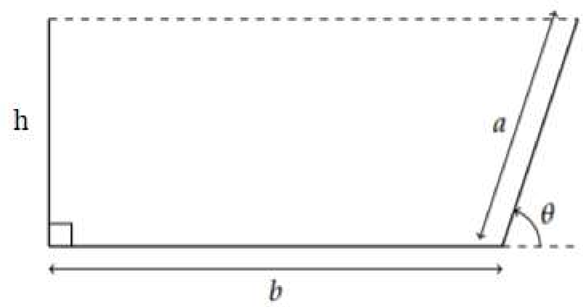
Problem 1:



1.1.1: Compute an expression for the cross-sectional area of the channel A as a function of a and θ only:

We have:

- $h + b + a = W = 3$
- $h = a * \sin\theta$
- $A = h * b + \frac{1}{2}h * (a * \cos\theta)$

$$= h(3 - a - h) + \frac{1}{2}h * a * \cos\ \theta$$

$$= a * \sin\theta * (3 - a - a * \sin\theta) + \frac{1}{2}a * \sin\theta * a * \cos\ \theta$$

$$= 3a * \sin\theta - a^2 * \sin\theta - a^2\sin^2\theta + \frac{1}{2}a^2\sin\theta\cos\theta \qquad (*)$$

1.1.2: Produce both a surface plot and a contour plot of the function A(a, θ), over the domain of physically-realistic values for a and θ.

We have:

- As a is length: $a > 0$
- As $h + b + a = 3$, and b, h are also lengths: $b, h > 0$
  $\Rightarrow a = 3 - b - h < 3 - 0 - 0 = 3$
  Hence, $0 < a < 3$
- θ is the angle between segment a and base b: it can take any values $0 < \theta < \pi$ (cannot equal 0 or π because if it is, this cannot be a channel)
  We can even says narrow down to $0 < \theta \leq \frac{\pi}{2}$ because if $\theta > \frac{\pi}{2}$:
  - If we take $\theta' = \pi - \theta$, we will have the same a, b, h but this channel has bigger cross-sectional area, which is our purpose. Hence, we can ignore $\theta > \frac{\pi}{2}$

The realistic domain we need to consider: $0 < a < 3$ and $0 < \theta \leq \frac{\pi}{2}$

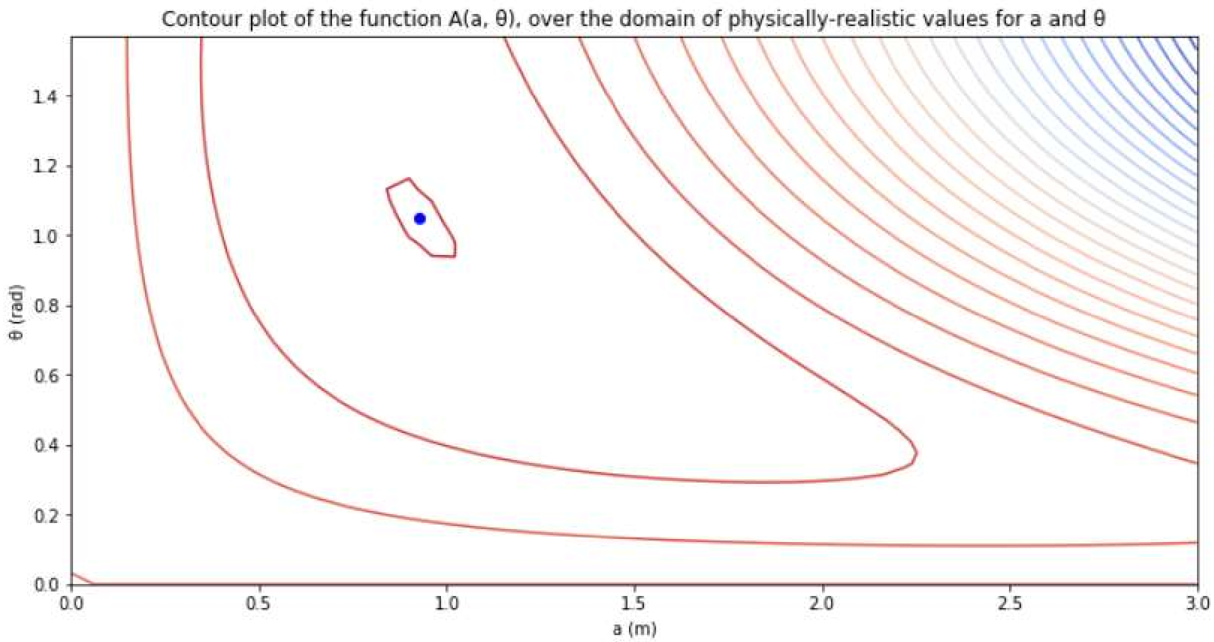


Fig. 1: The contour plot of the function $A(a, \theta)$ over the domain of physically realistic values for a and θ. The map setting is "cool warm", indicating the warmer the color, the bigger the values

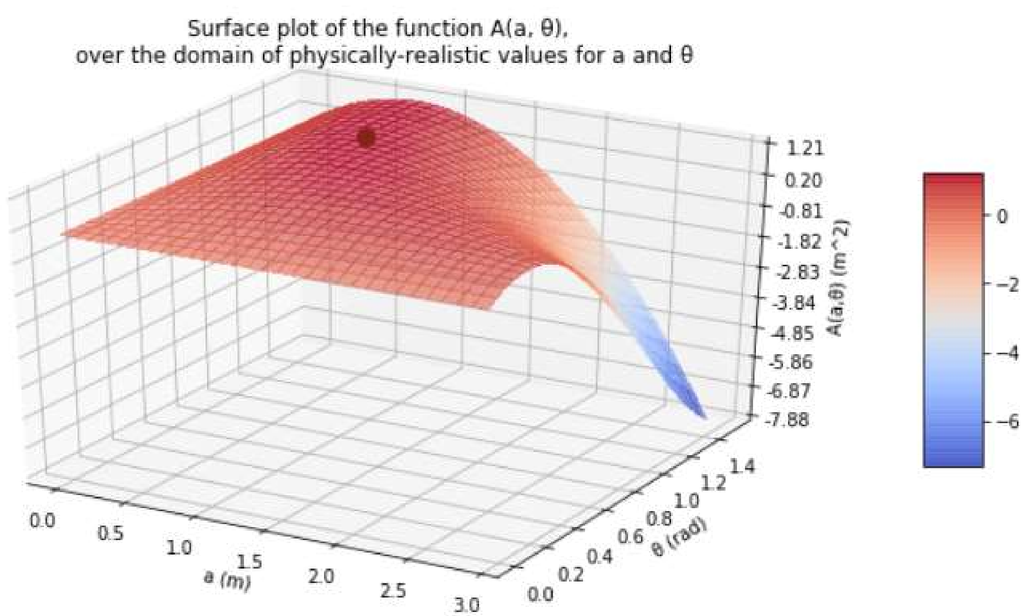


Fig. 2: The surface plot of the function $A(a, \theta)$ over the domain of physically realistic values for a and θ. The map setting is "cool warm", indicating the warmer the color, the bigger the values, coupled with the color bar on the right.

1.1.a. Estimate the values of a and θ that maximize the area from your plot

From the contour plot in Fig.1, we estimate a to be slightly less than 1, whereas θ to be around $\frac{\pi}{3}$ (because when we plot θ on degree scale, it seems to be around 60 degree)

The maximum looks unique over the domain of physically realistic values because in Fig.1 (the contour plot), the values of the functions seem to increase near (0.95, 1.05), which possibly near the peaks of the function. Furthermore, the contour map indicates the function is decently smooth. The contour plot infers there exist a maximum and the smoothness in the surface plot indicates we can reach it. We will analytically find it later:

1.1.b. The negated function −A(a, θ) is not coercive

The norm of the parameters: $\sqrt{a^2 + \theta^2}$

Function $-A(a,\theta) = -(3a * \sin\theta - a^2 * \sin\theta - a^2\sin^2\theta + \frac{1}{2}a^2\sin\theta\cos\theta)$ is coercive iff:

$$\lim_{\sqrt{a^2+\theta^2}\to\infty} -(3a * \sin\theta - a^2 * \sin\theta - a^2\sin^2\theta + \frac{1}{2}a^2\sin\theta\cos\theta) \to \infty$$

Consider one direction that $\sqrt{a^2 + \theta^2} \to \infty$:

Let $\theta = 0; a \to \infty$. Then $\lim_{\theta=0,a\to\infty} \sqrt{a^2+\theta^2} = \lim_{\theta=0,a\to\infty} \sqrt{a^2} = \lim_{\theta=0,a\to\infty} a \to \infty$: norm $\to \infty$

$$\lim_{\theta=0,a\to\infty} -(3a * \sin\theta - a^2 * \sin\theta - a^2\sin^2\theta + \frac{1}{2}a^2\sin\theta\cos\theta)$$

$$= \lim_{\theta=0,a\to\infty} -(3a * \sin 0 - a^2 * \sin 0 - a^2\sin^2 0 + \frac{1}{2}a^2\sin 0\cos\theta)$$

$$= \lim_{\theta=0,a\to\infty} 0 = 0 \quad \text{(because } \sin 0 = 0\text{)}$$

Hence, for a direction that norm $\to \infty$: $\theta = 0, a \to \infty$: $\lim_{\theta=0,a\to\infty} -A(a,\theta) = 0 \neq \infty$

Hence, $-A(a,\theta)$ is not coercive.

1.2.1. Using multivariable calculus to find exact values of $\boldsymbol{a}, \boldsymbol{\theta}$

$$A = h * b + \frac{1}{2}h * (a * \cos\theta) = a * \sin\theta * b + \frac{1}{2} * a * \sin\theta * a * \cos\theta$$

$$g(a,b,\theta) = a + b + h = a + b + a * \sin\theta$$

Constraint: $g(a,b,\theta) = 3$

Using Lagrange: Lagrange Function:

$$L(a,b,\theta,\lambda) = A - \lambda(g(a,b,\theta) - 3)$$
$$= a * \sin\theta * b + \frac{1}{2} * a * \sin\theta * a * \cos\theta - \lambda(a + b + a * \sin\theta - 3)$$

We set the 1st order partial derivative of the Lagrange function to 0

$$\frac{\partial L}{\partial a} = b * \sin\theta + a * \sin\theta * \cos\theta - \lambda - \lambda * \sin\theta = 0 \qquad (1)$$

$$\frac{\partial L}{\partial b} = a * \sin\theta - \lambda = 0 \Leftrightarrow \lambda = a * \sin\theta \qquad (2)$$

$$\frac{\partial L}{\partial \theta} = a * b * \cos\theta + \frac{1}{2}a^2 * (\cos^2\theta - \sin^2\theta) - \lambda * a * \cos\theta = 0 \qquad (3)$$

$$\frac{\partial L}{\partial \lambda} = a + b + a * \sin\theta - 3 = 0 \qquad (4)$$

Solve these equations (the steps and solution is in appendix 1), we have:
$\theta = \frac{\pi}{3}; a = 4\sqrt{3} - 6; \; b = 3 - \sqrt{3}; \lambda = 6 - 3\sqrt{3}$

As the Langrage multipliers in our case only return one value, it will either be the maximum or minimum point of this function. To check which one, we can compare it with the boundary points in the realistic domain of $A(a,\theta)$. We use function (*) $A(a,\theta)$ for easier control of parameters. We have: $a \in (0,3); \theta \in (0,\frac{\pi}{2}]$, and $A(a,\theta)$ is a continuous function (as all elements are elementary functions with no denominator).

Parameters value (a,θ)	$(4\sqrt{3} - 6\,,\frac{\pi}{3})$	(0,0)	$\left(0,\frac{pi}{2}\right)$	(3,0)	$\left(3,\frac{pi}{2}\right)$
Function values $A(a,\theta)$	$9 - \frac{9\sqrt{3}}{2} \approx 1.201$	0	0	0	-9

By comparing the point found using Lagrange multiplier $X(4\sqrt{3} - 6,\frac{\pi}{3})$ to the boundary points, we can see that at point $X\left(4\sqrt{3} - 6,\frac{\pi}{3}\right)$, function A has higher value. This implies X is the maximum point of A and $A(X) = A(4\sqrt{3} - 6,\frac{\pi}{3})$ is the maximum of the function.

Hence, at $X(4\sqrt{3} - 6,\frac{\pi}{3})$, the area of the channel reaches its maximum:
$A\left(4\sqrt{3} - 6,\frac{\pi}{3}\right) = 9 - \frac{9\sqrt{3}}{2} \approx 1.201\ (m^2)$.

1.3.

Gradient descent is the technique for finding the minimum. However, we are seeking the maximum. Hence, I use gradient descent for function $-A(a,\theta)$ (because the optimal point reaches minimum of $-A(a,\theta)$ is the optimal point reaches maximum of $A(a,\theta)$)

In my code, I used neg_f and neg_gradf as the negative of the function and its gradients. I used backtracking line search to find the optimal step size for gradient descent (or ascent in our case).

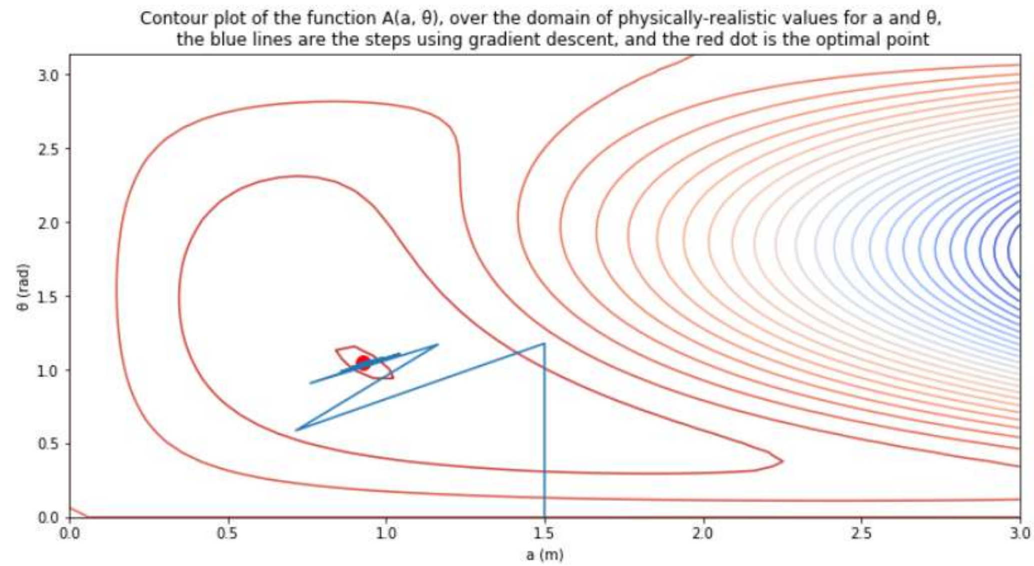


Fig. 3: The contour plot of the function $A(a, \theta)$ over the domain of physically realistic values for a and θ. Blue lines are the steps of gradient ascent + backtracking line search. The red dot is the optimal point. As we can see from the plot, we step closer and closer to the red dot and finally converge.

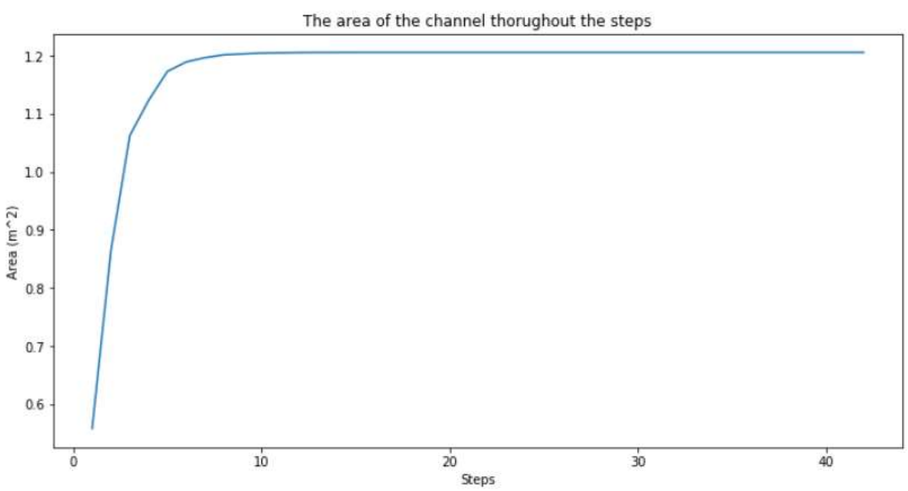


Fig. 4: The line plot of values the function $A(a, \theta)$ over the gradient ascent steps. We can see that the values increase and converge through the steps.

```
Iterations 0 . Current node: [1.5 0. ] . Value:  0.0
Iterations 10 . Current node: [0.90976936 1.03104415] . Value:  1.204655698385919
Iterations 20 . Current node: [0.9280385  1.04684741] . Value:  1.2057711464305945
Iterations 30 . Current node: [0.92819588 1.04717957] . Value:  1.2057713654156008
Iterations 40 . Current node: [0.92820291 1.04719662] . Value:  1.205771365938795
GD terminate:  43
Current node: [0.92820364 1.04719753] . Value:  1.2057713659398266
```

Fig 5: The table showing converging values using the algorithms

Appendix 1: The solutions for the Lagrange multipliers systems of equations:

$A = h * b + \frac{1}{2} h * (a * \cos\theta) = a * \sin\theta * b + \frac{1}{2} * a * \sin\theta * a * \cos\theta$

$g(a, b, \theta) = a + b + h = a + b + a * \sin\theta$

Constraint: $g(a, b, \theta) = 3$

Using Lagrange: Lagrange Function:

$$\begin{aligned} L(a, b, \theta, \lambda) &= A - \lambda(g(a, b, \theta) - 3) \\ &= a * \sin\theta * b + \frac{1}{2} * a * \sin\theta * a * \cos\theta - \lambda(a + b + a * \sin\theta - 3) \end{aligned}$$

We set the 1st order partial derivative of the Lagrange function to 0

$$\frac{\partial L}{\partial a} = b * \sin\theta + a * \sin\theta * \cos\theta - \lambda - \lambda * \sin\theta = 0 \quad (1)$$

$$\frac{\partial L}{\partial b} = a * \sin\theta - \lambda = 0 \Leftrightarrow \lambda = a * \sin\theta \quad (2)$$

$$\frac{\partial L}{\partial \theta} = a * b * \cos\theta + \frac{1}{2} a^2 * (\cos^2\theta - \sin^2\theta) - \lambda * a * \cos\theta = 0 \quad (3)$$

$$\frac{\partial L}{\partial \lambda} = a + b + a * \sin\theta - 3 = 0 \quad (4)$$

Plug (2) into (1):

$\mathrm{B} * \sin\theta + a * \sin\theta * \cos\theta - a * \sin\theta - a * \sin\theta * \sin\theta = 0$

$\Leftrightarrow \sin\theta * (b + a * \cos\theta - a - a * \sin\theta) = 0$

$\Leftrightarrow b + a * \cos\theta - a - a * \sin\theta = 0$ (5)

(Because in the domain $0 < \theta \leq \frac{\pi}{2}: 0 < \sin\theta \leq 1 \Rightarrow \sin\theta \neq 0$: we can divide both side by a non-zero value without affecting the equation)

From (4): $a + b + a * \sin\theta - 3 = 0$

(5) – (4) $\Rightarrow b + a * \cos\theta - a - a * \sin\theta - (a + b + a * \sin\theta - 3) = 0$

$\Leftrightarrow -2a - 2a * \sin\theta + a * \cos\theta + 3 = 0$

$\Leftrightarrow 2a - 3 = a * \cos\theta - 2a * \sin\theta$ (6)

(5) + (4) $\Rightarrow 2b + a * \cos\theta = 3 \Rightarrow 2b = 3 - a * \cos\theta$ (7)

From (3):

$$a*b*\cos\theta+\frac{1}{2}a^2*(\cos^2\theta-\sin^2\theta)-\lambda*a*\cos\theta=0$$

$$\Leftrightarrow\frac{1}{2}\ a*(2b)*\cos\theta+\frac{1}{2}a^2[2\cos^2\theta-(\cos^2\theta\ +\sin^2\theta)]\ -\lambda*a*\cos\theta=0$$

$$\Leftrightarrow\frac{1}{2}\ a*(3-a*\cos\theta)*\cos\theta+\frac{1}{2}a^2(2\cos^2\theta-1)\ -(a*\sin\theta)*a*\cos\theta=0$$

(Plug (7): $2b=3-a*\cos\theta$; (2): $\lambda=a*\sin\theta$; $\cos^2\theta\ +\sin^2\theta=1$ into the equation)

$$\Leftrightarrow\frac{1}{2}\ a*(3-a*\cos\theta)*\cos\theta+\frac{1}{2}a^2(2\cos^2\theta-1)-a^2*\sin\theta*\cos\theta=0$$

$$\Leftrightarrow a*(3-a*\cos\theta)*\cos\theta+a^2(2\cos^2\theta-1)-2a^2*\sin\theta*\cos\theta=0\quad\text{(multiply by 2)}$$

$$\Leftrightarrow 3a*\cos\theta-a^2*\cos^2\theta+2a^2\cos^2\theta-a^2-2a^2*\sin\theta*\cos\theta=0$$

$$\Leftrightarrow 3a*\cos\theta+a^2\cos^2\theta-a^2-2a^2*\sin\theta*\cos\theta=0$$

$$\Leftrightarrow 3a*\cos\theta-a^2+a*\cos\theta*(a*\cos\theta-2a*\sin\theta)=\ 0$$

$$\Leftrightarrow 3a*\cos\theta-a^2+a*\cos\theta*(2a-3)=0$$

(By plugging (6): $2a-\ 3=\ a*\cos\theta-2a*\sin\theta$)

$$\Leftrightarrow 3a*\cos\theta-a^2+2a^2*\cos\theta-3a*\cos\theta=0$$

$$\Leftrightarrow -a^2+2a^2*\cos\theta=0$$

$$\Leftrightarrow a^2(2*\cos\theta-1)=0$$

$\Rightarrow 2\cos\theta-1=0$ (because $a>0\Rightarrow$ we can divide both side by a non-zero value)

$$\Leftrightarrow\cos\theta=\frac{1}{2}\Rightarrow\theta=\pm\frac{\pi}{3}+k2\pi\text{ and }0<\theta<\frac{\pi}{2}\Rightarrow\theta=\frac{\pi}{3}\qquad(8)$$

Plug (8) in (6): $2a-\ 3=\ a*\cos\frac{\pi}{3}-2a*\sin\frac{\pi}{3}$

$$\Leftrightarrow 2a-\ 3=\ a*\frac{1}{2}-2a*\frac{\sqrt{3}}{2}\ \Leftrightarrow\left(\frac{3}{2}+\sqrt{3}\right)a=3$$

$$\Leftrightarrow a=\frac{3}{\frac{3}{2}+\sqrt{3}}=\frac{6}{3+2\sqrt{3}}=\frac{6\left(3-2\sqrt{3}\right)}{9-12}=4\sqrt{3}-6\qquad(9)$$

Plug (8), (9) into (2): $\lambda=a*\sin\theta\ =\left(4\sqrt{3}-6\right)*\frac{\sqrt{3}}{2}=6-3\sqrt{3}$ (10)

Plug (8), (9) into (7): $2b=3-a*\cos\theta=3-\left(4\sqrt{3}-6\right)*\frac{1}{2}=6-2\sqrt{3}$

$\Leftrightarrow b=3-\sqrt{3}$ (11)

Hence: from (8), (9), (10), (11): $\theta=\frac{\pi}{3}$; $a=\ 4\sqrt{3}-6$; $b=3-\sqrt{3}$; $\lambda=6-3\sqrt{3}$

Appendix 2: The optimal point would be easier to find if consider parameters $\boldsymbol{a}$ and $\boldsymbol{\theta}$ only:

$$\frac{\partial A}{\partial a} = 3\sin\theta - 2a * \sin\theta - 2a * \sin^2\theta + a * \sin\theta * \cos\theta$$
$$\frac{\partial A}{\partial \theta} = 3a * \cos\theta - a^2 * \cos\theta - a^2 * 2 * \sin\theta * \cos\theta + \frac{1}{2}a^2(\cos^2\theta - \sin^2\theta)$$

Now, we find the extrema of the function:

Let $\frac{\partial A}{\partial a} = 0$ and $\frac{\partial A}{\partial \theta} = 0$

$$\frac{\partial A}{\partial a} = 0$$
$$\Leftrightarrow\ 3\sin\theta - 2a * \sin\theta - 2a * \sin^2\theta + a * \sin\theta * \cos\theta = 0$$
$$\Leftrightarrow \sin\theta * (3 - 2a - 2a * \sin\theta + a * \cos\theta) = 0$$
$$\Leftrightarrow\ 3 - 2a - 2a * \sin\theta + a * \cos\theta = 0\ \ (\text{because } 0 < \theta \le \frac{\pi}{2} \Rightarrow 0 < \sin\theta \le 1 : \sin\theta \ne 0)$$
$$\Leftrightarrow a * (2 + 2\sin\theta - \cos\theta) = 3$$

$$\frac{\partial A}{\partial \theta} = 0$$

$$\Leftrightarrow\ 3a * \cos\theta - a^2 * \cos\theta - a^2 * 2 * \sin\theta * \cos\theta + \frac{1}{2}a^2(\cos^2\theta - \sin^2\theta) = 0$$
$$\Leftrightarrow\ 3a * \cos\theta - a^2 * \cos\theta - 2a^2 * \sin\theta * \cos\theta + \frac{1}{2}a^2[2\cos^2\theta - (\cos^2\theta + \sin^2\theta)] = 0$$
$$\Leftrightarrow\ 3a * \cos\theta - (2a^2 * \cos\theta - a^2 * \cos\theta) - 2a^2 * \sin\theta * \cos\theta + \frac{1}{2}a^2[2\cos^2\theta - 1] = 0$$
$$\Leftrightarrow\ 3a * \cos\theta + a^2 * \cos\theta - 2a^2 * \cos\theta - 2a^2 * \sin\theta * \cos\theta + a^2\cos^2\theta - \frac{1}{2}a^2 = 0$$
$$\Leftrightarrow\ 3a * \cos\theta + a^2 * \cos\theta - (2a^2 * \cos\theta + 2a^2 * \sin\theta * \cos\theta - a^2\cos^2\theta) - \frac{1}{2}a^2 = 0$$
$$\Leftrightarrow\ 3a * \cos\theta + a^2 * \cos\theta - a * \cos\theta\,(2a + 2a * \sin\theta - a * \cos\theta) - \frac{1}{2}a^2 = 0$$
$$\Leftrightarrow\ 3a * \cos\theta + a^2 * \cos\theta - a * \cos\theta * 3 - \frac{1}{2}a^2 = 0$$
$$\Leftrightarrow\ a^2 * \cos\theta - \frac{1}{2}a^2 = 0$$
$$\Leftrightarrow\ a^2 * (\cos\theta - \frac{1}{2}) = 0$$
$$\Leftrightarrow\ \cos\theta - \frac{1}{2} = 0$$
$$\Leftrightarrow \theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

Plug into $\Rightarrow\ a * \left(2 + 2\sin\frac{\pi}{3} - \cos\frac{\pi}{3}\right) = 3$

$$\Leftrightarrow a * \left(2 + 2 * \frac{\sqrt{3}}{2} - \frac{1}{2}\right) = 3$$
$$\Leftrightarrow a * \left(\frac{3}{2} + \sqrt{3}\right) = 3$$

$\Leftrightarrow a = \frac{3}{\frac{3}{2}+\sqrt{3}} = \frac{6}{3+2\sqrt{3}} = \frac{6*(2\sqrt{3}-3)}{12-9} = 2 * \left(2\sqrt{3} - 3\right) = 4\sqrt{3} - 6$: consistent with results from Appendix 1.

```
import matplotlib.pyplot as plt
import numpy as np
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm
from matplotlib.ticker import LinearLocator, FormatStrFormatter
```

Functions + Part 3

```
# define function
def f(X):
  x = X[0]
  y = X[1]
  return 3*x*np.sin(y) - (x**2)*np.sin(y) - (x**2)*(np.sin(y)**2) + 1/2*(x**2)*np.sin(y)*np.cos(y)

# the gradient
def gradf(X):
  x = X[0]
  y = X[1]
  return np.array([3*np.sin(y) - 2*x*np.sin(y) - 2*x*np.sin(y)**2 + x*np.sin(y)*np.cos(y),
                   3*x*np.cos(y) - x**2*np.cos(y) - 2*x**2*np.sin(y)*np.cos(y) + 1/2*x**2*(np.cos(y)**2 - np.sin(y)**2)])
```

```
# as we are doing gradient descent, but to look for the max, we change the objective function of -f(x)
def neg_f(X):
  return -f(X)

def neg_gradf(X):
  return -gradf(X)
```

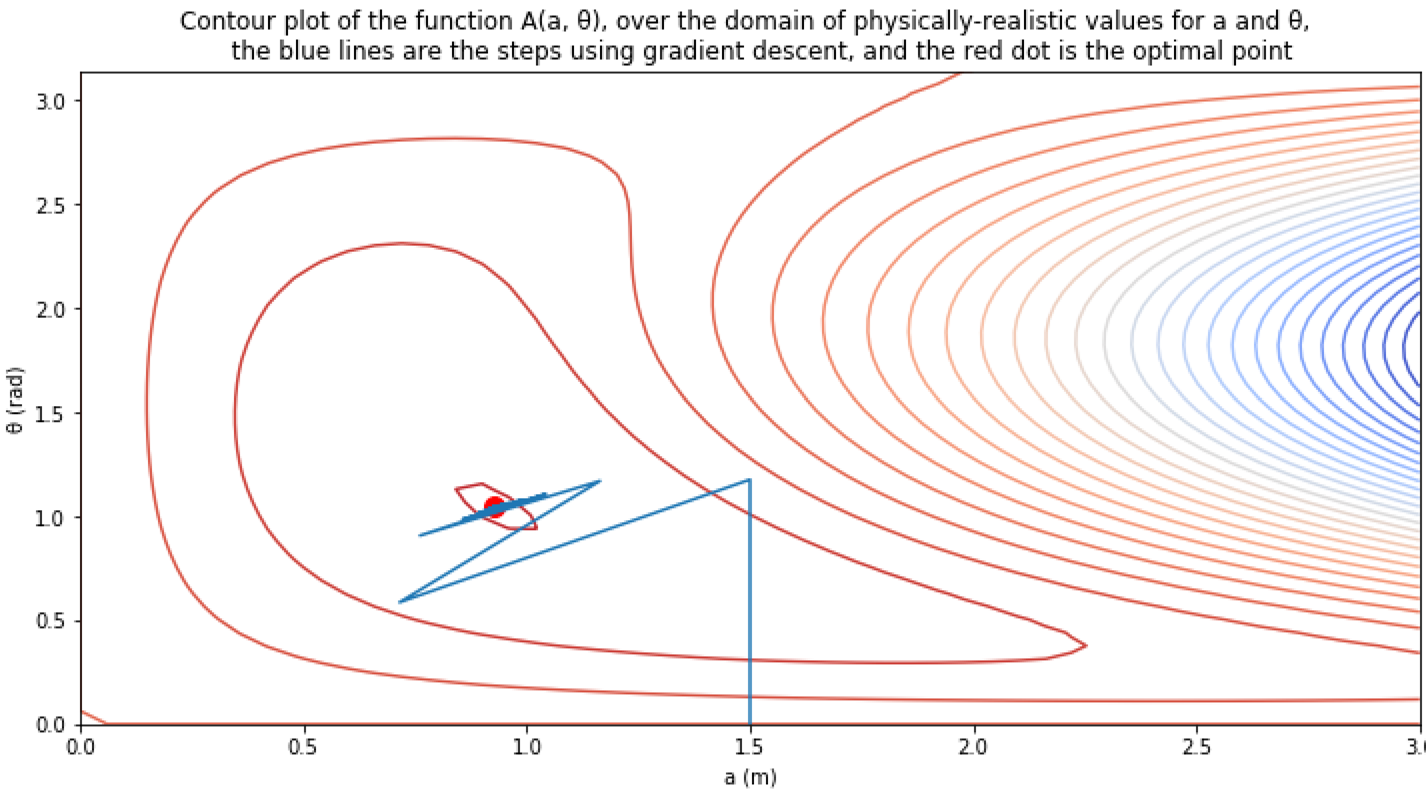
```
# backtracking line search
def backtrack(x0, alpha = 0.1, beta = 0.9):
  #initial step size = 1
  s = 1
  while True:
    # descent direction
    v = -neg_gradf(x0)
```

```
    theta = np.dot(neg_gradf(x0).T, v)
    # terminate condition
    if neg_f(x0 + s*v) <= neg_f(x0) + s*alpha*theta:
      break
    else:
      s = beta*s
  return s
```

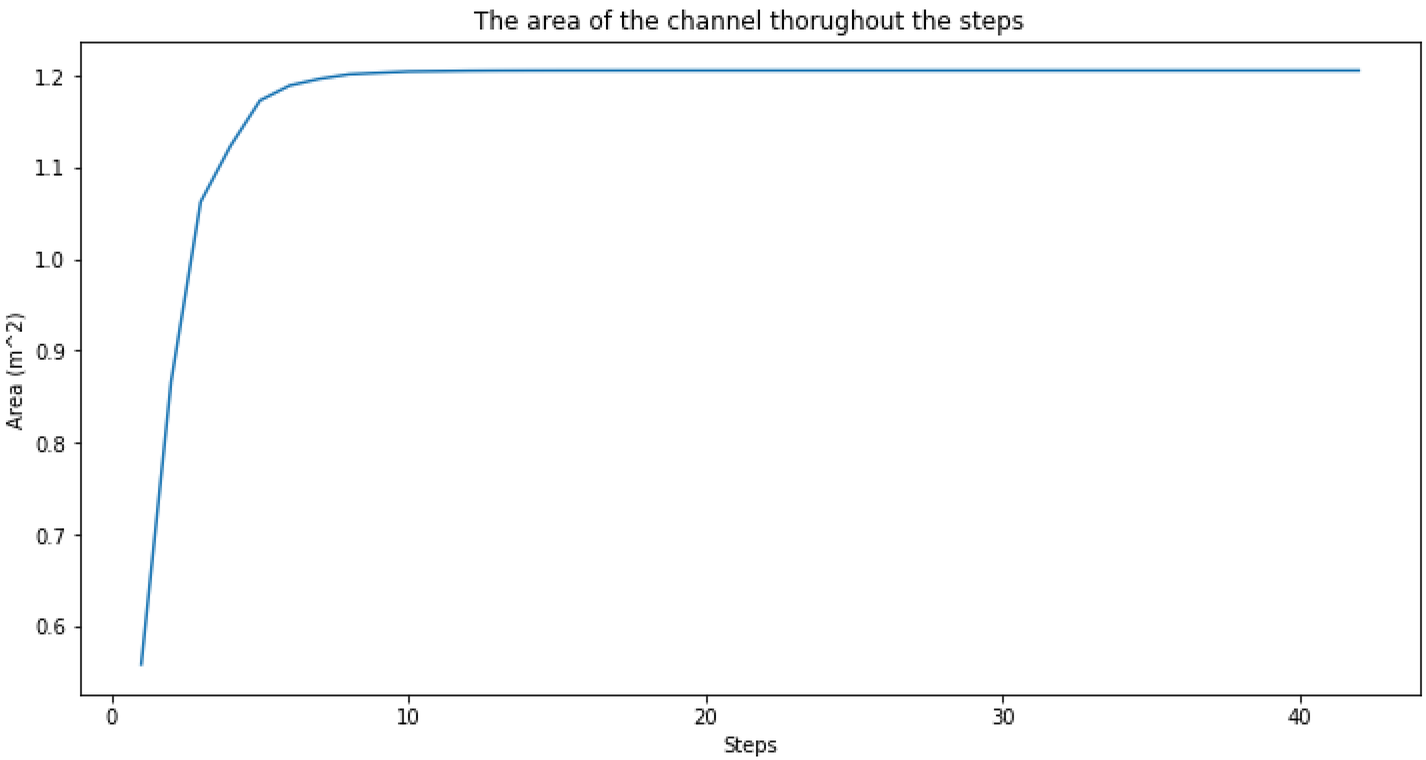
```
# initial point
X = np.array([1.5,0])
# lists to keep track of the values
z_axis = []
past = []
steps = []
for i in range(10000):
  # finding optimal step size
  step = backtrack(X)
  steps.append(step)
  # updating values
  X_new = X - step*neg_gradf(X)
  # termination condition: changes in value is small enough
  if np.linalg.norm(np.array(X) - np.array(X_new)) <= 1e-6:
    print("GD terminate: ", i+1)
    print("Current node:", X_new, ". Value: ", f(X_new))
    break
  if i % 10 == 0:
    print("Iterations", i, ". Current node:", X, ". Value: ", f(X))
  past.append(X)
  X = X_new
  z_axis.append(f(X))
```

```
Iterations 0 . Current node: [1.5 0. ] . Value:  0.0
Iterations 10 . Current node: [0.90976936 1.03104415] . Value:  1.204655698385919
Iterations 20 . Current node: [0.9280385  1.04684741] . Value:  1.2057711464305945
Iterations 30 . Current node: [0.92819588 1.04717957] . Value:  1.2057713654156008
Iterations 40 . Current node: [0.92820291 1.04719662] . Value:  1.205771365938795
GD terminate:  43
Current node: [0.92820364 1.04719753] . Value:  1.2057713659398266
```

```
# the lit of steps
coor_step = np.array(past).T

X, Y = np.meshgrid(np.linspace(0,3,51), np.linspace(0, np.pi, 51))
Z = f([X,Y])

# plot
plt.figure(figsize=(12, 6))
plt.contour(X, Y, Z, 30, cmap='coolwarm')
plt.plot(4*3**(1/2)-6, np.pi/3,  'bo', color = "red", markersize=10)
# plot the coordinate of the steps
plt.plot(coor_step[0], coor_step[1])
plt.xlabel("a (m)")
plt.ylabel("θ (rad)")
plt.title("Contour plot of the function A(a, θ), over the domain of physically-realistic values for a and θ, \n \
  the blue lines are the steps using gradient descent, and the red dot is the optimal point")
plt.show()
```

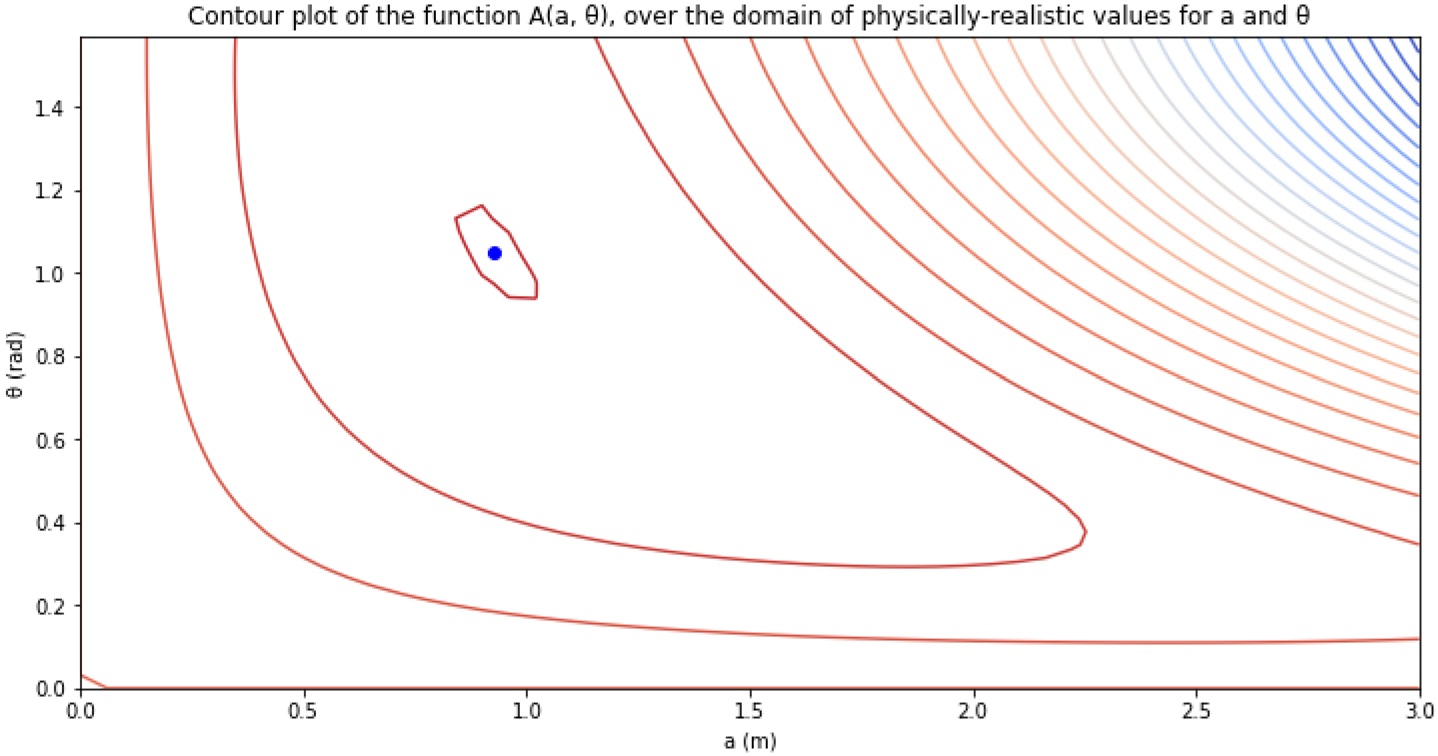



```
plt.figure(figsize=(12, 6))
plt.plot(list(range(1, i+1)), z_axis)
plt.xlabel("Steps")
plt.ylabel("Area (m^2)")
plt.title("The area of the channel thorughout the steps")
plt.show()
```



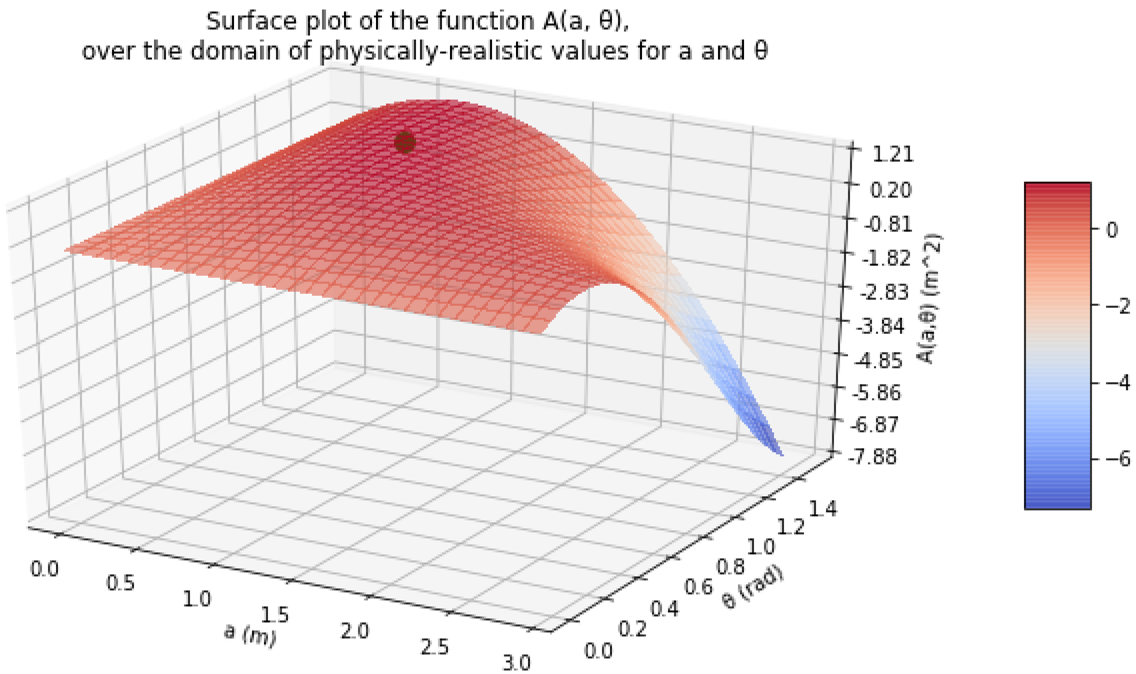
Part 1:

```
X, Y = np.meshgrid(np.linspace(0,3,51), np.linspace(0, np.pi/2, 51))
Z = f([X,Y])

# plot
plt.figure(figsize=(12, 6))
plt.contour(X, Y, Z, 30, cmap='coolwarm')
plt.plot(4*3**(1/2)-6, np.pi/3,  'bo')
plt.xlabel("a (m)")
plt.ylabel("θ (rad)")
plt.title("Contour plot of the function A(a, θ), over the domain of physically-realistic values for a and θ")
plt.show()
```



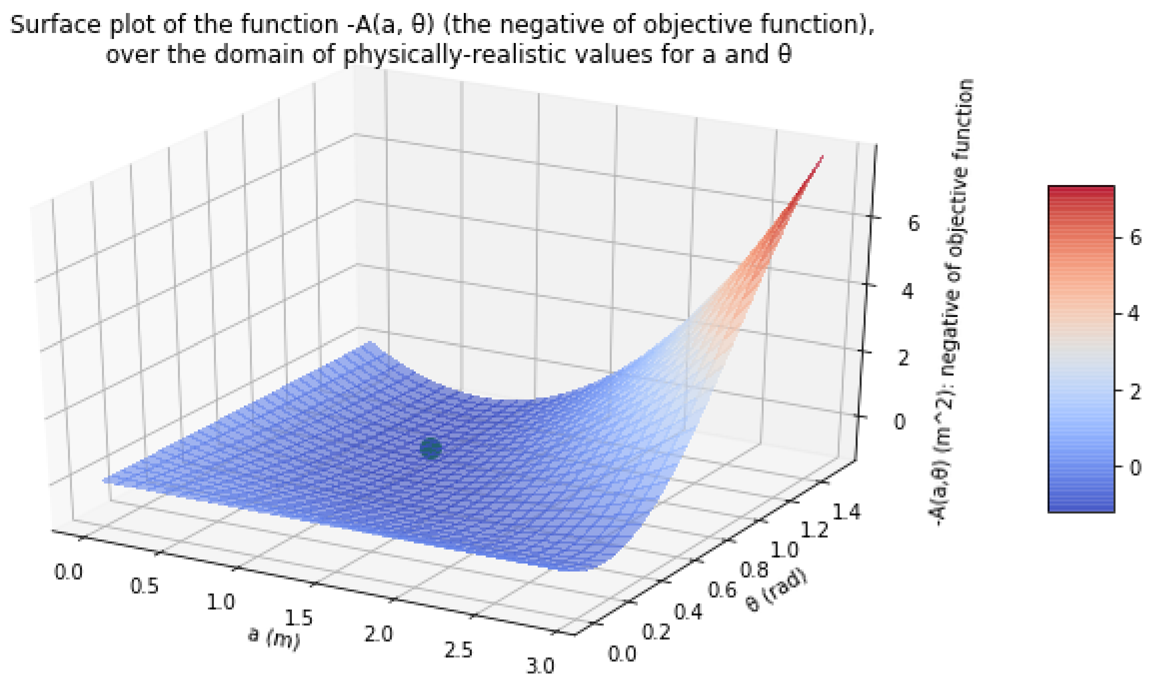

```
'''
=====================
3D surface (color map)
=====================

Demonstrates plotting a 3D surface colored with the coolwarm color map.
The surface is made opaque by using antialiased=False.

Also demonstrates using the LinearLocator and custom formatting for the
z axis tick labels.
'''

fig = plt.figure(figsize=(12, 6))
ax = fig.gca(projection='3d')

# Make data.
```

```
X = np.arange(0, 3, 0.1)
Y = np.arange(0, np.pi/2, np.pi/60)
X, Y = np.meshgrid(X, Y)
Z = f([X,Y])

# Plot the surface.
surf = ax.plot_surface(X, Y, Z, cmap=cm.coolwarm,
                       linewidth=0, antialiased=False, alpha = 0.6)

# Customize the z axis.
# ax.set_zlim(0, 1.5)
ax.zaxis.set_major_locator(LinearLocator(10))
ax.zaxis.set_major_formatter(FormatStrFormatter('%.02f'))

ax.plot([4*3**(1/2) - 6], [np.pi/3], [f([4*3**(1/2) - 6, np.pi/3])], markerfacecolor='g', markeredgecolor='g', marker='o', markersi

# Add a color bar which maps values to colors.
fig.colorbar(surf, shrink=0.5, aspect=5)
ax.set_xlabel('a (m)')
ax.set_ylabel('θ (rad)')
ax.set_zlabel('A(a,θ) (m^2)')
ax.set_title("Surface plot of the function A(a, θ), \n over the domain of physically-realistic values for a and θ")
plt.show()

```



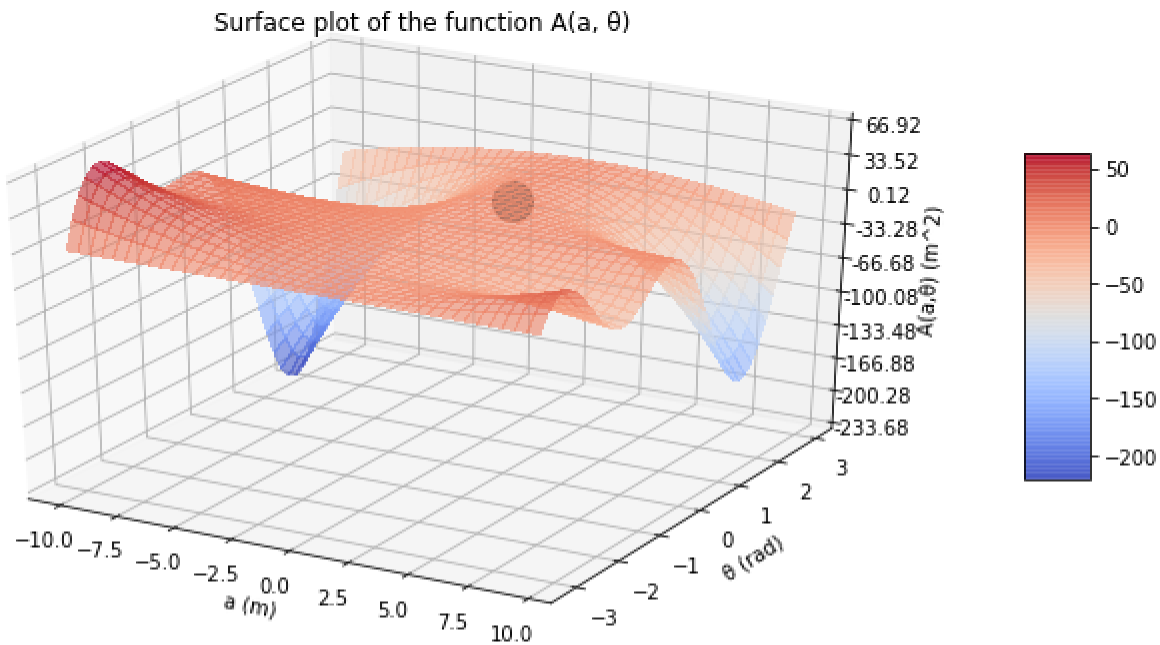

```

fig = plt.figure(figsize=(12, 6))
ax = fig.gca(projection='3d')

# Plot the surface.
surf = ax.plot_surface(X, Y, -Z, cmap=cm.coolwarm,
                       linewidth=0, antialiased=False, alpha = 0.6)

# Customize the z axis.
# ax.set_zlim(0, 1.5)
#ax.zaxis.set_major_locator(LinearLocator(10))
#ax.zaxis.set_major_formatter(FormatStrFormatter('%.02f'))

ax.plot([4*3**(1/2) - 6], [np.pi/3], [-f([4*3**(1/2) - 6, np.pi/3])], markerfacecolor='g', markeredgecolor='g', marker='o', markers

# Add a color bar which maps values to colors.
fig.colorbar(surf, shrink=0.5, aspect=5)
ax.set_xlabel('a (m)')
```

```
ax.set_ylabel('θ (rad)')
ax.set_zlabel('-A(a,θ) (m^2): negative of objective function')
ax.set_title("Surface plot of the function -A(a, θ) (the negative of objective function), \n over the domain of physically-realisti
plt.show()

```




```
fig = plt.figure(figsize=(12, 6))
ax = fig.gca(projection='3d')

# Make data.
X = np.arange(-10, 10, 0.5)
Y = np.arange(-np.pi, np.pi, np.pi/20)
X, Y = np.meshgrid(X, Y)
Z = f([X,Y])

# Plot the surface.
surf = ax.plot_surface(X, Y, Z, cmap=cm.coolwarm,
                       linewidth=0, antialiased=False, alpha = 0.6)

```

```
# Customize the z axis.
# ax.set_zlim(0, 1.5)
ax.zaxis.set_major_locator(LinearLocator(10))
ax.zaxis.set_major_formatter(FormatStrFormatter('%.02f'))

ax.plot([4*3**(1/2) - 6], [np.pi/3], [f([4*3**(1/2) - 6, np.pi/3])], markerfacecolor='k', markeredgecolor='k', marker='o', markersi

# Add a color bar which maps values to colors.
fig.colorbar(surf, shrink=0.5, aspect=5)
ax.set_xlabel('a (m)')
ax.set_ylabel('θ (rad)')
ax.set_zlabel('A(a,θ) (m^2)')
ax.set_title("Surface plot of the function A(a, θ)")
plt.show()
```



For fun: implement newton's method

```
def gradff(X):
    x = X[0]
```

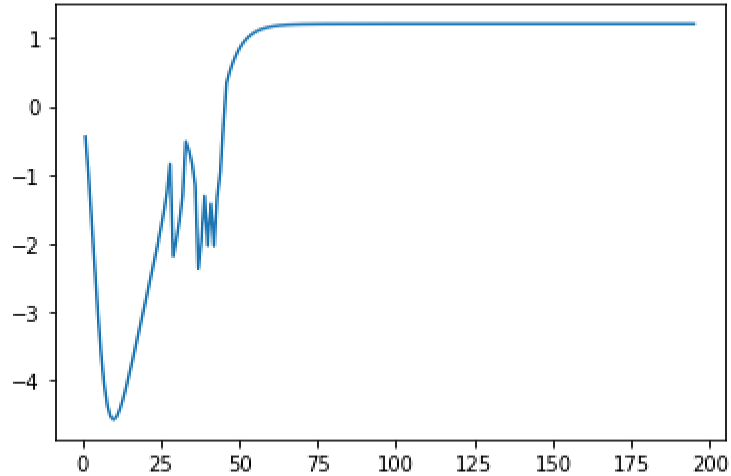
```
  y = X[0]
  return np.array([[-2*np.sin(y) - 2*np.sin(y)**2 + np.sin(y)*np.cos(y),
                  3*np.cos(y) - 2*x*np.cos(y) - 4*x*np.sin(y)*np.cos(y) + x*(np.cos(y)**2 - np.sin(y)**2)],
                 [3*np.cos(y) - 2*x*np.cos(y) - 4*x*np.sin(y)*np.cos(y) + x*(np.cos(y)**2 - np.sin(y)**2),
                  -3*x*np.sin(y) + x**2*np.sin(y) - 2*x**2*(np.cos(y)**2-np.sin(y)**2) - 2*x**2*np.cos(y)*np.sin(y)]])

def neg_gradff(X):
  return -gradff(X)
```

```
X = np.array([1.5,0]) # Starting state
step = 0.1
past_newt = []
z_axis = []
for i in range(0,10000):
    X_newt = X - step*np.linalg.inv(neg_gradff(X)).dot(neg_gradf(X))
    if np.linalg.norm(np.array(X) - np.array(X_newt)) <= 1e-6:
        print("Newton terminate: ", i+1)
        break
    X = X_newt
    past_newt.append(X)
    z_axis.append(f(X))
```

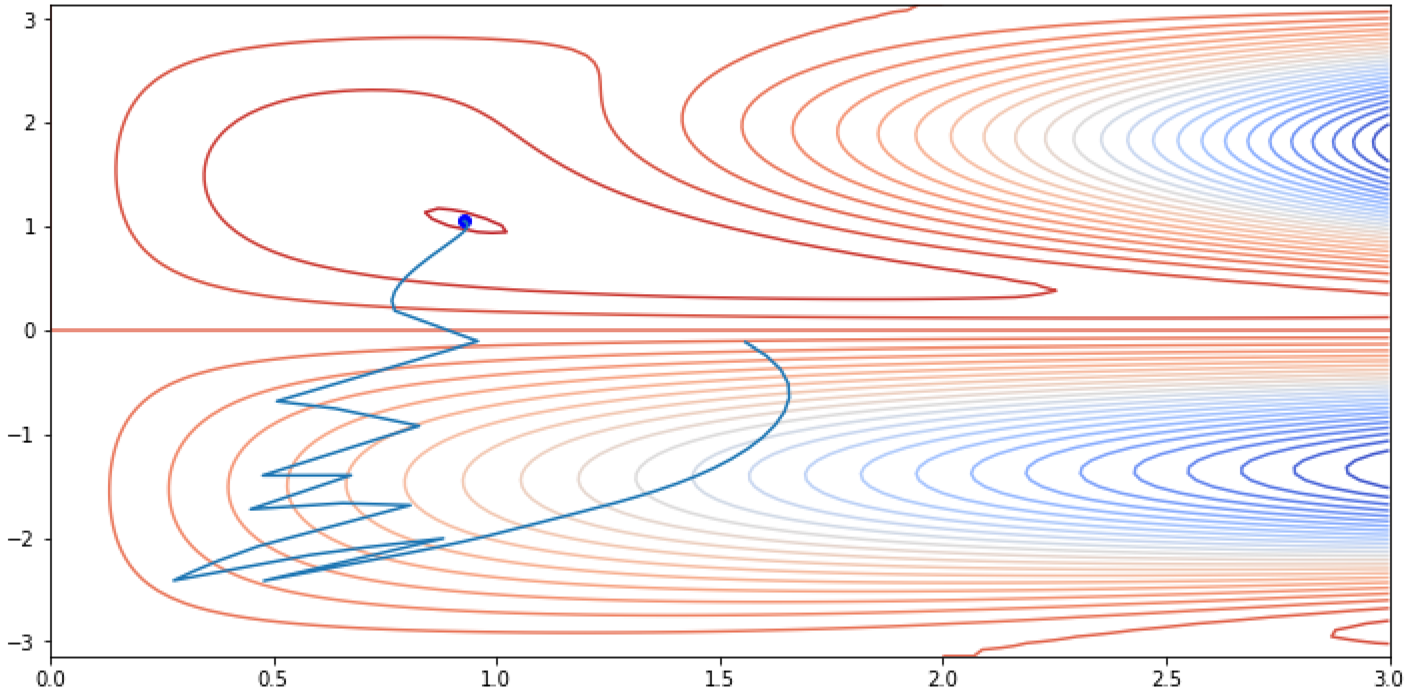
```
Newton terminate:  196
```

```
plt.plot(list(range(1, i+1)), z_axis)
```

```
[<matplotlib.lines.Line2D at 0x7f1be9411a90>]
```




```python
coor_step = np.array(past_newt).T

X, Y = np.meshgrid(np.linspace(0,3,101), np.linspace(-np.pi, np.pi, 101))
Z = f([X,Y])

# plot
plt.figure(figsize=(12, 6))
plt.contour(X, Y, Z, 30, cmap='coolwarm')
plt.plot(4*3**(1/2)-6, np.pi/3,  'bo')
plt.plot(coor_step[0], coor_step[1])
plt.show()
```


3
2
1
0
−1
−2
−3
0.0
0.5
1.0
1.5
2.0
2.5
3.0