3 Active and Reactive Power Flows

In this chapter the expressions for the active and reactive power flows in transmission lines, transformers, phase-shifting transformers, and unified branch models are derived.

THE SYSTEM COMPONENTS dealt with in this chapter are linear in the sense that the relations between voltages and currents are linear[1]. However, since one usually is interested rather in powers, active and reactive, than currents, the resulting equations will be non-linear, which introduces a complication when solving the resulting equations.

3.1 Transmission Lines

Consider the complex current I_{km} in a transmission line

$$I_{km} = y_{km}(E_k - E_m) + jb_{km}^{sh} E_k \tag{3.1}$$

with quantities defined according to Figure 2.2. The complex power, $S_{km} = P_{km} + jQ_{km}$, is

$$S_{km} = E_k I_{km}^* = y_{km}^* U_k e^{j\theta_k}(U_k e^{-j\theta_k} - U_m e^{-j\theta_m}) - jb_{km}^{sh} U_k^2 \tag{3.2}$$

where the conductance of y_{km}^{sh} has been neglected.

The expressions for P_{km} and Q_{km} can be determined by identifying the corresponding coefficients of the real and imaginary parts of eq. (3.2), which yields

$$P_{km} = U_k^2 g_{km} - U_k U_m g_{km} \cos\theta_{km} - U_k U_m b_{km} \sin\theta_{km} \tag{3.3}$$

$$Q_{km} = -U_k^2 (b_{km} + b_{km}^{sh}) + U_k U_m b_{km} \cos\theta_{km} - U_k U_m g_{km} \sin\theta_{km} \tag{3.4}$$

where the notation $\theta_{km} = \theta_k - \theta_m$ is introduced.

[1] This is at least true for the models analysed here. Different non-linear phenomena, e.g. magnetic saturation, can sometimes be important, but when studying steady state conditions the devices to be discussed in this chapter are normally within the region of linearity.

The active and reactive power flows in opposite directions, P_{mk} and P_{mk}, can be obtained in the same way, resulting in:

$$P_{mk} = U_m^2 g_{km} - U_k U_m g_{km} \cos\theta_{km} + U_k U_m b_{km} \sin\theta_{km} \tag{3.5}$$
$$Q_{mk} = -U_m^2 (b_{km} + b_{km}^{sh}) + U_k U_m b_{km} \cos\theta_{km} + U_k U_m g_{km} \sin\theta_{km} \tag{3.6}$$

From these expressions the active and reactive power losses of the lines are easily obtained as:

$$\begin{aligned} P_{km} + P_{mk} &= g_{km}(U_k^2 + U_m^2 - 2U_k U_m \cos\theta_{km}) \\ &= g_{km}|E_k - E_m|^2 \end{aligned} \tag{3.7}$$

$$\begin{aligned} Q_{km} + Q_{mk} &= -b_{km}^{sh}(U_k^2 + U_m^2) - b_{km}(U_k^2 + U_m^2 - 2U_k U_m \cos\theta_{km}) \\ &= -b_{km}^{sh}(U_k^2 + U_m^2) - b_{km}|E_k - E_m|^2 \end{aligned} \tag{3.8}$$

Note that $|E_k - E_m|$ represents the magnitude of the voltage drop across the line, $g_{km}|E_k - E_m|^2$ represents the active power losses, $-b_{km}^{sh}|E_k - E_m|^2$ represents the reactive power losses; and $-b_{km}^{sh}(U_k^2 + U_m^2)$ represents the reactive power generated by the shunt elements of the equivalent π-model (assuming actual transmission line sections, i.e. with $b_{km} < 0$ and $b_{km}^{sh} > 0$).

Example 3.1. *A 750 kV transmission line section has a series impedance of* $0.00072 + j0.0175$ *p.u., a total shunt impedance of 8.775 p.u., a voltage magnitude at the terminal buses of 0.984 p.u. and 0.962 p.u., and a voltage angle difference of* $22°$. *Calculate the active and reactive power flows.*

Solution The active and reactive power flows in the line are obtained by applying eqs. (3.3) and (3.4), where $U_k = 0.984$ p.u., $U_m = 0.962$ p.u., and $\theta_{km} = 22°$. The series impedance and admittances are as follows:

$$z_{km} = 0.00072 + j0.0175 \ p.u.$$

$$y_{km} = g_{km} + jb_{km} = z_{km}^{-1} = 2.347 - j57.05 \ p.u.$$

The π-model shunt admittances (100 MVA base) are:

$$b_{km}^{sh} = 8.775/2 = 4.387 \ p.u.$$

and

$$P_{km} = 0.984^2 \cdot 2.347 - 0.984 \cdot 0.962 \cdot 2.347 \cos 22° + 00.984 \cdot 0.962 \cdot 57.05 \sin 22° \ p.u.$$

$$Q_{km} = -0.984^2 \cdot (-57.05 + 4.39) - 0.984 \cdot 0.962 \cdot 57.05 \cos 22° - 00.984 \cdot 0.962 \cdot 2.347 \sin 22° \ p.u.$$

which yield

$$P_{km} = 2044 \ MW \qquad Q_{km} = 8.5 \ Mvar$$

In similar way one obtains:

$$P_{mk} = -2012\ MW \qquad Q_{mk} = -50.5\ Mvar$$

It should be noted that powers are positive when injected into the line. ♦

3.2 In-phase Transformers

The complex current I_{km} in an in-phase transformer is expressed as in eq. (2.13)

$$I_{km} = a_{km} y_{km} (a_{km} E_k - E_m)$$

The complex power, $S_{km} = P_{km} + jQ_{km}$, is given by

$$S_{km} = E_k I_{km}^* = y_{km}^* a_{km} U_k e^{j\theta_k} (a_{km} U_k e^{-j\theta_k} - U_m e^{-j\theta_m}) \tag{3.9}$$

Separating the real and imaginary parts of this latter expression yields the active and reactive power flow equations:

$$P_{km} = (a_{km}U_k)^2 g_{km} - a_{km}U_k U_m g_{km} \cos\theta_{km} - a_{km}U_k U_m b_{km} \sin\theta_{km} \tag{3.10}$$

$$Q_{km} = -(a_{km}U_k)^2 b_{km} + a_{km}U_k U_m b_{km} \cos\theta_{km} - a_{km}U_k U_m g_{km} \sin\theta_{km} \tag{3.11}$$

These same expressions can be obtained by comparing eqs. (3.9) and (3.2); in eq. (3.9) the term $jb_{km}^{sh} U_k^2$ is not present, and U_k is replaced by $a_{km}U_k$. Hence, the expressions for the active and reactive power flows on in-phase transformers are the same expressions derived for a transmission line, except the for two modifications: ignore b_{km}^{sh}, and replace U_k with $a_{km}U_k$.

Example 3.2. *A 500/750 kV transformer with a tap ratio of 1.050:1.0 on the 500 kV side, see Figure 2.4, has neglible series resistance and a leakage reactance of 0.00623 p.u., terminal voltage magnitudes of 1.023 p.u. and 0.968 p.u., and an angle spread of* $5.3°$. *Calculate the active and reactive power flows in the transformer.*

Solution The active and reactive power flows in the transformer are given by eqs. (3.10) and (3.11), where $U_k = 1.023$ p.u., $U_m = 0.968$ p.u., $\theta_{km} = 5.3°$, and $a_{km} = 1.0/1.05 = 0.9524$. The series reactance and susceptance are as follows:

$$x_{km} = 0.00623 \text{ p.u.}$$

$$b_{km} = -x_{km}^{-1} = -160.51 \text{ p.u.}$$

The active and reactive power flows can be expressed as

$$P_{km} = -0.9524 \cdot 1.023 \cdot 0.968 \cdot (-160.51) \sin 5.3^\circ \text{ p.u.}$$

$$Q_{km} = -(0.9524 \cdot 1.023)^2(-160.51) + 0.9524 \cdot 1.023 \cdot 0.968 \cdot (-160.51) \cos 5.3^\circ \text{ p.u.}$$

which yield

$$P_{km} = 1398 \text{ MW} \qquad Q_{km} = 163 \text{ Mvar}$$

The reader is encouraged to calculate P_{mk} and Q_{mk}. (The value of P_{mk} should be obvious.) ♦

3.3 Phase-Shifting Transformer with $a_{km} = 1$

The complex current I_{km} in a phase shifting transformer with $a_{km} = 1$ is as follows, see Figure 2.6:

$$I_{km} = y_{km}(E_k - e^{-j\varphi_{km}} E_m) = y_{km} e^{-j\varphi_{km}}(E_k e^{j\varphi_{km}} - E_m) \tag{3.12}$$

and the complex power, $S_{km} = P_{km} + jQ_{km}$, is thus

$$S_{km} = E_k I_{km}^* = y_{km}^* U_k e^{j(\theta_k + \varphi_{km})}(U_k e^{-j(\theta_k + \varphi_{km})} - U_m e^{-j\theta_m}) \tag{3.13}$$

Separating the real and imaginary parts of this expression, yields the active and reactive power flow equations, respectively:

$$\begin{aligned} P_{km} &= U_k^2 g_{km} - U_k U_m g_{km} \cos(\theta_{km} + \varphi_{km}) \\ &\quad - U_k U_m b_{km} \sin(\theta_{km} + \varphi_{km}) \end{aligned} \tag{3.14}$$

$$\begin{aligned} Q_{km} &= -U_k^2 b_{km} + U_k U_m b_{km} \cos(\theta_{km} + \varphi_{km}) \\ &\quad - U_k U_m g_{km} \sin(\theta_{km} + \varphi_{km}) \end{aligned} \tag{3.15}$$

As with in-phase transformers, these expressions could have been obtained through inspection by comparing eqs. (3.2) and (3.13): in eq. (3.13), the term $jb_{km}^{sh} U_k^2$ is not present, and θ_{km} is replaced with $\theta_{km} + \varphi_{km}$. Hence, the expressions for the active and reactive power flows in phase-shifting transformers are the same expressions derived for the transmission line, albeit with two modifiactions: ignore b_{km}^{sh} and replace θ_{km} with $\theta_{km} + \varphi_{km}$.

Example 3.3. *A Δ - Y, 230/138 kV transformer presents a 30° phase angle shift. Series resistance is neglected and series reactance is 0.0997 p.u. Terminal voltage magnitudes are 0.882 p.u. and 0.989 p.u., and the total angle difference is -16.0°. Calculate the active and reactive power flows in the transformer.*

Solution The active and reactive power flows in the phase-shifting transformer are given by eqs. (3.14) and (3.15), where $U_k = 0.882$ p.u., $U_m = 0.989$ p.u., $\theta_{km} = -16.6°$, and $\varphi_{km} = 30°$. The series reactance and susceptance are as follows:

$$x_{km} = 0.0997 \text{ p.u.}$$

$$b_{km} = -x_{km}^{-1} = -10.03 \text{ p.u.}$$

The active and reactive power flows can be expressed as

$$P_{km} = -0.882 \cdot 0.989 \cdot (-10.03) \cdot (-160.51) \sin(-16.6° + 30°) \text{ p.u.}$$

$$Q_{km} = -0.882^2(-10.03) + 0.882 \cdot 0.989 \cdot (-10.03) \cos(-16.6° + 30°) \text{ p.u.}$$

which yield

$$P_{km} = 203 \text{ MW} \qquad Q_{km} = -70.8 \text{ Mvar}$$

The reader is encouraged to calculate P_{mk} and Q_{mk}. (The value of P_{mk} should be obvious.) ♦

3.4 Unified Power Flow Equations

The expressions for active and reactive power flows on transmission lines, in-phase transformers, and phase shifting transformers, see Figure 2.9, can be expressed in the following unified forms:

$$\begin{aligned} P_{km} = {} & (a_{km}U_k)^2 g_{km} \\ & - (a_{km}U_k)(a_{mk}U_m) g_{km} \cos(\theta_{km} + \varphi_{km} - \varphi_{mk}) \\ & - (a_{km}U_k)(a_{mk}U_m) b_{km} \sin(\theta_{km} + \varphi_{km} - \varphi_{mk}) \end{aligned} \tag{3.16}$$

$$\begin{aligned} Q_{km} = {} & (a_{km}U_k)^2 (b_{km} + b_{km}^{sh}) \\ & + (a_{km}U_k)(a_{mk}U_m) b_{km} \cos(\theta_{km} + \varphi_{km} - \varphi_{mk}) \\ & - (a_{km}U_k)(a_{mk}U_m) g_{km} \sin(\theta_{km} + \varphi_{km} - \varphi_{mk}) \end{aligned} \tag{3.17}$$

Where, for the transmission lines like the one represented in Figure 2.2, $a_{km} = a_{mk} = 1$ and $\varphi_{km} = \varphi_{mk} = 0$; for in-phase transformers such as the one represented in Figure 2.4, $y_{km}^{sh} = y_{mk}^{sh} = 0$, $a_{mk} = 1$ and $\varphi_{km} = \varphi_{mk} = 0$; and for a phase-shifting transformer such as the one in Figure 2.6, $y_{km}^{sh} = y_{mk}^{sh} = 0$, $a_{mk} = 1$ and $\varphi_{mk} = 0$.