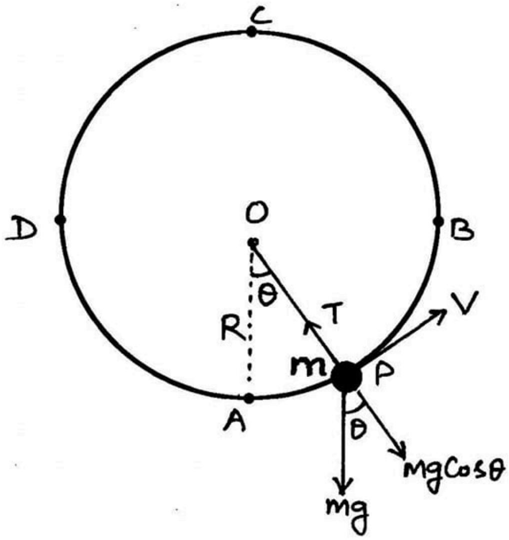
$$\therefore \quad \boxed{V = \sqrt{rg\tan\theta}} \qquad [V = \text{Linear Velocity}]$$

$$\therefore \text{ Time Period } T = \frac{2\pi r}{V} \Rightarrow T = \frac{2\pi r}{\sqrt{rg\tan\theta}}$$

$$\boxed{T = 2\pi\sqrt{\frac{r}{g\tan\theta}} = 2\pi\sqrt{\frac{L\cos\theta}{g}}}$$

Motion in a vertical Circle

Let a particle of mass m is attached to a light and inextensible string. The other end of the string is fixed at O and the particle moves in a verticle circle of radius R,.



Consider the particle when it is at the point P and the string makes an angle θ with the verticle.

Net force towards centre = Centripetal force

$$T - mg\cos\theta = \frac{mv^2}{R} \Rightarrow \boxed{T = \frac{mv^2}{R} + mg\cos\theta}$$

Since speed of the particle decreases with hight, hence tension is maximum at the bottom, where $\cos\theta = 1$ (as $\theta = 0°$).

Hence At bottom $\quad T_{max} = \frac{mv_b^2}{R} + mg$

Also Tension will be minimum at the top, where $\cos\theta = -1$

Hence At top $\quad T_{min} = \frac{mv_T^2}{R} - mg$

or $\quad \frac{mv_T^2}{R} = T_{min} + mg$

For V_T to be minimum, $T \approx 0$ $\quad \boxed{V_T = \sqrt{gR}}$

If V_b be the critical velocity of the particle at the bottom to just complete the circle, then from the Conservation of energy-

$$mg(2R) + \frac{1}{2}mV_T^2 = \frac{1}{2}mv_b^2$$

As $V_T = \sqrt{gR}$ $\Rightarrow$ $2mgR + \frac{1}{2}mgR = \frac{1}{2}mV_b^2$

$$\boxed{V_b = \sqrt{5gR}}$$

Also velocity of the particle at point B or D is by energy conservation $\frac{1}{2}mV_b^2 = \frac{1}{2}mV^2 + mgR$

$$\frac{1}{2}m(5gR) = \frac{1}{2}mV^2 + mgR$$

$$\boxed{V = \sqrt{3gR}}$$

Tension in the string at the bottom of circle

$$T = mg + \frac{mV_b^2}{R} \Rightarrow T = mg + \frac{m}{R}(5gR)$$

$$\boxed{T = 6mg}$$

Tension in the string at the point B or D is

$$T = \frac{mv^2}{R} \Rightarrow T = \frac{m}{R}(3gR)$$

$$\boxed{T = 3mg}$$

Note:

In case the particle is attached with a light rod of length l, at the highest point its minimum velocity may be zero. Then the critical velocity at the bottom is

$$\boxed{V = 2\sqrt{gl}}$$

It is clear from the table that -

$\boxed{T_A > T_B > T_C}$ and $\boxed{T_B = T_D}$

Various conditions for Vertical motion

Velocity at Lowest Point	Condition
$u > \sqrt{5gR}$	Tension in the string will not be zero at any point and particle will continue the circular motion.
$u = \sqrt{5gR}$	Tension at heighest point C will be zero and body will just complete the circle.
$\sqrt{2gl} < u < \sqrt{5gR}$	Particle will not complete the circular loop. Tension in string become zero somewhere between point B and C. Particle leaves circular path and follow parabolic trajectory.
$u = \sqrt{2gR}$	Both velocity and tension in the string becomes zero at B and particle will oscillate along semicircular path.
$u < \sqrt{2gR}$	Velocity of particle become zero between A and B but tension will not be zero and the particle will oscillate about the point A.

Note -

If the tension at C is zero then the particle will just complete the loop with critical velocity.

i.e. to complete the vertical circle the particle must be projected with velocity $u = V_A = \sqrt{5gR}$.