Supplementary materials for “Homogenization of fault frictional properties”

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1. Summary

The main document examines the quasistatic development of fault slip considering frictional properties to be nonuniform over many lengthscales. In particular, we focus when homogenizing —using an average estimate—of otherwise nonuniform frictional properties may be appropriate. The analysis considers fault frictional strength to be dependent on instantaneous slip rate and its history, referred as state. The parameters describing such rate-and state-dependent frictional strength are considered to be nonuniformly distributed. This supplementary information part focusses mainly on the nonlinear stability of the blowup solutions, special eigenmodes corresponding to spatial symmetry and bifurcations of blowup solutions, and their relevance in the homogenization limit.

2. Slip rate and state evolution

The analysis presented only explicitly considers thin-slab elastic configuration wherein the slip-traction relation is given by

$$\tau_{el}(x,t) := \bar{E}h\frac{\partial^2\delta(x,t)}{\partial x^2} \tag{1}$$

where the elastic modulus $\bar{E} = 2\mu/(1-\nu)$ and $\bar{E} = \mu$ for mode-II and mode-III sliding respectively. μ and ν are the shear modulus and Poisson's ratio, respectively. However, the results also apply to the conventional slip surface within two half-spaces case where the slip-traction relations is given by,

$$\tau_{el}(x,t) := \frac{\bar{\mu}}{2\pi}\int_{L^-}^{L^+} \frac{\partial\delta(\xi,t)/\partial\xi}{\xi - x} d\xi \tag{2}$$

where $\bar{\mu} = \mu/(1-\nu)$ and $\bar{\mu} = \mu$ for in- and anti-plane slip, respectively. The integrand is singular at $\xi = x$ and the integral is evaluated in a Cauchy principal-value sense.

We represent traction-slip relations for both the fault geometries using an operator $\mathcal{L}$ acting on the slip distribution $\delta(x,t)$, that is, $\tau_{el}(x,t) := \mathcal{L}[\delta(x,t);x]$. When and where fault has nonzero slip gradient the total shear traction ($\tau_{ex} + \mathcal{L}[\delta(x,t);x]$) is equal to the shear strength (τ_s) of the interface. In terms of their rates that equality can be expressed as

$$\frac{\partial\tau_{ex}}{\partial t} + \mathcal{L}[V(x,t);x] = \frac{\partial\tau_s}{\partial t} \tag{3}$$

where $V(x,t) = \partial\delta/\partial t$ is the slip rate.

We note that as slip attains an unbounded amplitude and the first term in the above equation becomes negligible. Considering a frozen spatial distribution of fault normal stress $\sigma(x)$ and using time derivative of friction coefficient (equation (5) in the main document), the evolution of the slip rate $V(x,t)$ can be expressed as

$$\frac{\partial V}{\partial t} = V(x,t)\frac{\mathcal{L}[V(x,t);x]}{\sigma(x)a(x)} + \frac{b(x)}{a(x)}\frac{V(x,t)^2}{D_c(x)}\left[-\frac{D_c(x)}{V(x,t)}\frac{\partial\theta/\partial t}{\theta(x,t)}\right] \tag{4}$$

A scaling analysis suggests the quantity within the bracket should be of order 1 ($\sim O(V^0)$) so that all the terms in the equation are comparable. This motivates to define a surrogate state variable $\Phi(x,t)$, given by,

$$\Phi(x,t) = -\frac{D_c(x)}{V(x,t)}\frac{\partial\theta/\partial t}{\theta(x,t)} \tag{5}$$

When aging-law of state evolution is considered the above definition assumes the form

$$\Phi(x,t) = 1 - \frac{D_c(x)}{V(x,t)\theta(x,t)} \tag{6}$$

With this definition, Φ can be interpreted as a measure of distance from steady-state sliding: $\Phi = 0$ for steady-state sliding ($V\theta/D_c = 1$) and $\Phi = 1$ when state of the slip is far from steady-state ($V\theta/D_c \gg 1$). The coupled system of quasistatic slip acceleration and evolution of $\Phi(x,t)$ with time, is given by

$$\frac{\partial V}{\partial t} = V(x,t)\frac{\mathcal{L}[V(x,t);x]}{\sigma(x)a(x)} + \frac{b(x)}{a(x)}\frac{V(x,t)^2}{D_c(x)}\Phi(x,t) \tag{7a}$$

$$\frac{\partial \Phi}{\partial t} = [1-\Phi(x,t)]\left[\frac{\mathcal{L}[V(x,t);x]}{\sigma(x)a(x)} + \left(\frac{b(x)}{a(x)} - 1\right)\frac{V(x,t)}{D_c(x)}\Phi(x,t)\right] \tag{7b}$$

The functional forms $\mathcal{R}[V(x,t),\Phi(x,t)]$ and $\mathcal{S}[V(x,t),\Phi(x,t)]$, in the system of equations (9) of the main document, are respectively given by the right sides of (7a) and (7b).

3. Blowup solutions and their attainability

Previous studies (Viesca, 2016a, 2016b; Ray & Viesca, 2017) showed that system of equations (7) has a solution of the form

$$V(x,t) = \frac{D_c(x)}{t_f}\mathcal{W}(x) \quad \text{and} \quad \Phi(x,s) = \mathcal{P}(x) \tag{8}$$

where the $t_f = t_{in} - t$ is the time remaining to instability and t_{in} as time of instability. The profile $\mathcal{W}(x)$, referred to as blowup solution, and its support on the fault are to be solved for. The profile $\mathcal{P}(x)$ is obtained from $\mathcal{W}(x)$ and shows the proximity of the instability from the steady state slip. On substituting above in the evolution equations (7) and choosing thin slab elastic configuration (1), we get,

$$\frac{\bar{E}hD_c}{\sigma(x)a(x)}\frac{d^2\mathcal{W}}{dx^2}+\frac{b(x)}{a(x)}\mathcal{W}(x)\mathcal{P}(x)=1 \tag{9a}$$

$$[1-\mathcal{P}(x)][1-\mathcal{W}(x)\mathcal{P}(x)]=0 \tag{9b}$$

We recall that Φ by its definition (6) is the measure of proximity to steady state sliding and lies between 0 (steady state) and 1 (far above and below steady state); and hence, the $\mathcal{P}(x)=1$ or $\mathcal{WP}=1$, respectively, for $\mathcal{W}(x)$ below or above unity (Viesca, 2016b, 2016a; Ray & Viesca, 2017).

3.1. Stability analysis of the blowup solutions

The blowup solutions, $\mathcal{W}(x)$, and its support on the fault, $2L$, are straightforward to be solved for; however, Ray and Viesca (2017) showed existence of multiple solutions under nonuniform parameter distributions. This naturally raises the question of attainability of a particular blowup solution; which, in turn, equivalent to asking if a preferential evolution of (7) is possible when provoked by an external or initial condition.

In order to address if a particular blowup solution is realizable, we perturb the blowup solution $\mathcal{W}(x)$ by a small function $\omega(x)$, and analyze when the perturbed velocity profile, given by,

$$V(x,t)=\frac{D_c(x)}{t_f}[\mathcal{W}(x)+\omega(x)(t_f/t_o)^{-\lambda}] \tag{10}$$

might converge to or diverge from the spatial profile $\mathcal{W}(x)$. That is, we analyze when and where on the fault the perturbation $\omega(x)$ could grow or decay as $t_f\to 0$, which is dictated by the sign of $\mathrm{Re}(\lambda)$. Likewise, we also consider whether a perturbation to the state variable in the form

$$\Phi(x,t)=\mathcal{P}(x)+\phi(x)(t_f/t_o)^{-\lambda} \tag{11}$$

might grow or shrink as $t_f \to 0$.

On substituting the above perturbations (10, 11) in (7) permits to consider the functions $\omega(x)$ and $\phi(x)$ and the scalar λ as solutions to an eigenvalue problem, given by,

$$\lambda \begin{bmatrix} \omega(x) \\ \phi(x) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \omega(x) \\ \phi(x) \end{bmatrix} \tag{12}$$

where the terms inside the matrix, for the thin-slab elastic configuration and with heterogeneity only in the parameter $a(x)$, are given by,

$$\begin{aligned} A_{11} &= \frac{\mathcal{W}(x)}{a(x)/b} \left[L_{bh}^2 \frac{d^2}{dx^2} + \mathcal{P}(x) \right] \\ A_{12} &= \frac{\mathcal{W}(x)^2}{a(x)/b} \\ A_{21} &= \frac{1-\mathcal{P}(x)}{a(x)/b} \left[L_{bh}^2 \frac{d^2}{dx^2} + (1 - a(x)/b)\, \mathcal{P}(x) \right] \\ A_{22} &= \frac{1 - a(x)/b}{a(x)/b} [1 - \mathcal{P}(x)] \mathcal{W}(x) - [1 - \mathcal{W}(x)\mathcal{P}(x)] \end{aligned}$$

The blowup solution $\mathcal{W}(x)$ is attainable when the maximum eigenvalue, $\lambda = \lambda_{max}$, has negative real and are referred to as stable solutions. In Figures 3 and 4 in the main document we plot the maximum eigenvalue, $\lambda = \lambda_{max}$, that determines the attainability of the blowup solutions $\mathcal{W}(x)$ in (8).

We note that the above eigenvalue problem is equivalent to that of the linear stability analysis of the fixed point solutions in (Viesca, 2016a, 2016b; Ray & Viesca, 2017). Considering solutions for which $\mathcal{P}(x) = 1$ in above allows us to focus on a reduced version of the eigenvalue problem wherein only the eigenmode $\omega(x)$ needs to be analyzed. A suitable rearrangement of the terms allow us to re-express the eigen equation in the simple form

$$L_{bh}^2 \frac{d^2\omega}{dx^2} + \omega(x) = \lambda \left[\frac{a(x)/b}{\mathcal{W}(x)} \right] \omega(x) \tag{13}$$

3.2. Spatial symmetry eigenmode

Here, we re-highight the existence of spatial-symmetry eigenmode for its relevance to the consideration of heterogeneous distributions of properties and its apparent connection with the observed stability reversals of blow-up solutions and homogenization.

We look for the eigenmode and eigenvalue indicating that a blow-up solution remains invariant when translated along the fault (Ray & Viesca, 2017). Presuming that shifting the origin of the blowup solution, by a small quantity ϵ, has no bearing on the form of the diverging slip rate (8), we may write

$$V(x,t) = \frac{D_c}{t_f}\mathcal{W}(x+\epsilon)$$

which may be expanded to first-order in the perturbation as

$$V(x,t) = \frac{D_c}{t_f}\left[\mathcal{W}(x) + \epsilon\mathcal{W}'(x)\right]$$

Comparing this last result with equation (10), we see that translational symmetry corresponds to the existence of an eigenmode and eigenvalue satisfying

$$\omega(x) = \mathcal{W}'(x) \quad \text{with} \quad \lambda = 0 \tag{14}$$

4. Blow-up solutions under variations of $a(x)$ with integer wavenumber

We recall that in Figure 3 of the main text, the blowup solutions at the maximum and minimum of the $a(x)/b$ distribution exchange their stability at integer wavenumbers $\kappa \neq 1$. In that figure, the stability curves (λ_{max} vs. κ) for the solutions at the extrema of $a(x)/b$ cross $\lambda_{max} = 0$ simultaneously. This is a signature of a transcritical bifurcation of the fixed-point solutions.

Here, we show how the stability reversals may be anticipated to occur at integer wavenumbers, if anywhere. Furthermore, we show how a translational symmetry mode is retrieved in the limit of large values of κ, the limit for which we found homogenization to be appropriate.

We consider the simple periodic variation of the parameter $a(x)/b = a_0 + a_1 \cos(\kappa x/L_{bh})$, with uniform σ and D_c. For solutions with $\mathcal{P}(x) = 1$, (9a) reduces to

$$L_{bh}^2 \frac{d^2\mathcal{W}}{dx^2} + \mathcal{W}(x) = a_0 + a_1 \cos(\kappa x/L_{bh}) \tag{15}$$

with the boundary contions $\mathcal{W}(\pm L) = 0$ and $\mathcal{W}'(\pm L) = 0$ determining both $\mathcal{W}(x)$ and L/L_{bh}. The blowup solutions corresponding to the maximum and minimum of $a(x)/b$ are obtained by switching the algebraic sign of a_1. For integer wavenumbers, $\kappa = n \neq 1$, the solution to (15) is

$$\mathcal{W}(x) = a_0 \left[1 + \cos(x/L_{bh})\right] + \frac{a_1}{1-n^2} \left[\cos(x/L_{bh}) + (-1)^n \cos(nx/L_{bh})\right], \quad L/L_{bh} = \pi \tag{16}$$

We note that the value of L/L_{bh} is equal to that for the homogeneous problem ($a_1 = 0$) for all $\kappa = n$. We also note that at large n the second term vanishes as n^{-2}, leaving only the solution to the homogeneous problem.

We may now proceed to find the eigenmodes associated with $\lambda = 0$ at integer κ following the linear stability analysis of these blow-up solutions. Specifically, we substitute $\lambda = 0$ in the reduced eigenvalue problem (13) and solve for the corresponding eigenfunction which satisfies

$$\frac{L_{bh}^2}{L^2} \frac{d^2\omega}{d\tilde{x}^2} + \omega(\tilde{x}) = 0 \tag{17}$$

where we have performed a change of the independent variable $\tilde{x} = x/L$. Given that $L/L_{bh} = \pi$ at integer κ, the general solution is a linear combination of $\sin(\pi\tilde{x})$ and $\cos(\pi\tilde{x})$. Imposing the boundary conditions $\omega(\tilde{x} = \pm 1) = 0$ eliminates the latter function and leaves

$$\omega(x) = \sin(\pi x/L) \quad \text{with} \quad \lambda = 0 \quad \text{for} \quad \kappa = n = 2, 3, 4, ... \tag{18}$$

Re-examining (16), we see that

$$\lim_{n\to\infty} \mathcal{W}'(x) = -a_0 \sin(x/L_{bh}) = -a_0 \sin(\pi x/L) \tag{19}$$

and upon finding that this result is, to within an arbitrary pre-factor, equal to the eigenmode (18), we find that we retrieve the translational symmetry mode condition (14) in the limit of large κ. Thus we find that, in the highly heterogeneous limit of large κ, the blow-up solutions regain the translational invariance that would be expected for a homogeneous fault.

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