6 Solution of the Power Flow Problem

In this chapter the basic methods to solve the non-linear power flow equations are reviewed. Solution methods based on the observation that active and reactive power flows are not so strongly coupled are introduced.

IN ALL REALISTIC CASES the power flow problem cannot be solved analytically, and hence iterative solutions implemented in computers must be used. In this chapter we will review two solutions methods, Gauss iteration with a variant called Gauss-Seidel iterative method, and the Newton-Raphson method.

6.1 Solution by Gauss-Seidel Iteration

Consider the power flow equations (5.1) and (5.2) which could be written in complex form as

$$S_k = E_k \sum_{m \in K} Y_{km}^* E_m^* \qquad k = 1, 2, \ldots N \tag{6.1}$$

which is a the same as eq. (4.10). The set K is the set of buses adjacent (connected) to bus k, including bus k, and hence shunt admittances are included in the summation. Furthermore $E_k = U_k e^{j\theta_k}$. This equation can be rewritten as

$$E_k^* = \frac{1}{Y_{kk}^*} \left[\frac{S_k}{E_k} - \sum_{m \in \Omega_k} Y_{km}^* E_m^* \right] \qquad k = 1, 2, \ldots N \tag{6.2}$$

where Ω_k is the set of all buses connected to bus k excluding bus k. Taking the complex conjugate of eq. (6.2) yields

$$E_k = \frac{1}{Y_{kk}} \left[\frac{S_k^*}{E_k^*} - \sum_{m \in \Omega_k} Y_{km} E_m \right] \qquad k = 1, 2, \ldots N \tag{6.3}$$

Thus we get $N-1$ algebraic (complex) equations in the complex variables E_k in the form

$$\begin{aligned} E_2 &= h_2(E_1, E_2, \dots, E_N) \\ E_3 &= h_3(E_1, E_2, \dots, E_N) \\ &\vdots \\ E_N &= h_N(E_1, E_2, \dots, E_N) \end{aligned} \tag{6.4}$$

where the functions h_i are given by eq. (6.3). It is here assumed that bus number 1 is the $U\theta$ bus, and hence is E_1 given and we have no equation for node 1. For PQ buses both the magnitude and angle of E_k are unknown, while for PU buses only the angle is unknown. For PQ buses S_k is known, while for PU buses only P_k is known. This will be discussed below in more detail. In vector form eq. (6.4) can be written as

$$\mathbf{x} = \mathbf{h}(\mathbf{x}) \tag{6.5}$$

and based on this equation the following iterative scheme is proposed

$$\mathbf{x}^{\nu+1} = \mathbf{h}(\mathbf{x}^{\nu}) \qquad \nu = 0, 1, \dots \tag{6.6}$$

where the superscript indicates the iteration number. Thus starting with an initial value $\mathbf{x}^0$, the sequence

$$\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \dots \tag{6.7}$$

is generated. If the sequence converges, i.e. $\mathbf{x}^{\nu} \to \mathbf{x}^*$, then

$$\mathbf{x}^* = \mathbf{h}(\mathbf{x}^*) \tag{6.8}$$

and $\mathbf{x}^*$ is a solution of eq. (6.5).

In practice the iteration is stopped when the changes in $\mathbf{x}^{\nu}$ becomes sufficiently small, i.e. when the norm of $\Delta\mathbf{x}^{\nu} = \mathbf{x}^{\nu+1} - \mathbf{x}^{\nu}$ is less than a pre-determined value ε.

To start the iteration a first guess of $\mathbf{x}$ is needed. Usually, if no a priori knowledge of the solution is known, one selects all unknown voltage magnitudes and phase angles equal to the ones of the reference bus, usually around 1 p.u. and phase angle = 0. This start solution is often called a *flat start*.

The difference between Gauss and Gauss-Seidel iteration can be explained by considering eq. (6.6) with all components written out explicitly[1]

$$\begin{aligned} x_2^{\nu+1} &= h_2(x_1, x_2^{\nu}, \dots, x_N^{\nu}) \\ x_3^{\nu+1} &= h_3(x_1, x_2^{\nu}, \dots, x_N^{\nu}) \\ &\vdots \\ x_N^{\nu+1} &= h_N(x_1, x_2^{\nu}, \dots, x_N^{\nu}) \end{aligned} \tag{6.9}$$

[1]In this particular formulation x_1 is the value of the complex voltage of the slack bus and consequently known. For completeness we have included it as a variable in the equations above, but it is actually known

In carrying out the computation (normally by computer) we process the equations from top to bottom. We now observe that when we solve for $x_3^{\nu+1}$ we already know $x_2^{\nu+1}$. Since $x_2^{\nu+1}$ is presumably a better estimate than x_2^{ν}, it seems reasonable to use the updated value. Similarly when we solve for $x_4^{\nu+1}$ we can use the values of $x_2^{\nu+1}$ and $x_3^{\nu+1}$. This is the line of reasoning called the Gauss-Seidel iteration:

$$\begin{aligned}
x_2^{\nu+1} &= h_2(x_1, x_2^{\nu}, \ldots, x_N^{\nu}) \\
x_3^{\nu+1} &= h_3(x_1, x_2^{\nu+1}, \ldots, x_N^{\nu}) \\
&\vdots \\
x_N^{\nu+1} &= h_N(x_1, x_2^{\nu+1}, \ldots, x_{N-1}^{\nu+1}, x_N^{\nu})
\end{aligned} \tag{6.10}$$

It is clear that the convergence of the Gauss-Seidel iteration is faster than the Gauss iteration scheme.

For PQ buses the complex power S_k is completely known and the calculation of the right hand side of eq. (6.3) is well defined. For PU buses however, Q is not defined but is determined so that the voltage magnitude is kept at the specified value. In this case we have to estimate the reactive power injection and an obvious choice is

$$Q_k^{\nu} = \Im\left[E_k^{\nu} \sum_{m \in K} Y_{km}^*(E_m^*)^{\nu}\right] \tag{6.11}$$

In the Gauss-Seidel iteration scheme one should use the latest calculated values of E_m. It should be clear that also for PU buses the above iteration scheme gives an solution if it converges.

A problem with the Gauss and Gauss-Seidel iteration schemes is that convergence can be very slow, and sometimes even the iteration does not converge despite that a solution exists. Furthermore, no general results are known concerning the the convergence characteristics and criteria. Therefore more efficient solution methods are needed, and one such method that is widely used in power flow computations is discussed in the subsequent sections.

6.2 Newton-Raphson Method

Before applying this method to the power flow problem we review the iteration scheme and some of its properties.

A system of nonlinear algebraic equations can be written as

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{6.12}$$

where $\mathbf{x}$ is an n vector of unknowns and $\mathbf{f}$ is an n vector function of $\mathbf{x}$. Given an appropriate starting value $\mathbf{x^0}$, the Newton-Raphson method solves this

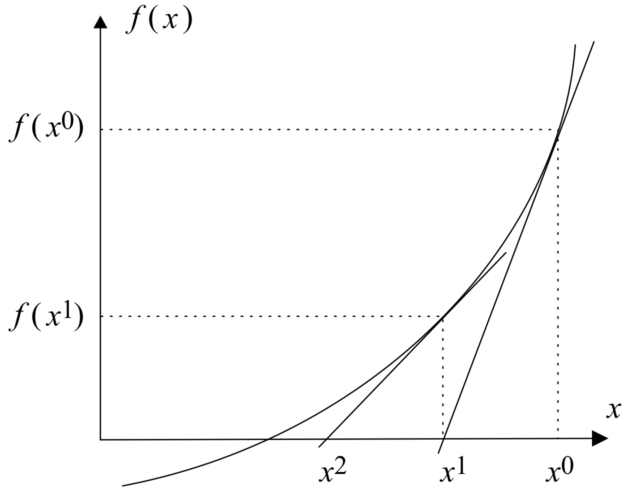

Figure 6.1. Newton-Raphson method in unidimensional case

vector equation by generating the following sequence:

$$\begin{aligned} \mathbf{J}(\mathbf{x}^\nu)\Delta\mathbf{x}^\nu &= -\mathbf{f}(\mathbf{x}^\nu) \\ \mathbf{x}^{\nu+1} &= \mathbf{x}^\nu + \Delta\mathbf{x}^\nu \end{aligned} \tag{6.13}$$

where $\mathbf{J}(\mathbf{x}^\nu) = \partial\mathbf{f}(\mathbf{x})/\partial\mathbf{x}$ is the Jacobian matrix with elements

$$J_{ij} = \frac{\partial f_i}{\partial x_j} \tag{6.14}$$

6.2.1 Unidimensional case

To get a better feeling for the method we first study the one dimensional case, and eq. (6.12) becomes

$$f(x) = 0 \tag{6.15}$$

where x is the unknown and $f(x)$ is a scalar function. Figure 6.1 illustrates a simple case in which there is a single solution to eq. (6.15). Under these circumstances, the following algorithm can be used to find the solution of eq. (6.15):

1. Set $\nu = 0$ and choose an appropriate starting value x^0;

2. Compute $f(x^\nu)$;

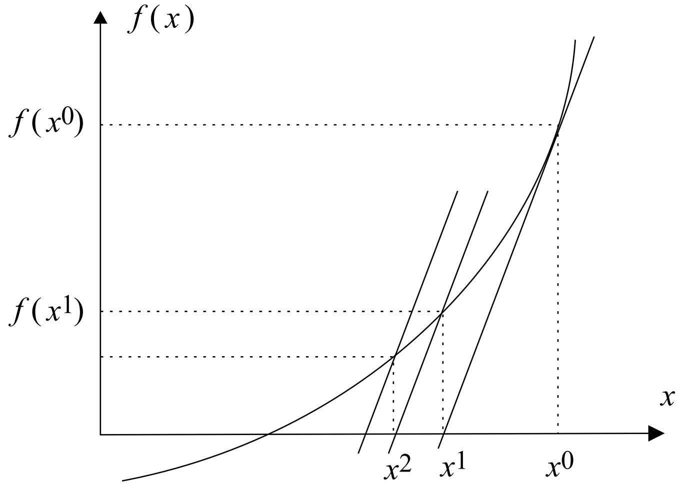

Figure 6.2. Dishonest Newton-Raphson method in unidimensional case

3. Compare $f(x^\nu)$ with specified tolerance ε;
   if $|f(x^\nu)| \leq \varepsilon$, then $x = x^\nu$ is the solution to eq. (6.15);
   Otherwise, $|f(x^\nu)| > \varepsilon$, go to the next step;

4. Linearize $f(x)$ at the current solution point $[x^\nu, f(x^\nu)]$, as shown in Figure 6.1. That is, $f(x^\nu + \Delta x^\nu) \approx f(x^\nu) + f'(x^\nu)\Delta x^\nu$, where $f'(x^\nu)$ is calculated at x^ν

5. Solve $f(x^\nu) + f'(x^\nu)\Delta x^\nu = 0$ for Δx^ν, and update the solution estimate, $x^{\nu+1} = x^\nu + \Delta x^\nu$, where $\Delta x^\nu = -f(x^\nu)/f'(x^\nu)$;

6. Update iteration counter $\nu + 1 \rightarrow \nu$ and go to step 2.

The dishonest Newton-Raphson method is illustrated in Figure 6.2. In this case at Step 4 of the algorithm, a constant derivative is assigned and $f'(x^\nu) = f'(x^0)$. Although the number of iterations required for convergence usually increases, it is not necessary to recalculate the derivatives for each iteration and hence the computation burden at each iteration is lower. When only limited accuracy is needed, the overall performance of the dishonest version may be better than that of the full Newton-Raphson method.

6.2.2 Quadratic Convergence

Close to the solution point x^*, the Newton-Raphson method normally presents a property called quadratic convergence. This can be proved for the unidimensional case discussed above if it is assumed that x^* is a simple (not a multiple) root and that its first and second derivatives are continuous.

Hence, $f'(x^*) = 0$, and for any x in a certain neighbourhood of x^*, $f'(x) \neq 0$. If ε_ν denotes the error at the ν-th iteration, i.e.

$$\varepsilon_\nu = x^* - x^\nu \tag{6.16}$$

the Taylor expansion about x^ν yields

$$\begin{aligned} f(x^*) &= f(x^\nu + \varepsilon_\nu) \\ &= f(x^\nu) + f'(x^\nu)\varepsilon_\nu + 1/2 f''(\bar{x})\varepsilon_\nu^2 \\ &= 0 \end{aligned} \tag{6.17}$$

where $\bar{x} \in [x^\nu, x^*]$. Dividing by $f'(x^\nu)$, this expression can be written as

$$\frac{f(x^\nu)}{f'(x^\nu)} + \varepsilon_\nu + 1/2 \frac{f''(\bar{x})}{f'(x^\nu)}\varepsilon_\nu^2 = 0 \tag{6.18}$$

Since,

$$\frac{f(x^\nu)}{f'(x^\nu)} + \varepsilon_\nu = \frac{f(x^\nu)}{f'(x^\nu)} + x^* - x^\nu = x^* - x^{\nu+1} = \varepsilon_{\nu+1} \tag{6.19}$$

the following relationship between ε_ν and $\varepsilon_{\nu+1}$ results:

$$\frac{\varepsilon_{\nu+1}}{\varepsilon_\nu^2} = -\frac{1}{2}\frac{f''(\bar{x})}{f'(x^\nu)} \tag{6.20}$$

In the vicinity of the root, i.e. as $x^\nu \to x^*$, $\bar{x} \to x^*$, and we thus have

$$\frac{|\varepsilon_{\nu+1}|}{\varepsilon_\nu^2} = \frac{1}{2}\frac{|f''(x^*)|}{|f'(x^*)|} \tag{6.21}$$

From eq. (6.21) it is clear that the convergence is quadratic with the assumptions stated above.

6.2.3 Multidimensional Case

Reconsider now the n-dimensional case

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{6.22}$$

where

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))^T \tag{6.23}$$

and

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \tag{6.24}$$

Thus $\mathbf{f}(\mathbf{x})$ and $\mathbf{x}$ are n-dimensional (column) vectors.

The Newton-Raphson method applied to to solve eq. (6.22) follows basically the same steps as those applied to the unidimensional case above, except that in Step 4, the Jacobian matrix $\mathbf{J}(\mathbf{x}^\nu)$ is used, and the linearization of $\mathbf{f}(\mathbf{x})$ at $\mathbf{x}^\nu$ is given by the Taylor expansion

$$\mathbf{f}(\mathbf{x}^\nu + \Delta\mathbf{x}^\nu) \approx \mathbf{f}(\mathbf{x}^\nu) + \mathbf{J}(\mathbf{x}^\nu)\Delta\mathbf{x}^\nu \tag{6.25}$$

where the Jacobian matrix has the general form

$$\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{pmatrix} \tag{6.26}$$

The correction vector $\Delta\mathbf{x}$ is the solution to

$$\mathbf{f}(\mathbf{x}^\nu) + \mathbf{J}(\mathbf{x}^\nu)\Delta\mathbf{x}^\nu = 0 \tag{6.27}$$

Note that this is the linearized version of the original problem $\mathbf{f}(\mathbf{x}^\nu + \Delta\mathbf{x}^\nu) = 0$. The solution of eq. (6.27) involves thus the solution of a system of linear equations, which usually is done by Gauss elimination (LU factorization).

The Newton-Raphson algorithm for the n-dimensional case is thus as follows:

1. Set $\nu = 0$ and choose an appropriate starting value $\mathbf{x}^0$;

2. Compute $\mathbf{f}(\mathbf{x}^\nu)$;

3. Test convergence:
   If $|f_i(\mathbf{x}^\nu)| \leq \varepsilon$ for $i = 1, 2, \ldots, n$, then $\mathbf{x}^\nu$ is the solution
   Otherwise go to 4;

4. Compute the Jacobian matrix $\mathbf{J}(\mathbf{x}^\nu)$;

5. Update the solution

$$\begin{aligned} \Delta\mathbf{x}^\nu &= -\mathbf{J}^{-1}(\mathbf{x}^\nu)\mathbf{f}(\mathbf{x}^\nu) \\ \mathbf{x}^{\nu+1} &= \mathbf{x}^\nu + \Delta\mathbf{x}^\nu \end{aligned} \tag{6.28}$$

6. Update iteration counter $\nu + 1 \to \nu$ and go to step 2.

6.3 Newton-Raphson applied to the Power Flow Equations

In this section we will now formulate the Newton-Raphson iteration of the power flow equations. Firstly, the state vector of unknown voltage angles and magnitudes is ordered such that

$$\mathbf{x} = \begin{pmatrix} \theta \\ \mathbf{U} \end{pmatrix} \tag{6.29}$$

and the nonlinear function $\mathbf{f}$ is ordered so that the first components correspond to active power and the last ones to reactive power:

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \mathbf{\Delta P}(\mathbf{x}) \\ \mathbf{\Delta Q}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \mathbf{P}(\mathbf{x}) - \mathbf{P}^{(\mathbf{s})} \\ \mathbf{Q}(\mathbf{x}) - \mathbf{Q}^{(\mathbf{s})} \end{pmatrix} \tag{6.30}$$

with

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} P_2(\mathbf{x}) - P_2^{(s)} \\ \vdots \\ P_m(\mathbf{x}) - P_m^{(s)} \\ ------- \\ Q_2(\mathbf{x}) - Q_2^{(s)} \\ \vdots \\ Q_n(\mathbf{x}) - Q_n^{(s)} \end{pmatrix} \tag{6.31}$$

In eq . (6.31) the functions $P_k(\mathbf{x})$ are the active power flows out from bus k given by eq. (4.11) and the $P_k^{(s)}$ are the known active power injections into bus k from generators and loads, and the functions $Q_k(\mathbf{x})$ are the reactive power flows out from bus k given by eq. (4.12) and $Q_k^{(s)}$ are the known reactive power injections into bus k from generators and loads. The first $m-1$ equations are formulated for PU and PQ buses, and the last $n-1$ equations can only be formulated for PQ buses. If there are N_{PU} PU buses and N_{PQ} PQ buses, $m-1 = N_{PU} + N_{PQ}$ and $n-1 = N_{PQ}$. The load flow equations can now be written as

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \mathbf{\Delta P}(\mathbf{x}) \\ \mathbf{\Delta Q}(\mathbf{x}) \end{pmatrix} = 0 \tag{6.32}$$

and the functions $\mathbf{\Delta P}(\mathbf{x})$ and $\mathbf{\Delta Q}(\mathbf{x})$ are called active and reactive (power) mismatches. The updates to the solutions are determined from the equation

$$\mathbf{J}(\mathbf{x}^\nu) \begin{pmatrix} \Delta\theta^\nu \\ \Delta\mathbf{U}^\nu \end{pmatrix} + \begin{pmatrix} \mathbf{\Delta P}(\mathbf{x}^\nu) \\ \mathbf{\Delta Q}(\mathbf{x}^\nu) \end{pmatrix} = 0 \tag{6.33}$$

The Jacobian matrix $\mathbf{J}$ can be written as

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial \mathbf{\Delta P}}{\partial \theta} & \dfrac{\partial \mathbf{\Delta P}}{\partial \mathbf{U}} \\ \dfrac{\partial \mathbf{\Delta Q}}{\partial \theta} & \dfrac{\partial \mathbf{\Delta Q}}{\partial \mathbf{U}} \end{pmatrix} \tag{6.34}$$

which is equal to

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial \mathbf{P}(\mathbf{x})}{\partial \theta} & \dfrac{\partial \mathbf{P}(\mathbf{x})}{\partial \mathbf{U}} \\ \dfrac{\partial \mathbf{Q}(\mathbf{x})}{\partial \theta} & \dfrac{\partial \mathbf{Q}(\mathbf{x})}{\partial \mathbf{U}} \end{pmatrix} \tag{6.35}$$

or simply

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial \mathbf{P}}{\partial \theta} & \dfrac{\partial \mathbf{P}}{\partial \mathbf{U}} \\ \dfrac{\partial \mathbf{Q}}{\partial \theta} & \dfrac{\partial \mathbf{Q}}{\partial \mathbf{U}} \end{pmatrix} \tag{6.36}$$

In eq. (6.34) the matrices $\partial \mathbf{P}/\partial \theta$ and $\partial \mathbf{Q}/\partial \mathbf{U}$ are always quadratic, and so is of course $\mathbf{J}$.

6.4 $P\theta - QU$ Decoupling

The ac power flow problem above involves four variables associated with each network node k:

- U_k, the voltage magnitude
- θ_k, the voltage angle
- P_k, the net active power (generation - load)
- Q_k, the net reactive power (generation - load)

For transmission systems, a strong coupling is normally observed between P and θ, as well as between Q and U. This property will in this section be employed to simplify and speed up the computations. In the next section we will derive a linear approximation called dc power flow (or dc load flow). This linear model relates the active power P to the bus voltage angle θ.

Let us consider a π-model of a transmission line, where the series resistance and the shunt admittance both are neglected and put to zero. In this case, the active and reactive power flows are given by the following simplified expressions of eqs. (3.3) and (3.4)

$$P_{km} = \frac{U_k U_m \sin \theta_{km}}{x_{km}} \tag{6.37}$$

$$Q_{km} = \frac{U_k^2 - U_k U_m \cos\theta_{km}}{x_{km}} \tag{6.38}$$

where x_{km} is the series reactance of the line.

The sensitivities between power flows P_{km} and Q_{km} and the state variables U and θ are for this approximation given by

$$\frac{\partial P_{km}}{\partial \theta_k} = \frac{U_k U_m \cos\theta_{km}}{x_{km}} \qquad \frac{\partial P_{km}}{\partial U_k} = \frac{U_m \sin\theta_{km}}{x_{km}} \tag{6.39}$$

$$\frac{\partial Q_{km}}{\partial \theta_k} = \frac{U_k U_m \sin\theta_{km}}{x_{km}} \qquad \frac{\partial Q_{km}}{\partial U_k} = \frac{2U_k - U_m \cos\theta_{km}}{x_{km}} \tag{6.40}$$

When $\theta_{km} = 0$, perfect decoupling conditions are observed, i.e.

$$\frac{\partial P_{km}}{\partial \theta_k} = \frac{U_k U_m}{x_{km}} \qquad \frac{\partial P_{km}}{\partial U_k} = 0 \tag{6.41}$$

$$\frac{\partial Q_{km}}{\partial \theta_k} = 0 \qquad \frac{\partial Q_{km}}{\partial U_k} = \frac{2U_k - U_m}{x_{km}} \tag{6.42}$$

As illustrated in Figure 6.3, in the usual range of operations (relatively small voltage angles), a strong coupling between active power and voltage angle as well as between reactive power and voltage magnitudes exists, while a much weaker coupling between reactive power and voltage angle, and between voltage magnitude and active power exists. Notice, however, that for larger angles this is no longer true. In the the neighbourhood of $\theta_{km} = 90°$, there is strong coupling between P and U as well as between Q and θ.

Example 6.1. *A 750 kV transmission line has 0.0175 p.u. series reactance (the series reactance and the shunt admittance are ignored in this example). The terminal bus voltage magnitudes are 0.984 and 0.962 p.u. and the angle difference is* $10°$. *Calculate the sensitivities of the active and reactive power flows with respect to voltage magnitude and phase angle.*

Solution The four sensitivities are calculated by using eqs. (6.39) and (6.40):

$$\frac{\partial P_{km}}{\partial \theta_k} = \frac{U_k U_m \cos\theta_{km}}{x_{km}} = \frac{0.984 \cdot 0.962 \cos 10°}{0.0175} = 54.1$$

$$\frac{\partial P_{km}}{\partial U_k} = \frac{U_m \sin\theta_{km}}{x_{km}} = \frac{0.962 \sin 10°}{0.0175} = 9.5$$

$$\frac{\partial Q_{km}}{\partial \theta_k} = \frac{U_k U_m \sin\theta_{km}}{x_{km}} = \frac{0.984 \cdot 0.962 \sin 10°}{0.0175} = 9.4$$

$$\frac{\partial Q_{km}}{\partial U_k} = \frac{2U_k - U_m \cos\theta_{km}}{x_{km}} = \frac{2 \cdot 0.984 - 0.962 \cos 10°}{0.0175} = 58.3$$

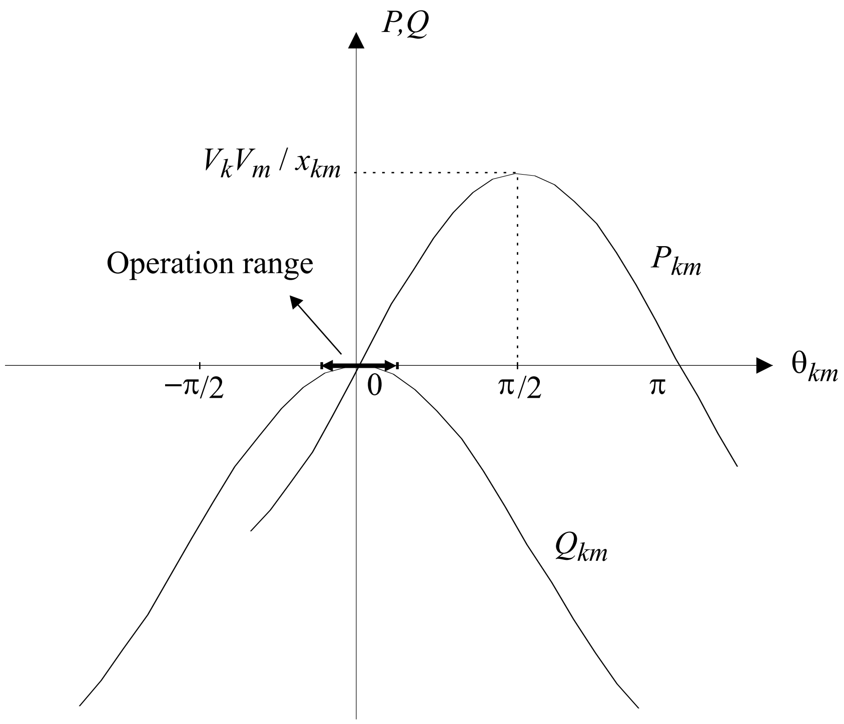

Figure 6.3. $P - \theta$ and $Q - \theta$ curves for a line with a series resistance and a shunt admittance of zero and considering terminal voltages $U_k = U_m = 1.0$ p.u.

As seen the $P - \theta$ and $Q - U$ couplings are much greater than the other couplings. ♦

If the couplings $Q-\theta$ and $P-U$ are neglected the Newton-Raphson iteration scheme can be simplified. With this assumption the Jacobian Matrix can be written as

$$\mathbf{J}_{DEC} = \begin{pmatrix} \dfrac{\partial \mathbf{P}}{\partial \theta} & 0 \\ & \\ 0 & \dfrac{\partial \mathbf{Q}}{\partial \mathbf{U}} \end{pmatrix} \tag{6.43}$$

Thus there is no coupling between the updates of voltage magnitudes and angles and eq. (6.33) can be written as two uncoupled equations:

$$\frac{\partial \mathbf{P}}{\partial \theta} \Delta\theta^{\nu} + \mathbf{\Delta P}(\theta^{\nu}, \mathbf{U}^{\nu}) = 0 \tag{6.44}$$

$$\frac{\partial \mathbf{Q}}{\partial \mathbf{U}} \Delta\mathbf{U}^{\nu} + \mathbf{\Delta Q}(\theta^{\nu+1}, \mathbf{U}^{\nu}) = 0 \tag{6.45}$$

In this formulation two systems of linear equations have to be solved instead of one system. Of course, the total number of equations to be solved is the same, but since the needed number of operations to solve a system of linear equations increases more than linearly with the dimension, it takes less operations to solve eqs. (6.44) and (6.45) than eq. (6.33) with the complete Jacobian matrix, J. It should be noted that if the iterations of eqs. (6.44) and (6.45) converge, it converges to a correct solution of the load flow equations. No approximations have been introduced in the functions $\mathbf{P}(\mathbf{x})$ or $\mathbf{Q}(\mathbf{x})$, only in the way we calculate the updates. The convergence of the decoupled scheme is somewhat slower than the full scheme, but often the faster solution time for the updates compensates for slower convergence, giving as faster overall solution time. For not too heavily loaded systems a faster overall solution time is almost always obtained. The two equations in eqs. (6.44) and (6.45) are solved sequentially, and then the updated unknowns of the first equation, eq. (6.44), can be used to calculate the mismatches of the second system of equations, eq. (6.45), resulting in an increased speed of convergence.

A number of approximations can be made to calculate the matrix elements of the the two sub-matrices of the Jacobian in eqs. (6.44) and (6.45). This will only influence the speed of convergence of the solution. If the method converges to a solution, this is the correct solution as long as the accurate expression for $\mathbf{\Delta P}$ and $\mathbf{\Delta Q}$ are used.

If approximation regarding the active and reactive power mismatches are used, the solution can be even faster, but then only a approximative solution will be obtained. This is further elaborated in next section.

6.5 Approximative Solutions of the Power Flow Problem

In the previous section the exact expressions of the power flow equations were used. However, since the power flow equations are solved frequently in the operation and planning of electric power systems there is a need that the equations can be solved fast, and for this purpose the approximations introduced in this chapter have proved to be of great value. Often the approximations described here are used together with exact methods. Approximative methods could be used to identify the most critical cases, which then are further analysed with the full models. The fast, approximative methods can also used to provide good initial guesses for a complete solution of the equations.

6.5.1 Linearization

In this subsection the linearized dc power flow equations will be derived.

Transmission Line

Consider again expressions for the active power flows (P_{km} and P_{mk}) in a transmission line:

$$P_{km} = U_k^2 g_{km} - U_k U_m g_{km} \cos\theta_{km} - U_k U_m b_{km} \sin\theta_{km} \tag{6.46}$$
$$P_{mk} = U_m^2 g_{km} - U_k U_m g_{km} \cos\theta_{km} + U_k U_m b_{km} \sin\theta_{km} \tag{6.47}$$

These equations can be used to determine the real power losses in a transmission line

$$P_{km} + P_{mk} = g_{km}(U_k^2 + U_m^2 - 2U_k U_m \cos\theta_{km}) \tag{6.48}$$

If the terms corresponding to the active power losses are ignored in eqs. (6.46) and (6.47), the result is

$$P_{km} = -P_{mk} = -U_k U_m b_{km} \sin\theta_{km} \tag{6.49}$$

The following additional approximations are often valid, particularly during light load conditions

$$U_k \approx U_m \approx 1 \text{ p.u.} \tag{6.50}$$

$$\sin\theta_{km} \approx \theta_{km} \tag{6.51}$$

And since

$$b_{km} = -1/x_{km} \tag{6.52}$$

we can simplify the expression for the active power flow P_{km} to

$$P_{km} = \theta_{km}/x_{km} = \frac{\theta_k - \theta_m}{x_{km}} \tag{6.53}$$

This equation is analogous to Ohm's law applied to a resistor carrying a dc current:

- P_{km} is the dc current;
- θ_k and θ_m are the dc voltages at the resistor terminals;
- x_{km} is the resistance.

This is illustrated in Figure 6.4.

Series Capacitor

For a given voltage angle spread, the active power flow in a transmission line decreases with the line reactance (and series reactance normally increases with line length). Series compensation aims at reducing the effective electric length of the line: a series capacitor connected in series with the line. If, for example, a 40% compensation corresponds to a capacitor with a reactance

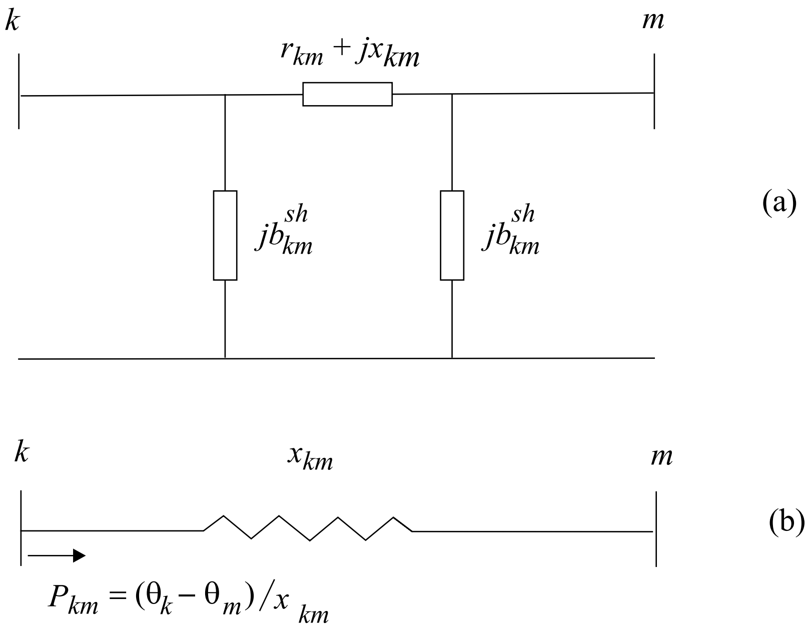

Figure 6.4. Transmission line. (a) Equivalent π-model. (b) DC power flow model

of 40% of the original line reactance, but with opposite sign, the resulting reactance of the compensated line becomes 60% of the original value. Thus

$$x_{km}^{comp} = x_{km} - x_{sc} \tag{6.54}$$

with obvious notation. In the dc power flow model the series capacitor can thus be regarded as a negative resistance inserted in series with the equivalent line resistance.

In-Phase Transformer

The active power flows, P_{km} and P_{mk}, in an in-phase transformer are given by eq. (3.10)

$$P_{km} = (a_{km}U_k)^2 g_{km} - a_{km}U_kU_m g_{km} \cos\theta_{km} - a_{km}U_kU_m b_{km} \sin\theta_{km} \tag{6.55}$$

Neglecting the terms associated with losses and introducing the same approximations used for transmission lines yields

$$P_{km} = \frac{\theta_{km}}{x_{km}/a_{km}} \tag{6.56}$$

where further approximating $a_{km} \approx 1$, i.e. the transformer tap ratio is close to the relation between the nominal voltages of the two sides, yields the

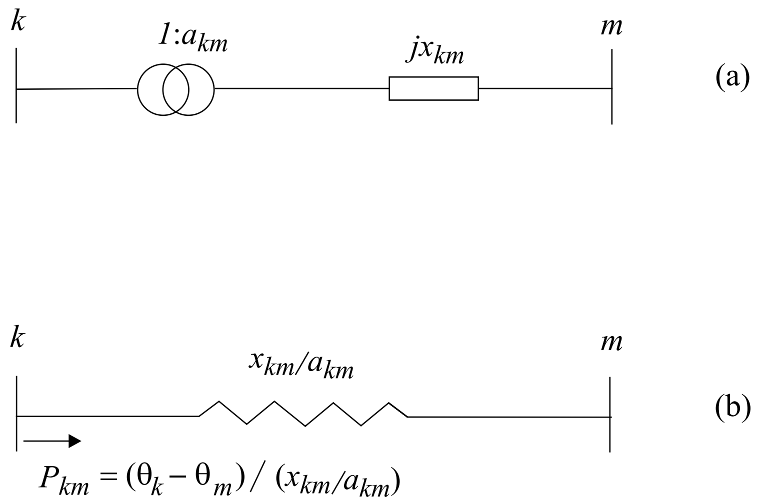

Figure 6.5. In-phase transformer. (a) Transformer comprising ideal transformer and series reactance. (b) DC power flow model

same expression as for transmission lines

$$P_{km} = \frac{\theta_{km}}{x_{km}} \tag{6.57}$$

This is illustrated in Figure 6.5.

Phase Shifter

Let us consider again the expression for the active power flow P_{km} in a phase-shifting transformer of the type represented in Figure 2.6 with $a_{km} = 1$ (eq. (3.14)):

$$P_{km} = U_k^2 g_{km} - U_k U_m g_{km} \cos(\theta_{km} + \varphi_{km}) - U_k U_m b_{km} \sin(\theta_{km} + \varphi_{km})$$

As with transmission lines and in-phase transformers, if the terms associated with active power losses are ignored and $U_k = U_m = 1$ p.u. and $b_{km} = -x_{km}^{-1}$, the result is

$$P_{km} = \frac{\sin(\theta_{km} + \varphi_{km})}{x_{km}} \tag{6.58}$$

and if $(\theta_{km} + \varphi_{km}) << \pi/2$, then linear approximation can be used, giving

$$P_{km} = \frac{(\theta_{km} + \varphi_{km})}{x_{km}} \tag{6.59}$$

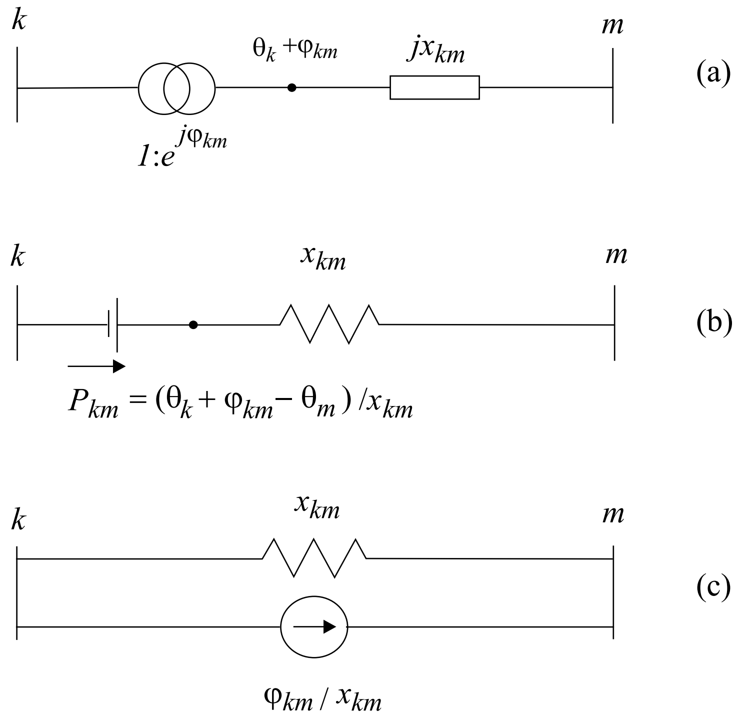

Figure 6.6. Phase-shifting transformer. (a) Phase-shifting transformer model (b) Thévénin dc power flow model. (c) Norton dc power flow model

Note that P_{km} has two components, the first depending on the terminal bus voltage angles, θ_{km}/x_{km}, and the other depending only on the phase-shifting transformer angle, φ_{km}/x_{km}. If φ_{km} is considered to be a constant, eq. (6.59) can be represented by the linearized model shown in Figure 6.6, where the constant part of the active power flow, φ_{km}/x_{km}, appears as an extra load on the terminal bus k and an extra generation on the terminal bus m if $\varphi_{km} > 0$, or vice-verse if $\varphi_{km} < 0$.

6.5.2 Matrix Formulation of DC Power Flow Equations

In this section, the dc model developed above is expressed in the form $\mathbf{I} = \mathbf{YE}$. According to the dc model, the active power flow in a branch is given by

$$P_{km} = x_{km}^{-1}\theta_{km} \tag{6.60}$$

where x_{km} is the series reactance of the branch (parallel equivalent of all the circuits existing in the branch).

The active power injection at bus k is thus given by

$$P_k = \sum_{m \in \Omega_k} x_{km}^{-1} \theta_{km} = (\sum_{m \in \Omega_k} x_{km}^{-1}) \theta_k + \sum_{m \in \Omega_k} (-x_{km}^{-1} \theta_m) \tag{6.61}$$

for $k = 1, 2, \ldots, N$, where N is the number of buses in the network. This can be put into matrix form as follows:

$$\mathbf{P} = \mathbf{B}' \theta \tag{6.62}$$

where

- $\mathbf{P}$ is the vector of the net injections P_k
- $\mathbf{B}'$ is the nodal admittance matrix with the following elements:

$$B'_{km} = -x_{km}^{-1}$$
$$B'_{kk} = \sum_{m \in \Omega_k} x_{km}^{-1}$$

- θ is the vector of voltage angles θ_k

The matrix $\mathbf{B}'$ in eq. (6.62) is singular, i.e. with a determinant equal to zero. This means that the system of equations in eq. (6.62) has no unique solution. Since the power losses have been ignored, the sums of the components of $\mathbf{P}$ is equal to zero. This means that the rows of $\mathbf{B}'$ are linearly dependent. To make the system solvable, one of the equations in the system is removed, and the bus associated with that row is chosen as the angle reference, i.e. $\theta_{ref} = 0$.

In forming the matrix $\mathbf{B}'$, in-phase and phase-shifting transformers are treated like transmission lines. The phase-shifting transformers also contribute to the construction of the independent vector $\mathbf{P}$ with the Norton equivalent injections shown in Figure 6.6(c).

Example 6.2. *Consider the network given in Figure 6.7 in which the reference angle is $\theta_1 = 0$. Use the dc power flow method to calculate the power flows in the lines.*

Solution In this case, the elements of the matrix $\mathbf{B}'$ are calculated as

$$\begin{aligned}
B_{22} &= x_{21}^{-1} + x_{23}^{-1} = (1/3)^{-1} + (1/2)^{-1} = 5 \\
B_{23} &= -x_{23}^{-1} = -(1/2)^{-1} = -2 \\
B_{32} &= -x_{32}^{-1} = -(1/2)^{-1} = -2 \\
B_{33} &= x_{31}^{-1} + x_{32}^{-1} = (1/2)^{-1} + (1/2)^{-1} = 4
\end{aligned}$$

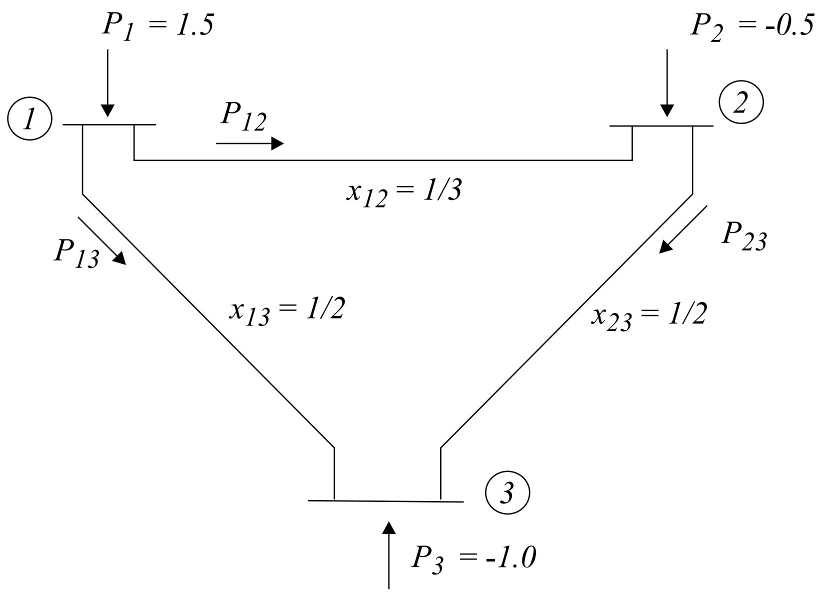

Figure 6.7. 3-bus network. (Active power in p.u.; branch reactances in p.u.)

and thus

$$\mathbf{B}' = \begin{pmatrix} 5 & -2 \\ -2 & 4 \end{pmatrix}$$

and

$$(\mathbf{B}')^{-1} = \begin{pmatrix} 1/4 & 1/8 \\ 1/8 & 5/16 \end{pmatrix}$$

The nodal voltage angles (in radians) can now easily be calculated

$$\begin{aligned} \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} &= (\mathbf{B}')^{-1}\mathbf{P} \\ &= \begin{pmatrix} 1/4 & 1/8 \\ 1/8 & 5/16 \end{pmatrix} \begin{pmatrix} -0.5 \\ -1.0 \end{pmatrix} = \begin{pmatrix} -0.250 \\ -0.375 \end{pmatrix} \end{aligned}$$

The power flows in the transmission lines are according to the dc power flow model

$$\begin{aligned} P_{12} &= x_{12}^{-1}\theta_{12} = 3 \cdot 0.25 = 0.75 \text{ p.u.} \\ P_{13} &= x_{13}^{-1}\theta_{13} = 2 \cdot 0.375 = 0.75 \text{ p.u.} \\ P_{23} &= x_{23}^{-1}\theta_{23} = 2 \cdot 0.125 = 0.25 \text{ p.u.} \end{aligned}$$

♦

6.5.3 DC Power Flow Model

The linearized model $\mathbf{P} = \mathbf{B}'\theta$ can be interpreted as the model for a network of resistors fed by dc current sources where $\mathbf{P}$ is the vector of nodal current injections, θ is the nodal vector of dc voltages, and $\mathbf{B}'$ is the nodal conductance matrix, as illustrated in Figure 6.8.

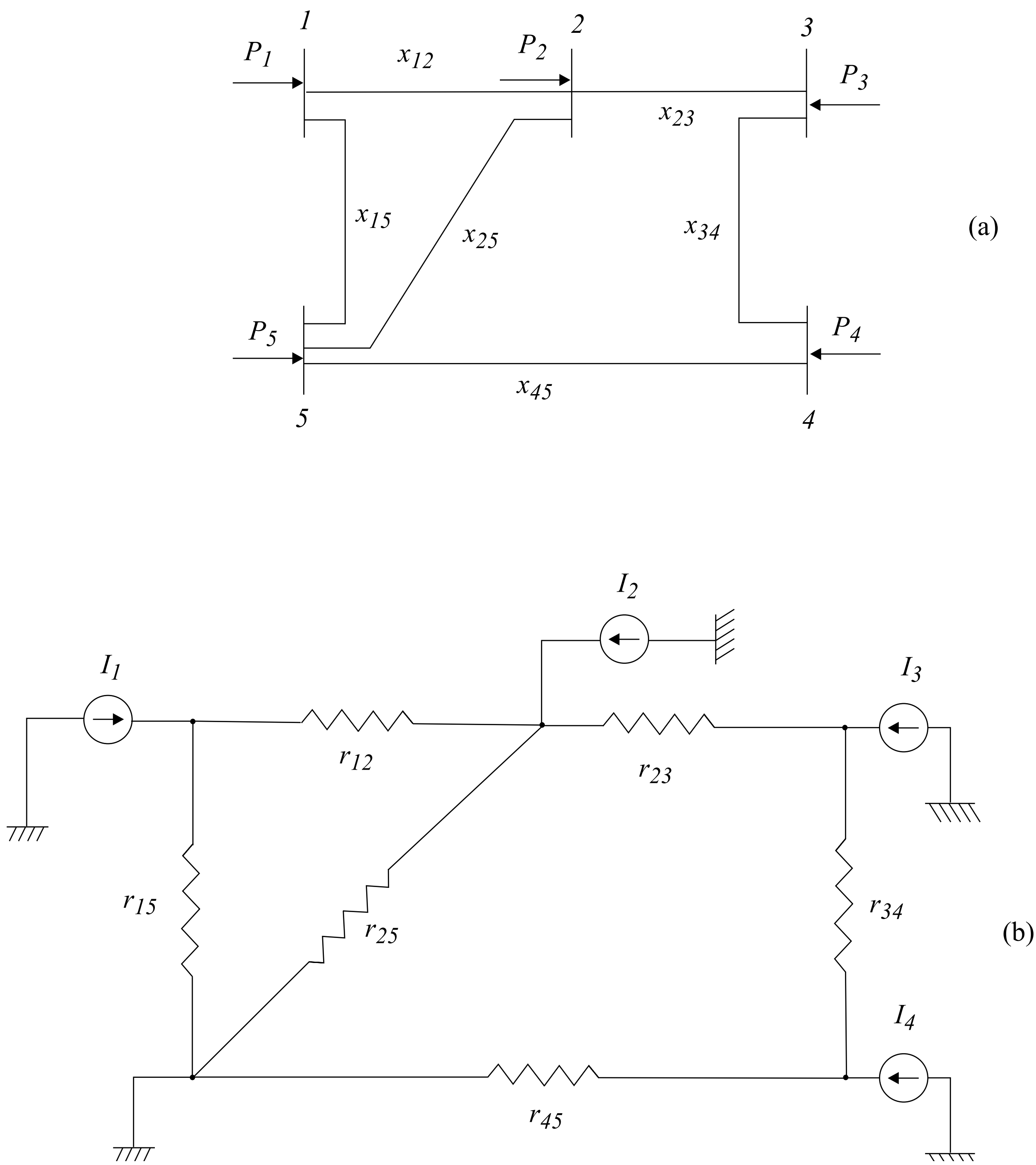

Figure 6.8. 6-bus network. (a) power network. (b) dc power flow model.