Probability
Lecture 3: Discrete Random Variables

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Content

Discrete Random Variables

- Recall that: $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space with Ω: set of all possible outcomes.
- Observe that: we are also interested in the value of the outcomes.
+ For example: lottery (loto) game.

Definition (Intuitive definition)

A random variable is a function $X : \Omega \to \mathbb{R}$.

Definition (Precise definition)

A random variable is a measurable function $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, i.e. for any $a \in \mathbb{R}$,

$$\{\omega \in \Omega : X(\omega) \leq a\} \in \mathcal{F}.$$

- Examples: height, weight, salary, grade mark, ...

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A random variable X is called discrete if the set of possible values is finite or countable.

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Let the set of possible values be $\{x_1, \ldots, x_n, \ldots\}$. The distribution of the discrete random variable X is given through the **distribution table**

X	x_1	x_2	$\ldots$	x_n	$\ldots$
$\mathbb{P}$	p_1	p_2	$\ldots$	p_n	$\ldots$

where $p_i = \mathbb{P}(X = x_i)$. Hence we can define the **probability mass function (PMF)** p_X.

Theorem

We have $\sum_{x \in X(\Omega)} p_X(x) = 1.$

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Examples

- Vietlot $6/45$

X	$0k$	$30k$	$300k$	$10m$	$40b$
$\mathbb{P}$	$\dots$	$\frac{C_6^3 C_{39}^3}{C_{45}^6}$	$\frac{C_6^4 C_{39}^2}{C_{45}^6}$	$\frac{C_6^5 C_{39}^1}{C_{45}^6}$	$\frac{1}{C_{45}^6}$

- Throw a dice twice. Let X be the max of two times. Find the distribution of X.

X	1	2	3	4	5	6
$\mathbb{P}$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

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Examples

Consider a game in which we flip a coin 3 times. The reward of the game is
\$1 if there are 2 heads
\$8 if there are 3 heads
\$0 if there are 0 or 1 head.
Find the distribution of the reward X.

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Property

Theorem

We have

$$\sum_{x \in X(\Omega)} p_X(x) = 1.$$

- Example: let $p_X(k) = \frac{c}{2^k}$ for $k = 1, 2, \ldots$. Find the value of c.

$$c \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \quad \Rightarrow \quad c = 1.$$

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Distribution table and Histogram

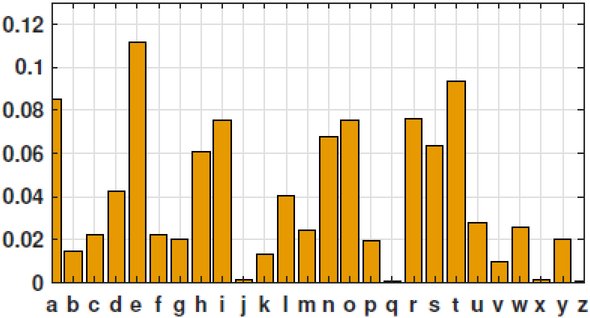
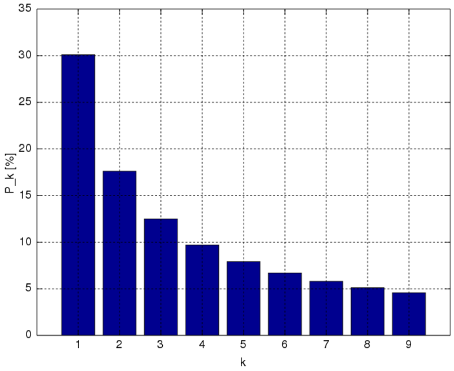


Figure: The frequency of the 26 English letters. Data source: Wikipedia.

Benford's law

- A set of numbers is said to satisfy Benford's law if the leading digit $d \in \{1, ..., 9\}$ occurs with probability

$$p_X(d) = \log_{10}(d+1) - \log_{10}(d) = \log_{10}\left(\frac{d+1}{d}\right).$$



Expectation: Motivation

- Consider the statistics of Probability marks of a class of 40 students. What is the average (mean) mark?

$$\frac{a_1 + a_2 + \ldots + a_{40}}{40} = \frac{1 \times f_1 + 2 \times f_2 + \ldots + 10 \times f_{10}}{40},$$

where f_i the frequency of the mark i.
- Draw randomly a student in a class and let X be the mark of him/her.

X	1	2	$\ldots$	10
$\mathbb{P}$	$\frac{f_1}{40}$	$\frac{f_2}{40}$		$\frac{f_{10}}{40}$

$$Mean = 1 \times \frac{f_1}{40} + 2 \times \frac{f_2}{40} + \ldots + 10 \times \frac{f_{10}}{40}.$$

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The expectation of X, denoted as $\mathbb{E}(X)$ is defined as

$$\mathbb{E}(X) = \sum_{x_i \in X(\Omega)} x_i p_i.$$

- Example: Let X be a random variable with PMF $p_X(0) = 1/4$, $p_X(1) = 1/2$ and $p_X(2) = 1/4$. Find $\mathbb{E}(X)$.

$$\mathbb{E}(X) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1.$$

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- Flip an unfair coin, where the probability of getting a head is 3/4. Let X be a random variable such that $X = 1$ means getting a head and $X = 0$ otherwise. Find $\mathbb{E}(X)$.

$$\mathbb{E}(X) = 1.\frac{3}{4} + 0.\frac{1}{4} = \frac{3}{4}.$$

- Let $p_X(k) = \frac{1}{2^k}$ for $k = 1, 2, \ldots$. Find $\mathbb{E}(X)$.

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{k}{2^{k-1}} = \frac{1}{2}\left(\frac{1}{1-r}\right)' |_{r=1/2} = 2.$$

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Existence of expectation

- Let $p_X(k) = \frac{6}{\pi^2 k^2}$ for $k = 1, 2, \ldots$.

$$\mathbb{E}(X) = \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k} \to \infty.$$

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The discrete random variable X is called **absolutely summable** if

$$\mathbb{E}(|X|) = \sum_{x_i \in X(\Omega)} |x_i| p_i < \infty.$$

We say that X has an expectation if it is absolutely summable.

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Properties of expectation

Theorem

The expectation of a random variable X has the following properties:

(i). **Function**. *For any function g,*

$$\mathbb{E}[g(X)] = \sum g(x_i)p_i.$$

(ii). **Linearity**. *For any function g and h,*

$$\mathbb{E}[g(X) + h(X)] = \mathbb{E}[g(X)] + \mathbb{E}[h(X)].$$

(iii). **Scale**. *For any constant c,*

$$\mathbb{E}[cX] = c\mathbb{E}[X].$$

(iv). **DC Shift**. *For any constant c,*

$$\mathbb{E}[X + c] = \mathbb{E}[X] + c.$$

- Let X be a random variable with four equally probable states $0, 1, 2, 3$. We want to compute the expectation $\mathbb{E}[\cos(\pi X/2)]$.
hint: $g(X) = \cos(\pi X/2)$.

$$\mathbb{E}[\cos(\pi X/2)] = \cos 0.\frac{1}{4} + \cos\frac{\pi}{2}.\frac{1}{4} + \cos\pi.\frac{1}{4} + \cos\frac{3\pi}{2}.\frac{1}{4} = 0.$$

- Let X be a random variable with $\mathbb{E}[X] = 1$ and $\mathbb{E}[X^2] = 3$. We want to find the expectation $\mathbb{E}[(aX+b)^2]$.
hint: $(aX+b)^2 = a^2X^2 + 2abX + b^2$

$$\mathbb{E}[(aX+b)^2] = \mathbb{E}(a^2X^2) + \mathbb{E}(2abX) + \mathbb{E}(b^2) = 3a^2 + 2ab + b^2.$$

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Moment and Variance

Definition

The k-th moment of a discrete random variable X is

$$\mathbb{E}[X^k] = \sum x_i^k p_i.$$

Definition

The variance of a discrete random variable X, denoted by $\mathrm{Var}(X)$ (or $V(X)$, $\mathrm{D}(X)$), is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}X)^2].$$

Meaning: measure the dispersion (variation) around the mean value.

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Example

- Find the k-moment and variance of Bernoulli random variable

X	0	1
$\mathbb{P}$	$1-p$	p

+ $\mathbb{E}[X^k] = 0^k.(1-p) + 1^k.p = p.$
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Properties of variance

Theorem

The variance of a random variable X has the following properties:

(i). **Moment**.

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2.$$

(ii). **Scale**. *For any constant c,*

$$Var(cX) = c^2\, Var(X).$$

(iii). **DC Shift**. *For any constant c,*

$$Var(X + c) = Var(X).$$

Common Discrete Random Variables

- Bernoulli (p).
- Binomial (n, p).
- Geometric (p).
- Poisson (λ).

Bernoulli Random Variables

Definition

Let X be a Bernoulli random variable. Then, the PMF of X is

$$p_X(0) = 1 - p, p_X(1) = p,$$

where $0 < p < 1$ is called the Bernoulli parameter. We write

$$X \sim Bernoulli(p)$$

to say that X is drawn from a Bernoulli distribution with a parameter p.

- Idea: do an experimental trial with success proba p and failure proba $1 - p$.
- For $p = 1/2$, flip a fair coin.
- $\mathbb{E}X = \mathbb{E}[X^2] = p$ and $\mathrm{Var}(X) = p(1-p)$.
- Rademacher random variable $p_X(1) = p_X(-1) = 1/2$.

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Erdos-Renyi random graph

- The Erdos-Renyi graph model says that the probability of getting an edge is an independent Bernoulli random variable.

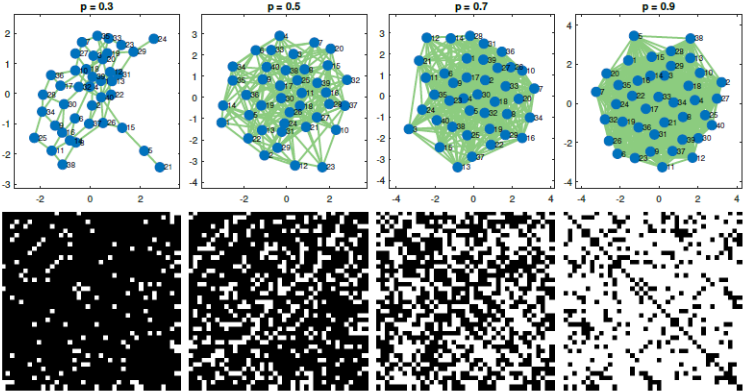
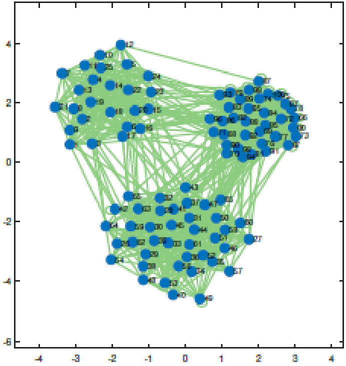


Figure: Simulations of Erdos-Renyi graphs and corresponding adjacency matrices.

Erdos-Renyi random graph



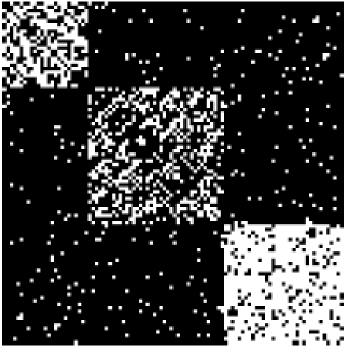


Figure: Cluster model.

Binomial Random Variables

Definition

Let X be a Binomial random variable. Then, the PMF of X is

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

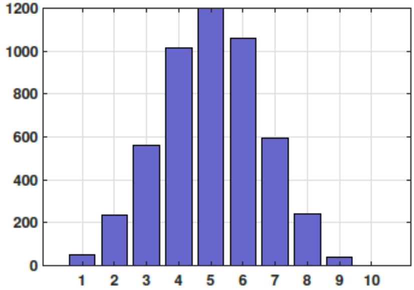
where $0 < p < 1$ is the binomial parameter, and n is the total number of states. We write

$$X \sim B(n, p)$$

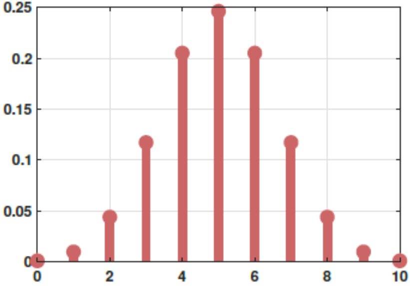
to say that X is drawn from a binomial distribution with a parameter p of size n.

- Idea: represent number of success trials in n independent experimental trials.
- For $n = 1$, Bernoulli random variable.

Binomial Random Variables



(a) Histogram based on 5000 samples



(b) PMF

Figure: An example of a binomial distribution with $n = 10, p = 0.5$.

Binomial Random Variables

Theorem

Let $X \sim B(n,p)$ with distribution

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

Then

$$\mathbb{E}X = np, \ \mathbb{E}[X^2] = np(np+1-p), \ Var(X) = np(1-p).$$

Binomial Random Variables

$$\mathbb{E}X = \sum_{k=0}^{n} k p_X(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-}$$

$$= np \sum_{i=0}^{n-1} \frac{(n-1)!}{i!(n-1-i)!} p^i (1-p)^{n-1-i} = np[p + (1-p)]^{n-1} = np.$$

$$\mathbb{E}X^2 = \sum_{k=0}^{n} k^2 p_X(k) = \sum_{k=0}^{n} (k^2 - k + k) \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k(k-1) \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} + \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= n(n-1)p^2 + np.$$

Geometric Random Variables

Definition

Let X be a geometric random variable. Then, the PMF of X is

$$p_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots$$

where $0 < p < 1$ is the geometric parameter. We write

$$X \sim \ Geometric(p)$$

to say that X is drawn from a geometric distribution with a parameter p.

-Idea: number of trials until the first success.

Theorem

If $X \sim \ Geometric(p)$, then

$$\mathbb{E}X = \frac{1}{p}, \ \mathbb{E}[X^2] = \frac{2}{p^2} - \frac{1}{p}, \ Var(X) = \frac{1-p}{p^2}.$$

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Geometric Random Variables

$$\mathbb{E}X = \sum_{k=0}^{\infty} k p_X(k) = \sum_{k=0}^{\infty} kp(1-p)^{k-1} = p\left[\sum_{k=0}^{\infty} x^k\right]'_{x=1-p}$$

$$= p\left[\frac{1}{1-x}\right]'_{x=1-p} = p\left[\frac{1}{(1-x)^2}\right]_{x=1-p} = \frac{1}{p}.$$

$$\mathbb{E}X^2 = \sum_{k=0}^{\infty} k^2 p_X(k) = \sum_{k=0}^{\infty} k^2 p(1-p)^{k-1} = \sum_{k=0}^{\infty} [(k^2-k)+k]p(1-p)^{k-1}$$

$$= p(1-p)\left[\sum_{k=0}^{\infty} k(k-1)x^{k-2}\right]_{x=1-p} + p\left[\sum_{k=0}^{\infty} kx^{k-1}\right]_{x=1-p}$$

$$= p(1-p)\left[\frac{1}{1-x}\right]''_{x=1-p} + p\left[\frac{1}{(1-x)}\right]'_{x=1-p}.$$

Poisson Random Variables

Definition

. We write

$$X \sim \ Poisson(\lambda)$$

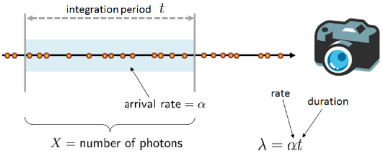
to say that X is drawn from a Poisson distribution with a parameter λ,

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

where λ is the Poisson rate.

- Idea: number of occurrences in a unit of time.
- Number of accidents in a month (week, day). Number of cars on a road during a day. Number clients visiting a supermarket on a day.

Applications in Computational photography



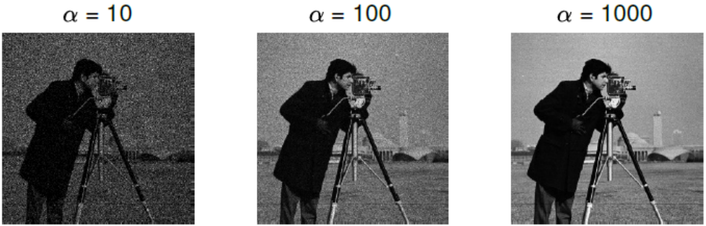
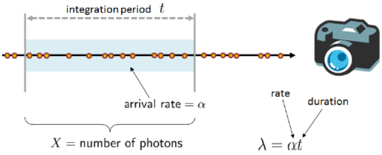


Figure: α denotes the gain level of the sensor: Larger α means that there are more photons coming to the sensor.

Applications in Computational photography



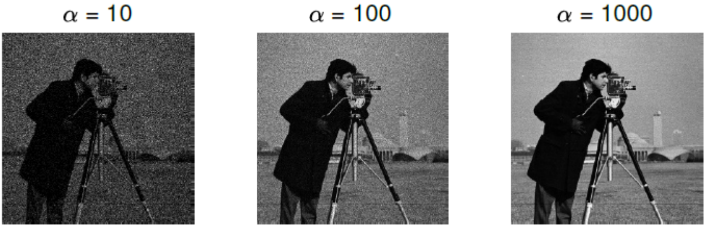


Figure: α denotes the gain level of the sensor: Larger α means that there are more photons coming to the sensor.

Theorem

If $X \sim Poi(\lambda)$, *then*

$$\mathbb{E}X = \lambda,\ \mathbb{E}[X^2] = \lambda + \lambda^2,\ \ Var(X) = \lambda.$$

Poisson Random Variables

$$\mathbb{E}X = \sum_{k=0}^{\infty} k p_X(k) = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!}$$
$$= \lambda \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = \lambda.$$

$$\mathbb{E}X^2 = \sum_{k=0}^{\infty} k^2 p_X(k) = \sum_{k=0}^{\infty} [(k^2 - k) + k] e^{-\lambda} \frac{\lambda^k}{k!}$$
$$= \sum_{k=0}^{\infty} k(k-1) e^{-\lambda} \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!}$$
$$= \lambda^2 + \lambda.$$

Approximation a Binomial distribution by a Poisson distribution

Theorem

For small p and large n such that $np \to \lambda$ as $n \to \infty$, and for a fixed integer k,

$$\binom{n}{k} p^k (1-p)^{n-k} \to e^{-\lambda} \frac{\lambda^k}{k!}.$$

Homework

+ Stanley Chan: Chapter3. 2, 3, 4, 5, 6, 8, 12, 13.
+ Ngo Hoang Long: Chapter1. 4, 6, 10, 16, 19, 20, 26.

Thank you!