Probability
First lecture: Probability space

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Content

Motivation: Deterministic vs Random

- Consider the following ODE:

$$\begin{cases} y'(t) = ky(t) \\ y(0) = T \end{cases}$$

What is the meaning of this equation?

Motivation: Deterministic vs Random

- Consider the following ODE:

$$\begin{cases} y'(t) = ky(t) \\ y(0) = T \end{cases}$$

+ growth of population.
+ carbon C14.
- ODE, PDE: describe the natural phenomenon: fluid, air,....

17 Equations That Changed The World

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1.	Pythagora's Theorem	$a^2 + b^2 = c^2$	Pythagoras, 530 BC
2.	Logarithms	$\log xy = \log x + \log y$	John Napier, 1610
3.	Calculus	$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	Newton, 1668
4.	Law of Gravity	$F = G\frac{m_1 m_2}{d^2}$	Newton, 1687
5.	The Square Root of Minus One	$i^2 = -1$	Euler, 1750
6.	Euler's Formula for Polyhedra	$F - E + V = 2$	Euler, 1750
7.	Normal Distribution	$\Phi(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	C.F. Gauss, 1810

17 Equations That Changed The World

Wave Equation	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	J. d'Almbert, 1746
Fourier Transform	$f(\omega) = \int_{\infty}^{\infty} f(x) e^{-2\pi i x \omega} \mathrm{d}\, x$	J. Fourier, 1822
Navier-Stokes Equation	$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{T} + \mathbf{f}$	C. Navier, G. Stokes, 1845
Maxwell's Equations	$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$ $\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial E}{\partial t}$	J.C. Maxwell, 1865
Second Law of Thermodynamics	$\mathrm{d}S \geq 0$	L. Boltzmann, 1874
Relativity	$E = mc^2$	Einstein, 1905
Schrodinger's Equation	$i\hbar \frac{\partial}{\partial t} \Psi = H \Psi$	E. Schrodinger, 1927
Information Theory	$H = -\sum p(x) \log p(x)$	C. Shannon, 1949
Chaos Theory	$x_{t+1} = k x_t (1 - x_t)$	Robert May, 1975
Black-Scholes Equation	$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0$	F. Black, M. Scholes, 1990

Randomness

- Randomness: one input, many outcomes.
- Aim: to describe and control the randomness.

+ How many outcomes are there?
+ How odd does an property appears?

Recall of Set theory

- A set is a collection of elements, $A = \{x_1, \ldots, x_n\}$.
- Subset $B \subset A$, property, condition.
- Complement: $\overline{B}$ or B^c.
- Union: $B \cup C$.
- Intersection: $B \cap C$.
- A and B are disjoint if $A \cap B = \emptyset$.
- Difference $A \setminus B = A \cap B^c$.
- De Morgan Law

$$(A_1 \cap \ldots \cap A_n)^c = A_1^c \cup \ldots \cup A_n^c,$$

$$(A_1 \cup \ldots \cup A_n)^c = A_1^c \cap \ldots \cap A_n^c.$$

1 Motivation

2 Probability space

3 Nice examples

Sample space

Definition

A sample space Ω is the set of all possible outcomes from an experiment.

- Discrete outcomes

- Coin flip: $\Omega = \{Head, Tail\}$.
- Flip two coins: $\Omega = \{HH, HT, TH, TT\}$.
- Throw a die (dice): $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Paper/scissor/stone: $\Omega = \{paper, scissor, stone\}$.
- Vietlott 6/45 or 5/45.

- Continuous outcomes

- Draw a number: $\Omega = [0, 1]$.
- Waiting time for a bus $\Omega = [0, 30]$ (minutes).

Exercises

- Exer 1: There are 8 processors on a computer. A computer job scheduler chooses one processor randomly. What is the sample space? If the computer job scheduler can choose two processors at once, what is the sample space then?
- Exer 2: A cell phone tower has a circular average coverage area of radius of 10 km. We observe the source locations of calls received by the tower. What is the sample space of all possible source locations?

Exercises

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$$\Omega_1 = \{1, 2, \ldots, 8\}.$$

$$\Omega_2 = \{(1,2), (1,3), \ldots, (7,8)\}.$$

- Exer 2: A cell phone tower has a circular average coverage area of radius of 10 km. We observe the source locations of calls received by the tower. What is the sample space of all possible source locations?

$$\Omega = \{(x, y) \mid (x-a)^2 + (y-b)^2 \leq 10^2\}.$$

Events and Probability

Definition

An event E is a subset of the sample space Ω, (satisfying property E). The set of all possible events is denoted by $\mathcal{F}$.

- Examples. Throw a die.

- $E_1 = \{\text{even numbers}\} = \{2, 4, 6\}$.
- $E_2 = \{\text{odd numbers}\} = \{1, 3, 5\}$.
- $E_3 = \{\text{prime numbers}\} = \{2, 3, 5\}$.
- $E_4 = \{\text{odd prime numbers}\} = \{3, 5\} = E_2 \cap E_3$.

- Examples. Waiting time.

- $E_1 = \{\text{less than 10 minutes}\} = [0, 10)$.
- $E_2 = \{\text{at most 10 minutes}\} = [0, 10]$.
- $E_3 = \{\text{at least 10 minutes}\} = [10, 30]$.
- $E_4 = \{\text{more than 10 minutes}\} = (10, 30]$.

Probability: Classical definition

Pierre de Fermat and Blaise Pascal in 1654.

Definition

If the sample space Ω is finite and the outcomes are uniform, then the probability of an event E is

$$\mathbf{P}(E) = \frac{|E|}{|\Omega|}.$$

- Examples. Throw a die.

- even numbers. $\mathbf{P}(E_1) = 3/6 = 1/2.$
- odd numbers. $\mathbf{P}(E_2) = 3/6 = 1/2.$
- prime numbers. $\mathbf{P}(E_3) = 3/6 = 1/2.$
- odd prime numbers. $\mathbf{P}(E_4) = 2/6 = 1/3.$

Exercises

- There are 8 processors with labels 1,2,...,8 on a computer. A computer job scheduler chooses two processors at once randomly. What is the probability that the labels of the two chosen processors are not consecutive?

- Winning jackpot probability of Vietlott 6/45 and 6/55?

Probability for infinite sample space

- Examples. Waiting time.

- $E_1 = \{\text{less than 10 minutes}\} = [0, 10)$. $\Rightarrow \mathbf{P}(E_1) = 10/30 = 1/3$.
- $E_2 = \{\text{at most 10 minutes}\} = [0, 10]$. $\Rightarrow \mathbf{P}(E_2) = 10/30 = 1/3$.
- $E_3 = \{\text{at least 10 minutes}\} = [10, 30]$. $\Rightarrow \mathbf{P}(E_3) = 20/30 = 2/3$.
- $E_4 = \{\text{more than 10 minutes}\} = (10, 30]$. $\Rightarrow \mathbf{P}(E_4) = 20/30 = 2/3$.

- A cell phone tower has a circular average coverage area of radius 10 km. We observe the source locations of calls received by the tower. What is the probability of the event when the source location of a call is within 5 km from the tower?

$$\mathbf{P}(E) = \frac{\pi 5^2}{\pi 10^2} = 1/4.$$

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By Kolmogorov definition

Definition

A probability space is a triplet $(\Omega, \mathcal{F}, \mathbf{P})$, where

- Ω is the sample space.
- $\mathcal{F}$ is a collection of subsets of Ω such that $\mathcal{F}$ is a σ-field. An element A in $\mathcal{F}$ is called an event.
- $\mathbf{P}$ is a probability measure, i.e. $\mathbf{P}: \mathcal{F} \to [0,1]$.

$\mathcal{F}$ is a σ-field if it satisfies the following conditions:

1. If $A \in \mathcal{F}$ then $A^c = \in \mathcal{F}$;
2. For any sequence of events $A_1, \ldots, A_n \in \mathcal{F}$, we have $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Properties of probability measure

1. $\mathbf{P}(\emptyset) = 0$;
2. $\mathbf{P}(A^c) = 1 - \mathbf{P}(A)$;
3. If $A, B \in \mathcal{F}$ and $A \subset B$ then $\mathbf{P}(A) \leq \mathbf{P}(B)$;
4. If A and B are disjoint, then $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$;
5. In general, $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$;

Inclusion-exclusion principle

Theorem

We have

$$\mathbf{P}(A_1 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbf{P}(A_i) - \sum_{i<j} \mathbf{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbf{P}(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1}\mathbf{P}(A_1 \cap A_2 \cap \ldots A_n).$$

Example: One randomly puts n letters into n envelopes. Find the probability that none letter is in its right envelope.

Hint: - let A_i be the event the i-th letter is in the right envelope (the i-th envelope). $\mathbf{P}(A_i) =$?

- what is the meaning of the event $A_1 \cup A_2 \cup \ldots \cup A_n$?
- what is the meaning of the event $A_{i_1} \cap \ldots A_{i_k}$?

Monty hall problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats.
You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat.
He then says to you, "Do you want to pick door No. 2?"
Is it to your advantage to switch your choice?

Behind door 1	Behind door 2	Behind door 3	Result if staying at door #1	Result if switching to the door offered
Goat	Goat	**Car**	Wins goat	**Wins car**
Goat	**Car**	Goat	Wins goat	**Wins car**
Car	Goat	Goat	**Wins car**	Wins goat

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Stick break

You have a stick. You randomly break it into three small parts. What is the probability that these three parts can form a triangle?

Solution.

Consider the stick of unit length.

$$\Omega = \{(x, y, z) \mid 0 \le x, y, z \le 1,\ x + y + z = 1\}.$$

$$E = \{(x, y, z) \mid 0 \le x, y, z \le 1/2,\ x + y + z = 1\}.$$

proba$= 1/4$.

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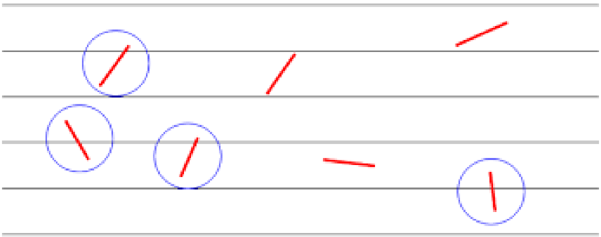
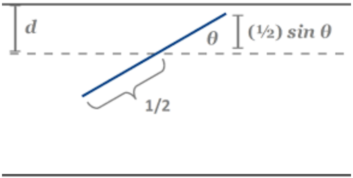
Buffon needlet

Suppose we have a floor made of parallel strips of wood of width h. We randomly drop a needlet of length $l \leq h$ onto the floor. What is the probability that the needle will lie across a line among the strips?



Buffon needlet



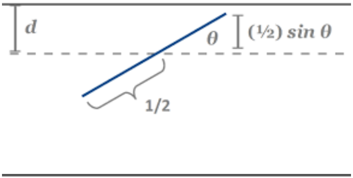
$$\Omega = \{(d,\theta) : \; d \in [0, h/2], \theta \in [0,\pi]\}.$$

$$E = \{(d,\theta) : \; d \in [0, h/2], \theta \in [0,\pi], d \leq \frac{l}{2}\sin\theta\}.$$

Then the probability is

$$\frac{1}{h\pi/2}\int_0^{\pi}\left(\int_0^{\frac{l}{2}\sin\theta} dx\right) d\theta = \frac{2l}{h\pi}.$$

Buffon needlet



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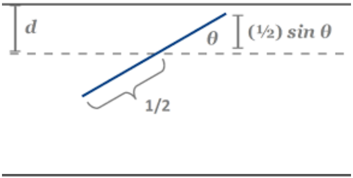

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Homework

+ Stanley Chan: Chapter2. 1, 5-16.
+ Ngo Hoang Long: Chapter1. 9, 10, 27, 20, 21, 23, 30, 31, 32.